On path partitions of the divisor graph

Paul Melotti, Eric Saias

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Paul Melotti
Sorbonne Université
LPSM
4 place Jussieu
F-75005 PARIS
FRANCE
E-mail: paul.melotti@upmc.fr

Éric Saias
Sorbonne Université
LPSM
4 place Jussieu
F-75005 PARIS
FRANCE
E-mail: eric.saias@upmc.fr

Abstract

It is known that the longest simple path in the divisor graph that uses integers $\leq N$ is of length $\asymp N/\log N$. We study the partitions of $\{1, 2, \ldots, N\}$ into a *minimal* number of paths of the divisor graph, and we show that in such a partition, the longest path can have length asymptotically $N^{1-o(1)}$.

1 Introduction

The *divisor graph* is the unoriented graph whose vertices are the positive integers, and edges are the $\{a, b\}$ such that $a < b$ and a divides b. A *path* of length l in the divisor graph is a finite sequence $n_1, \ldots, n_l$ of pairwise distinct positive integers such that n_i is either a divisor or a multiple of n_{i+1}, for all i such that $1 \leq i < l$. Let $F(x)$ be the minimal cardinal of a partition of $\{1, 2, \ldots, \lfloor x \rfloor\}$ into paths of the divisor graph.

The asymptotic behaviour of $F(x)$ has been studied in [3, 8, 4, 1]. Thanks to the works of Mazet and Chadozeau, we know that there is a constant $c \in (\frac{1}{6}, \frac{1}{4})$ such that

$$F(x) = cx \left(1 + O\left(\frac{1}{\log\log x \ \log\log\log x}\right)\right). \tag{1}$$

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A partition of $\{1, 2, \ldots, N\}$ into paths of the divisor graph is said to be *optimal* if its cardinal is $F(N)$. We are interested in the length of the paths in an optimal partition.

Let us take the example $N = 30$ that was considered in [5, 8]. It is known (see [8]) that $F(30) = 5$, so that the following partition is optimal:

$13, 26, 1, 11, 22, 2, 14, 28, 7, 21, 3, 27, 9, 18, 6, 12, 24, 8, 16, 4, 20, 10, 30, 15, 5, 25$

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Four of these five paths are singletons. In fact, at the end of the proof of Theorem 2 of [3], it is proven that the number of singletons in a (not necessarily optimal) partition is $\asymp N$ for N large enough.

Let us look at the longest paths in an optimal partition of $\{1, 2, \ldots, N\}$. Let $L(N)$ be the maximal path length, among all paths of all *optimal* partitions of $\{1, 2, \ldots, N\}$ into paths of the divisor graph. Let also $f(N)$ denote the maximal length of a path of the divisor graph that uses integers $\leq N$.

It is known that (Theorem 2 of [7])

$$f(N) \asymp \frac{N}{\log N}. \tag{2}$$

Of course $L(N) \leq f(N)$. In the previous example, four of the five paths are singletons, which implies that the longest path has maximal length. In other words $L(30) = f(30) = 26$. More generally, we know that for all $N \geq 1$,

$$F(N) \geq N - \lfloor N/2 \rfloor - \lfloor N/3 \rfloor \tag{3}$$

(see [8]). Inspired by the case $N = 30$, for any $N \in [1, 33]$ it is easy to construct a partition of $\{1, \ldots, N\}$ into $N - \lfloor N/2 \rfloor - \lfloor N/3 \rfloor$ paths, all of them but one being singletons. This shows that for $1 \leq N \leq 33$, (3) is an equality and $L(N) = f(N) = \lfloor N/2 \rfloor + \lfloor N/3 \rfloor + 1$.

However for larger N the situation becomes more complicated. For N large enough there is no optimal partition with all paths but one being singletons. This can be deduced from (2) and the fact that the constant c in (1) is less than 1. Still, it is natural to wonder if the equality $L(N) = f(N)$ holds for any $N \geq 1$.

We were unable to answer this question, but we looked for lower bounds on $L(N)$ and proved the following:

Theorem. *There is a constant* $A \geq 0$ *such that for all* $N \geq 3$,

$$L(N) \geq \frac{N}{(\log N)^A \exp\left[\frac{(\log\log N)^2}{\log 2}\right]}.$$

To prove this we introduce a new function $H(x)$. For a real number $x \geq 1$ and two distinct integers $a, b \in [1, x]$, let $L_{a,b}(x)$ be the maximal length of a path having a and b as endpoints and belonging to an *optimal* partition of $\{1, 2, \ldots, \lfloor x \rfloor\}$. If there is no such path, we set $L_{a,b}(x) = 0$. Then we set

$$H(x) = \min L_{r',r}(x)$$

where the min is over all couples (r', r) of *prime* numbers such that

$$\frac{x}{3} < r \leq \frac{x}{2} < r' \leq x.$$

The theorem will be an easy consequence of the following.

Proposition. *There is a constant* N_0 *such that for any* $N \geq N_0$, *there is a set* $\mathcal{P}(N)$ *of prime numbers in* $(3\sqrt{N \log N}, 4\sqrt{N \log N}]$, *of cardinal* $|\mathcal{P}(N)| \geq \frac{\sqrt{N}}{19(\log N)^{3/2}}$, *such that*

$$H(N) \geq \sum_{p \in \mathcal{P}(N)} H\left(\frac{N}{p}\right). \tag{4}$$

The technique used here is analogous to that of [6] in the study of the longest path. More precisely, in [6], $f^*(N)$ denotes the maximal length of a path that uses integers in $[\sqrt{N}, N]$. A quantity h^* is introduced, which is to f^* what H is to L in our case. The inequality (4) is analogous to Buchstab's unequality (40) from [6]. The corresponding lower bounds led to the proof that $f^*(N) \asymp N/\log N$ (Theorem 2 in [7]).

The analogy can be pushed further: in both the proof of (4) and of (40) in [6], we borrow a technique used by Erdős, Freud and Hegyvári who proved the following asymptotic behaviour:

$$\min \max_{1 \leq i \leq N-1} \operatorname{lcm}(a_i, a_{i+1}) = \left(\frac{1}{4} + o(1)\right) \frac{N^2}{\log N},$$

where the min is over all permutations $(a_1, a_2, \ldots, a_N)$ of $\{1, 2, \ldots, N\}$; see Theorem 1 of [2]. In [2] as in [6] or in the present work, the proof goes through the construction of a sequence of integers by concatenating blocks whose largest prime factor is constant, and linking blocks together with separating integers. In [6] as in the present work, these blocks take the form of sub-paths $p\mathcal{C}_{N/p}$, where the $\mathcal{C}_{N/p}$ is a path of integers $\leq N/p$ whose largest prime factor is $\leq p$.

It is worth mentioning that the article [2] of Erdős, Freud and Hegyvári is the origin of all works related to the divisor graph.

2 Notations

The letters p, q, q', r, r' will always denote generic prime numbers. For an integer $m \geq 2$, $P^-(m)$ denotes the smallest prime factor of m.

Let $N \geq 1$. A *path* of integers $\leq N$ of length l is a l-uple $\mathcal{C} = (a_1, a_2, \ldots, a_l)$ of pairwise distinct positive integers $\leq N$, such that for all i with $1 \leq i \leq l-1$, a_i is either a divisor or a multiple of a_{i+1}. For convenience, we take $\mathcal{C}$ up to global flip, *i.e.* we identify $(a_1, \ldots, a_l)$ with $(a_l, \ldots, a_1)$. We will denote this path by $a_1 - a_2 - \cdots - a_l$ (or $a_l - \cdots - a_2 - a_1$). If b and c are integers such that $b = a_i$ and $c = a_{i\pm1}$ for some i, we say that b and c are *neighbours* (in $\mathcal{C}$).

When a partition $\mathcal{A}(N)$ of $\{1, 2, \ldots, N\}$ is fixed, for any $n \in \{1, 2, \ldots, N\}$ we will simply denote by $\mathcal{C}(n)$ the path that contains n in $\mathcal{A}(N)$.

A partition of $\{1, 2, \ldots, N\}$ into paths is said to be *optimal* if it contains $F(N)$ paths (see the Introduction for the definition of F).

Let $\mathcal{C}$ be a path of integers $\leq N$ and $1 \leq n \leq N$. Then $\mathcal{C}$ is said to be *n-factorizable* if all the integers of $\mathcal{C}$ are multiple of n. Then $\mathcal{C}$ can be written as $\mathcal{C} = n\mathcal{D}$ where $\mathcal{D}$ is a path of integers $\leq N/n$.

For integers $1 \leq n \leq N$ and a partition $\mathcal{A}(N)$ of $\{1, 2, \ldots, N\}$, we say that n is *factorizing* for $\mathcal{A}(N)$ if every path of $\mathcal{A}(N)$ that contains a multiple of n is n-factorizable.

3 Lemmas

Lemma 1. *Let $N \geq 1$ and $\mathcal{A}(N)$ be an optimal partition.*

(i) Let $1 \leq n \leq N$ with n factorizing for $\mathcal{A}(N)$. Let $k = \lfloor N/n \rfloor$. There are exactly $F(k)$ paths in $\mathcal{A}(N)$ that contain a multiple of n. They are of the form $n\mathcal{D}_1, n\mathcal{D}_2, \ldots, n\mathcal{D}_{F(k)}$ where $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{F(k)}$ is an optimal partition of $\{1, 2, \ldots, k\}$.

(ii) Let $z > 1$ be a real number. Let $M_z(N)$ be the set of integers $m \leq N$

that are not *factorizing for $\mathcal{A}(N)$ and such that*

$$m > \frac{N}{z} \quad \textit{and} \quad P^-(m) > z.$$

Then

$$|M_z(N)| < \frac{2N}{z}.$$

Proof. (i) The set of paths that contain a multiple of n is of the form $\{n\mathcal{D}_1, n\mathcal{D}_2, \dots, n\mathcal{D}_g\}$ where $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_g$ is a partition of the integers $\leq k = \lfloor N/n \rfloor$. Since $\mathcal{A}(N)$ is optimal, $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_g$ is optimal, hence $g = F(k)$.

(ii) Let $m \in M_z(N)$. There is a path $\mathcal{C}_m$ in $\mathcal{A}(N)$ that contains multiples and non-multiples of m. Hence there is an integer $c(m)$ in $\mathcal{C}_m$ that is not a multiple of m, and is neighbour to an integer $b(m)$ which is a multiple of m. Then $c(m)$ has to be a divisor of $b(m)$. More precisely, if $b(m) = am$, then $c(m)$ can be written as $c(m) = \tilde{a}\tilde{m}$ with $\tilde{a}$ a divisor of a and $\tilde{m}$ a strict divisor of m. Since $P^-(m) > z$, $c(m) < N/z$.

Moreover, if m, m' are two distinct elements of $M_z(N)$, then

$$\operatorname{lcm}(m, m') \geq \min(mP^-(m'), m'P^-(m)) > \frac{N}{z} z = N.$$

As a result the map

$$\begin{aligned} b : M_z(N) &\to \{1, 2, \dots, N\} \\ m &\mapsto b(m) \end{aligned}$$

is an injection.

Moreover, any integer $c < N/z$ has at most two neighbours in $\mathcal{C}(c)$. Consequently the map

$$\begin{aligned} c : M_z(N) &\to \{1 \leq n < N/z\} \\ m &\mapsto c(m) \end{aligned}$$

is at-most-two-to-one. Thus

$$|M_z(N)| < \frac{2N}{z}.$$

□

Lemma 2. *There exists a constant N_1 such that for any $N \geq N_1$, there is a set $\widetilde{\mathcal{P}}(N)$ of prime numbers in $(3\sqrt{N\log N}, 4\sqrt{N\log N}]$ of cardinal*

$$|\widetilde{\mathcal{P}}(N)| \geq \sqrt{\frac{N}{\log N}}, \tag{5}$$

such that for any prime numbers r, r' with

$$\frac{N}{3} < r \leq \frac{N}{2} < r' \leq N,$$

there exists an optimal partition $\mathcal{A}(N)$ of $\{1, 2, \ldots, N\}$ that contains the paths r' and $2r - r$ and for which all the integers in $\widetilde{\mathcal{P}}(N)$ are factorizing.

Proof. Let N_1 be such that for any $N \geq N_1$,

$$\pi\left(4\sqrt{N\log N}\right) - \pi\left(3\sqrt{N\log N}\right) - \frac{2}{3}\sqrt{\frac{N}{\log N}} \geq \sqrt{\frac{N}{\log N}}, \tag{6}$$

$$\pi\left(\frac{N}{2}\right) - \pi\left(\frac{N}{3}\right) \geq 8. \tag{7}$$

The existence of such a N_1 comes from the prime number theorem (more precisely the left-hand-side of (6) is equivalent to $\frac{4}{3}\sqrt{\frac{N}{\log N}}$). We also take N_1 large enough so that

$$\left(3\sqrt{N\log N}, 4\sqrt{N\log N}\right] \cap \left(\frac{N}{3}, \frac{N}{2}\right] = \emptyset. \tag{8}$$

Let $N \geq N_1$. We start by fixing an optimal partition $\mathcal{A}'(N)$. We apply Lemma 1 (ii) to $\mathcal{A}'(N)$ with $z = 3\sqrt{N\log N}$. All the prime numbers p in $(3\sqrt{N\log N}, 4\sqrt{N\log N}]$ that are not factorizing are in $M_z(N)$, since they satisfy $p > 3\sqrt{N\log N} \geq \frac{N}{z}$ and $P^-(p) = p > z$, so there are at most $\frac{2}{3}\sqrt{\frac{N}{\log N}}$ of them. By removing these and using (6), we get a set $\widetilde{\mathcal{P}}(N)$ of prime numbers in $(3\sqrt{N\log N}, 4\sqrt{N\log N}]$ that are factorizing in $\mathcal{A}'(N)$, with cardinality

$$|\widetilde{\mathcal{P}}(N)| \geq \sqrt{\frac{N}{\log N}}.$$

We now change notations slightly and fix two prime numbers r_0, r_0' such that

$$\frac{N}{3} < r_0 \leq \frac{N}{2} < r_0' \leq N.$$

Our goal is to go from $\mathcal{A}'(N)$ to a new optimal partition $\mathcal{A}(N)$ that contains the paths r_0' and $2r_0 - r_0$ *while maintaining the fact that the elements of* $\widetilde{\mathcal{P}}(N)$ *are factorizing.*

Let us denote the set of prime numbers

$$\mathcal{R} = \left\{ \frac{N}{3} < r \leq \frac{N}{2} \right\},$$

and $\mathcal{R}^*(\mathcal{A}'(N))$ the subset of $r \in \mathcal{R}$ such that r does not have 1 as a neighbour in $\mathcal{C}(r)$ and $2r$ does not have 1 nor 2 has a neighbour in $\mathcal{C}(2r)$. Then for any $r \in \mathcal{R}^*(\mathcal{A}'(N))$, since the only possible neighbour of r is $2r$ and reciprocally, by optimality the path $\mathcal{C}(r)$ is equal to $r - 2r$. Moreover, since 1 and 2 have at most two neighbours,

$$|\mathcal{R} \setminus \mathcal{R}^*(\mathcal{A}'(N))| \leq 4. \tag{9}$$

Now we make it so that r_0' is a path. If it is not the case, since the only possible neighbour of r_0' is 1, $\mathcal{C}(r_0')$ is of the form $\mathcal{D} - r_0'$ with $\mathcal{D}$ a path ending in 1. We split this path into $\mathcal{D}$ on one side and r_0' on the other side. By (9) and (7), there is at least one element $r^* \in \mathcal{R}^*(\mathcal{A}'(N))$. We stick $\mathcal{D}$ to $\mathcal{C}(r^*)$, thus forming the path $\mathcal{D} - \mathcal{C}(r^*)$. This is possible because $\mathcal{D}$ ends in 1. Let $\mathcal{A}''(N)$ be this new partition. The total number of paths has not changed so $\mathcal{A}'(N)$ is still optimal, furthermore it contains the path r_0', and the elements of $\widetilde{\mathcal{P}}(N)$ are still factorizing because the integers in the paths that changed were not multiples of any $p \in \widetilde{\mathcal{P}}(N)$.

The subset $\mathcal{R}^*(\mathcal{A}''(N))$ might differ from $\mathcal{R}^*(\mathcal{A}'(N))$ by one element, but it still satisfies (9) and its elements r still satisfy that $\mathcal{C}(r)$ is equal to $r - 2r$. If $r_0 \in \mathcal{R}^*(\mathcal{A}''(N))$, we can set $\mathcal{A}(N) = \mathcal{A}''(N)$ and the proof is over. We now suppose that $r_0 \notin \mathcal{R}^*(\mathcal{A}''(N))$.

By (9) and (7), there are at least four elements r_1, r_2, r_3, r_4 in $\mathcal{R}^*(\mathcal{A}''(N))$. We cut the path $\mathcal{C}(1)$ into one, two or three paths, one of them being the singleton 1 (we will see later that we get in fact three paths). Such a move will be called an *extraction* of the integer 1. We similarly *extract* the integer 2. We now use these integers 1 and 2 to stick together the paths $r_i - 2r_i$ by forming

$$r_1 - 2r_1 - 1 - 2r_2 - r_2 \quad \text{and} \quad r_3 - 2r_3 - 2 - 2r_4 - r_4.$$

We thus get a new partition $\mathcal{A}(N)$. Its number of paths is less or equal to that of $\mathcal{A}''(N)$, so it is still optimal (this shows in particular that 1 and 2 were not endpoints of their paths). It also satisfies $r_0 \in \mathcal{R}^*(\mathcal{A}(N))$ since 1 and 2 are not linked to r_0 nor $2r_0$, so that it contains the path $r_0 - 2r_0$, as well as r_0', and the elements of $\widetilde{\mathcal{P}}(N)$ are still factorizing. $\square$

4 Proof of the Proposition

Let N_1 be the constant of Lemma 2. We fix a N_0 such that

$$N_0 \geq N_1^4 \tag{10}$$

and such that for all $N \geq N_0$,

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{N}{\log N}} &\geq \pi\left(\frac{1}{4}\sqrt{\frac{N}{\log N}}\right) - \pi\left(\frac{1}{6}\sqrt{\frac{N}{\log N}}\right) \\ &\geq \pi\left(\frac{1}{8}\sqrt{\frac{N}{\log N}}\right) - \pi\left(\frac{1}{9}\sqrt{\frac{N}{\log N}}\right) \\ &\geq \left\lfloor \frac{\sqrt{N}}{37(\log N)^{3/2}} \right\rfloor \geq \frac{\sqrt{N}}{38(\log N)^{3/2}} + \frac{1}{2} \geq 5 \end{aligned} \tag{11}$$

and

$$4\sqrt{\log N} \leq N^{1/4}. \tag{12}$$

The existence of such a N_0 is again an easy consequence of the prime number theorem. Also note that since $N_0 \geq N_1$, (8) still holds.

Let $N \geq N_0$. We chose a set $\widetilde{\mathcal{P}}(N)$ according to Lemma 2. Let us denote

$$I = \left\lfloor \frac{1}{37}\frac{\sqrt{N}}{(\log N)^{3/2}} \right\rfloor.$$

By (5) and (11) we can chose $2I$ elements in $\widetilde{\mathcal{P}}(N)$, which we denote as

$$p_1, p_2, \ldots, p_{2I}.$$

We set $\mathcal{P}(N) = \{p_1, \ldots, p_{2I-1}\}$. By (11) again, $|\mathcal{P}(N)| \geq \frac{\sqrt{N}}{19(\log N)^{3/2}}$. It remains to prove that this set $\mathcal{P}(N)$ satisfies (4).

Let r, r' be two prime numbers such that

$$\frac{N}{3} < r \leq \frac{N}{2} < r' \leq N. \tag{13}$$

By the property of $\widetilde{\mathcal{P}}(N)$ in Lemma 2, there exists an optimal partition $\mathcal{A}'(N)$, that contains the paths r' and $2r - r$, for which the elements of $\widetilde{\mathcal{P}}(N)$ (and in particular the elements of $\mathcal{P}(N)$) are factorizing.

We denote two sets of prime numbers

$$\mathcal{Q}(N) = \left\{ \frac{1}{9}\sqrt{\frac{N}{\log N}} < q \leq \frac{1}{8}\sqrt{\frac{N}{\log N}} \right\},$$
$$\mathcal{Q}'(N) = \left\{ \frac{1}{6}\sqrt{\frac{N}{\log N}} < q' \leq \frac{1}{4}\sqrt{\frac{N}{\log N}} \right\}.$$

For all $(p, q, q') \in \widetilde{\mathcal{P}}(N) \times \mathcal{Q}(N) \times \mathcal{Q}'(N)$ we have

$$\frac{N}{3} < pq \leq \frac{N}{2}, \tag{14}$$
$$\frac{N}{2} < pq' \leq N. \tag{15}$$

We focus on the factorizing prime number p_{2I}. For any $q \in \mathcal{Q}(N)$, because of (14) the only possible neighbours of $p_{2I}q$ are p_{2I} and $2p_{2I}q$. Similarly, the only possible neighbours of $2p_{2I}q$ are $p_{2I}, 2p_{2I}$ or $p_{2I}q$. But p_{2I} and $2p_{2I}$ can be linked to at most 4 elements of type $p_{2I}q$ or $2p_{2I}q$. By (11) we know that $|\mathcal{Q}(N)| \geq 5$, so there exists a $q_{2I} \in \mathcal{Q}(N)$ for which neither $p_{2I}q_{2I}$ nor $2p_{2I}q_{2I}$ is a neighbour of p_{2I} or $2p_{2I}$. As a result, the only possible neighbour for $p_{2I}q_{2I}$ is $2p_{2I}q_{2I}$, and reciprocally. By optimality, $\mathcal{A}'(N)$ contains the path $p_{2I}q_{2I} - 2p_{2I}q_{2I}$.

Using (11) we can chose

- I elements of $\mathcal{Q}'(N)$ which we write as

$$q_1, q_3, \ldots, q_{2I-1}; \tag{16}$$

- $I - 1$ elements of $\mathcal{Q}(N) \setminus \{q_{2I}\}$ which we write as

$$q_2, q_4, \ldots, q_{2I-2}. \tag{17}$$

Let i be such that $1 \leq i \leq 2I-1$. Then the prime number p_i is factorizing for $\mathcal{A}'(N)$ so by Lemma 1 (i) the paths of $\mathcal{A}'(N)$ that contain multiples of p_i are of the form

$$p_i\mathcal{C}_{i,1}, p_i\mathcal{C}_{i,2}, \ldots, p_i\mathcal{C}_{i,F(N/p_i)}$$

where $\mathcal{C}_{i,1}, \mathcal{C}_{i,2}, \ldots, \mathcal{C}_{i,F(N/p_i)}$ is an optimal partition of $\{1, 2, \ldots, \lfloor N/p_i \rfloor\}$. By our choice of indices (16),(17), one of the elements q_i, q_{i+1} is in $\mathcal{Q}'(N)$, we rename it $\widetilde{q_i}$, and the other is in $\mathcal{Q}(N)$, we rename it $\widetilde{q_{i+1}}$. Using (14),(15) we get

$$\frac{N}{3p_i} < \widetilde{q_{i+1}} \leq \frac{N}{2p_i} < \widetilde{q_i} \leq \frac{N}{p_i}.$$

Using (12) and (10), we have $N/p_i \geq N^{1/4} \geq N_0^{1/4} \geq N_1$. Hence we can apply Lemma 2 with N/p_i instead of N. We deduce that there exists an optimal partition of $\{1, 2, \ldots, \lfloor N/p_i \rfloor\}$ that contains the paths $\widetilde{q_i}$ and $\widetilde{q_{i+1}} - 2\widetilde{q_{i+1}}$. By extracting 1 in that partition, we can stick these two paths together into $\widetilde{q_i} - 1 - 2\widetilde{q_{i+1}} - \widetilde{q_{i+1}}$ while keeping an optimal partition. To sum up, we know now that there is an optimal partition of the integers $\leq N/p_i$ containing a path that has q_i and q_{i+1} as endpoints.

Let $\mathcal{D}_{i,1}, \mathcal{D}_{i,2}, \ldots, \mathcal{D}_{i,F(N/p_i)}$ be an optimal partition of the integers $\leq N/p_i$, with $\mathcal{D}_{i,1}$ having q_i, q_{i+1} as endpoints and of maximal length $L_{q_i,q_{i+1}}(N/p_i)$. We can transform $\mathcal{A}'(N)$ by replacing the paths $(p_i\mathcal{C}_{i,j})_{1\leq j\leq F(N/p_i)}$ by $(p_i\mathcal{D}_{i,j})_{1\leq j\leq F(N/p_i)}$. In this way we get a new optimal partition $\mathcal{A}''(N)$ that contains all the paths $p_i\mathcal{D}_{i,1}$ for $1 \leq i \leq 2I-1$, as well as r', $2r - r$, and $p_{2I}q_{2I} - 2p_{2I}q_{2I}$.

By extracting the integers 1, 2 and the q_i for $2 \leq i \leq 2I$, we construct the path of Figure 1 while keeping an optimal partition of $\{1, 2, \ldots, N\}$.

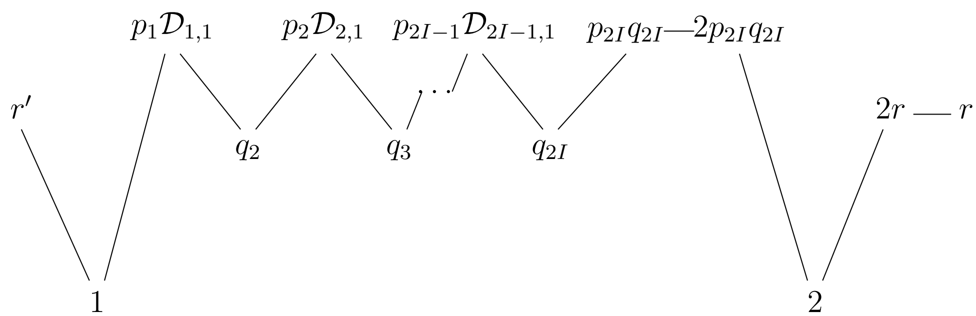

Figure 1: A long path with endpoints r', r.

Its length is larger than

$$\sum_{i=1}^{2I-1} L_{q_i,q_{i+1}}(N/p_i) \geq \sum_{p\in\mathcal{P}(N)} H(N/p_i).$$

This being true for any r, r' satisfying (13), we get

$$H(N) \geq \sum_{p\in\mathcal{P}(N)} H(N/p_i).$$

□

5 Proof of the Theorem

Let us fix a constant $N_2 = 2^{2^{k_0}} \geq N_0$, where N_0 is the constant from the Proposition. We chose a constant B such that for all $N \leq 2^{2^{k_0+2}}$,

$$N \leq 4(\log N)^B \exp\left[\frac{(\log\log N)^2}{\log 2}\right] \tag{18}$$

and

$$B \geq 8. \tag{19}$$

We show by induction on $k \geq k_0 + 2$ that for all N such that

$$2^{2^{k_0}} < N \leq 2^{2^k},$$

we have

$$H(N) \geq \frac{N}{(\log N)^B \exp\left[\frac{(\log\log N)^2}{\log 2}\right]} \tag{20}$$

Base case

Let N be such that $2^{2^{k_0}} < N \leq 2^{2^{k_0+2}}$, then we have $N > N_2 \geq N_0 \geq N_1^4$ (see (10)) with N_1 the constant of Lemma 2. Let r, r' be two prime numbers such

$$\frac{N}{3} < r \leq \frac{N}{2} < r' \leq N.$$

Lemma 2 implies that there is an optimal partition $\mathcal{A}(N)$ of $\{1, 2, \ldots, N\}$ which contains the paths r' and $2r - r$. By extracting 1, we can stick them into $r' - 1 - 2r - r$ while keeping an optimal partition. This implies that $H(N) \geq 4$, and (18) yields the base case.

Induction step

Let $k \geq k_0 + 2$. We suppose that (20) holds for all $N \in \left(2^{2^{k_0}}, 2^{2^k}\right]$.

Let N be such that $2^{2^k} < N \leq 2^{2^{k+1}}$. Since $k \geq k_0 + 2$, we also have $N^{1/4} > 2^{2^{k_0}}$.

Let $p \in \left(3\sqrt{N \log N}, 4\sqrt{N \log N}\right]$. By (12), we have

$$2^{2^{k_0}} < N^{1/4} \leq \frac{N}{p} \leq \sqrt{N} \leq 2^{2^k}.$$

By using the induction hypothesis on N/p, we get

$$
\begin{aligned}
H\left(\frac{N}{p}\right) &\geq \frac{N}{p(\log(N/p))^B \exp\left[\frac{(\log\log(N/p))^2}{\log 2}\right]} \\
&\geq \frac{N}{p(\log\sqrt{N})^B \exp\left[\frac{(\log\log\sqrt{N})^2}{\log 2}\right]} \\
&= \frac{2^{B-1}(\log N)^2 N}{p(\log N)^B \exp\left[\frac{(\log\log N)^2}{\log 2}\right]}.
\end{aligned}
$$

Hence by using the Proposition and (19),

$$
\begin{aligned}
H(N) &\geq \sum_{p\in\mathcal{P}(N)} H\left(\frac{N}{p}\right) \\
&\geq \frac{|\mathcal{P}(N)|}{\max\mathcal{P}(N)} \frac{2^{B-1}(\log N)^2 N}{(\log N)^B \exp\left[\frac{(\log\log N)^2}{\log 2}\right]} \\
&\geq \frac{2^{B-1}}{76} \frac{N}{(\log N)^B \exp\left[\frac{(\log\log N)^2}{\log 2}\right]} \\
&\geq \frac{N}{(\log N)^B \exp\left[\frac{(\log\log N)^2}{\log 2}\right]}.
\end{aligned}
$$

This concludes the induction step.

Finally, since $L(N) \geq 1$ for all $N \geq 1$, we get the Theorem by chosing $A = \max(B, A_0)$ where A_0 is a constant such that for all $3 \leq N < N_0$,

$$
N \leq (\log N)^{A_0} \exp\left[\frac{(\log\log N)^2}{\log 2}\right].
$$

□

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