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Abstract

We introduce the Maker-Breaker domination game, a two player game on a graph. At his turn, the first player, Dominator, select a vertex in order to dominate the graph while the other player, Staller, forbids a vertex to Dominator in order to prevent him to reach his goal. Both players play alternately without missing their turn. This game is a particular instance of the so-called Maker-Breaker games, that is studied here in a combinatorial context. In this paper, we first prove that deciding the winner of the Maker-Breaker domination game is $PSPACE$-complete, even for bipartite graphs and split graphs. It is then showed that the problem is polynomial for cographs and trees. In particular, we define a strategy for Dominator that is derived from a variation of the dominating set problem, called the *pairing dominating set* problem.

Key words: positional games; Maker-Breaker domination game; domination game; complexity; tree; cograph;

AMS Subj. Class: 05C57, 05C69, 91A43

1 Introduction

Since their introduction by Erdős and Selfridge in [9], positional games have been widely studied in the literature (see [11] for a recent survey book on the topic). These games are played on an hypergraph of vertex set X, with a finite set $\mathcal{F} \subseteq 2^X$ of hyperedges. The set X is often called the *board* of the game, and an element of $\mathcal{F}$ a *winning set*. The game involves two players that alternately occupy a previously unoccupied vertex of X. The winner is determined by a convention: in the *Maker-Maker* convention, the first player to occupy all the vertices of a winning set is the winner. Such games may end in a draw, as it is the case in Tic-Tac-Toe. In the *Maker-Breaker* convention, the objectives are opposite: one player (the *Maker*) aims to occupy all the vertices of a winning set, whereas *Breaker* wins if he occupies a vertex in every winning set. In view of the complexity of solving both kinds of games, Maker-Breaker instances are generally more considered in the literature as by definition, there is always a winner. In addition, rulesets of such games are often built from a graph. For example, one can mention the famous *Shannon switching game* (popularized as the game BRIDG-IT) [15], where, given a graph $G = (V, E)$ and two particular vertices u and v, the board X corresponds to E, and winning sets correspond to all the subsets of E corresponding to a $u - v$ path in G. In the *Hamiltonicity* game [6], the winning sets correspond to all the sets of edges containing an Hamiltonian cycle.

In view of such examples, converting a graph property into a 2-player game is a natural operation. Hence it is not surprising that it has also been done for dominating sets. More precisely, several games having different rulesets and known as *domination game* have been defined in the literature. For example, in [1, 10], a move consists in orienting an edge of a given graph G and the two players try to maximize (resp. minimize) the domination number of the resulting digraph. In [5], the rules require two colors during the play. In [3], the domination game is defined in a sense where the players both select vertices and try to maximize (resp. minimize) the length of the play before building a dominating set. Since then, this version has become the standard one for the domination game, with regular progress on it [4, 8, 14, 13]. However, among the different variants of the domination game , the natural Maker-Breaker version (in the sense of Erdős and Selfridge) has never been considered in the literature. In this paper, we consider the so-called *Maker Breaker Domination game*, where, given a graph $G = (V, E)$, the board X is the set V, and $\mathcal{F}$ is the set of all the dominating sets of G. In other words, the two players alternately occupy a not yet occupied vertex of G. Maker wins if he manages to build a dominating set of G, whereas Breaker wins if she manages to occupy a vertex and all its neighbors. In what follows and in order to be consistent with the standard domination game, Maker will be called *Dominator*, and Breaker will be the *Staller*.

When dealing with Maker-Breaker games, there are two main questions that naturally arise:

- Given a graph G, which player has a winning strategy for the Maker-Breaker domination game on G ?
- If Dominator has a winning strategy on G, what is the minimum number of turns needed to win?

The current paper is about the first question. In the next section, we give definitions for the different cases about the winner, together with first general results. Section 3 deals with the algorithmic complexity of the problem, where the $PSPACE$-completeness is proved. In Section 4, a so-called *pairing strategy* is given, yielding a strategy for Dominator in graphs having certain properties. The last section is about graph operators that lead to polynomial strategies on trees and cographs.

2 Preliminaries

A *position* of the Maker-Breaker domination game is denoted by a triplet $G = (V, E, c)$, where V is a set of vertices, E is a set of edges on V and c is a function $c : V \rightarrow \{Dominator, Staller, Unplayed\}$. In other words, the function c allows to describe any game position encountered during the play. If, for all u in V, $c(u) = Unplayed$, then G is said to be a *starting position.* In this case, we will identify G with the graph (V, E).

As by definition, Maker-Breaker games have no draw, there are four cases - also called *outcomes* - to characterize the winner of the game, according to who starts. We define $\mathcal{D}$, $\mathcal{S}$, $\mathcal{N}$ and $\mathcal{P}$ as the different possible outcomes for a position of the Maker-Breaker domination game.

Definition 1. A position G has four possible outcomes:

- $\mathcal{D}$ if Dominator has a winning strategy as first and second player,
- $\mathcal{S}$ is Staller has a winning strategy as first and second player,

- $\mathcal{N}$ if the next player (i.e., the one who starts) has a winning strategy,
- $\mathcal{P}$ otherwise (i.e., the second player wins).

Note that for proximity reasons, the notion of outcome and the last two notations are derived from combinatorial game theory [17]. In addition, the outcome of G is denoted $o(G)$.

If Staller has a winning strategy, then she manages to occupy the closed neighborhood of a certain vertex u. In this case, we say that she *isolates* u.

The following proposition is a direct application of a general result on Maker-Breaker games stated in [11, 2]. It ensures that the outcome $\mathcal{P}$ never occurs. For the sake of completeness, we here give a proof of this result adapted to our particular case.

Proposition 2 (Imagination strategy)**.** *There is no position G of the Maker-Breaker domination game such that $o(G) = \mathcal{P}$.*

Proof. Assume Dominator wins playing second on G. When he is the first player, he plays any vertex and then imagines he did not. He thus considers himself as the second player, seeing this vertex as an extra vertex. Whenever his winning strategy (as a second player) requires to play the extra vertex, he plays any other unoccupied vertex u, and considers u as the new extra vertex. If Dominator was winning before all the vertices were chosen, he still wins no later than his last move in the game where he was playing second. Otherwise, when Staller chooses the last vertex of the graph, her strategy asks her to play the extra vertex since it is the only one available in the imagined game, but it means that Dominator had already won on the previous turn.

Hence if Dominator wins as second player he also wins as first player and the outcome of the game is $\mathcal{D}$. If he does not win as second player, then the outcome can be either $\mathcal{S}$ or $\mathcal{N}$. $\square$

In other words, this proof ensures that a player has no interest to miss his/her turn. Figure 1 gives an example of graphs for the three remaining outcomes.

According to the three possible outcomes of a position, we now introduce an order relation on the outcomes derived from combinatorial game theory: $\mathcal{S} \prec \mathcal{N} \prec \mathcal{D}$. This allows us to state the following proposition.

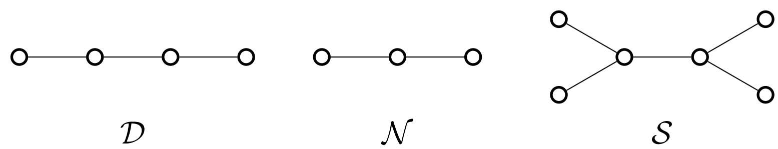

Figure 1: Example of a graph for each possible outcome.

Proposition 3. *Let $G = (V, E, c)$ be a position of the Maker-Breaker domination game and let $H = (V, E', c)$ be another position, with $E' \subseteq E$. Then $o(H) \preceq o(G)$.*

Proof. A reformulation of the proposition is that if Dominator has a winning strategy on H as first or second player, then he also have a winning strategy on G.

Assume Dominator has a winning strategy on H. A winning strategy for Dominator on G is to apply the same strategy as on H. Indeed, for every possible sequence of moves of Staller, Dominator is able to dominate H. Since every edge of H is also in G, Dominator is also able to dominate G. □

In other words, this result says that adding edges to a position can only benefit Dominator , and removing edges can only benefit Staller. Note that this property does not hold in the standard domination game.

Another result can be derived from Maker-Breaker games. The following theorem is a well known result from the early studies about positional games.

Theorem 4 (Erdős-Selfridge Criterion [9])**.** *Given a Maker-Breaker game G on an hypergraph $(X, \mathcal{F})$, we have*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2} \Rightarrow G \textit{ is a Breaker's win}.$$

In order to apply this theorem to the Maker-Breaker domination game, we need to consider a reverse version of it. Indeed, as the set $\mathcal{F}$ corresponds to the dominating sets of G, the sizes of the winning sets are not easy to control. Thus, we can also consider the Maker-Breaker domination game as the Maker-Breaker game where $\mathcal{F}$ is the set of the closed neighborhoods of every vertex of G. In that case, Dominator is the Breaker, and Staller is the Maker. Nom Theorem 4 can be applied on this game:

Proposition 5. *Let G be a starting position of the Maker-Breaker domination game and let δ be the minimum degree of G. If $|V| < 2^{\delta}$ then Dominator has a winning strategy for the Maker-Breaker domination game on G.*

Proof. As stated before, the Maker-Breaker domination game on G is a Maker-Breaker game played on $\mathcal{H} = (V, \mathcal{F})$ where $\mathcal{F}$ is the set of the closed neighborhoods of G, and Staller plays the role of Maker in this game. Applying the Erdős-Selfridge Criterion, we know that if $\sum_{u \in \mathcal{V}} 2^{-|N[u]|} < \frac{1}{2}$ then Dominator has a winning strategy. For all u in V, we have $N[u] \geq \delta + 1$, hence $2^{-|N[u]|} \leq 2^{-(\delta+1)}$. Thus if $|V| \times 2^{-(\delta+1)} < \frac{1}{2}$ then Dominator has a winning strategy. □

Although this result can only be applied for very particular graphs (hypercubes with at least an additional edge for example), it suggests that highly connected graphs are more advantageous for Dominator.

3 Complexity

In this section, we consider the computational complexity of deciding whether a game position of the Maker-Breaker domination game is $\mathcal{S}$, $\mathcal{N}$, or $\mathcal{D}$. First remark that in the general case, deciding the outcome of a Maker-Breaker game $(X, \mathcal{F})$ is $PSPACE$-complete. Indeed, this game exactly corresponds to the game POS CNF that was proved to be $PSPACE$-complete in [16].

POS CNF is played on a formula F in conjunctive normal form, with variables $x_1 \cdots x_n$, where each variable is positive, that is $F = C_1 \wedge \cdots \wedge C_m$ with $C_i = x_{i_1} \vee \cdots \vee x_{i_{k_i}}$. Two players, Prover and Disprover, alternate turns in choosing a variable that has not been chosen yet. When all variables have been chosen, variables chosen by Prover are set to true, while variables chosen by Disprover are set to false. Prover wins if F is true under this valuation and Disprover wins otherwise. Without loss of generality, we can consider that each variable appears in the formula, otherwise we consider the formula $F' = F \wedge (x_1 \vee \cdots \vee x_n)$. Clearly, any Maker-Breaker game $(X, \mathcal{F})$ is equivalent to a POS CNF game, as X corresponds to the set of variables, and the winning sets correspond to the clauses. Prover has the same role as Breaker, and Maker the role of Disprover.

The next result shows that the complexity of this game remains $PSPACE$-complete when reduced to instances of the domination game.

Theorem 6. *Deciding the outcome of a Maker-Breaker domination game position is $PSPACE$-complete on bipartite graphs.*

Proof. We reduce the problem from POS CNF. Let $F = C_1 \wedge \cdots \wedge C_m$ be a positive formula in conjunctive normal form using n variables $X_1 \cdots X_n$. We build a bipartite graph $G = (V, E)$ from F as follows: $V = \{x_i | 1 \leq i \leq n\} \cup \{c_j^k | 1 \leq j \leq m, 0 \leq k \leq 1\}$, $E = \{(x_i, c_j^k) | 1 \leq i \leq n, 1 \leq j \leq m, 0 \leq k \leq 1, X_i \in C_j\}$. Figure 2 shows an example of such a construction, from the example where $F = (X_1 \vee X_2) \wedge (X_1 \vee X_n) \wedge \ldots \wedge (X_2 \vee X_3 \vee X_n)$.

We now show that Prover has a winning strategy in F if and only if Dominator has a winning strategy in G.

Assume first Prover has a winning strategy in F. Dominator builds his strategy in G as follows: Whenever Prover's strategy requires to choose a variable X_i, Dominator chooses the vertex x_i. Whenever Staller chooses a vertex c_j^k, Dominator answers by choosing the vertex c_j^{1-k}. Whenever Staller chooses a vertex x_i, Dominator assumes Disprover chose the variable X_i. When all vertices are chosen, since Prover was winning in F, for each vertex c_j^k, there is a neighbor x_i that was chosen by Dominator. As all variables are in a clause, and for each j, Dominator chose either c_j^0 or c_j^1, all vertices of the form x_i are also dominated by Dominator's choice of vertices. Hence Dominator wins the game.

Assume now Disprover has a winning strategy in F. Staller builds his strategy in G as follows: Whenever Disprover's strategy requires to choose a variable X_i, Staller chooses the vertex x_i. Whenever Dominator chooses a vertex c_j^k, Staller answers by choosing the vertex c_j^{1-k}. Whenever Dominator chooses a vertex x_i, Staller assumes Prover chose the variable X_i. When all vertices are chosen, since Disprover was winning in F, there exists a couple of vertices (c_j^0, c_j^1) such that none of their neighbours was chosen by Dominator. As Staller managed to choose one among these two, this particular one is not dominated by Dominator's choice of vertices. Hence Staller wins the game. $\square$

Corollary 7. *Deciding the outcome of a Maker-Breaker domination game position is $PSPACE$-complete on chordal graphs, and also in particular on split graphs.*

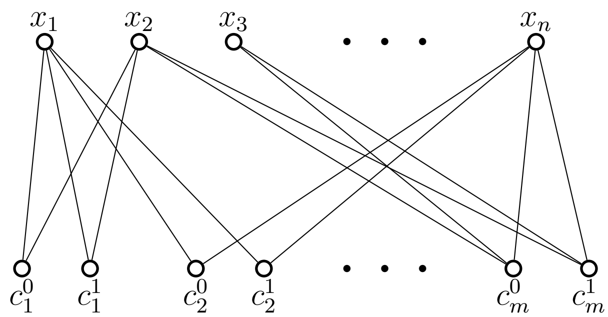

Figure 2: Reduction from POS CNF on $(X_1 \vee X_2) \wedge (X_1 \vee X_n) \wedge \ldots \wedge (X_2 \vee X_3 \vee X_n)$.

Proof. This proof of Theorem 6 remains valid by adding edges between the vertices x_i. In particular, it works if they form a clique, so that the resulting graph is a split graph, that is special case of chordal graphs. $\square$

In view of these complexity results, the question of the threshold between $PSPACE$-completeness and polynomiality is of natural interest. The following section is a first step towards it, with a characterization of a certain structure in the graph that induces a natural winning strategy for Dominator .

4 Pairing strategy

A natural winning strategy for Breaker in a Maker-Breaker game is the so-called *pairing strategy* as defined in [11]. This strategy can be applied when a subset of the board X can be partitioned into pairs such that each winning set contains one of the pairs. In that case, a strategy for Breaker as a second player consists in occupying the other element of the pair that has been just occupied by Maker. By doing so, Breaker will occupy at least one element in each winning set and thus win the game. In the context of the Maker-Breaker domination game , such a subset can be described as follows:

Definition 8. Given a graph $G = (V, E)$, a subset $\{(u_1, v_1), \ldots, (u_k, v_k)\}$ of V is a pairing dominating set if by choosing any vertex in (u_i, v_i) for each $1 \leq i \leq k$, the resulting set (of size k) is a dominating set of G.

Figure 3 shows an example of a pairing dominating set.

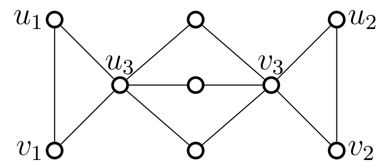

Figure 3: The set $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ is a pairing dominating set.

Another way to consider a pairing dominating set is from the union of the closed neighborhoods of each pair:

Proposition 9. *Given a graph $G = (V, E)$, a set $D = \{(u_1, v_1), \ldots, (u_k, v_k)\}$ is a pairing dominating set of G if and only if it satisfies*

$$V = \bigcup_{i=1}^{k} N[u_i] \cap N[v_i]$$

.

Proof. Let $S = \bigcup_{i=1}^{k} N[u_i] \cap N[v_i]$. If $S = V$, then by definition of S, choosing any element in (u_i, v_i) for all i builds a dominating set of G. On the contrary, if $S \neq V$, then there exists a vertex $w \in V \setminus S$ such that for all i, w is not a neighbor of either u_i or v_i. By choosing these non-adjacent vertices to w, they do not form a dominating set of G as w is not dominated. $\square$

From this property, we will say that a vertex w is *pairing dominated* if there exists a pair (u, v) from a pairing dominated set such that $w \in N[u] \cap N[v]$. In addition, all the pairs (u, v) satisfying $N[u] \cap N[v] = \emptyset$ are useless in the construction of a pairing dominating set.

Hence the pairing strategy applied to the Maker-Breaker domination game can be translated into a strategy on a pairing dominating set:

Proposition 10. *If a graph G admits a pairing dominating set, then $o(G) = \mathcal{D}$.*

Proof. If G admits a pairing dominating set, then Dominator applies the following strategy as a second player: each time Staller occupies a vertex of a pair (u_i, v_i) for some i, Dominator answers by occupying the other vertex of the same pair if it is not yet occupied. Otherwise, Dominator plays randomly.

By definition of a pairing dominating set, it ensures that the vertices chosen by Dominator form a dominating set of G. $\square$

This result induces the following corollary that ensures a winning strategy for Dominator as a first player.

Corollary 11. *Given a graph G, if there exists a vertex u of G such that $G \setminus N[u]$ admits a pairing dominating set, then $\mathcal{N} \preceq o(G)$.*

Proof. If such a vertex exists, then Dominator starts and occupy it. He then applies his pairing strategy on $G \setminus N[u]$ as a second player to dominate the rest of the graph. $\square$

From this property, a natural question that arises is the detection of graphs having a pairing dominating set. An example of such graphs is when the vertices of the graph can be partitioned into cliques of size at least 2. In that case, a trivial pairing dominating set consists in choosing any two vertices in each clique. Note that the question of the existence of such a partition is often referred to as the *packing by cliques* problem (with cliques of size at least 2). It was proved to be polynomial by Hell and Kirkpatrick in [12]. A particular case of this decomposition is when the graph admits a perfect matching. As an example, Proposition 10 ensures that paths or cycles of even size are $\mathcal{D}$ as they have a perfect matching.

Remark 12. The condition of Proposition 10 is not necessary. Indeed, the graphs of Figure 4 are examples with outcome $\mathcal{D}$ and it can be shown that they do not admit a pairing dominating set. Yet, we will see in Section 5 two families of graphs (cographs and trees) for which there is an equivalence between the existence of a winning strategy for Dominator and the existence of a pairing dominating set.

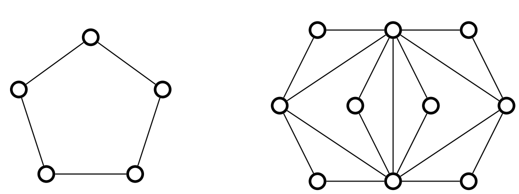

Figure 4: Graphs with outcome $\mathcal{D}$ and without a pairing dominating set.

We conclude this section with a study of the complexity of the pairing dominating set problem.

Theorem 13. *Given a graph G, it is NP-complete to decide whether G admits a pairing dominating set.*

Proof. Let $G = (V, E)$ be a graph. According to Proposition 9, the problem is clearly in NP. It remains to prove the NP-hardness of the problem by reducing it from 3-SAT. Let $F = C_1 \vee \cdots \vee C_m$ be an instance of 3-SAT over the variables $X_1, \ldots, X_n$. Without loss of generality, one can assume that all the variables appear in both their positive and negative version in F, but not in the same clause. From F, we build the following graph G as illustrated by Figure 5.

- For $1 \leq j \leq m$, to each clause C_j we associate a vertex c_j.
- Each variable X_i, $1 \leq i \leq n$ is associated to a gadget over seven vertices $\{x_i, y_i, z_i, x'_i, y'_i, z'_i, t_i\}$ such that $x_i y_i z_i$ and $x'_i y'_i z'_i$ are two triangles, and t_i is adjacent to both x_i and x'_i. The pairs (x_i, y_i) and (x'_i, y'_i) will be denoted e_i and $\overline{e_i}$ respectively.
- For each variable X_i, we add the two edges $c_j x_i$ and $c_j y_i$ (resp. $c_j x'_i$ and $c_j y'_i$) each time X_i appears in clause C_j in its positive (resp. negative) form.

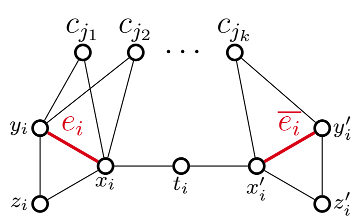

Figure 5: Gadget around a variable X_i for the proof of NP-completeness. The clauses $C_{j_1}, \ldots, C_{j_k}$ are those where the variable X_i appears.

We first claim that any assignment of the variables $X_1, \ldots, X_n$ that makes F satisfiable induces a pairing dominating set in G. Let σ be such an assignment. Now build the following set D of pairs of vertices: for each variable X_i, we add the pairs $\{(x_i, y_i), (t_i, x'_i), (y'_i, z'i)\}$ to D if X_i is TRUE in σ , and the pairs $\{(x'_i, y'_i), (t_i, x_i), (y_i, z_i)\}$ otherwise. It now suffices to check that D is a pairing dominating set. First of all, one can easily remark that all the vertices of the gadgets (i.e., vertices different from the clauses c_j) are pairing dominated by D. In addition, as each clause C_j is satisfied by σ, each vertex c_j is adjacent to at least one pair (x_i, y_i) or (x'_i, y'_i) of D. Hence any choice of vertex in such a pair allows to dominate c_j.

Now consider a pairing dominating set D of G. We first show that for each gadget associated a variable X_i, up to symmetry, there are only four cases to dominate the vertices t_i, z_i and z'_i, depicted by Figure 6. Indeed, for each vertex t_i, as it is of degree 2, there are three cases for it to be pairing dominated by D: either the pair (t_i, x'_i), or (t_i, x_i), or (x_i, x'_i) belongs to D.
(i) the pair (t_i, x'_i) belongs to D (cases (a), (b) and (c) of Figure 6). Then, by considering the vertex z'_i, the pair (y'_i, z'_i) must belong to D. Indeed, if z'_i is not in a pair of D, then one can always find vertices from D different from x'_i and y'_i such that z'_i is not dominated. The same argument holds if z'_i is in D, but not in a pair with y'_i. Concerning the vertex z_i, it is necessarily dominated by vertices from the triangle $x_i y_i z_i$, leading to the three cases (a), (b) and (c) of Figure 6.
(ii) the pair (t_i, x_i) belongs to D. By symmetry of the gadget, this case is similar to the previous one and we get the symmetric pairs from figures (a), (b) and (c).
(iii) the pair (x_i, x'_i) belongs to D (Figure 6 (d)). Then both vertices z_i and z'_i must belong to D in the pairs (y_i, z_i) and (y'_i, z'_i). Indeed, if z_i (or, by symmetry z'_i) is not in D, then one can always choose vertices from D different from x_i and y_i such that z_i is not dominated. The same argument holds if z_i is in D, but not in a pair with y_i.

In order to find an assignment for F, we now show that D can be transformed into a pairing dominating set where each pair is as in Figure 6 (a) (or its symmetrical, according to case (ii)). Consider first that for the gadget associated to some variable X_i, the pairs of D are those depicted by Figure 6 (b). As the vertex z_i has no other neighbor than x_i and y_i, replacing a pair (x_i, z_i) by the pair (x_i, y_i) in D remains a valid pairing dominating set since both x_i and y_i are adjacent to z_i. This operation is clearly possible if y_i is not in D. In the case where y_i is already in D, say in a pair (y_i, u), remark that removing this pair from D does not break the pairing dominating property of D if (x_i, y_i) is added. Indeed, as by definition of G,

x_i and y_i have the same neighborhood (except t_i, that is already in a pair), we have that $N[u] \cap N[y_i] \subseteq N[x_i] \cap N[y_i]$. Since x_i and y_i play a symmetrical role, we can use the same argument to replace the pairs of Figure 6 (c) by those of (a) in D. The last case is when the pairs of D are those of Figure 6 (d) for the variable X_i. Since $N[y_i] \cap N[z_i] \subseteq N[y_i] \cap N[x_i]$ and $N[x_i] \cap N[x'_i] = \{t_i\} \subset N[t_i] \cap N[x'_i]$ (as X_i and $\overline{X_i}$ cannot be in the same clause), we can replace the pairs of Figure 6 (d) by those of Figure 6 (a) without breaking the pairing dominating property of D. In case t_i was already in D, say in a pair (t_i, u), once again this pair can be removed from D as $N[t_i] \cap N[u]$ is either empty or at most a subset of $\{x_i, x'_i\}$, which is already pairing dominated by the pairs of Figure 6 (a).

Hence we have transformed D such that all the vertices different from the c_j are pairing dominated by the pairs of vertices of Figure 6 (a). In addition, if D admits other pairs than those depicted by Figure 6 (a), then these pairs are necessarily of the form (c_j, c_l), (z_i, u), or (z'_i, u). The last two types of pairs can be removed from D as $N[z_i]$ and $N[z'_i]$ is already pairing dominated. Concerning the pairs (c_j, c_l), they can also be removed from D as the sets $N[c_j] \cap N[c_l]$ belong to the gadgets (and are different from the clause vertices), and are thus already pairing dominated.

We now build the following assignment of the variables of F: for all $1 \leq i \leq n$, the variable X_i is set to TRUE if and only if the pair e_i belongs to D. As each vertex c_j is pairing dominated in D by at least a pair e_i or $\overline{e_i}$ for some i, it means that each corresponding clause C_j has at least a variable equal to TRUE, which concludes the proof.

□

5 Graph operations

In the first part of this section, we study the outcome of operations of graphs for which the outcome is already known. This will lead to polynomial time algorithms to solve the Maker-Breaker domination game on cographs and forests, as these families can be built from joins, unions and by adjoining hanging edges.

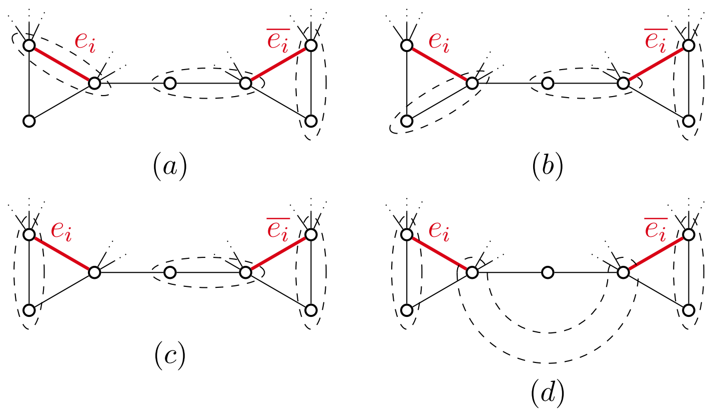

Figure 6: Possible pair dominating sets for the gadget of the proof of Theorem 13 (up to symmetry).

5.1 Union and join

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs on disjoint sets of vertices. The *union* $G \cup H$ of G and H is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. The *join* $G \bowtie H$ of G and H is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H \cup \{uv | u \in V_G, v \in V_H\}$.

Theorem 14. *Let G and H be two starting positions of the Maker-Breaker domination game.*

- *If $o(G) = \mathcal{S}$ or $o(H) = \mathcal{S}$ then $o(G \cup H) = \mathcal{S}$.*
- *If $o(G) = o(H) = \mathcal{N}$ then $o(G \cup H) = \mathcal{S}$.*
- *If $o(G) = o(H) = \mathcal{D}$ then $o(G \cup H) = \mathcal{D}$.*
- *Otherwise, $o(G \cup H) = \mathcal{N}$.*

This result is summarized in Table 1. Note that the outcome $\mathcal{S}$ is absorbing for the union, while the outcome $\mathcal{D}$ is neutral.

Proof. Assume Staller has a winning strategy on G or H. Then she has a winning strategy on $G \cup H$. Indeed, without loss of generality assume that

$o(G)$ \ $o(H)$	$\mathcal{D}$	$\mathcal{N}$	$\mathcal{S}$
$\mathcal{D}$	$\mathcal{D}$	$\mathcal{N}$	$\mathcal{S}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{S}$	$\mathcal{S}$
$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$

Table 1: Outcomes of the Maker-Breaker domination game played on the union of G and H.

she has a winning strategy on G. Her strategy on $G \cup H$ is to play only on G following her winning strategy. If at some point Dominator is playing on H, this can be considered as a passing move in G and by Proposition 2 this does not compromise Staller's strategy. At some point she will isolate a vertex in G and thus in $G \cup H$.

Thus if G or H has outcome $\mathcal{S}$, then whatever Dominator plays as a first move, Staller still has a winning strategy on this graph. If both positions have outcome $\mathcal{N}$ then after Dominator's first move, Staller can play on the other component and also wins. This proves the first two points.

If both positions have outcome $\mathcal{D}$, then Dominator has a winning strategy on both graphs playing second. He can answer to every move of Staller on the component she plays with his winning strategy on this component. At the end, Dominator dominates both components and so $G \cup H$ has outcome $\mathcal{D}$.

Finally, assume without loss of generality that $o(G) = \mathcal{N}$ and $o(H) = \mathcal{D}$. If Staller plays first, as in the first case, by applying her winning strategy as the first player in G she will be able to isolate a vertex and to win. On the other hand, if Dominator plays first, he can play his winning move on G and then answers to Staller on the component she has played on with his winning strategy. So the first player has a winning strategy and the outcome is $\mathcal{N}$.

□

Theorem 15. *Let G and H be two starting positions of the Maker-Breaker domination game.*

(i) If $G = K_1$ and $o(H) = \mathcal{S}$ (or $H = K_1$ and $o(G) = \mathcal{S}$) then $o(G \bowtie H) = \mathcal{N}$.

(ii) Otherwise, $o(G \bowtie H) = \mathcal{D}$.

Proof. (i) Assume that $G = K_1$ and $o(H) = \mathcal{S}$. If Dominator starts, he will win by playing on the only vertex of G and dominates the join, so he has a winning strategy as a first player. However, since $o(H) = \mathcal{S}$, if Staller starts, she can play on the only vertex of G and then apply her winning strategy as second player on H. So she wins on $G \bowtie H$ as first player as well as Dominator. So $o(G \bowtie H) = \mathcal{N}$.

(ii) Since we are not in the first case, there are two possibilities : Either both G and H have at least two vertices or, without loss of generality, $G = K_1$ and $o(H) \succeq \mathcal{N}$.

Assume first that both G and H have more than two vertices. Let u_1, v_1 be two vertices of G and u_2, v_2 two vertices of H. Since every vertex of G is a neighbor of every vertex of H and conversely, $\{(u_1, v_1), (u_2, v_2)\}$ forms a pairing dominating set for $G \bowtie H$ and the outcome is $\mathcal{D}$ according to Proposition 10.

Assume now that $G = K_1$ and $o(H) \succeq \mathcal{N}$. Note that Dominator has a winning strategy on H as first player. Assume that Staller is the first player. If on her first move she does not play on the vertex of G, then Dominator wins immediately by playing on it. If she does play on it, then Dominator will apply his winning strategy as first player on H. This will allow him to dominate H and, since each vertex of H dominates G, all the vertices of $G \bowtie H$ will be dominated. Dominator has a winning strategy as second player, hence $o(G \bowtie H) = \mathcal{D}$. □

The combination of these two results gives a complexity result on the class of cographs. Recall that cographs (or P_4-free graphs) can be inductively built from a single vertex by taking the union of two cographs or the join of two cographs. In addition, from a given cograph, recovering this construction from unions and joins can be found with a linear time algorithm [7]. Since we know the outcome of Maker-Breaker domination game for K_1 and for the union and the join operators, we can deduce the following corollary.

Corollary 16. *Deciding the outcome of the Maker-Breaker domination game on cographs can be done in polynomial time.*

As stated in Remark 12, for some families of graphs the outcome of a starting position is $\mathcal{D}$ if and only if it admits a pairing dominating set. We show that the family of cographs satisfies this property.

Theorem 17. *A cograph G has outcome $\mathcal{D}$ if and only if it admits a pairing dominating set.*

Proof. We know from Proposition 10 that if a graph admits a pairing dominating set, then it has outcome $\mathcal{D}$. It remains to prove that all cographs with outcome $\mathcal{D}$ admits a pairing dominating set.

The proof is done by induction on the number n of vertices of G.

First note that the result is true when $n \leq 2$. The only such cographs are K_1, K_2 and $K_1 \cup K_1$, and among them the only graph with outcome $\mathcal{D}$ is K_2. K_2 admits a perfect matching and thus a pairing dominating set.

Assume now that every cograph of outcome $\mathcal{D}$ with a number of vertices less or equal to n admits a pairing dominating set. Let G be a cograph of outcome $\mathcal{D}$ with $n+1$ vertices. By definition of a cograph, G is either the union or the join of two smaller cographs.

If G is the union of two cographs G_1 and G_2, they necessarily have outcome $\mathcal{D}$ by Theorem 14. By induction hypothesis, they both admit a pairing dominating set, which union also is a pairing dominating set for G.

Assume now that G is the join of two cographs G_1 and G_2.

If both G_1 and G_2 have more than two vertices, then if u_1, v_1 are any two vertices of G_1 and u_2, v_2 are any two vertices of G_2, $\{(u_1, v_1), (u_2, v_2)\}$ forms a pairing dominating set for G. Indeed, both u_1 and v_1 are neighbors of every vertices of G_2 and both u_2 and v_2 are neighbors of every vertices of G_1.

Assume now that $G_1 = K_1$ and let x be its unique vertex. Then G_2 has either outcome $\mathcal{N}$ or $\mathcal{D}$ by Theorem 15. If G_2 has outcome $\mathcal{D}$ then by induction hypothesis, it admits a pairing dominating set. Every vertex of this pairing dominating set is a neighbor of x and it remains also a pairing dominating set for G.

Assume now that $o(G_2) = \mathcal{N}$. G_2 is either the union of two cographs or the join of two cographs.

If G_2 is the join of two cographs, by Theorem 15, it must be the join of a graph K_1 with vertex y and of a graph H with outcome $\mathcal{S}$. Notice that x and y are both universal vertices so $\{(x, y)\}$ is a pairing dominating set for G. If G_2 is the union of H_1 and H_2 then, without loss of generality, by Theorem 14 $o(H_1) = \mathcal{D}$ and $o(H_2) = \mathcal{N}$. By induction hypothesis, H_1 admits a pairing dominating set S_1. Note also that by Theorem 15, $x \bowtie H_2$ has outcome $\mathcal{D}$, so by induction hypothesis it admits a pairing dominating set S_2. Since S_1 dominates H_1 and S_2 dominates $\{x\} \cup H_2$, $S_1 \cup S_2$ forms a pairing dominating set for G. $\square$

5.2 Glue operator and trees

We now study the operator consisting of gluing two graphs on a vertex. This operator will be useful in the study of trees. A more formal definition is the following:

Definition 18. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs and let $u \in V_G$ and $v \in V_H$ be two vertices. The *glued graph* of G and H at u and v is the graph $G{}_u\!\bowtie_v H$ with vertex set $(V_G \setminus \{u\}) \cup (V_H \setminus \{v\}) \cup \{w\}$ (where w is a new vertex) and for which xy is an edge if and only if xy is an edge of G or H or $y = w$ and xu is and edge of G or xv is an edge of H.

If the vertex u is clear from the context or does not matter, the glue will be denoted by $G \bowtie_v H$. Similarly if the vertex v is not useful in the notation, we might also remove it. Figure 7 gives a representation of the glued of two graphs.

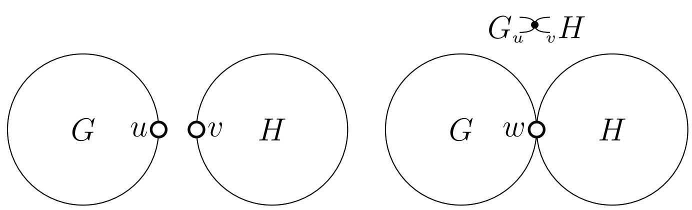

Figure 7: Representation of the glued graph of G and H on u and v.

Let H be a graph and v a vertex of H. We say that the couple (H, v) is *neutral* for the glue operator if for every graph G and every vertex u of G, $o(G{}_u\!\bowtie_v H) = o(G)$.

Theorem 19. *Let H be a graph and v be a vertex of H. (H, v) is neutral for the glue operator if and only if $o(H) = \mathcal{N}$ and $o(H \setminus \{v\}) = \mathcal{D}$.*

Proof. First, let H be a graph and v be a vertex of H. Assume that (H, v) is neutral. Then $o(K_1 \bowtie_v H) = o(K_1)$. Notice that $K_1 \bowtie_v H = H$ and since $o(K_1) = \mathcal{N}$, we necessarily have $o(H) = \mathcal{N}$.

Now consider the graph G that consists of H with a pendant vertex v' attached to v. This corresponds to $K_2 \bowtie_v H$. Since (H, v) is neutral, G has the same outcome as K_2 that is $\mathcal{D}$. In particular, Dominator has a winning strategy on G by playing second. If Staller plays first on v, Dominator has to play on v'. His remaining winning strategy is a winning strategy on $H \setminus \{v\}$.

This proves that the conditions are necessary for (H, v) to be neutral. We now prove that they are sufficient.

Let H be a graph, v be a vertex of H and $H' = H \setminus \{v\}$, such that $o(H) = \mathcal{N}$ and $o(H') = \mathcal{D}$. Let G be a graph and u a vertex of G. In the following, we identify the vertices u and v to w and the glued graph of G and H will be denoted by $G \bowtie H$.

Since $o(H') = \mathcal{D}$, $o(G \cup H') = o(G)$ by Theorem 14. Note that $G \cup H'$ is a subgraph of $G \bowtie H$ where only edges are removed so, by Proposition 3, $o(G \bowtie H) \succeq o(G \cup H') = o(G)$.

We now show that $o(G \bowtie H) \preceq o(G)$ to conclude the proof. Note that if $o(G) = \mathcal{D}$ we necessarily have $o(G \bowtie H) \preceq o(G)$.

Assume that $o(G) \preceq \mathcal{N}$. This means that Staller has a winning strategy on G as first player. Since $o(H) = \mathcal{N}$, Staller also has a winning strategy on H as first player. The following strategy is a winning strategy on $G \bowtie H$ for Staller as first player. Staller begins by applying her winning strategy on H until the strategy requires her to play on w. If during this stage Dominator plays on w, by following her strategy, Staller will isolate a vertex on H different from w. This vertex is not connected to G so she wins. If Dominator plays on $G \setminus \{w\}$ then Staller can imagine that Dominator has played on w and will win similarly. So we can assume that Dominator always answers in H'.

When Staller's strategy on H is to play on w, she switches to her winning strategy on G instead. Similarly as before, if Dominator does not answer in $G \setminus \{w\}$, Staller will win by isolating a vertex of G different from w. She continues to apply her winning strategy on G until this strategy requires her to play on w. Note that at this point w is a winning move for Staller both in G and H.

Staller now plays w and answers to every move of Dominator with her strategy in the same component. Since she follows her winning strategy in G and H she will isolate a vertex in each of these graphs. If one of those two vertices is not w, then Staller wins because this vertex is isolated in $G \bowtie H$. If both of these vertices are w, then w and its whole neighborhood are played by Staller in the glued graph and Staller wins. So Staller has a winning strategy as first player in $G \bowtie H$ and $o(G \bowtie H) \preceq \mathcal{N}$.

Assume now that $o(G) = \mathcal{S}$, i.e. Staller has a winning strategy on G as second player. If Dominator begins by playing on w, then Staller can apply her winning strategy in G, she will isolate a vertex different from w will win. If Dominator begins by playing in H', then Staller can imagine that he played on w, apply her winning strategy on G and win similarly as before. So we can assume that Dominator begins by playing in $G \setminus \{w\}$. Then Staller can follow the same strategy as when she when playing first and she wins. Thus $o(G \bowtie H) = \mathcal{S}$.

These three cases prove that $o(G {}_u\!\bowtie_v H) \preceq o(G)$. Since we also have $o(G {}_u\!\bowtie_v H) \succeq o(G)$, this prove that $o(G {}_u\!\bowtie_v H) = o(G)$. $\square$

A question that could be asked is whether or not neutral graphs exist. We solve it by exhibiting an infinite family of neutral graphs:

Definition 20. For $n \geq 2$, the *hanging split graph* of size n, H_n, is the graph composed of a clique of size n with vertex set $\{u, u_1, \ldots, u_{n-1}\}$ and an independent of size $n-1$ with vertex set $\{v_1, \ldots, v_{n-1}\}$. Add an edge u_iv_i for all $1 \leq i \leq n-1$.

Figure 8 gives a representation of the first two hanging split graphs and of the general case.

Proposition 21. *For all $n \geq 2$, (H_n, u) is neutral for the glue operator.*

Proof. Note that $H_n \setminus \{u\}$ has a perfect matching so it has outcome $\mathcal{D}$ by Proposition 10.

If Dominator plays first on H_n, a winning strategy is to start on u, then the remaining graph has a perfect matching and he will win.

If Staller plays first on H_n, a winning strategy is to play on each u_i. Dominator has to answer on v_i otherwise Staller wins immediately by isolating this vertex. When every u_i is played, she can play on u and isolate it.

So both players have a winning strategy when playing first and thus $o(H_n) = \mathcal{N}$. By Theorem 19, (H_n, u) is neutral. $\square$

An interest of the neutral graphs is that if a graph G is of the form $G' \bowtie_v H$ with (H, v) being neutral, then we can restrict the study of G to the study of G'. In the following, we apply this idea to trees by noticing that P_3 is isomorphic to H_2 and thus neutral.

We define a *P_2-irreducible* graph as a graph without pendant P_2, where a pendant P_2 is a P_2 attached to a graph by an edge.

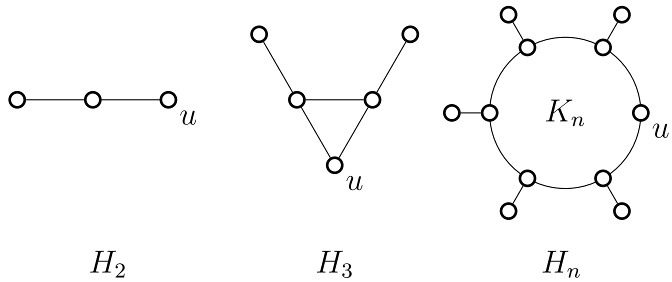

Figure 8: Examples of hanging split graphs.

Lemma 22. *Every P_2-irreducible tree has one of the following form :*

- K_1
- P_2
- $K_{1,n}$ *with* $n \geq 3$
- *Trees where there are at least two vertices with more than two leaves as children.*

Figure 9 shows a representation of these different cases.

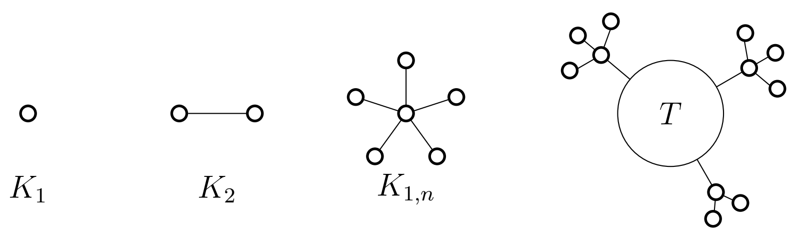

Figure 9: Different possible reductions for trees.

Proof. Let T be a P_2-irreducible graph. If T has only one vertex, $T = K_1$. Otherwise T must have leaves. If a leaf is connected to a vertex of degree 2, then there is a pendant P_2 and T is not irreducible. If there is a leaf connected to a vertex of degree 1, then the only possible case is that $T = K_2$. If we are in none of the previous cases, each leaf parent is connected to at least two leaves. If there is only one leaf parent, then $T = K_{1,n}$ with $n \geq 3$

and otherwise there are at least two leaf parents connected to at least two leaves.

□

Theorem 23. *Deciding the outcome of the Maker-Breaker domination game on trees is polynomial.*

Proof. The following algorithm solves the Maker-Breaker domination game on trees in polynomial time:

For a tree T, iteratively remove a pendant P_2 until it is not possible anymore. Let T' be the obtained tree. If $T' = P_2$, return the answer $\mathcal{D}$. If $T' = K_1$ or $K_{1,n}$ with $n \geq 3$, then return $\mathcal{N}$. Otherwise, return $\mathcal{S}$.

Note that the above algorithm is polynomial. Indeed, removing pendant P_2's can be done in polynomial time by keeping in memory the set of leaves at each time and updating it when necessary. Verifying that a tree is K_1, P_2 or a star can also be done in polynomial time.

We now prove the correctness of the algorithm. Let $T_1, \ldots, T_k$ be the intermediary trees obtained after removing a pendant P_2. From Proposition 21, we know that P_3 is neutral and a pendant P_2 can be seen as the glue with a P_3. So $o(T) = o(T_1) = \ldots = o(T_k) = o(T')$, and the outcome of T is the same as the outcome of T'. As T' is P_2-irreducible so it corresponds to one of the situations described in Lemma 22. If it is a P_2, the outcome is $\mathcal{D}$. If it is K_1 or $K_{1,n}$ with $n \geq 3$, the first player wins by playing on the central vertex and thus the outcome is $\mathcal{N}$. In the last case, two distinct vertices are attached to two leaves or more. Assume that Staller plays second on T'. After Dominator's first move, one of these two vertices and its leaves are unplayed by Dominator. Staller can play this vertex and will isolate one of its leaves after her next move. Hence T' is indeed $\mathcal{S}$ in this last case.

We conclude that the outcome of T is the same as the outcome of T' and the algorithm correctly returns the right output. □

Remark 24. Note that a tree has outcome $\mathcal{D}$ only if by removing pendant P_2's the remaining graph is a P_2. This means that a tree has outcome $\mathcal{D}$ if and only if it admits a perfect matching and thus if and only if there is a pairing dominating set.

6 Conclusion and perspectives

In this paper, the complexity of the Maker-Breaker domination game is studied for different classes of graphs. $PSPACE$-completeness is proved for split and bipartite graphs, whereas polynomial algorithms are given for cographs and trees. An interesting equivalence property is that in these last two cases, the outcome is $\mathcal{D}$ if and only if the graph admits a pairing dominating set. The study of the pairing dominating set problem might be a key in the study of the threshold between $PSPACE$ and P for the Maker-Breaker domination game.

As stated in the introduction, another problem that might be relevant to consider is the number of moves needed by Dominator to win. In particular, it could be worth studying the correlation of this value with the dominating number or the game dominating number.

Also, this game has been built from the dominating set problem. Other remarkable structures in graphs could have been chosen, such as total dominating sets. Another variant would be to consider the game in an oriented version.

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