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Nonconvex Spectral Optimization Algorithms for reduced-order $\mathcal{H}_\infty$ LPV-LFT controllers

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SUMMARY

A novel sequential semidefinite programming method is developed for optimization subject to rank-constraints on matrix-valued nonlinear functions of semi-definite matrix decision variables, which arise in reduced order LPV-LFT (linear parameter varying linear fractional transformational) control synthesis. The global convergence of the method is easily proved without any step size control. An intensive simulation shows the clear advantage of the proposed method over the state-of-the-art solvers. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: reduced-order LPV-LFT $\mathcal{H}_\infty$ control; matrix-rank constraint; nonlinear programming

1. INTRODUCTION

Rank-constrained optimization is referred to optimization involving rank constraints on matrix-valued functions of the decision variables. Initialized by the pioneering work [1], which reformulates the reduced order $\mathcal{H}_\infty$ control synthesis for linear time invariant (LTI) systems as linear matrix inequality (LMI) optimization subject to a rank constraint on a matrix-valued affine function of the Lyapunov matrix variables, many other important and difficult problems in robust control are also reformulated in similar matrix-rank constrained optimizations [2]. The simplest approach is to relax or just to drop that rank constraints with hope that the optimal solution of the relaxed (convex) optimization would satisfy these matrix-rank constraints. For instance, matrix trace minimization and nuclear norm minimization were proposed to obtain low matrix rank of positive semi-definite matrix and rectangular matrices, respectively [3, 4]. These techniques are unable to address the matrix-rank constraints as they are. Indeed, just a trace of a matrix or its nuclear norm don't give any adequate indication on the matrix rank. Another attempt is to use a Newton-like method to find a projection of a positive semi-definite matrix to the manifold of fixed rank matrices [5, 6], which is equally computationally difficult optimization due to complex geometry of this manifold [7], especially for lower fixed rank matrices of larger size. Realizing the challenge by these matrix-rank constraints on Lyapunov matrix variables, most later developments in robust control preferred to avoid them in favor of alternative bilinear matrix inequality (BMI) [8, 2, 9, 10, 11, 12, 13, 14]. The state-of-the-art BMI solvers [12, 14] initialize from a reduced-order stabilizing controller and then move within a convex feasibility subset containing this initialized point. There are a few difficulties arisen with this kind of feasibility algorithms. Firstly, finding a good reduced-order stabilizing controller is not an easy task because its computation is still an NP-hard problem [15].

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Secondly, the feasibility set of reduced order stabilizing controllers are highly nonconvex, which is disconnected in general. This means moving within a convex neighborhood of such reduced-order stabilizing controller may be trapped by local minima. Thirdly, usually the convergence of this kind of algorithms is slow and dependent very much on the local geometry around such initial point [16], which may be unpredictable. Within the past few years, it has been realized that all BMI solvers [12, 14], which treat the problems in state space with Lyapunov functions, could hardly compete with the nonsmooth optimization solver developed earlier in [17], which directly treats the control synthesis for LTI systems in the frequency domain to bypass the Lyapunov variables of high dimension. Nowadays, the Matlab `systune` command [18], which is based on [17], is the most powerful tool for control synthesis of LTI systems and is widely used in industry. This means that rank-constrained optimization and BMI should seek applications outside uncertain linear time-invariant systems such as linear parameter varying (LPV) systems [19], where Lyapunov function is irreplaceable.

Meanwhile, for solution of indefinite quadratic optimization in signal processing applications, [20, 21, 22] developed an approach for optimization on the rank-one constrained positive semi-definite outer product of decision vector variable. Intensive simulations even for large scale indefinite quadratic optimization [22] show that the rank-one matrices can be quickly located, which are global optimal solutions of the considered indefinite quadratic problems in most cases. Reduced order robust LPV controller synthesis is more difficult than indefinite quadratic programming. The matrix-rank constraints in the former are much more challenging than the rank-one constraint in the latter. Indeed, they are lower fixed rank constraints on matrix-valued affine functions of larger size with very complex geometry. For instance, k -order robust control synthesis for a LPV plant of order n leads to rank- $(n + k)$ constraint on the positive semi-definite matrix-valued affine function of size $(2n) \times (2n)$ [23, 24]. A novel approach proposed in the present paper is to equivalently express these rank- $(n + k)$ constraints on the positive semi-definite matrix-valued affine function by rank- k constraint on the matrix-valued nonlinear function of size n , which are then exactly expressed by spectral nonlinear functions. We then show a simple but effective optimization technique leading to a path-following optimization procedure for these problems. To the author's best knowledge, spectral nonlinear function optimization was not quite considered in the literature.

The paper is organized as follows. After the Introduction, section II is devoted to algorithmic solutions for reduced-order LPV $\mathcal{H}_\infty$ controllers while section III is devoted to static output feedback LPV controllers. An intensive simulation is provided in Section IV to support the algorithmic development of the previous sections. Section V concludes the paper.

Notation. Notation used in this paper is standard. Particularly, $X \succeq 0$, $X \succ 0$, $X \preceq 0$ and $X \prec 0$ mean that a symmetric matrix X is positive semi-definite, positive definite, negative semi-definite and negative definite, respectively, while $\langle X, Y \rangle$ is the dot product of the matrices X and Y . For simplicity, we also denote $\text{tr}(X)$ as the trace of X . I is the identity matrix but when needed we also use I_n to emphasize the size $n \times n$ of I . In symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for terms that are induced by symmetry, e.g.,

$$K \begin{bmatrix} S + S^T & M^T \\ M & Q \end{bmatrix} K^T = K \begin{bmatrix} S + (*) & * \\ M & Q \end{bmatrix} *$$

The matrix variables are typed boldfaced in the paper.

2. DYNAMIC REDUCED ORDER $\mathcal{H}_\infty$ LPV CONTROL SYNTHESIS

Consider a continuous LPV system in linear fractional transformation (LFT) [23, 25, 24]

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ z_\Delta(t) \\ z(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} A & B_\Delta & B_1 & B_2 \\ C_\Delta & D_{\Delta\Delta} & D_{\Delta 1} & D_{\Delta 2} \\ C_1 & D_{1\Delta} & D_{11} & D_{12} \\ C_2 & D_{2\Delta} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w_\Delta(t) \\ w(t) \\ u(t) \end{bmatrix} \\ w_\Delta(t) &= \Delta(\alpha(t))z_\Delta(t) \end{aligned} \quad (1)$$

where

$$\Delta(\alpha(t)) = \sum_{i=1}^L \alpha_i(t) \Delta_i, \alpha_i(t) \geq 0, \sum_{i=1}^L \alpha_i(t) = 1. \quad (2)$$

Here, $x(t) \in R^n$, $y(t) \in R^{n_y}$, $z(t) \in R^{n_z}$, $w(t) \in R^{n_z}$, $z_\Delta(t) \in R^{n_\Delta}$, $w_\Delta(t) \in R^{n_\Delta}$. Note that we assume without loss of generality that $z(t)$ and $w(t)$ ($z_\Delta(t)$ and $w_\Delta(t)$, resp.) have the same dimension. The pair (w_Δ, z_Δ) is regarded as the gain-scheduling channel. All matrices in (1)-(2) are given with appropriate size. Parameters $\alpha_i(t)$ are measured online and exploited by the controller. The standard $\mathcal{H}_\infty$ LPV control design is to find k -order controller in LFT

$$\begin{aligned} \begin{bmatrix} \dot{x}_K(t) \\ u(t) \\ z_K(t) \end{bmatrix} &= \begin{bmatrix} A_K & B_{K1} & B_{K\Delta} \\ C_{K1} & D_{K11} & D_{K1\Delta} \\ C_{K\Delta} & D_{K\Delta 1} & D_{K\Delta\Delta} \end{bmatrix} \begin{bmatrix} x_K(t) \\ y(t) \\ w_K(t) \end{bmatrix} \\ &= \Delta_K(\alpha(t)) z_K(t) \end{aligned} \quad (3)$$

with

$$\Delta_K(\alpha(t)) = \sum_{i=1}^L \alpha_i \Delta_{Ki} \quad (4)$$

such that the closed-loop system is internally stable and satisfies

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|w(t)\|^2 dt \quad \forall w(\cdot) \in L_2, T < +\infty, \quad (5)$$

initialized from $x(0) = 0$. Here $x_K(t) \in R^k$, $z_K(t) \in R^{n_\Delta}$ and $w_K(t) \in R^{n_\Delta}$. k is called the control order and the pair (w_K, z_K) is regarded as the control's gain-scheduling channel.

Note that (1) and (3) are the following LPV LFTs

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} &= \left(\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} + \begin{bmatrix} B_\Delta \\ D_{1\Delta} \\ D_{2\Delta} \end{bmatrix} (I - \Delta(\alpha(t)) D_{\Delta\Delta})^{-1} \Delta(\alpha(t)) \right. \\ &\quad \left. \times \begin{bmatrix} C_\Delta & D_{\Delta 1} & D_{\Delta 2} \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \end{aligned} \quad (6)$$

and

$$\begin{aligned} \begin{bmatrix} \dot{x}_K(t) \\ u(t) \end{bmatrix} &= \left(\begin{bmatrix} A_K & B_{K1} \\ C_{K1} & D_{K11} \end{bmatrix} + \begin{bmatrix} B_{K\Delta} \\ D_{K1\Delta} \end{bmatrix} (I - \Delta_K(\alpha(t)) D_{K\Delta\Delta})^{-1} \Delta_K(\alpha(t)) \right. \\ &\quad \left. \times \begin{bmatrix} C_{K\Delta} & D_{K\Delta 1} \end{bmatrix} \right) \begin{bmatrix} x_K(t) \\ y(t) \end{bmatrix}, \end{aligned} \quad (7)$$

respectively.

Let's state the following result adapted from [24]: the feasibility of the following matrix inequality in $\mathbf{X} \in R^{n \times n}$, $\mathbf{Y} \in R^{n \times n}$, $\mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{Ki}$ and

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{A}_K & \mathbf{B}_{K1} & \mathbf{B}_{K\Delta} \\ \mathbf{C}_{K1} & \mathbf{D}_{K11} & \mathbf{D}_{K1\Delta} \\ \mathbf{C}_{K\Delta} & \mathbf{D}_{K\Delta 1} & \mathbf{D}_{K\Delta\Delta} \end{bmatrix} \quad (8)$$

is sufficient for the existence of such controller

$$\begin{bmatrix} \mathbf{LMI}_1 & * \\ \mathbf{LMI}_2 & \mathbf{LMI}_3 \end{bmatrix} \prec 0, \quad (9)$$

$$\begin{bmatrix} \mathbf{R} & I & \Delta_i^T Q & \Delta_i^T \\ I & \mathbf{H} & \Delta_{\mathbf{K}i}^T & \mathbf{H} \Delta_i^T \\ Q \Delta_i & \Delta_{\mathbf{K}i} & -\mathbf{Q} & -I \\ \Delta_i & \Delta_i \mathbf{H} & -I & -\mathbf{E} \end{bmatrix} \succ 0, \quad i = 1, 2, \dots, L.$$

$$\begin{pmatrix} \mathbf{X} & I_n \\ I_n & \mathbf{Y} \end{pmatrix} \succeq 0, \quad (10)$$

$$\text{rank}(\mathbf{X} - \mathbf{Y}^{-1}) \leq k, \quad (11)$$

where

$$\begin{aligned} \mathbf{LMI}_1 &:= \begin{bmatrix} \mathbf{X}A + \mathbf{B}_{\mathbf{K}1}C_2 + (*) & * & * & * \\ \mathbf{A}_{\mathbf{K}}^T + A + B_2\mathbf{D}_{\mathbf{K}11}C_2 & (A\mathbf{Y} + B_2\mathbf{C}_{\mathbf{K}1}) + (*) & * & * \\ B_{\Delta}^T\mathbf{X} + D_{2\Delta}^T\mathbf{B}_{\mathbf{K}1}^T & B_{\Delta}^T D_{2\Delta}^T \mathbf{D}_{\mathbf{K}11}^T B_2^T & Q & * \\ \mathbf{B}_{\mathbf{K}\Delta}^T & \mathbf{E}B_{\Delta}^T + \mathbf{D}_{\mathbf{K}1\Delta}^T B_2^T & -I & \mathbf{E} \end{bmatrix} \\ \mathbf{LMI}_2 &:= \begin{bmatrix} B_1^T\mathbf{X} + D_{21}^T\mathbf{B}_{\mathbf{K}1}^T & B_1^T + D_{21}^T\mathbf{D}_{\mathbf{K}11}^T B_2^T & 0 & 0 \\ \mathbf{R}C_{\Delta} + \mathbf{D}_{\mathbf{K}\Delta 1}C_2 & \mathbf{C}_{\mathbf{K}\Delta} & \mathbf{R}D_{\Delta\Delta} + \mathbf{D}_{\mathbf{K}\Delta 1}D_{2\Delta} & \mathbf{D}_{\mathbf{K}\Delta\Delta} \\ C_{\Delta} + D_{\Delta 2}\mathbf{D}_{\mathbf{K}11}C_2 & C_{\Delta}\mathbf{Y} + D_{\Delta 2}\mathbf{C}_{\mathbf{K}1} & D_{\Delta\Delta} + D_{\Delta 2}\mathbf{D}_{\mathbf{K}11}D_{2\Delta} & D_{\Delta\Delta}\mathbf{E} + D_{\Delta 2}\mathbf{D}_{\mathbf{K}1\Delta} \\ C_1 + D_{12}\mathbf{D}_{\mathbf{K}11}C_2 & C_1\mathbf{Y} + D_{12}\mathbf{C}_{\mathbf{K}1} & D_{1\Delta} + D_{12}\mathbf{D}_{\mathbf{K}11}D_{2\Delta} & D_{1\Delta}\mathbf{E} + D_{12}\mathbf{D}_{\mathbf{K}1\Delta} \end{bmatrix} \\ \mathbf{LMI}_3 &:= \begin{bmatrix} -\gamma I & * & * & * \\ \mathbf{R}D_{\Delta 1} + \mathbf{D}_{\mathbf{K}\Delta 1}D_{21} & -\mathbf{R} & * & * \\ D_{\Delta 1} + D_{\Delta 2}\mathbf{D}_{\mathbf{K}11}D_{21} & -I & -\mathbf{H} & * \\ D_{11} + D_{12}\mathbf{D}_{\mathbf{K}11}D_{21} & 0 & 0 & -\gamma I \end{bmatrix}. \end{aligned} \quad (12)$$

Note that (10)-(9) implies that

$$\text{rank}(I_n - \mathbf{X}\mathbf{Y}) \leq k.$$

Without loss of generality, assume $\text{rank}(I_n - \mathbf{X}\mathbf{Y}) = k$. Then factorize

$$I_n - \mathbf{X}\mathbf{Y} = \mathbf{M}\mathbf{N}^T$$

with full-rank $\mathbf{M} \in R^{n \times k}$ and $\mathbf{N} \in R^{n \times k}$. Their left-inverse matrices are

$$\mathbf{M}^+ = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T, \mathbf{N}^+ = (\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T.$$

Also factorize

$$I - \mathbf{R}\mathbf{H} = \mathbf{R}_{12}\mathbf{H}_{12}^T \quad \text{and} \quad I - \mathbf{Q}\mathbf{E} = \mathbf{Q}_{12}\mathbf{E}_{12}^T$$

with invertible matrices $\mathbf{R}_{12}$, $\mathbf{H}_{12}$, $\mathbf{Q}_{12}$ and $\mathbf{E}_{12}$. Accordingly, the controller (3) can be recovered as follows [23]:

$$D_{K11} = \mathbf{D}_{\mathbf{K}11} \quad (13)$$

$$B_{K1} = \mathbf{M}^+(\mathbf{B}_{\mathbf{K}1} - \mathbf{X}B_2D_{K11}) \quad (14)$$

$$C_{K1} = (\mathbf{C}_{\mathbf{K}1} - D_{K11}C_2\mathbf{Y})(\mathbf{N}^+)^T, \quad (15)$$

$$A_K = \mathbf{M}^+[\mathbf{A}_{\mathbf{K}} - (\mathbf{X}A\mathbf{Y} + \mathbf{M}B_{K1}C_2\mathbf{Y} + \mathbf{X}B_2C_{K1}\mathbf{N}^T + \mathbf{X}B_2D_{K11}C_2\mathbf{Y})](\mathbf{N}^+)^T \quad (16)$$

$$D_{K1\Delta} = (\mathbf{D}_{\mathbf{K}1\Delta} - D_{K11}D_{2\Delta}\mathbf{E})(\mathbf{E}_{12}^{-1})^T \quad (17)$$

$$D_{K\Delta 1} = \mathbf{R}_{12}^{-1}(\mathbf{D}_{\mathbf{K}\Delta 1} - \mathbf{R}D_{\Delta 2}D_{K11}) \quad (18)$$

$$B_{K\Delta} = \mathbf{M}^+[\mathbf{B}_{\mathbf{K}\Delta} - (\mathbf{X}B_{\Delta}\mathbf{E} + \mathbf{M}B_{k1}D_{2\Delta}\mathbf{E} + \mathbf{X}B_2D_{K11}D_{2\Delta}\mathbf{E} + \mathbf{X}B_2D_{K1\Delta}\mathbf{E}_{12}^T)](\mathbf{E}_{12}^{-1})^T \quad (19)$$

$$C_{K\Delta} = \mathbf{R}_{12}^{-1}[\mathbf{C}_{\mathbf{K}\Delta} - (\mathbf{R}C_{\Delta}\mathbf{Y} + \mathbf{R}D_{\Delta 2}D_{K11}C_2\mathbf{Y} + \mathbf{R}_{12}D_{K\Delta 1}C_2\mathbf{Y} + \mathbf{R}D_{\Delta 2}C_{K1}\mathbf{N}^T)](\mathbf{N}^+)^T \quad (20)$$

$$D_{K\Delta\Delta} = \mathbf{R}_{12}^{-1}[\mathbf{D}_{\mathbf{K}\Delta\Delta} - (\mathbf{R}D_{\Delta\Delta}\mathbf{E} + \mathbf{R}D_{\Delta 2}D_{K11}D_{2\Delta}\mathbf{E} + \mathbf{R}_{12}D_{K\Delta 1}D_{2\Delta}\mathbf{E} + \mathbf{R}D_{\Delta 2}D_{K1\Delta}\mathbf{E}_{12}^T)](\mathbf{E}_{12}^{-1})^T. \quad (21)$$

It should be noted that (10)-(11) are equivalent to (10) and

$$\text{rank} \left(\begin{pmatrix} \mathbf{X} & I_n \\ I_n & \mathbf{Y} \end{pmatrix} \right) \leq n + k \quad (22)$$

which is a lower fixed rank constraint on a matrix-valued affine function of $(\mathbf{X}, \mathbf{Y})$. Although our below developed algorithms still work for this constraint (22), we will see that in fact (11) is more efficiently handled. The difficulty degree of formulations in [26, 12, 14] is proportional to the dimension of the control variable (A_K, B_K, C_K, D_K) in (3), i.e it is proportional to the control order k . In contrast, by exploring (11), the difficulty degree in our formulation is proportional to $\min\{k, n - k\}$, i.e. the computational difficulty with k -order and $(n - k)$ -order controllers is the same.

The k -order LPV-LFT $\mathcal{H}_\infty$ control is therefore formulated as

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{Ki}, \hat{\mathbf{K}}} \gamma \quad \text{s.t.} \quad (9), (10), (11), \quad (23)$$

where all the nonconvexity of the problem is concentrated in the rank constraint (11), which is automatically satisfied for the (full) n -order control. For $k < n$, as expected (11) is a highly nonconvex and discontinuous constraint. Consequently, the feasibility set (9)-(11) is disconnected in general, for which locating a feasible point is already not an easy task.

The function $\text{rank}(\mathbf{X} - \mathbf{Y}^{-1})$ in (11) seems to be very complicated. However, we will see shortly that it can be efficiently handled from the following observation. Suppose $f_{[k]}(\mathbf{X} - \mathbf{Y}^{-1})$ is the sum of the k largest eigenvalues of $\mathbf{X} - \mathbf{Y}^{-1}$, which is positive-definite ($\mathbf{X} - \mathbf{Y}^{-1} \succ 0$) thanks to (10). Then (11) holds true if and only if $\text{tr}(\mathbf{X} - \mathbf{Y}^{-1}) = f_k(\mathbf{X} - \mathbf{Y}^{-1})$ because it implies that $\mathbf{X} - \mathbf{Y}^{-1}$ has at least $(n - k)$ zero eigenvalues. Under the positive semi-definiteness condition (10), the quantity $\text{tr}(\mathbf{X} - \mathbf{Y}^{-1}) - f_k(\mathbf{X} - \mathbf{Y}^{-1})$ is always nonnegative and can therefore be used to measure the degree of satisfaction of the matrix rank constraint (11). Instead of handling nonconvex constraint (11) we incorporate it into the objective, resulting in the following alternative formulation to (23):

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{Ki}, \hat{\mathbf{K}}} F_\mu(\mathbf{X}, \mathbf{Y}, \gamma) := \gamma + \mu(\text{tr}(\mathbf{X} - \mathbf{Y}^{-1}) - f_{[k]}(\mathbf{X} - \mathbf{Y}^{-1})) \quad \text{s.t.} \quad (9) - (10), \quad (24)$$

where $\mu > 0$ is a penalty parameter. Without square on the factor of μ , the above penalization is exact, meaning that the constraint (11) can be satisfied by a minimizer of (24) with a finite value of μ (see e.g. [27, Chapter 16]). This is generally considered as a sufficiently nice property to make such exact penalization attractive. On the other hand, any feasible $(\mathbf{X}, \mathbf{Y}, \gamma)$ to (23) is also feasible to (24), implying that the optimal value of (24) for any $\mu > 0$ is upper bounded by the optimal value of (23).

Suppose $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ is a feasible point to the convex feasibility set (9)-(10). Using the following variational principle [28, p. 191]

$$f_{[k]}(\mathbf{X} - \mathbf{Y}^{-1}) = \max_{\text{orthonormal } x_1, \dots, x_k} \sum_{i=1}^k x_i^H (\mathbf{X} - \mathbf{Y}^{-1}) x_i$$

it follows that

$$f_{[k]}(\mathbf{X} - \mathbf{Y}^{-1}) \geq \sum_{i=1}^k (x_i^{(\kappa)})^H (\mathbf{X} - \mathbf{Y}^{-1}) x_i^{(\kappa)}, \quad (25)$$

where $x_i^{(\kappa)}$, $i = 1, \dots, k$ are the orthonormal eigenvectors corresponding to k largest eigenvalues of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$. On the other hand, as $\text{tr}(\mathbf{Y}^{-1})$ is convex in $\mathbf{Y} \succ 0$, it is true that

$$\begin{aligned} \text{tr}(\mathbf{Y}^{-1}) &\geq \text{tr}((Y^{(\kappa)})^{-1}) - \text{tr}((Y^{(\kappa)})^{-1}(\mathbf{Y} - Y^{(\kappa)})Y^{(\kappa)})^{-1} \\ &= 2\text{tr}((Y^{(\kappa)})^{-1}) - \text{tr}((Y^{(\kappa)})^{-1}\mathbf{Y}Y^{(\kappa)})^{-1}. \end{aligned} \quad (26)$$

The following convex optimization is majorant minimization for (24)

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{K^i}, \hat{\mathbf{K}}} F_{\mu}^{(\kappa)}(\mathbf{X}, \mathbf{Y}, \gamma) := \gamma + \mu(\text{tr}(\mathbf{X} - 2(\mathbf{Y}^{(\kappa)})^{-1}) + \text{tr}((\mathbf{Y}^{(\kappa)})^{-1} \mathbf{Y} (\mathbf{Y}^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (\mathbf{X} - \mathbf{Y}^{-1}) x_i^{(\kappa)}) \quad \text{s.t.} \quad (9) - (10). \quad (27)$$

because by (25) and (26), function $F_{\mu}^{(\kappa)}$ obeys the two following crucial properties

$$F_{\mu}^{(\kappa)}(\mathbf{X}, \mathbf{Y}, \gamma) \geq F(\mathbf{X}, \mathbf{Y}, \gamma) \quad \forall (\mathbf{X}, \mathbf{Y}, \gamma) \quad \text{on} \quad (9) - (10)$$

and

$$F_{\mu}^{(\kappa)}(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)}) = F(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)}).$$

Therefore, for the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$ of the convex program (27), it is true that

$$\begin{aligned} F_{\mu}(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)}) &\leq F_{\mu}^{(\kappa)}(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)}) \\ &\leq F_{\mu}^{(\kappa)}(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)}) \\ &= F_{\mu}(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)}), \end{aligned}$$

implying that $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$ is better than $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ toward optimizing the objective in (24). By using [29], we can prove the following result of global convergence.

Proposition 1

Initialized by any feasible point $(X^{(0)}, Y^{(0)}, \gamma^{(0)})$ of SDP (9)-(10), $\{(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})\}$ is a sequence of improved feasible points of the nonconvex program (24), which converges to a point satisfying first-order necessary optimality conditions.

Proof The sequence $\{(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})\}$ terminates (whenever $F_{\mu}(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)}) = F_{\mu}^{(\kappa)}(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$) or convergence to $\{(\bar{X}, \bar{Y}, \bar{\gamma})\}$, which is the optimal solution of the convex program

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{K^i}} \bar{F}_{\mu}(\mathbf{X}, \mathbf{Y}, \gamma) := \gamma + \mu(\text{tr}(\mathbf{X} - 2\bar{\mathbf{Y}}^{-1}) + \text{tr}(\bar{\mathbf{Y}}^{-1} \mathbf{Y} \bar{\mathbf{Y}}^{-1}) - \sum_{i=1}^k (\bar{x}_i^H (\mathbf{X} - \mathbf{Y}^{-1}) \bar{x}_i) \quad \text{s.t.} \quad (9) - (10). \quad (28)$$

where $\bar{x}_i$, $i = 1, \dots, k$ are the orthonormal eigenvectors corresponding to k largest eigenvalues of $\bar{X} - \bar{Y}^{-1}$. Therefore $(\bar{X}, \bar{Y}, \bar{\gamma})$ satisfies Kuh-Tucker condition for the convex program (28), which is also the first-order necessary optimality condition for the nonconvex program (24). $\square$

The algorithm 1 is pseudo-code for the above procedure.

Alternatively, we can also use the following formulation instead of (24)

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{K^i}, \hat{\mathbf{K}}} \gamma + \mu[\text{tr}(\mathbf{Y} - \mathbf{X}^{-1}) - f_{[k]}(\mathbf{Y} - \mathbf{X}^{-1})] \quad \text{s.t.} \quad (9) - (10), \quad (31)$$

for which the Algorithm 1 can be easily adjusted for solution. Usually, the initial point $(X^{(0)}, Y^{(0)}, \gamma^{(0)})$ is taken as the optimal solution of the full-order controller program

$$\min_{X, Y, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{K^i}, \hat{\mathbf{K}}} \gamma \quad \text{s.t.} \quad (9), (10). \quad (32)$$

and the preference of using (24) or (31) goes to whichever smaller among $\text{tr}(X^{(0)} - Y^{(0)}) - f_{[k]}(X^{(0)} - Y^{(0)})$ and $\text{tr}(Y^{(0)} - X^{(0)}) - f_{[k]}(Y^{(0)} - X^{(0)})$.

On the other hand, we also fix γ and consider

$$\min_{\mathbf{X}, \mathbf{Y}, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{K^i}, \hat{\mathbf{K}}} F(\mathbf{X}, \mathbf{Y}) := \text{tr}(\mathbf{X} - \mathbf{Y}^{-1}) - f_{[k]}(\mathbf{X} - \mathbf{Y}^{-1}) \quad \text{s.t.} \quad (9) - (10), \quad (33)$$

Algorithm 1 Nonconvex Spectral Optimization Algorithm for k -order $\mathcal{H}_\infty$ controllers

- 1: Initialize $\kappa := 0$ and solve SDP (32) to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$. For k normalized eigenvectors corresponding to k largest eigenvalues of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$ stop the algorithm if

$$\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (X^{(\kappa)} - (Y^{(\kappa)})^{-1}) x_i^{(\kappa)} \leq \epsilon_2$$

and accept $(X^{(0)}, Y^{(0)}, \gamma^{(0)})$ as the optimal solution of the nonconvex program (23). Otherwise set $\mu = 0.5$.

- 2: **repeat**

- 3: **if**

$$\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (X^{(\kappa)} - (Y^{(\kappa)})^{-1}) x_i^{(\kappa)} \geq \epsilon_2 \quad (29)$$

for k normalized eigenvectors corresponding to k largest eigenvalues of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$ **then** reset $\mu \rightarrow 2\mu$ and solve SDP (27) to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$.

- 4: **else** Solve SDP (27) with additional convex constraint

$$\text{tr}(\mathbf{X} - 2(Y^{(\kappa)})^{-1}) + \text{tr}((Y^{(\kappa)})^{-1} \mathbf{Y} Y^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (\mathbf{X} - Y^{-1}) x_i^{(\kappa)} \leq \epsilon_2 \quad (30)$$

- 5: **end if**

- 6: Set $\kappa := \kappa + 1$.

- 7: **until** $\gamma^{(\kappa)} - \gamma^{(\kappa-1)} \leq \epsilon_1$

- 8: **Accept** $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ as a found solution of (23) if $\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - f_{[k]}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leq \epsilon_2$.

which seeks a k -order control to satisfy the $\mathcal{H}_\infty$ -gain condition (5) for given γ . The pseudo-code for (33) is provided by Algorithm 2. Again, an alternative formulation to (33)

$$\min_{\mathbf{X}, \mathbf{Y}, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\kappa i}, \hat{\mathbf{K}}} F(\mathbf{X}, \mathbf{Y}) := \text{tr}(\mathbf{Y} - \mathbf{X}^{-1}) - f_{[k]}(\mathbf{Y} - \mathbf{X}^{-1}) \quad \text{s.t.} \quad (9) - (10), \quad (34)$$

is preferred if $\text{tr}(Y^{(0)} - X^{(0)}) - f_{[k]}(Y^{(0)} - X^{(0)})$ is smaller than $\text{tr}(X^{(0)} - Y^{(0)}) - f_{[k]}(X^{(0)} - Y^{(0)})$, where $(X^{(0)}, Y^{(0)})$ is the optimal solution of the following full-order controller program

$$\min_{\mathbf{X}, \mathbf{Y}, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\kappa i}, \hat{\mathbf{K}}} \text{tr}(\mathbf{X} + \mathbf{Y}) \quad \text{s.t.} \quad (9), (10). \quad (35)$$

Remarks on less reduced order controllers. The computational difficulty degree in the formulation in [12, 14] (for time variant systems) is proportional to the control order k . Particularly, less reduced order controllers may pose more computational challenges than highly reduced order ones. In contrast, we now show that using the rank constraints (11) helps us solve them at the same computational efficiency.

Indeed, less reduced order k means that $n - k$ is small. Suppose $\lambda_{[n-k]}(\mathbf{X} - \mathbf{Y}^{-1})$ is the sum of the $n - k$ smallest eigenvalues of $\mathbf{X} - \mathbf{Y}^{-1}$. Then (11) holds true if and only if $\lambda_{[n-k]}(\mathbf{X} - \mathbf{Y}^{-1}) = 0$. Therefore, we propose the following alternative formulation for (23):

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\kappa i}, \hat{\mathbf{K}}} \gamma + \mu \lambda_{[n-k]}(\mathbf{X} - \mathbf{Y}^{-1}) \quad \text{s.t.} \quad (9) - (10). \quad (37)$$

Algorithm 2 Nonconvex Spectral Optimization Algorithm for feasible k -order $\mathcal{H}_\infty$ controllers

- 1: Initialize $\kappa := 0$ and solve SDP (35) to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)})$. Stop the algorithm if

$$\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (X - Y^{-1}) x_i^{(\kappa)} \leq \epsilon_2$$

and accept $(X^{(0)}, Y^{(0)})$ as the solution of the nonconvex program (33).

- 2: **repeat**

- 3: Solve SDP

$$\min_{\mathbf{X}, \mathbf{Y}} \text{tr}(\mathbf{X} - 2(Y^{(\kappa)})^{-1}) + \text{tr}((Y^{(\kappa)})^{-1} \mathbf{Y} (Y^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (\mathbf{X} - Y^{-1}) x_i^{(\kappa)} \quad \text{s.t.} \quad (9), (10) \quad (36)$$

to find its optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)})$

- 4: Set $\kappa := \kappa + 1$.

- 5: **until** $\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - \sum_{i=1}^k (x_i^{(\kappa)})^H (X - Y^{-1}) x_i^{(\kappa)} \leq \epsilon_2$ or $F(X^{(\kappa-1)}, Y^{(\kappa-1)}) - F(X^{(\kappa)}, Y^{(\kappa)}) \leq \epsilon_1$

- 6: Accept $(X^{(\kappa)}, Y^{(\kappa)})$ as a found feasible solution of (23) under fixed γ if $\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) - f_{[k]}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leq \epsilon_2$.
-

Using the following variational principle

$$\lambda_{[n-k]}(X - Y^{-1}) = \min_{\text{orthonormal } x_1, \dots, x_{n-k}} \sum_{i=1}^{n-k} x_i^H (\mathbf{X} - \mathbf{Y}^{-1}) x_i$$

the following optimization is majorant optimization for (37)

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\kappa i}, \hat{\mathbf{K}}} \gamma + \mu \sum_{i=1}^{n-k} (x_i^{(\kappa)})^H (\mathbf{X} - \mathbf{Y}^{-1}) x_i^{(\kappa)} \quad \text{s.t.} \quad (9) - (10), \quad (38)$$

where $x_i^{(\kappa)}$, $i = 1, \dots, n - k$ are the orthonormal eigenvectors corresponding to $(n - k)$ smallest eigenvectors of $X^{(\kappa)} - (Y^{(\kappa)})^{-1}$. Since each $(x_i^{(\kappa)})^H Y^{-1} x_i^{(\kappa)}$ is convex in $Y > 0$, the following convex optimization is majorant minimization for (38) and (37)

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\kappa i}, \hat{\mathbf{K}}} \gamma + \mu \sum_{i=1}^{n-k} (x_i^{(\kappa)})^H (\mathbf{X} - 2(Y^{(\kappa)})^{-1} + (Y^{(\kappa)})^{-1} \mathbf{Y} (Y^{(\kappa)})^{-1}) x_i^{(\kappa)} \quad \text{s.t.} \quad (9) - (10), \quad (39)$$

which provides an alternative to κ -th iteration (27). This iteration is more efficient than (27) for larger $n - k$, i.e. for higher order k of the controllers.

3. STATIC OUTPUT FEEDBACK LPV-LFT $\mathcal{H}_\infty$ CONTROLLER

The static output feedback LPV-LFT controller corresponds to $k = 0$, i.e. the control in (7) is in the form

$$u(t) = (D_{K11} + D_{K1\Delta}(I - \Delta_K(\alpha(t))D_{K\Delta\Delta})^{-1}\Delta_K(\alpha(t))D_{K\Delta1})y(t) \quad (40)$$

leading to the following optimization formulation for its synthesis

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\kappa i}, \hat{\mathbf{K}}} \gamma \quad \text{s.t.} \quad (9) - (10), \quad (41)$$

$$\mathbf{X} = \mathbf{Y}^{-1}, \quad (42)$$

for

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{D}_{\mathbf{K}11} & \mathbf{D}_{\mathbf{K}1\Delta} \\ \mathbf{D}_{\mathbf{K}\Delta 1} & \mathbf{D}_{\mathbf{K}\Delta\Delta} \end{bmatrix}$$

and setting $\mathbf{A}_{\mathbf{K}} = 0$, $\mathbf{B}_{\mathbf{K}1} = 0$, $\mathbf{B}_{\mathbf{K}\Delta} = 0$, $\mathbf{C}_{\mathbf{K}1} = 0$, $\mathbf{C}_{\mathbf{K}\Delta} = 0$ in (12). The controller (40) is recovered by (13), (17), (18) and (21).

The first attractive reformulation of (42) is given in [30]

$$(10), \text{Trace}(\mathbf{X}\mathbf{Y}) \leq n \quad (43)$$

where the nonconvexity is concentrated at the last constraint which is still convex on $\mathbf{X}$ or $\mathbf{Y}$ when $\mathbf{Y}$ or $\mathbf{X}$ is held fixed. Alternating optimization between $\mathbf{X}$ and $\mathbf{Y}$ is applied in handling (43).

Later, [14] also addressed the static output feedback controller problem for LTI systems by developing the so called convex-concave inequality approach for a solution of the corresponding BMI reformulation. All these results must start from a feasible point of a nonconvex feasible set, which is not easily located.

Note that $\mathbf{X} \succeq \mathbf{Y}^{-1}$ by (10) so (42) holds if and only if $\text{tr}(\mathbf{X}) = \text{tr}(\mathbf{Y}^{-1})$ or $\lambda_{[n]}(\mathbf{X} - \mathbf{Y}^{-1}) = \text{tr}(\mathbf{X} - \mathbf{Y}^{-1}) = 0$, so as a particular case of the above approach we consider

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\mathbf{K}i}, \hat{\mathbf{K}}} \gamma + \mu(\text{tr}(\mathbf{X}) - \text{tr}(\mathbf{Y}^{-1})) \quad \text{s.t.} \quad (9) - (10) \quad (44)$$

with κ th iteration

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\mathbf{K}i}, \hat{\mathbf{K}}} \gamma + \mu(\text{tr}(\mathbf{X}) + \text{tr}((\mathbf{Y}^{(\kappa)})^{-1} \mathbf{Y} (\mathbf{Y}^{(\kappa)})^{-1})) \quad \text{s.t.} \quad (9) - (10). \quad (45)$$

The pseudo-code for this iterative procedure is given by Algorithm 3.

Alternatively, instead of (44) we consider

$$\min_{\mathbf{X}, \mathbf{Y}, \gamma, \mathbf{R}, \mathbf{H}, \mathbf{Q}, \mathbf{E}, \Delta_{\mathbf{K}i}, \hat{\mathbf{K}}} \gamma + \mu(\text{tr}(\mathbf{Y}) - \text{tr}(\mathbf{X}^{-1})) \quad \text{s.t.} \quad (9) - (10) \quad (46)$$

for which Algorithm 3 can be easily adjusted for solution whenever

$$\text{tr}(\mathbf{Y}^{(0)}) - \text{tr}((\mathbf{X}^{(0)})^{-1}) < \text{tr}(\mathbf{X}^{(0)}) - \text{tr}((\mathbf{Y}^{(0)})^{-1}) \quad (47)$$

for the initial point $(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)})$.

Again, we can fix γ to consider the following problem

$$\min_{\mathbf{X}, \mathbf{Y}} \text{tr}(\mathbf{X}) - \text{tr}(\mathbf{Y}^{-1}) \quad \text{s.t.} \quad (9) - (10) \quad (50)$$

which seeks a static output feedback $\mathcal{H}_\infty$ controller to satisfy the $\mathcal{H}_\infty$ -gain condition (5) for given γ . Its κ th iteration is

$$\min_{\mathbf{X}, \mathbf{Y}} \text{tr}(\mathbf{X}) + \text{tr}((\mathbf{Y}^{(\kappa)})^{-1} \mathbf{Y} (\mathbf{Y}^{(\kappa)})^{-1}) \quad \text{s.t.} \quad (9) - (10), \quad (51)$$

and Algorithm 3 is the pseudo-code for the implementation. Alternatively, whenever (47), we can consider

$$\min_{\mathbf{X}, \mathbf{Y}} \text{tr}(\mathbf{Y}) - \text{tr}(\mathbf{X}^{-1}) \quad \text{s.t.} \quad (9) - (10) \quad (52)$$

for which Algorithm 4 can be easily adjusted for computation.

4. SIMULATION RESULTS

The hardware and software facilities for our computational implementation are:

- Processor: Intel(R) Core(TM) i5-3470 CPU @3.20GHz;
- Software: Matlab version R2015b;
- Matlab toolbox: Yalmip[31] with SeDumi 1.3 [32] solver for SDP;
- Criterion: The stop and rank check criterion ϵ_1 and ϵ_2 are all set as 10^{-4} .

Algorithm 3 Nonconvex Spectral Optimization Algorithm for static LPV-LFT $\mathcal{H}_\infty$ controllers

- 1: Initialize $\kappa := 0$ and solve SDP (41) to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$. Stop the algorithm if

$$\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leq \epsilon_2 \quad (48)$$

and accept $(X^{(0)}, Y^{(0)}, \gamma^{(0)})$ as the optimal solution of the nonconvex program (23). Otherwise set $\mu = 0.5$.

2: **repeat**

- 3: **if** $\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \geq \epsilon_2$ **then** reset $\mu \rightarrow 2\mu$ and solve SDP (45) to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$.
4: **else** Solve SDP (45) with additional convex constraint

$$\text{tr}(\mathbf{X} - 2(Y^{(\kappa)})^{-1}) + \text{tr}((Y^{(\kappa)})^{-1} \mathbf{Y} Y^{(\kappa)})^{-1}) \leq \epsilon_2 \quad (49)$$

to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)}, \gamma^{(\kappa+1)})$

5: **end if**

6: Set $\kappa := \kappa + 1$.

7: **until** $\gamma^{(\kappa)} - \gamma^{(\kappa-1)} \leq \epsilon_1$

8: Accept $(X^{(\kappa)}, Y^{(\kappa)}, \gamma^{(\kappa)})$ as a found suboptimal solution of (41)-(42) if (48) is fulfilled.

Algorithm 4 Nonconvex Spectral Optimization Algorithm for feasible static output feedback LPV-LFT $\mathcal{H}_\infty$ controllers

- 1: Initialize $\kappa := 0$ and solve SDP (41) (for fixed $\gamma = \bar{\gamma}$ to find its optimal solution $(X^{(\kappa)}, Y^{(\kappa)})$. Stop the algorithm if

$$\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leq \epsilon_2 \quad (53)$$

and accept $(X^{(0)}, Y^{(0)})$ as the optimal solution of the nonconvex program (23). Otherwise set $\mu = 0.5$.

2: **repeat**

3: Solve SDP (51) to find the optimal solution $(X^{(\kappa+1)}, Y^{(\kappa+1)})$.

4: Set $\kappa := \kappa + 1$.

5: **until** $\text{tr}(X^{(\kappa)} - (Y^{(\kappa)})^{-1}) \leq \epsilon_2$ or $\text{tr}(X^{(\kappa-1)} - Y^{(\kappa-1)}) - \text{tr}(X^{(\kappa)} - Y^{(\kappa)}) \leq \epsilon_1$

6: Accept $(X^{(\kappa)}, Y^{(\kappa)})$ as a found feasible solution of (41)-(42) under fixed γ if (53) is fulfilled.

4.1. RTAC control

Consider the nonlinear benchmark model [33] of rotational-translational actuator (RTAC). The regulated output is the tracking performance of the translational and angular positions and control

$$z = (0.1x_1, 0.1x_3, u)^T$$

The system can be represented by LPV-LFT (1)-(2) [24, Appendix] with the numerical values of the matrices in (1)-(2) recalled in Appendix A.

By solving SDP (32) for Step 1 of Algorithm 1, we found $\gamma^{(0)} = 8.1909$ with $X^0 - (Y^0)^{-1}$ of rank-four, which lead to full-order control (3)-(4) [24]. Implementing Algorithm 1 with $\mu = 1$ for the first-order controller, $\gamma = 9.3785$ was found and the following numerical data for control (3)-(4) are obtained

$$A_K = -4.2617, B_{K1} = [0.4767 \quad -2.0207], C_{K1} = -0.1327, D_{K11} = [-0.0670 \quad 0.0284],$$

$$B_{K\Delta} = [0 \quad -0.0030 \quad -0.0132], D_{K1\Delta} = [0 \quad 0 \quad 0.0013], D_{K\Delta\Delta} = \begin{bmatrix} -0.0002 & -0.3509 & -1.4991 \\ 0 & -0.0003 & -0.0015 \\ 0 & 0.0002 & 0.0008 \end{bmatrix},$$

$$C_{K\Delta} = \begin{bmatrix} -30.6379 \\ 4.2867 \\ -1.6715 \end{bmatrix}, D_{K\Delta 1} = \begin{bmatrix} -11.5584 & -2.5580 \\ 0.1977 & 0.0127 \\ 0.0034 & -0.0107 \end{bmatrix},$$

$$\Delta_{K1} = \begin{bmatrix} -0.3118 & 4.8064 & -25.2717 \\ 0.3425 & 4.8514 & 64.0377 \\ -0.0932 & -27.9300 & -84.1243 \end{bmatrix}, \Delta_{K2} = \begin{bmatrix} -0.3125 & 3.5657 & 51.1251 \\ 0.3434 & 6.5093 & -39.3112 \\ -0.0934 & -28.3084 & -59.9568 \end{bmatrix},$$

$$\Delta_{K3} = \begin{bmatrix} 0.3118 & -4.8064 & 25.2717 \\ -0.3425 & -4.8514 & -64.0377 \\ 0.0932 & 27.9300 & 84.1243 \end{bmatrix}, \Delta_{K4} = \begin{bmatrix} 0.3125 & -3.5657 & -51.1251 \\ -0.3434 & -6.5093 & 39.3112 \\ 0.0934 & 28.3084 & 59.9568 \end{bmatrix},$$

Under the condition $x(0) = (0.5, 0, 0, 0)^T$, the simulation given by Figs. 1-3 clearly show that our first-order LPV-LFT stabilizes the system well.

Figure 1. Order-1 RTAC Tracking performance in the absence of disturbance

Figure 2. Order-1 RTAC Tracking performance with the disturbance $w = 0.1 \sin 5\pi t$

4.2. Reduced order LPV-LFT controllers

We modify the LTI examples in [12, Sec. 10] by adding the gain-scheduling channel (w_Δ, z_Δ to have (1). The randomly generated matrix sets for the gain-scheduling channel are provided in Appendix B.

The computational results by implementing Algorithm 1 are provided by Table I. The system state dimension is given in the second row with the the initial γ obtained by solving (32) (for full-order LPV-LFT control) given in the third row. The fourth column indicates the value of initial μ used. The fifth column is the found value of γ for the controller of order indicated in the sixth column.

Figure 3. Order-1 RTAC Control performance in the absence of disturbance (dot) and with disturbance $w = 0.1 \sin 5\pi t$ (solid).

Table I. Simulation Results of Dynamic Cases

	State dimension	Lower bound by (32)	μ	γ	order
VTOL helicopter	4	0.0871	0.05	0.2958	1
Chemical reactor	4	0.8653	0.05	0.8653	1
Transport airplane	10	1.7042	10	2.5302	1

Table II. Simulation Results of Static Cases

	state dimension	Lower bound by (32)	μ	γ
AC1	5	2.76E-08	0.1	6.68E-07
AC2	5	0.117675	1	2.113598
AC3	5	3.095253	1	4.891439
AC6	7	3.683339	5	4.948768
AC8	9	8.727383	20	9.57084
AC9	10	1.000091	5	1.002586
AC15	4	16.30176	5	18.91505
AC17	4	6.686188	5	6.688397
HE4	8	28.81522	5	37.26205

4.3. Static output feedback LPV-LFT controller

We modify the LTI examples in [14, Tab. III] by adding the matrices relating to z_Δ and w_Δ provided in Appendix C. The computational results by implementing Algorithm 3 are provided in Table II, which is formatted similarly to Table I.

4.4. LTI systems

In LTI systems there are no gain-scheduling channel (w_Δ, z_Δ) in system (1) and no gain-scheduling channel (w_K, z_K) in controller (3). Accordingly,

$$\hat{\mathbf{K}} := \begin{bmatrix} \mathbf{A}_K & \mathbf{B}_{K1} \\ \mathbf{C}_{K1} & \mathbf{D}_{K11} \end{bmatrix}$$

in (8) and LMI (9) becomes

$$\begin{bmatrix} \mathbf{X}\mathbf{A} + \mathbf{B}_{\mathbf{K}1}\mathbf{C}_2 + (*) & * & * & * \\ \mathbf{A}_{\mathbf{K}}^T + \mathbf{A} + \mathbf{B}_2\mathbf{D}_{\mathbf{K}11}\mathbf{C}_2 & (\mathbf{A}\mathbf{Y} + \mathbf{B}_2\mathbf{C}_{\mathbf{K}1}) + (*) & * & * \\ \mathbf{B}_1^T\mathbf{X} + \mathbf{D}_{21}^T\mathbf{B}_{\mathbf{K}1}^T & \mathbf{B}_1^T + \mathbf{D}_{21}^T\mathbf{D}_{\mathbf{K}11}^T\mathbf{B}_2^T & -\gamma\mathbf{I} & * \\ \mathbf{C}_1 + \mathbf{D}_{12}\mathbf{D}_{\mathbf{K}11}\mathbf{C}_2 & \mathbf{C}_1\mathbf{Y} + \mathbf{D}_{12}\mathbf{C}_{\mathbf{K}1} & \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{D}_{\mathbf{K}11}\mathbf{D}_{21} & -\gamma\mathbf{I} \end{bmatrix} \prec 0 \quad (54)$$

with control recovered by (13)-(16) [34].

As mentioned in the Introduction, the Matlab command `systune` [18] is the most efficient tool for LTI systems. Nevertheless, to show the efficiency of our algorithms in handling the rank-reduced constraints, we also consider numerical examples from [12, Sec. 10] and [14, Tab. III] for LTI systems.

4.4.1. Dynamic and static output feedback controllers in [12] All cases in [12, Section 10] were tested. The computational results are summarized in Table III, where the first column is the cases name, the second column is the initial γ obtained by solving (32), i.e. it is the optimal $\mathcal{H}_\infty$ by the full-order controller, the third column is the objective function, which is either (24) or (31) for k -order controller and either (44) or (46) for static output feedback controller. The fourth column is the value of initial μ . The fifth column is the found value of γ . The sixth column provides the controller order. The seventh column is the iterations by our method; the last two column are the found γ and corresponding order in [12], respectively.

Compared with [12], it can be seen that, our optimal γ are better than [12] in all cases provided. For the Transport aircraft example, [12] failed to obtain 2-order controller, though it was found by our Algorithm after 6 iterations. The Piezoelectric actuator example poses the most difficulty for [12] but it is easily solved by our algorithm with 3 iterations for both order-2 controllers and statistic output feedback controller.

Table III. Numerical results compared with [12]

Cases	Lower bound	Obj	μ	γ	order	# iter	γ in [12]	order in [12]
VTOL helicopter	0.0737	(24)	1	0.118713	2	1	0.133	2
VTOL helicopter	0.0737	(44)	0.7	0.1539	0	20	0.1542	0
Chemical reactor	0.8617	(24)	1	0.8617	2	1	1.142	2
Chemical reactor	0.8617	(44)	1	0.8937	0	28	1.183	0
Transport aircraft	0.0417	(31)	10	0.349	1	42	2.86	1
Transport aircraft	0.0417	(31)	1	0.2167	2	6	failed	2
Piezoelectric actuator	3.11E-05	(31)	5	0.0048	2	3	0.03	2
Piezoelectric actuator	3.11E-05	(46)	100	0.0213	0	3	0.0578	0
Coupled springs model	0.0737	(24)	1	0.01993	2	4	0.0235	4

The last example in [12] for static output feedback for a plant with state dimension 82. The computational results by implementing Algorithm 2 are summarized in Table IV, whose format is similar to Table IV but the second column is the fixed value of γ , which is better than the value provided by [12] in the 7-th column. The sixth column is the value of $\text{trace}(\mathbf{X} - \mathbf{Y}^{-1})$ at the last iteration.

Table IV. Distillation tower case compared with [12] with γ fixed

Cases	Fixed γ	Obj	order	# iter	$\text{trace}(\mathbf{X} - \mathbf{Y}^{-1})$	γ in [12]	order in [12]
Distillation tower	0.8000	(44)	0	64	2.24E-05	1.0722	0

4.4.2. Static output feedback controllers in [14] There are 45 cases in [14, Table III]. The computational results are summarized in Tables V and VI, whose format is in similar style to Table

III. For AC9, EB1, EB2 and EB3, the results by [14] are obviously incorrect as its provided values of γ in the seventh column is even smaller than their lower bound in the second column. As for case TG1, the value of μ is increased to regulate the convergence speed, but a larger μ results in larger γ as well. According to [14], the iteration threshold to stop its solver is 300. The solver [14] is trapped by local minima in AC4, TG1, MFP as its found value is much bigger than that found by our Algorithm. The former is also heading to a wrong minima in the case AC12 as the value found after 300 iteration is still very far from that found by the later. In AC1, AC2, AC8, AC11, AC16, AC17, HE1, HE2, HE3, REA2, DIS3, DIS4, WEC2, WEC3, IH, PSM, NN1 the later clearly outperforms the former in both computational performance and convergence. For all other cases our simulation results are better than or consistent with [14].

Table V. Numerical results of static output feedback controllers compared with [14]

Cases	Lower bound	Obj	μ	γ	# iter	γ in [14]	# iter in [14]
AC1	2.68E-06	(46)	0.1	3.42E-05	7	0.0177	93
AC2	0.1115	(46)	0.1	0.1115	1	0.1140	99
AC3	2.9675	(46)	0.15	3.4696	300	3.4859	210
AC4	0.5573	(44)	0.3	1.0064	400	69.9900	2
AC6	3.4275	(46)	0.1	4.1208	132	4.1954	167
AC7	0.0396	(46)	0.1	0.0657	150	0.0548	300
AC8	1.6165	(46)	2	2.0508	16	3.052	247
AC9	1.0000	(46)	1	1.003	1	0.9237 (wrong)	300
AC11	2.8079	(44)	10	2.9261	400	3.0104	68
AC12	0.0225	(44)	1	0.4706	14	2.3025	300
AC15	14.8628	(46)	0.2	15.1730	116	15.1995	105
AC16	14.8556	(44)	0.09	15.0012	24	14.9881	186
AC17	6.6124	(46)	1	6.61214	1	6.6373	129
HE1	0.0737	(44)	0.7	0.1539	20	0.1807	300
HE2	2.4181	(46)	10	4.4162	272	6.7846	177
HE3	0.7990	(44)	0.6	0.9999	2	0.9243	105
HE4	22.8382	(46)	3	22.8431	203	22.8713	252
REA1	0.8617	(46)	0.2	0.8911	189	0.8815	96
REA2	1.1341	(44)	1	1.1895	1	1.4188	300
REA3	74.2513	(46)	1	74.2513	4	74.5478	2
DIS1	4.1593	(44)	5	4.5625	276	4.1943	93
DIS2	0.9476	(46)	0.1	1.0556	27	1.1546	54
DIS3	1.0423	(46)	0.1	1.0933	150	1.1382	285
DIS4	0.7315	(44)	0.1	0.7556	64	0.7498	126
TG1	3.4652	(44)	4000	31.3940	333	12.9336	45
AGS	8.1732	(46)	1	8.1732	5	8.1732	24
WEC2	3.5981	(46)	100	5.9166	128	6.6082	300
WEC3	3.7685	(46)	100	6.2305	107	6.8402	300
BDT1	0.2653	(46)	0.1	0.331	195	0.8562	29
MFP	4.1865	(46)	1000	33.4841	300	31.6079	171
IH	1.26E-06	(46)	1	1.40E-05	1	1.1858	114
CSE1	0.02	(44)	1	0.02	1	0.022	3
PSM	0.9202	(44)	0.1	0.9206	15	0.9227	87

Table VI. Numerical results of static output feedback controllers compared with [14]

Cases	Lower bound	Obj	μ	γ	# iter	γ in [14]	# iter in [14]
EB1	3.1041	(44)	20	3.142	1	2.2076 (wrong)	300
EB2	1.7676	(44)	1	2.0205	24	0.8148 (wrong)	84
EB3	1.7976	(44)	1	2.058	26	0.8153 (wrong)	84
NN1	13.1299	(46)	1	17.2732	4	18.4813	300
NN2	1.7645	(44)	1	2.2217	27	2.2216	9
NN4	1.2862	(44)	2	1.4091	280	1.3802	156
NN8	2.3576	(46)	0.47	3.074	312	2.9345	180
NN9	13.6461	(46)	40	30.0387	1000	32.1222	300
NN11	0.0181	(46)	50	0.1981	648	0.1566	9
NN15	0.0977	(44)	1	0.0993	2	0.1194	6
NN16	0.9556	(44)	1	0.9714	157	0.9656	48
NN17	2.6386	(44)	20	11.6537	34	11.2381	117

5. CONCLUSIONS

We have proposed new algorithms for solving matrix-rank constrained optimization arising in reduced-order $\mathcal{H}_\infty$ LPV-LFT controller design. Unlike the previous developments, we formulate the problem as minimization of nonconvex objective function over a convex feasibility set. The global convergence of the proposed Algorithms follows immediately from their path-following nature, while there is no difficulty for initial solutions. The numerical results reported for the benchmark collections have shown their performance superior over the state-of-art controller solvers. Their application to solutions of reduced-order generalized $\mathcal{H}_2$ LPV-LFT controllers is obvious. Their extensions to multi-objective and structured controller design are currently under development.

APPENDIX A: LPV-LFT DATA OF RTAC SYSTEM

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.0365 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.1946 & 0 & 0 & 0 \end{bmatrix}, B_\Delta = \begin{bmatrix} 0 & 0 & 0 \\ -0.5 & 0.5 & 1.0365 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & -0.1946 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1.0365 \\ 0 \\ -0.1946 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -0.1946 \\ 0 \\ 1.0365 \end{bmatrix} \\
C_\Delta &= \begin{bmatrix} 1.5157 & 0 & 0 & 0 \\ 0.7088 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D_{\Delta\Delta} = \begin{bmatrix} 1.2311 & 0 & -1.5157 \\ 0 & -0.8419 & -0.7088 \\ 0 & 0 & 0 \end{bmatrix}, D_{\Delta 1} = \begin{bmatrix} -1.5157 \\ -0.7088 \\ 0 \end{bmatrix}, D_{\Delta 2} = \begin{bmatrix} 1.5157 \\ -0.7088 \\ 0 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 0.31622 & 0 & 0 & 0 \\ 0 & 0 & 0.3162 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Delta_1 &= \begin{bmatrix} 0.01224 & 0 & 0 \\ 0 & 0.01224 & 0 \\ 0 & 0 & 0.04794 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.01224 & 0 & 0 \\ 0 & 0.01224 & 0 \\ 0 & 0 & -0.04794 \end{bmatrix} \\
\Delta_3 &= \begin{bmatrix} -0.01224 & 0 & 0 \\ 0 & -0.01224 & 0 \\ 0 & 0 & -0.04794 \end{bmatrix}, \Delta_4 = \begin{bmatrix} -0.01224 & 0 & 0 \\ 0 & -0.01224 & 0 \\ 0 & 0 & 0.04794 \end{bmatrix} \\
\alpha_1(t) &= \frac{1}{4a_2a_5}(a_2 - \delta_1(t))(a_5 - \delta_2(t)), \alpha_2(t) = \frac{1}{4a_2a_5}(a_2 - \delta_1(t))(a_5 + \delta_2(t)) \\
\alpha_3(t) &= \frac{1}{4a_2a_5}(a_2 + \delta_1(t))(a_5 + \delta_2(t)), \alpha_4(t) = \frac{1}{4a_2a_5}(a_2 + \delta_1(t))(a_5 - \delta_2(t))
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \epsilon \frac{\cos 0.5 + 1}{2}, a_2 = \epsilon \frac{1 - \cos 0.5}{2}, a_3 = 1 - a_1, a_4 = 1 + a_1, a_5 = \epsilon 0.5 \sin 0.5 \\
\delta_1 &= \epsilon \cos x_3 - a_1, -a_2 \leq \delta_1 \leq a_2, \delta_2 = \epsilon x_4 \sin x_3, -a_5 \leq \delta_2 \leq a_5, \epsilon = 0.2.
\end{aligned}$$

APPENDIX B

Modified VTOL helicopter system

$$B_{\Delta} = \begin{bmatrix} 0.5243 & 0.4413 \\ 0.3440 & 0.1393 \\ 0.1109 & 0.5365 \\ 0.2478 & 0.1764 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.3031 & 0.5501 & 0.5605 & 0.0255 \\ 0.3292 & 0.2303 & 0.3367 & 0.1256 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0740 & 0.0618 \\ 0.0217 & 0.0152 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0305 & 0.0611 \\ 0.0097 & 0.0724 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0.0477 & 0.0562 \\ 0.0371 & 0.0565 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0506 & 0.0704 \\ 0.0474 & 0.0156 \end{bmatrix}.$$

Modified chemical reactor system

$$B_{\Delta} = \begin{bmatrix} 0.3992 & 0.2357 & 0.1968 & 0.2560 \\ 0.3125 & 0.0069 & 0.3433 & 0.0130 \\ 0.2363 & 0.0491 & 0.0851 & 0.2498 \\ 0.3772 & 0.3506 & 0.2244 & 0.1473 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.1869 & 0.3511 & 0.2284 & 0.2440 \\ 0.3123 & 0.2240 & 0.0962 & 0.2576 \\ 0.0888 & 0.3294 & 0.2597 & 0.2844 \\ 0.1941 & 0.2879 & 0.0325 & 0.3472 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0032 & 0.0133 & 0.0410 & 0.0322 \\ 0.0316 & 0.0095 & 0.0182 & 0.0346 \\ 0.0124 & 0.0122 & 0.0348 & 0.0287 \\ 0.0079 & 0.0028 & 0.0449 & 0.0080 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0232 & 0.0099 & 0.0097 & 0.0067 \\ 0.0403 & 0.0267 & 0.0448 & 0.0079 \\ 0.0413 & 0.0303 & 0.0039 & 0.0294 \\ 0.0128 & 0.0197 & 0.0050 & 0.0271 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0019 & 0.0024 & 0.0319 & 0.0148 \\ 0.0346 & 0.0320 & 0.0292 & 0.0050 \\ 0.0271 & 0.0347 & 0.0191 & 0.0011 \\ 0.0274 & 0.0366 & 0.0066 & 0.0349 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0144 & 0.0309 & 0.0408 & 0.0085 \\ 0.0141 & 0.0012 & 0.0166 & 0.0316 \\ 0.0159 & 0.0402 & 0.0213 & 0.0158 \\ 0.0223 & 0.0267 & 0.0026 & 0.0429 \end{bmatrix}.$$

Modified transport airplane system

$$B_{\Delta}^T = \begin{bmatrix} 0.1454 & 0.0308 & 0.2485 & 0.2151 & 0.0160 & 0.0712 & 0.1049 & 0.1605 & 0.2354 & 0.1063 \\ 0.1822 & 0.2487 & 0.1115 & 0.1627 & 0.2309 & 0.2232 & 0.0366 & 0.0514 & 0.1590 & 0.1985 \\ 0.0632 & 0.0946 & 0.0874 & 0.1169 & 0.0438 & 0.1910 & 0.0251 & 0.2280 & 0.0210 & 0.2002 \\ 0.2174 & 0.1155 & 0.1550 & 0.0908 & 0.1944 & 0.1510 & 0.1566 & 0.2160 & 0.1170 & 0.2261 \end{bmatrix},$$

$$C_{\Delta} = \begin{bmatrix} 0.1991 & 0.0018 & 0.2215 & 0.2496 & 0.1639 & 0.1131 & 0.0108 & 0.0643 & 0.2234 & 0.1669 \\ 0.0184 & 0.2225 & 0.1165 & 0.2177 & 0.1031 & 0.0067 & 0.1392 & 0.1817 & 0.1950 & 0.2448 \\ 0.2484 & 0.2162 & 0.0930 & 0.0000 & 0.1192 & 0.1064 & 0.1530 & 0.1785 & 0.1238 & 0.0293 \\ 0.1942 & 0.1746 & 0.0208 & 0.1702 & 0.0972 & 0.0265 & 0.1849 & 0.1261 & 0.1906 & 0.2175 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.0191 & 0.0212 & 0.0033 & 0.0055 \\ 0.0241 & 0.0303 & 0.0386 & 0.0324 \\ 0.0217 & 0.0173 & 0.0281 & 0.0281 \\ 0.0279 & 0.0353 & 0.0216 & 0.0161 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0136 & 0.0429 & 0.0119 & 0.0158 \\ 0.0156 & 0.0105 & 0.0376 & 0.0239 \\ 0.0289 & 0.0252 & 0.0393 & 0.0138 \\ 0.0342 & 0.0211 & 0.0192 & 0.0078 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0181 & 0.0437 & 0.0088 & 0.0238 \\ 0.0130 & 0.0085 & 0.0322 & 0.0011 \\ 0.0111 & 0.0416 & 0.0298 & 0.0010 \\ 0.0294 & 0.0260 & 0.0152 & 0.0365 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0037 & 0.0241 & 0.0188 & 0.0333 \\ 0.0234 & 0.0321 & 0.0223 & 0.0058 \\ 0.0263 & 0.0101 & 0.0321 & 0.0320 \\ 0.0269 & 0.0338 & 0.0076 & 0.0325 \end{bmatrix}.$$

APPENDIX C

Modified AC1 system

$$B_{\Delta} = \begin{bmatrix} 0.2972 & 0.1612 \\ 0.4326 & 0.0854 \\ 0.1907 & 0.2157 \\ 0.4383 & 0.2318 \\ 0.5171 & 0.3084 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.2360 & 0.5001 & 0.1001 & 0.5307 & 0.0602 \\ 0.1685 & 0.2330 & 0.4901 & 0.2377 & 0.1398 \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0.1028 & 0.0531 \\ 0.0173 & 0.1622 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0046 & 0.1142 \\ 0.1453 & 0.0764 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0.0936 & 0.0742 \\ 0.0384 & 0.1558 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.1177 & 0.0499 \\ 0.1122 & 0.1052 \end{bmatrix}.$$

Modified AC2 system

$$\begin{aligned}
B_\Delta &= \begin{bmatrix} 0.1860 & 0.1471 & 0.1931 & 0.1790 & 0.3075 \\ 0.3197 & 0.2706 & 0.0325 & 0.0686 & 0.2026 \\ 0.2126 & 0.2984 & 0.0404 & 0.1789 & 0.3360 \\ 0.0559 & 0.2856 & 0.0493 & 0.0534 & 0.2518 \\ 0.0722 & 0.1151 & 0.2453 & 0.0199 & 0.2107 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.0063 & 0.2641 & 0.1275 & 0.2541 & 0.2679 \\ 0.1064 & 0.1382 & 0.2599 & 0.3052 & 0.2471 \\ 0.1364 & 0.2854 & 0.2427 & 0.1053 & 0.0538 \\ 0.0869 & 0.1257 & 0.1213 & 0.2158 & 0.2771 \\ 0.0633 & 0.2472 & 0.0694 & 0.1410 & 0.3182 \end{bmatrix}, \\
\Delta_1 &= \begin{bmatrix} 0.0464 & 0.0349 & 0.0130 & 0.0420 & 0.0356 \\ 0.0500 & 0.0563 & 0.0284 & 0.0334 & 0.0376 \\ 0.0563 & 0.0300 & 0.0513 & 0.0140 & 0.0415 \\ 0.0000 & 0.0273 & 0.0327 & 0.0379 & 0.0507 \\ 0.0493 & 0.0456 & 0.0481 & 0.0048 & 0.0559 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0630 & 0.0099 & 0.0452 & 0.0041 & 0.0120 \\ 0.0476 & 0.0706 & 0.0516 & 0.0401 & 0.0155 \\ 0.0760 & 0.0396 & 0.0026 & 0.0158 & 0.0035 \\ 0.0475 & 0.0692 & 0.0503 & 0.0101 & 0.0520 \\ 0.0014 & 0.0171 & 0.0297 & 0.0168 & 0.0231 \end{bmatrix}, \\
\Delta_3 &= \begin{bmatrix} 0.0396 & 0.0091 & 0.0153 & 0.0697 & 0.0457 \\ 0.0511 & 0.0361 & 0.0415 & 0.0060 & 0.0422 \\ 0.0367 & 0.0627 & 0.0471 & 0.0078 & 0.0038 \\ 0.0394 & 0.0643 & 0.0307 & 0.0104 & 0.0685 \\ 0.0327 & 0.0199 & 0.0151 & 0.0122 & 0.0536 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0496 & 0.0578 & 0.0090 & 0.0224 & 0.0376 \\ 0.0043 & 0.0528 & 0.0021 & 0.0314 & 0.0574 \\ 0.0579 & 0.0345 & 0.0632 & 0.0436 & 0.0234 \\ 0.0628 & 0.0119 & 0.0203 & 0.0017 & 0.0300 \\ 0.0662 & 0.0268 & 0.0199 & 0.0566 & 0.0036 \end{bmatrix}.
\end{aligned}$$

Modified AC3 system

$$\begin{aligned}
B_\Delta &= \begin{bmatrix} 0.0421 & 0.1351 & 0.2372 & 0.3017 & 0.4218 \\ 0.0606 & 0.0929 & 0.0787 & 0.2381 & 0.0728 \\ 0.0718 & 0.1072 & 0.0905 & 0.1338 & 0.1101 \\ 0.0838 & 0.3812 & 0.0330 & 0.0710 & 0.1694 \\ 0.1355 & 0.3002 & 0.3901 & 0.2657 & 0.0316 \end{bmatrix}, C_\Delta = \begin{bmatrix} 0.2106 & 0.2908 & 0.1364 & 0.0399 & 0.2294 \\ 0.3321 & 0.1855 & 0.3337 & 0.0034 & 0.0470 \\ 0.1710 & 0.1951 & 0.3008 & 0.1827 & 0.2729 \\ 0.1179 & 0.2634 & 0.0541 & 0.2831 & 0.0546 \\ 0.0161 & 0.0257 & 0.0562 & 0.3122 & 0.0580 \end{bmatrix}, \\
\Delta_1 &= \begin{bmatrix} 0.0495 & 0.0112 & 0.0254 & 0.0433 & 0.0260 \\ 0.0291 & 0.0276 & 0.0497 & 0.0243 & 0.0404 \\ 0.0712 & 0.0117 & 0.0213 & 0.0217 & 0.0538 \\ 0.0291 & 0.0549 & 0.0384 & 0.0328 & 0.0307 \\ 0.0450 & 0.0631 & 0.0603 & 0.0306 & 0.0311 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0086 & 0.0659 & 0.0523 & 0.0319 & 0.0113 \\ 0.0017 & 0.0644 & 0.0510 & 0.0146 & 0.0459 \\ 0.0200 & 0.0315 & 0.0512 & 0.0068 & 0.0616 \\ 0.0219 & 0.0166 & 0.0073 & 0.0567 & 0.0356 \\ 0.0450 & 0.0526 & 0.0469 & 0.0121 & 0.0484 \end{bmatrix}, \\
\Delta_3 &= \begin{bmatrix} 0.0107 & 0.0562 & 0.0380 & 0.0014 & 0.0640 \\ 0.0663 & 0.0520 & 0.0277 & 0.0642 & 0.0552 \\ 0.0376 & 0.0084 & 0.0289 & 0.0454 & 0.0401 \\ 0.0472 & 0.0365 & 0.0126 & 0.0648 & 0.0306 \\ 0.0025 & 0.0226 & 0.0178 & 0.0114 & 0.0179 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0471 & 0.0448 & 0.0199 & 0.0398 & 0.0395 \\ 0.0143 & 0.0402 & 0.0510 & 0.0595 & 0.0222 \\ 0.0040 & 0.0262 & 0.0494 & 0.0278 & 0.0624 \\ 0.0480 & 0.0245 & 0.0534 & 0.0038 & 0.0140 \\ 0.0420 & 0.0511 & 0.0317 & 0.0543 & 0.0409 \end{bmatrix}.
\end{aligned}$$

Modified AC6 system

$$\begin{aligned}
B_\Delta^T &= \begin{bmatrix} 0.0327 & 0.0447 & 0.3191 & 0.3238 & 0.1948 & 0.0202 & 0.0795 \\ 0.1196 & 0.2781 & 0.0052 & 0.0146 & 0.0572 & 0.2198 & 0.2478 \\ 0.2194 & 0.1527 & 0.1853 & 0.1004 & 0.2522 & 0.0640 & 0.2326 \\ 0.0622 & 0.1248 & 0.2119 & 0.2642 & 0.0275 & 0.3148 & 0.2627 \end{bmatrix}, \\
C_\Delta &= \begin{bmatrix} 0.3139 & 0.3081 & 0.1451 & 0.0664 & 0.3260 & 0.1220 & 0.2818 \\ 0.2094 & 0.2247 & 0.0274 & 0.0867 & 0.3412 & 0.3250 & 0.1408 \\ 0.1986 & 0.1267 & 0.0866 & 0.1507 & 0.1773 & 0.1334 & 0.0873 \\ 0.0523 & 0.1854 & 0.0445 & 0.0179 & 0.1767 & 0.0402 & 0.1459 \end{bmatrix}, \\
\Delta_1 &= \begin{bmatrix} 0.0394 & 0.0411 & 0.0521 & 0.0284 \\ 0.0352 & 0.0413 & 0.0306 & 0.0759 \\ 0.0361 & 0.0661 & 0.0656 & 0.0708 \\ 0.0248 & 0.0643 & 0.0431 & 0.0445 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0677 & 0.0512 & 0.0246 & 0.0338 \\ 0.0638 & 0.0251 & 0.0186 & 0.1004 \\ 0.0226 & 0.0918 & 0.0248 & 0.0468 \\ 0.0328 & 0.0212 & 0.0474 & 0.0201 \end{bmatrix}, \\
\Delta_3 &= \begin{bmatrix} 0.0883 & 0.0252 & 0.0588 & 0.0290 \\ 0.0956 & 0.0399 & 0.0694 & 0.0311 \\ 0.0428 & 0.0581 & 0.0216 & 0.0414 \\ 0.0108 & 0.0256 & 0.0115 & 0.0496 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0075 & 0.0820 & 0.0209 & 0.0460 \\ 0.0232 & 0.0645 & 0.0405 & 0.0204 \\ 0.0707 & 0.0431 & 0.0850 & 0.0431 \\ 0.0026 & 0.0511 & 0.0483 & 0.0551 \end{bmatrix}.
\end{aligned}$$

Modified AC8 system

$$\begin{aligned}
B_\Delta^T &= \begin{bmatrix} 0.0964 & 0.3048 & 0.2852 & 0.3392 & 0.3409 & 0.1208 & 0.1274 & 0.2153 & 0.1339 \\ 0.3424 & 0.3336 & 0.2293 & 0.1083 & 0.2802 & 0.0962 & 0.1879 & 0.1583 & 0.2217 \end{bmatrix}, \\
C_\Delta &= \begin{bmatrix} 0.4599 & 0.0189 & 0.0804 & 0.3147 & 0.1684 & 0.2111 & 0.3167 & 0.0977 & 0.1679 \\ 0.2118 & 0.3876 & 0.0747 & 0.1322 & 0.2087 & 0.2006 & 0.0867 & 0.0492 & 0.3997 \end{bmatrix}, \\
\Delta_1 &= \begin{bmatrix} 0.1237 & 0.1223 \\ 0.0942 & 0.0293 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0958 & 0.1372 \\ 0.0093 & 0.1091 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0.1035 & 0.1122 \\ 0.1193 & 0.0495 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0001 & 0.0671 \\ 0.1746 & 0.0708 \end{bmatrix}
\end{aligned}$$

Modified AC9 system

$$B_{\Delta}^T = \begin{bmatrix} 0.2911 & 0.3110 & 0.1462 & 0.2964 & 0.3551 & 0.0966 & 0.0516 & 0.0896 & 0.1393 & 0.1143 \\ 0.3691 & 0.0204 & 0.2359 & 0.0648 & 0.3337 & 0.0667 & 0.1999 & 0.3977 & 0.1414 & 0.0187 \end{bmatrix}$$

$$C_{\Delta} = \begin{bmatrix} 0.1202 & 0.2758 & 0.2005 & 0.3071 & 0.3533 & 0.1315 & 0.1279 & 0.2939 & 0.0463 & 0.0093 \\ 0.0354 & 0.2762 & 0.1248 & 0.2039 & 0.3294 & 0.2016 & 0.2298 & 0.2752 & 0.3034 & 0.1529 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0704 & 0.1100 \\ 0.1312 & 0.0757 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.1194 & 0.1227 \\ 0.0871 & 0.0558 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0.1365 & 0.0886 \\ 0.0008 & 0.1163 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0315 & 0.1032 \\ 0.1260 & 0.1117 \end{bmatrix}$$

Modified AC15 system

$$B_{\Delta}^T = \begin{bmatrix} 0.0637 & 0.3367 & 0.0498 & 0.3052 \\ 0.2307 & 0.3636 & 0.3338 & 0.4218 \\ 0.4159 & 0.1560 & 0.3262 & 0.0923 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.3155 & 0.4589 & 0.4242 & 0.1216 \\ 0.1837 & 0.0175 & 0.3698 & 0.1558 \\ 0.1707 & 0.4112 & 0.0459 & 0.3157 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0031 & 0.0480 & 0.0618 \\ 0.0745 & 0.0905 & 0.0860 \\ 0.0500 & 0.0610 & 0.0806 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0672 & 0.1033 & 0.0196 \\ 0.0213 & 0.0033 & 0.1141 \\ 0.0280 & 0.0571 & 0.0831 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0726 & 0.0990 & 0.0757 \\ 0.0684 & 0.0062 & 0.0140 \\ 0.0087 & 0.0104 & 0.1187 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0804 & 0.0649 & 0.0639 \\ 0.0711 & 0.0510 & 0.0788 \\ 0.0147 & 0.0957 & 0.0447 \end{bmatrix}$$

Modified AC17 system

$$B_{\Delta} = \begin{bmatrix} 0.1462 & 0.2304 & 0.3153 & 0.0251 \\ 0.3333 & 0.1848 & 0.2733 & 0.1295 \\ 0.1619 & 0.3464 & 0.2865 & 0.0017 \\ 0.0925 & 0.3603 & 0.3468 & 0.3192 \end{bmatrix}, C_{\Delta} = \begin{bmatrix} 0.0576 & 0.1217 & 0.3314 & 0.4104 \\ 0.0607 & 0.3968 & 0.1556 & 0.3572 \\ 0.3994 & 0.0267 & 0.1841 & 0.0318 \\ 0.2767 & 0.2101 & 0.2418 & 0.1279 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0563 & 0.0321 & 0.0455 & 0.0393 \\ 0.0406 & 0.0076 & 0.0137 & 0.0133 \\ 0.0252 & 0.0094 & 0.0492 & 0.0632 \\ 0.0594 & 0.0076 & 0.0998 & 0.0971 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0319 & 0.0677 & 0.0523 & 0.0801 \\ 0.0038 & 0.0418 & 0.0575 & 0.0428 \\ 0.0130 & 0.0611 & 0.0803 & 0.0129 \\ 0.0070 & 0.0640 & 0.0606 & 0.0062 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0572 & 0.0426 & 0.0399 & 0.0193 \\ 0.0247 & 0.0573 & 0.0750 & 0.0549 \\ 0.0538 & 0.0405 & 0.0312 & 0.0673 \\ 0.0523 & 0.0391 & 0.0094 & 0.0777 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0192 & 0.0456 & 0.0270 & 0.0693 \\ 0.0679 & 0.0443 & 0.0080 & 0.0160 \\ 0.0522 & 0.0582 & 0.0738 & 0.0098 \\ 0.0360 & 0.0665 & 0.0348 & 0.0809 \end{bmatrix}$$

Modified HE4 system

$$B_{\Delta}^T = \begin{bmatrix} 0.1677 & 0.0014 & 0.0621 & 0.1371 & 0.1744 & 0.1203 & 0.2021 & 0.1830 \\ 0.2395 & 0.0546 & 0.1572 & 0.1196 & 0.3053 & 0.2925 & 0.2790 & 0.1386 \\ 0.2348 & 0.2842 & 0.1166 & 0.0363 & 0.1641 & 0.1066 & 0.2654 & 0.0582 \\ 0.0652 & 0.1657 & 0.0263 & 0.0752 & 0.1983 & 0.0347 & 0.1898 & 0.2780 \end{bmatrix}$$

$$C_{\Delta} = \begin{bmatrix} 0.1227 & 0.2765 & 0.1709 & 0.1227 & 0.0710 & 0.0542 & 0.0562 & 0.1699 \\ 0.2542 & 0.1658 & 0.2155 & 0.1823 & 0.1005 & 0.0276 & 0.1711 & 0.2431 \\ 0.1595 & 0.1418 & 0.1693 & 0.0498 & 0.0855 & 0.1956 & 0.2001 & 0.1906 \\ 0.2060 & 0.1499 & 0.2037 & 0.2471 & 0.2728 & 0.0271 & 0.2510 & 0.2213 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.0433 & 0.0570 & 0.0391 & 0.0547 \\ 0.0277 & 0.0428 & 0.0431 & 0.0079 \\ 0.0673 & 0.0387 & 0.0760 & 0.0099 \\ 0.0842 & 0.0164 & 0.0632 & 0.0508 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0.0472 & 0.0643 & 0.0811 & 0.0197 \\ 0.0557 & 0.0492 & 0.0298 & 0.0826 \\ 0.0039 & 0.0164 & 0.0519 & 0.0068 \\ 0.0592 & 0.0771 & 0.0243 & 0.0200 \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} 0.0838 & 0.0086 & 0.0144 & 0.0314 \\ 0.0495 & 0.0756 & 0.0556 & 0.0346 \\ 0.0502 & 0.0767 & 0.0155 & 0.0077 \\ 0.0817 & 0.0184 & 0.0100 & 0.0588 \end{bmatrix}, \Delta_4 = \begin{bmatrix} 0.0444 & 0.0540 & 0.0086 & 0.0591 \\ 0.0801 & 0.0604 & 0.0718 & 0.0232 \\ 0.0173 & 0.0307 & 0.0362 & 0.0482 \\ 0.0746 & 0.0386 & 0.0543 & 0.0301 \end{bmatrix}$$

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