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Domain decomposition methods in a geometrical multi-scale domain using finite volume schemes.

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Abstract. The heat equation is solved by using a finite volume discretization in a domain that consists of a two-dimensional (2D) central node and several one-dimensional (1D) outgoing branches. Several interface connection options to match the submodels with or without continuity are looked at. For each of them, monolithic schemes are defined, and existence and uniqueness of solutions is proved. New schemes are deduced from them, which are obtained through domain decomposition methods in the form of interface systems, with one or two interface unknowns. A comparative systematic study is carried out from an algebraic and numerical point of view to learn more about accuracy of these schemes according to the interface conditions : Dirichlet, Neumann or Robin, considering especially the impact of the Robin parameter. It is shown that some of the properties of classical iterative substructuring methods remain valid in the multi-scale case.

Résumé. On approche la solution de l'équation de la chaleur avec une discrétisation de type volumes finis dans un domaine constitué d'un noeud bi-dimensionnel (2D) d'où partent plusieurs branches uni-dimensionnelles (1D). Plusieurs types de raccordements, continus ou non, sont envisagés aux interfaces entre les sous-modèles 1D et 2D. Pour chacun, des schémas monolithiques sont construits et l'existence et l'unicité d'une solution est démontrée. On en déduit de nouveaux schémas obtenus par décomposition de domaine sous la forme de systèmes d'interface, avec une ou deux inconnues par interface. Une étude comparative systématique est menée d'un point de vue algébrique et numérique pour connaître la précision de ces schémas en fonction des conditions d'interface : Dirichlet, Neumann ou Robin, en regardant notamment l'influence du paramètre de Robin. On montre que certaines propriétés des méthodes itératives classiques sont encore vraies dans le contexte multi-échelle.

Mathematics Subject Classification : 35K05, 65F10, 65N08, 65N12, 65N55.

Mots clés : finite volume scheme, parabolic problem, multi-scale domain, domain decomposition, stability and convergence of numerical methods, Robin interface condition.

1 Introduction

The aim of the present paper is to use a domain decomposition approach to define, analyze and compare, new finite volume schemes to solve a model problem set in a geometrical multi-scale domain. This kind of domains derive for instance from a finite rod structure, which is a connected union of cylinders (rectangles in the two-dimensional (2D) case). Arterial trees in the cardiovascular system (see [15]), systems of pipes in industrial installations, or canal systems, are classical examples of rod structures. Partial differential equation solving in such domains results in high computational costs. To reduce them, rods are often considered as

one-dimensional (1D) at some distance from the junctions. This leads to work in a geometrical multi-scale domain where the original space scale of the problem is kept in the regions of interest.

The main application area is computational hemodynamics. Dimensionally-heterogeneous modelling in that context can be found for instance in [2, 3, 4, 5, 6, 7, 13, 14, 25, 30]. In [19], the authors deal with dimensionally-heterogeneous hydraulic networks. Coupling of 1D and 2D CFD codes is discussed in [17] (Euler system). This work has been extended recently in [9] to the coupling of a density-based 3D Euler code to a 1D version of the code (for instance, an application is the simulation of diesel injectors). In [2], the applications are the heat transfer problem with pure diffusion phenomena and the linear elasticity problem.

The model problem, in this paper, is the heat equation in a 1D-2D domain. Three options are considered for the transmission conditions : the solution and the mean value of the flux (normal derivative) are continuous, the flux and the mean value of the solution are continuous, and the mean value of the solution or of the flux is continuous supplemented by a Robin condition. A finite volume scheme of hybrid dimension for solving the model problem with the continuity of the solution on the interfaces was studied in [24]. The problem was fully coupled and treated as a monolithic entity, it was solved by using a direct method, an error estimate was obtained. So far, however, few theoretical results for dimensionally-heterogeneous problems have been produced. For instance, in [2], the authors work in a general theoretical framework and prove existence and uniqueness of the solution for both the continuous problem and its finite element approximation by writing an augmented variational formulation where Lagrange multipliers derive from the coupling conditions. On the contrary, in this paper, we deal exclusively with finite volume approximations. Hybrid dimension monolithic schemes are built, following the same structure as [24], to approximate the solution of the model problem with the two other kinds of transmission conditions. Existence and uniqueness of the solution of these schemes is proved. Then a domain decomposition method is used (as in [2, 3, 13] for instance). This partitioned approach, in which the submodels are solved separately, was not covered in [24].

We use the iterative strong coupling method because this is a decomposition strategy that works with subdomains which do not overlap. The methodology is developed in [2, 3, 4, 18, 19, 32]. This technique can be understood as a domain decomposition approach where the partitioning takes place at the interfaces of models of different geometric dimensions, the interface system is solved by using a Krylov method. The procedure involves two times more interface unknowns than classical methods but it is not a drawback here since there is a small number of interface unknowns. In this paper, a systematic review is carried out by choosing Dirichlet, Neumann or Robin interface conditions in pairs, for the three kinds of interface conditions matching the submodels. Each time, it is shown how to write restriction operators, with or without the mean value of the solution or its normal derivative, for schemes to be consistent with transmission conditions. All schemes are analyzed from an algebraic point of view. We compare the numerical results with the ones obtained by solving the monolithic schemes with a direct method. In particular, the way the Robin parameter influences the accuracy is looked at. We show that approximations are not equally accurate for all interface conditions.

Classical domain decomposition methods are generally amenable to procedures for an interface problem, for instance the Schur complement system or the flux equation, by eliminating the unknowns in the interior of the subdomains (see [26],[29]). The same is true in the geome-

trical multi-scale case and definitions of these procedures are proposed for each of the three transmission conditions. Another procedure referred to as "the Robin equation" is defined in the paper, which can be considered as a dual method in the same way as the flux equation.

Then, we look at some iterative substructuring methods. Many classical domain decomposition methods are preconditioned Richardson methods for the Schur complement system or for the flux equation (see [29] or [10] for instance). We explain the extent to which this property remains true for the Dirichlet-Neumann method and for the Neumann-Dirichlet method (adaptation to the geometrical multi-scale case). The relationship with the Gauss-Seidel algorithm is highlighted, as we see in the classical case for instance in [28].

This work is organized as follows. Section 2 presents the model problem and its variants. The hybrid monolithic scheme developed in [24] is recalled in Section 3 and two other versions are defined that allow discontinuity of the solution across the interfaces. Existence and uniqueness of the solution of each scheme is proved. In Section 4, the iterative strong coupling procedure is applied, some classical domain decomposition strategies are adapted and compared, the Robin equation is introduced. We give numerical experiments in Section 5 illustrating the difference between the three monolithic schemes and giving the error between the approximations obtained by each monolithic scheme and its associated interface systems deriving from the strong coupling method, testing many different interface conditions.

2 The model problems

2.1 Description of the geometrical multi-scale 1D-2D domain

The geometrical multi-scale 1D-2D domain on which our model problems are set derives from a finite rod structure, which consists of one node and p branches. This construction is done in [23] and is reminded below.

Let $e_j = [O, O_j], j = 1, \dots, p$, be p closed segments in $\mathbb{R}^2$, having a common end point denoted by O , with length $l_j = OO_j, j = 1, \dots, p$.

Let (x, y) denote the coordinates in the canonical basis of $\mathbb{R}^2$, and (x^{e_j}, y^{e_j}) denote the local coordinates associated with the segment $e_j, j = 1, \dots, p$. This local system is orthonormal and such that x^{e_j} is the coordinate in the direction e_j .

Let $\varepsilon > 0$. Let $\theta_1, \dots, \theta_p$ be positive numbers independent of ε .

Let $B_j^\varepsilon = \{(x, y) \mid x^{e_j} \in (0, l_j), y^{e_j} \in (-\frac{\varepsilon\theta_j}{2}, \frac{\varepsilon\theta_j}{2})\}$, and $\beta_j^\varepsilon = \{(x, y) \mid x^{e_j} = l_j, y^{e_j} \in (-\frac{\varepsilon\theta_j}{2}, \frac{\varepsilon\theta_j}{2})\}$.

Let ω_0 be a bounded domain in $\mathbb{R}^2$ with smooth boundary containing O (see [23]). Let $\omega_0^\varepsilon = \{(x, y) \mid \frac{(x, y) - O}{\varepsilon} \in \omega_0\}$. We assume that $B_j^\varepsilon \setminus \omega_0^\varepsilon \cap B_i^\varepsilon \setminus \omega_0^\varepsilon = \emptyset, i \neq j$. The domain ω_0^ε (see the dotted line in Figure 1 (a)) is added in order to smooth the boundary of the final structure by removing the corners. We assume that $\omega_0^\varepsilon \setminus \cup_{j=1}^p B_j^\varepsilon$ is *not too large*.

Let $\Omega_\varepsilon = \cup_{j=1}^p B_j^\varepsilon \cup \omega_0^\varepsilon$.

Now, let us describe the 1D-2D domain under consideration. Let $\delta > 0$, such that $\delta < \min \{l_j, j = 1, \dots, p\}$ and such that ω_0^ε is in the ball of center O and radius δ .

FIGURE 1 – (a) The rod structure Ω_ε and (b) The geometrical multi-scale domain D_ε .

Denote $B_j^\varepsilon = B_j^\varepsilon \cap \{(x, y) \mid x^{e_j} \in (0, \delta)\}$, $j = 1, \dots, p$. Denote $\Omega(0) = \cup_{j=1}^p B_j^\varepsilon \cup \omega_0^\varepsilon$. Therefore $\Omega(0)$ is a truncated part of the initial domain Ω_ε .

Let $S_j = \{(x, y) \mid y^{e_j} = 0, x^{e_j} \in (\delta, l_j)\}$, $j = 1, \dots, p$, be segments such that $S_j \subset e_j$.

We denote by $\gamma_j = \{(x, y) \mid x^{e_j} = \delta, y^{e_j} \in (-\frac{\varepsilon\theta_j}{2}, \frac{\varepsilon\theta_j}{2})\}$, $j = 1, \dots, p$, the 1D-2D interfaces.

Let us define $D_\varepsilon = \Omega(0) \cup (\cup_{j=1}^p S_j)$. The set D_ε is the so-called geometrical multi-scale domain.

2.2 The model problems

The first boundary value problem in the domain D_ε , which is considered in this paper, is the following :

$$\left\{ \begin{array}{l} \frac{\partial v_j}{\partial t}(x^{e_j}, t) - \frac{\partial^2 v_j}{\partial x^{e_j 2}}(x^{e_j}, t) = f_j(x^{e_j}, t), x^{e_j} \in (\delta, l_j), \\ j = 1, \dots, p, t \in (0, T) \quad (a) \\ v_j(x^{e_j}, 0) = 0, x^{e_j} \in (\delta, l_j), j = 1, \dots, p \\ v_j(l_j, t) = 0, t \in (0, T) \\ \frac{\partial u}{\partial t}(x, y, t) - \Delta u(x, y, t) = f(x, y, t), (x, y) \in \Omega(0), t \in (0, T) \quad (b) \\ u(x, y, 0) = 0, (x, y) \in \Omega(0) \\ \frac{\partial u}{\partial n}(x, y, t) = 0, (x, y) \in \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j), t \in (0, T) \\ u(x, y, t) = v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \\ \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(\cdot, t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), j = 1, \dots, p, t \in (0, T) \quad (c) \end{array} \right. \quad (1)$$

We assume that the functions f_j (respectively f) are smooth and independent of ε , are constant with respect to (x, y) in some neighborhood of O_j , $j = 1, \dots, p$ (respectively O), and vanish for $t \leq t_0$, $t_0 > 0$. In addition, we assume they are such that the partial derivatives of u in $\overline{\Omega(0)}$ and v_j in $[\delta, l_j]$, $j = 1, \dots, p$, exist and are continuous till the order two. This allows us to obtain the error estimate below.

More precisely, we define a global solution u^d of (1) by letting

$$u^d(x, y, t) = \begin{cases} u(x, y, t) & \text{if } (x, y) \in \Omega(0), t \in (0, T) \\ v_j(x^{e_j}, t) & \text{if } (x, y) \in B_j^\varepsilon, x^{e_j} \in (\delta, l_j), j = 1, \dots, p, t \in (0, T) \end{cases} \quad (2)$$

The solution u^d is defined in Ω_ε but $u^d(x, y, t)$ does not depend on y^{e_j} when $(x, y) \in B_j^\varepsilon \setminus B_j^{\prime\varepsilon}$. Defining the solution on Ω_ε allows us to use standard norms in a 2D domain, and more broadly to easily compare u^d with u_ε (see below).

Remark 1 *Problem (1) was introduced in [24] in the framework of the method of asymptotic partial domain decomposition (MAPDD) (see [22]). The following result was proved in [24].*

For any $J > 0$, there is M , independent of ε , such that if $\delta = M\varepsilon|\ln\varepsilon|$, then

$$\|u_\varepsilon - u^d\|_{H^1(\Omega_\varepsilon \times (0, T))} = O(\varepsilon^J),$$

where u_ε is the unique solution of the following parabolic linear model equation

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f, & \text{in } \Omega_\varepsilon, t \in (0, T) \\ u_\varepsilon = 0, & \text{on } \beta_j^\varepsilon, j = 1, \dots, p, t \in (0, T) \\ \frac{\partial u_\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon \setminus (\cup_{j=1}^p \beta_j^\varepsilon), t \in (0, T) \\ u_\varepsilon(x, y, 0) = 0, & (x, y) \in \Omega_\varepsilon, t \in (0, T) \end{cases} \quad (3)$$

where f is a smooth function defined in $\Omega_\varepsilon \times [0, T]$ such that $f(x, y, t) = f_j(x^{e_j}, t)$, if $(x, y) \in B_j^\varepsilon, j = 1, \dots, p, t \in (0, T)$; where f_j are independent of ε C^{J+5} -smooth functions, $J \geq 5$, and they vanish in some neighborhood of the node O and vertices $O_j, j = 1, \dots, p$. We assume that $f_j(\cdot, t) = 0$ for $t \leq t_0, t_0 > 0$.

Of course, in reality, the problem which arises in the first instance is (3). Then, some assumptions are made that give the problem (1). For instance, the 1D simplified model can be viewed as a reduction of the complete 2D model by means of kinematical restrictions (see [3]) introduced on a part of the branches. Then, boundary conditions based upon mean quantities are prescribed on the coupling interfaces. Therefore, it makes sense to require the continuity of the solution and its normal derivative in an average sense (as in [18]) across $\gamma_j, j = 1, \dots, p$, that is to say to require that

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u(\cdot, t) d\gamma = v_j(\delta, t), j = 1, \dots, p, t \in (0, T) \\ \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(\cdot, t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), j = 1, \dots, p, t \in (0, T) \end{cases} \quad (4)$$

instead of

$$\begin{cases} u(x, y, t) = v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \\ \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(\cdot, t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), j = 1, \dots, p, t \in (0, T) \end{cases} \quad (5)$$

However, the problem (1) with (4) instead of (5) is not well posed : some supplementary conditions are needed. It is clear that (1) was obtained by supplementing (4) by the continuity of the solution across the interfaces. There are further alternatives.

In the following, we consider also the problem (1)"dsa", to indicate a discontinuous selected alternative, which is obtained when the problem (1) is changed such that (5) is replaced by

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u(., t) d\gamma = v_j(\delta, t), j = 1, \dots, p, t \in (0, T) \\ \frac{\partial u}{\partial n}(x, y, t) = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \end{cases} \quad (d) \quad (6)$$

that is to say (4) is supplemented by the continuity of the flux on the interfaces.

Other options can be considered. The problem (1)"rsa", to indicate a Robin selected alternative, is obtained when the problem (1) is changed such that (5) is replaced by

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u(., t) d\gamma = v_j(\delta, t), j = 1, \dots, p, t \in (0, T) \\ \frac{\partial u}{\partial n}(x, y, t) + q_j u(x, y, t) = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) + q_j v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \end{cases} \quad (7)$$

or

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(., t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), j = 1, \dots, p, t \in (0, T) \\ \frac{\partial u}{\partial n}(x, y, t) + q_j u(x, y, t) = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) + q_j v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \end{cases} \quad (8)$$

that is to say (4) is supplemented by Robin conditions at the interfaces. Here, $q_j, j = 1, \dots, p$, are any non-negative values. The conditions (7) and (8) are equivalent if $q_j \neq 0$ for all $j = 1, \dots, p$. If $q_j = 0$ for all $j = 1, \dots, p$, then the problem (1)"rsa" with (7) is equivalent to (1)"dsa", while (1)"rsa" with (8) is not well posed as soon as only one value of q_j is nil.

It is worth noting that the solution of the problem (1) is continuous whereas the solution of the problems (1)"dsa" or (1)"rsa" may be discontinuous. Furthermore, all the versions which may be imagined by using average values in the Robin condition lead to problems which are not well defined or to problems which are already defined above. To mention but a few examples :

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(., t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), j = 1, \dots, p, t \in (0, T) \\ \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(., t) d\gamma + q_j \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u(., t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) + q_j v_j(\delta, t), (x, y) \in \gamma_j, \\ j = 1, \dots, p, t \in (0, T) \end{cases} \quad (9)$$

is not well defined.

$$\begin{cases} \frac{\partial u}{\partial n}(x, y, t) = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \\ \frac{\partial u}{\partial n}(x, y, t) + q_j \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u(., t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) + q_j v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \end{cases} \quad (10)$$

leads to (1)"dsa".

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(., t) d\gamma = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) \text{ or } \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u(., t) d\gamma = v_j(\delta, t) \text{ or } u(x, y, t) = v_j(\delta, t), (x, y) \in \gamma_j, \\ j = 1, \dots, p, t \in (0, T) \\ \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u}{\partial n}(., t) d\gamma + q_j u(x, y, t) = \frac{\partial v_j}{\partial x^{e_j}}(\delta, t) + q_j v_j(\delta, t), (x, y) \in \gamma_j, j = 1, \dots, p, t \in (0, T) \end{cases} \quad (11)$$

all lead to (1).

Remark 2 An error estimate proved in [24] is reminded. Let $\omega_{0,poly}^\varepsilon$ be a polygonal domain included in ω_0^ε , and $B_{\varepsilon,poly} = (\cup_{j=1}^p B_j^\varepsilon) \cup \omega_{0,poly}^\varepsilon$. It is stated that, if u_ε is the solution of the heat equation (3) and $u_{d,\mathcal{T},k}$ is the approximate finite volume solution of the problem (1), if h is the size of the mesh and k the time step, then the following estimate holds under the assumptions of Remark 1 :

$$\|u_\varepsilon - u_{d,\mathcal{T},k}\|_{L^2(B_{\varepsilon,poly} \times (0,T))} = O\left(\frac{h}{\sqrt{\varepsilon}} + k\right) + O(\varepsilon^J)$$

The theoretical study of the problems (1)"dsa" and (1)"rsa" is not considered in this paper. Here the focus is on only one aspect : domain decomposition and accuracy with which the solutions of the monolithic schemes can be approximated.

3 Monolithic numerical schemes

We recall here the monolithic numerical scheme which is constructed in [24] to approach the solution of (1). Other monolithic schemes are defined to solve (1)"dsa" and (1)"rsa". Implicit methods are used whether it is for (1), (1)"dsa" or (1)"rsa".

3.1 The meshes

First, a mesh of S_j on the axis $Ox^{e_j}, j = 1, \dots, p$, is defined. For each value of j , we choose $N_j \in \mathbb{N}^*$, and $N_j + 1$ distinct and increasing values $x_{i+1/2}^{e_j}, i = 0, \dots, N_j$, such that $x_{1/2}^{e_j} = \delta, x_{N_j+1/2}^{e_j} = l_j$. Denote $I_i^{e_j} = (x_{i-1/2}^{e_j}, x_{i+1/2}^{e_j})$, and $h_i^{e_j} = x_{i+1/2}^{e_j} - x_{i-1/2}^{e_j}, i = 1, \dots, N_j$. Set $h^{e_j} = \max\{h_i^{e_j}, i = 1, \dots, N_j\}$ the size of the mesh of the interval (δ, l_j) . Then we choose N_j points $x_i^{e_j}, i = 1, \dots, N_j$, such that $x_i^{e_j} \in I_i^{e_j}$. Set $x_0^{e_j} = \delta, x_{N_j+1}^{e_j} = l_j$, and $h_{i+1/2}^{e_j} = x_{i+1}^{e_j} - x_i^{e_j}, i = 0, \dots, N_j$.

Then an admissible mesh over $\Omega(0)$, denoted by $\mathcal{T}$, is constructed. We assume in the following that $\Omega(0)$ is polygonal. We recall (see the definition in [12]) that such a mesh consists in a family of open polygonal convex subsets K of $\Omega(0)$ called control volumes, a family of edges σ of the control volumes denoted by $\mathcal{E}$, and a family of points x_K chosen in each control volume K denoted by $\mathcal{P}$. The mesh $\mathcal{T}$ satisfies the following properties

- 1) The closure of the union of all the control volumes is $\overline{\Omega(0)}$.
- 2) For any $K \in \mathcal{T}$, there is a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$, and $\bigcup_{K \in \mathcal{T}} \mathcal{E}_K = \mathcal{E}$.
- 3) For any $(K, L) \in \mathcal{T}^2, K \neq L$, one of three following assertions holds :
either $\bar{K} \cap \bar{L} = \emptyset$, or $\bar{K} \cap \bar{L}$ is a common vertex of K and L ,
or $\bar{K} \cap \bar{L} = \bar{\sigma}, \sigma$ being a common edge of K and L denoted by $\sigma_{K/L}$.
- 4) The family $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$ is such that for any $K \in \mathcal{T}$, $x_K \in \bar{K}$.
For any $(K, L) \in \mathcal{T}^2, K \neq L$, it is assumed that $x_K \neq x_L$ and that the straight line going through x_K and x_L is orthogonal to $\sigma_{K/L}$.
- 5) For any $\sigma \in \mathcal{E}$, if $\sigma \subset \partial\Omega(0), \sigma \in \mathcal{E}_K$ and $x_K \notin \sigma$, the orthogonal projection of x_K on the straight line containing the edge σ , belongs to σ .

(12)

Let $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}, \sigma \not\subset \partial\Omega(0)\}$.

For any $(K, L) \in \mathcal{T}^2, K \neq L$, if $\sigma = \sigma_{K/L}$, let d_σ be the distance between x_K and x_L . For any $K \in \mathcal{T}$, if $\sigma \in \mathcal{E}_K$ and if $\sigma \subset \partial\Omega(0)$, let d_σ be the distance between x_K and σ .

We assume that for any $\sigma \in \mathcal{E}, d_\sigma \neq 0$.

For any $K \in \mathcal{T}$, let $m(K)$ be the area of K . For any $\sigma \in \mathcal{E}$, let $m(\sigma)$ be the length of σ . Let h_0 be the size of the mesh $\mathcal{T}$, $h_0 = \max\{\text{diam}(K), K \in \mathcal{T}\}$, where diam is the abbreviation for diameter.

The number of edges of γ_j is called $n_{\sigma,j}, j = 1, \dots, p$, and the number of edges of $\cup_{j=1}^p \gamma_j$ is called $n_\sigma = \sum_{j=1}^p n_{\sigma,j}$.

We denote by $\mathcal{TS}$ the global 1D-2D mesh of D_ε . Let h be the size of the 1D-2D mesh of $D_\varepsilon : h = \max\{h_0, h^{e_j}, j = 1, \dots, p\}$.

3.2 The monolithic schemes

A constant time step $k \in (0, T)$ is merely chosen for the time discretization. Let $N_k \in \mathbb{N}^*$ such that $N_k = \max\{n \in \mathbb{N}, nk < T\}$. Let $t_n = nk$, for $n \in \{0, \dots, N_k + 1\}$. The value $v_{j,i}^n$ is an approximation of $v_j(x_i^{e_j}, t_n), i = 0, \dots, N_j + 1$. The value u_K^n is an approximation of $u(x_K, t_n), K \in \mathcal{T}, n \in \{0, \dots, N_k + 1\}$.

$$\left\{ \begin{array}{l} h_i^{e_j} \frac{v_{j,i}^{n+1} - v_{j,i}^n}{k} + F_{i+1/2}^{j,n+1} - F_{i-1/2}^{j,n+1} = h_i^{e_j} f_i^{e_j,n}, \quad i = 1, \dots, N_j, j = 1, \dots, p, \\ n \in \{0, \dots, N_k\} \quad (a) \\ F_{i+1/2}^{j,n} = -\frac{v_{j,i+1}^n - v_{j,i}^n}{h_{i+1/2}^{e_j}}, \quad i = 0, \dots, N_j, \quad v_{j,N_j+1}^n = 0, j = 1, \dots, p, \\ n \in \{1, \dots, N_k + 1\} \\ f_i^{e_j,n} = \frac{1}{h_i^{e_j}} \int_{x_{i-1/2}^{e_j}}^{x_{i+1/2}^{e_j}} f_j(x_1, t_{n+1}) dx_1, \quad i = 1, \dots, N_j, j = 1, \dots, p, \\ n \in \{0, \dots, N_k\} \\ m(K) \frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = m(K) f_K^n, \quad \forall K \in \mathcal{T}, n \in \{0, \dots, N_k\} \quad (b) \\ F_{K,\sigma}^n = -\frac{m(\sigma)}{d_\sigma} (u_L^n - u_K^n), \quad \forall \sigma \in \mathcal{E}_{int}, \text{ if } \sigma = \sigma_{K/L}, n \in \{1, \dots, N_k + 1\} \\ F_{K,\sigma}^n = \begin{cases} -\frac{m(\sigma)}{d_\sigma} (v_{j,0}^n - u_K^n) & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \\ 0 & , \forall \sigma \subset \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \end{cases} \quad (d) \\ f_K^n = \frac{1}{m(K)} \int_K f(x, t_{n+1}) dx, \quad \forall K \in \mathcal{T}, n \in \{0, \dots, N_k\} \\ \frac{v_{j,1}^n - v_{j,0}^n}{h_{1/2}^{e_j}} = \frac{1}{\theta_j \varepsilon} \sum_{\substack{\sigma \subset \gamma_j \\ \sigma \in \mathcal{E}_K}} \frac{m(\sigma)}{d_\sigma} (v_{j,0}^n - u_K^n), j = 1, \dots, p, \\ n \in \{1, \dots, N_k + 1\} \quad (c) \end{array} \right. \quad (13)$$

with the initial condition

$$\begin{cases} u_K^0 = 0, \forall K \in \mathcal{T} \\ v_{j,i}^0 = 0, i = 1, \dots, N_j, j = 1, \dots, p \end{cases}$$

Let us recall that $v_{j,0}^n$ is a convex combination of the approximated values of the solution on each side of $\gamma_j, j = 1, \dots, p$. Indeed (13c) can be rewritten in the following way :

$$v_{j,0}^n = \left(\frac{v_{j,1}^n}{h_{1/2}^{e_j}} + \frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} u_K^n \right) \left(\frac{1}{h_{1/2}^{e_j}} + \frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j} \frac{m(\sigma)}{d_\sigma} \right)^{-1} \quad (14)$$

For the sake of simplicity, in (13) and (14), the summation is done for $\sigma \subset \gamma_j$, and for each of them, K is the control volume such that $\sigma \in \mathcal{E}_K$.

Let us define on Ω_ε the function $u_{d,\mathcal{T}}^n$ for $n \in \{0, \dots, N_k + 1\}$ by

$$u_{d,\mathcal{T}}^n(x, y) = \begin{cases} u_K^n, & (x, y) \in K, K \in \mathcal{T} \\ v_{j,i}^n, & (x, y) \in B_j^\varepsilon, x^{e_j} \in (x_{i-1/2}^{e_j}, x_{i+1/2}^{e_j}), i = 1, \dots, N_j, \\ j = 1, \dots, p. \end{cases} \quad (15)$$

The approximate solution of (1) is defined on $\Omega_\varepsilon \times (0, (N_k + 1)k)$ by

$$u_{d,\mathcal{T},k}(x, y, t) = u_{d,\mathcal{T}}^{n+1}(x, y), (x, y) \in \Omega_\varepsilon, t \in (nk, (n+1)k), n \in \{0, \dots, N_k\}$$

The scheme (13) leads to a linear system in the form of $AU^{n+1} = BU^n + F^n$ in which U^n is known and U^{n+1} is unknown, $n \in \{0, \dots, N_k\}$, the initial value U^0 being given ;

$$(U^n)^T = (\{v_{j,i}^n, i = 1, \dots, N_j\}, j = 1, \dots, p, \{u_K^n, K \in \mathcal{T}\}).$$

It is proved in [24] that there is a unique solution (U^{n+1}) to equations (13), (U^n) being given.

To solve (1), we assume that ε and δ are fixed given parameters. Let us remark that a numerical approximation of the solution of (1) is also a numerical approximation of the solution of (3) under some conditions about ε and δ . Indeed, in view of Remark 1, (1) is a reasonable approximation for (3) if ε is small and δ of order $\varepsilon \ln \varepsilon$.

Let us define now the finite volume schemes corresponding with the alternatives (1)"dsa" and (1)"rsa". In both cases, the value of the solution is not assumed to be constant on the interfaces, and the approximate values of the solution on the edge σ on the interface γ_j at time t_n is called $u_\sigma^n, \sigma \subset \gamma_j, j = 1, \dots, p$. For both schemes, the equation (13d) is replaced by

$$F_{K,\sigma}^n = -\frac{m(\sigma)}{d_\sigma} (u_\sigma^n - u_K^n), \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \quad (16)$$

Then, to obtain the scheme for (1)"dsa", the equation (13c) is replaced by

$$\begin{cases} \frac{1}{\theta_j \varepsilon} \sum_{\substack{\sigma \subset \gamma_j \\ \sigma \in \mathcal{E}_K}} m(\sigma) u_\sigma^n = v_{j,0}^n, j = 1, \dots, p, & (d) \\ \frac{u_\sigma^n - u_K^n}{d_\sigma} = \frac{v_{j,1}^n - v_{j,0}^n}{h_{1/2}^{e_j}}, \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \end{cases} \quad (17)$$

that gives

$$v_{j,0}^n = \left(\left(\sum_{\sigma \subset \gamma_j} m(\sigma) d_\sigma \right) v_{j,1}^n + h_{1/2}^{e_j} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} m(\sigma) u_K^n \right) \left(\sum_{\sigma \subset \gamma_j} m(\sigma) d_\sigma + \theta_j \varepsilon h_{1/2}^{e_j} \right)^{-1} \quad (18)$$

It is pointed out that $v_{j,0}^n$ remains a convex combination of the approximated values of the solution on each side of the interface (this is also the case for (1)"rsa" below). The scheme resulting from the equations (13) with (16) and (17) in place of (13c) and (13d) is the monolithic scheme that is used to solve (1)"dsa".

Lemma 3 *There is a unique solution (U^{n+1}) to the linear system that corresponds to the monolithic scheme to solve (1)"dsa", (U^n) being given.*

Proof. For a given $n \in \{0, \dots, N_k\}$, set $f_i^{e_j,n} = 0, v_{j,i}^n = 0, j = 1, \dots, p, i = 1, \dots, N_j$ and $f_K^n = 0, u_K^n = 0, K \in \mathcal{T}$, then the system becomes in the form of $AU^{n+1} = 0$. Let us prove that $U^{n+1} = 0$.

Let us multiply (13a) by $v_{j,i}^{n+1}$ and sum over i , then multiply by $\theta_j \varepsilon$ and sum over j . In the same way, let us multiply (13b) by u_K^{n+1} and sum over K . Reordering the summations over the set of edges in $\Omega(0)$, considering separately the interior edges and the edges on $\gamma_j, j = 1, \dots, p$, we get

$$\begin{aligned} & \sum_{j=1}^p \theta_j \varepsilon \sum_{i=1}^{N_j} \frac{h_i^{e_j}}{k} (v_{j,i}^{n+1})^2 + \sum_{j=1}^p \theta_j \varepsilon \left(\sum_{i=1}^{N_j} F_{i+1/2}^{j,n+1} v_{j,i}^{n+1} - \sum_{i=0}^{N_j-1} F_{i+1/2}^{j,n+1} v_{j,i+1}^{n+1} \right) \\ & + \sum_{K \in \mathcal{T}} \frac{m(K)}{k} (u_K^{n+1})^2 + \sum_{\sigma \in \mathcal{E}_{int}, \sigma = \sigma_{K|L}} F_{K,\sigma}^{n+1} (u_K^{n+1} - u_L^{n+1}) \\ & + \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} u_K^{n+1} = 0 \end{aligned}$$

Replacing the numerical fluxes by their values results in

$$\begin{aligned} & \sum_{j=1}^p \theta_j \varepsilon \sum_{i=1}^{N_j} \frac{h_i^{e_j}}{k} (v_{j,i}^{n+1})^2 + \sum_{K \in \mathcal{T}} \frac{m(K)}{k} (u_K^{n+1})^2 \\ & + \sum_{j=1}^p \theta_j \varepsilon \left(\sum_{i=1}^{N_j} \frac{(v_{j,i+1}^{n+1} - v_{j,i}^{n+1})^2}{h_{i+1/2}^{e_j}} + \frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} v_{j,1}^{n+1} \right) \\ & + \sum_{\sigma \in \mathcal{E}_{int}, \sigma = \sigma_{K|L}} \frac{m(\sigma)}{d_\sigma} (u_K^{n+1} - u_L^{n+1})^2 - \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1}) u_K^{n+1} = 0 \end{aligned} \quad (19)$$

It follows from (17) that

$$- \sum_{j=1}^p \theta_j \varepsilon \frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} v_{j,0}^{n+1} + \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1}) u_\sigma^{n+1} = 0.$$

Adding to the above equality leads to

$$\begin{aligned}
& \sum_{j=1}^p \theta_j \varepsilon \sum_{i=1}^{N_j} \frac{h_i^{e_j}}{k} (v_{j,i}^{n+1})^2 + \sum_{K \in \mathcal{T}} \frac{m(K)}{k} (u_K^{n+1})^2 \\
& + \sum_{j=1}^p \theta_j \varepsilon \sum_{i=0}^{N_j} \frac{(v_{j,i+1}^{n+1} - v_{j,i}^{n+1})^2}{h_{i+1/2}^{e_j}} + \sum_{\sigma \in \mathcal{E}_{int}, \sigma = \sigma_{K|L}} \frac{m(\sigma)}{d_\sigma} (u_K^{n+1} - u_L^{n+1})^2 \\
& + \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1})^2 = 0
\end{aligned} \tag{20}$$

From (20), it easily follows that all the components of U^{n+1} are nil. Thereby A is a regular matrix.

For $n \in \{0, N_k + 1\}$, let us consider the function $\underline{u}_{d,\mathcal{T}}^n$ defined as the restriction of $u_{d,\mathcal{T}}^n$ (see (15)) to D_ε . It is an interesting point that (20) can be written as $\frac{1}{k} \|\underline{u}_{d,\mathcal{T}}^{n+1}\|_{2,\mathcal{T}}^2 + \|\underline{u}_{d,\mathcal{T}}^{n+1}\|_{1,\mathcal{T}}^2 = 0$. See below the definitions of $\|\cdot\|_{2,\mathcal{T}}$ and $\|\cdot\|_{1,\mathcal{T}}$ for the problem (1)"dsa". The same result occurs for the problem (1). ■

And finally, to obtain the scheme for (1)"rsa" with the conditions (7), the equation (13c) is replaced by

$$\left\{ \begin{array}{l} \frac{1}{\theta_j \varepsilon} \sum_{\substack{\sigma \subset \gamma_j \\ \sigma \in \mathcal{E}_K}} m(\sigma) u_\sigma^n = v_{j,0}^n, j = 1, \dots, p, \quad (d) \\ \frac{u_\sigma^n - u_K^n}{d_\sigma} + q_j u_\sigma^n = \frac{v_{j,1}^n - v_{j,0}^n}{h_{1/2}^{e_j}} + q_j v_{j,0}^n, \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p, \quad (r) \end{array} \right. \tag{21}$$

that gives

$$v_{j,0}^n = \left(\left(\sum_{\sigma \subset \gamma_j} \frac{m(\sigma) d_\sigma}{1 + q_j d_\sigma} \right) v_{j,1}^n + h_{1/2}^{e_j} \sum_{\sigma \subset \gamma_j} \frac{m(\sigma)}{1 + q_j d_\sigma} u_K^n \right) \left(\left(\sum_{\sigma \subset \gamma_j} \frac{m(\sigma) d_\sigma}{1 + q_j d_\sigma} \right) (1 - q_j h_{1/2}^{e_j}) + \theta_j \varepsilon h_{1/2}^{e_j} \right)^{-1} \tag{22}$$

If $q_j = 0$ for all $j = 1, \dots, p$, (21) is brought back to (17) and (22) to (18). This was exactly what was expected, since the problem (1)"rsa" with (7) is equivalent to (1)"dsa" in this case.

The problem (1)"rsa" can be written with the conditions (7) or (8), but each condition yields a specific scheme. The equation (21r) describes both. The equation (21d) corresponds only to (7). A scheme which discretizes (8) is written by keeping (21r) supplemented by

$$\frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} m(\sigma) \frac{u_\sigma^n - u_K^n}{d_\sigma} = \frac{v_{j,1}^n - v_{j,0}^n}{h_{1/2}^{e_j}}, j = 1, \dots, p \tag{23}$$

and this time

$$v_{j,0}^n = q_j \left(\left(\sum_{\sigma \subset \gamma_j} \frac{m(\sigma) d_\sigma}{1 + q_j d_\sigma} \right) v_{j,1}^n + h_{1/2}^{e_j} \sum_{\sigma \subset \gamma_j} \frac{m(\sigma)}{1 + q_j d_\sigma} u_K^n \right) \left(\left(\sum_{\sigma \subset \gamma_j} \frac{m(\sigma)}{1 + q_j d_\sigma} \right) (q_j h_{1/2}^{e_j} - 1) + \theta_j \varepsilon \right)^{-1} \tag{24}$$

It has been said that (1)"rsa" with (8) is not well posed if there is $j \in \{1, \dots, p\}$ such that $q_j = 0$. That is found in (24) whose denominator cancels in this case.

The scheme resulting from the equations (13) with (16) and (21r), supplemented by (21d) to describe (7) or (23) to describe (8), in place of (13c) and (13d), is the monolithic scheme that is used to solved (1)"rsa". If $q_j \neq 0$ for all $j = 1, \dots, p$, then (21) is equivalent to (21r) and (23).

Lemma 4 *There is a unique solution (U^{n+1}) to the linear system that corresponds to the monolithic scheme to solve (1)"rsa", (U^n) being given.*

Proof. Let us assume that $q_j \neq 0$ for all $j = 1, \dots, p$ to use interchangeably (21d) and (23) (when it comes to dealing with $j \in \{1, \dots, p\}$ such that $q_j = 0$, we proceed as for the problem (1)"dsa" to address the associated term). The proof is almost the same as above. Considering again (19), it follows from (23) that

$$-\sum_{j=1}^p \theta_j \varepsilon \frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} v_{j,0}^{n+1} + \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1}) v_{j,0}^{n+1} = 0.$$

Writing that $v_{j,0}^{n+1} = u_\sigma^{n+1} + (v_{j,0}^{n+1} - u_\sigma^{n+1})$ and adding to the above equality leads to

$$\begin{aligned} & \sum_{j=1}^p \theta_j \varepsilon \sum_{i=1}^{N_j} \frac{h_i^{e_j}}{k} (v_{j,i}^{n+1})^2 + \sum_{K \in \mathcal{T}} \frac{m(K)}{k} (u_K^{n+1})^2 \\ & + \sum_{j=1}^p \theta_j \varepsilon \sum_{i=0}^{N_j} \frac{(v_{j,i+1}^{n+1} - v_{j,i}^{n+1})^2}{h_{i+1/2}^{e_j}} + \sum_{\sigma \in \mathcal{E}_{int}, \sigma = \sigma_{K|L}} \frac{m(\sigma)}{d_\sigma} (u_K^{n+1} - u_L^{n+1})^2 \\ & + \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1})^2 \\ & + \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1}) (v_{j,0}^{n+1} - u_\sigma^{n+1}) = 0 \end{aligned} \quad (25)$$

To derive from (25) that all the components of U^{n+1} are nil, let us prove by using (21r) that the last term is non-negative :

$$\begin{aligned} & \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} (u_\sigma^{n+1} - u_K^{n+1}) (v_{j,0}^{n+1} - u_\sigma^{n+1}) \\ & = \sum_{j=1}^p \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} m(\sigma) \left(\frac{v_{j,1}^{n+1} - v_{j,0}^{n+1}}{h_{1/2}^{e_j}} + q_j (v_{j,0}^{n+1} - u_\sigma^{n+1}) \right) (v_{j,0}^{n+1} - u_\sigma^{n+1}) \\ & = \sum_{j=1}^p q_j \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} m(\sigma) (v_{j,0}^{n+1} - u_\sigma^{n+1})^2 \geq 0 \end{aligned}$$

by means of (21d). Thereby A is a regular matrix.

In contrast, in the present "rsa" case, (25) can be written as $\frac{1}{k} \|\underline{u}_{d,\mathcal{T}}^{n+1}\|_{2,\mathcal{T}}^2 + \|\underline{u}_{d,\mathcal{T}}^{n+1}\|_{1,\mathcal{T}}^2 + \text{"interface term"} = 0$. See below the definition of $\|\cdot\|_{1,\mathcal{T}}$ for the problem (1)"rsa". This "interface term" measures the gap between the limit values of the solution on the left and right sides of the interfaces. It does not appear in the proofs corresponding to (1) and (1)"dsa". ■

3.3 The specific norms

Three norms are defined in the three spaces of piecewise constant functions on D_ε satisfying the same discrete boundary and interface conditions as the solutions of (1), (1)"dsa" and (1)"rsa" respectively. The discrete L^2 and H^1 norms for solutions of (1) are previously defined in [31].

Definition 5 We define $X(\mathcal{TS})$ the set of the functions from D_ε to $\mathbb{R}$ which are constant over each control volume of $\mathcal{TS}$. Let $W \in X(\mathcal{TS})$,

$$W(x, y) = \begin{cases} W_K, (x, y) \in K, K \in \mathcal{T} \\ W_{j,i}, (x, y) \in S_j, x^{e_j} \in (x_{i-1/2}^{e_j}, x_{i+1/2}^{e_j}), i = 1, \dots, N_j, j = 1, \dots, p. \end{cases}$$

We define and we denote the functions

$$W \mapsto \|W\|_{2,\mathcal{T}} = \left(\sum_{K \in \mathcal{T}} m(K) W_K^2 + \sum_{j=1}^p \theta_j \varepsilon \sum_{i=1}^{N_j} h_i^{e_j} W_{j,i}^2 \right)^{\frac{1}{2}} \text{ and}$$

$$W \mapsto |W|_{1,\mathcal{T}} = \left(\sum_{\sigma \in \mathcal{E}_{int}, \sigma \subset \cup_{j=1}^p \gamma_j} m(\sigma) d_\sigma \left(\frac{D_\sigma W}{d_\sigma} \right)^2 + \sum_{j=1}^p \theta_j \varepsilon \sum_{i=0}^{N_j} \frac{(W_{j,i+1} - W_{j,i})^2}{h_{i+1/2}^{e_j}} \right)^{1/2}$$

where, for the problem (1), $D_\sigma W = \begin{cases} |W_K - W_L|, \sigma \in \mathcal{E}_{int}, \sigma = \sigma_{K|L} \\ |W_K - W_{j,0}|, \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \end{cases}$

$$\text{and } W_{j,0} = \left(\frac{W_{j,1}}{h_{1/2}^{e_j}} + \frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} W_K \right) \left(\frac{1}{h_{1/2}^{e_j}} + \frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j} \frac{m(\sigma)}{d_\sigma} \right)^{-1}$$

and $W_{j,N_j+1} = 0, j \in \{1, \dots, p\}$.

For the problem (1)"dsa", $\frac{D_\sigma W}{d_\sigma} = \frac{|W_{j,1} - W_{j,0}|}{h_{1/2}^{e_j}}, \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p$ and $W_{j,0}$ is given by (18).

For the problem (1)"rsa", $D_\sigma W = |W_\sigma - W_K|, \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p$ where W_σ is deduced from (21r) and $W_{j,0}$ is given by (22) or (24).

The function $|\cdot|_{1,\mathcal{T}}$ is a norm in the discrete space related to the solutions of (1). It is a semi-norm in the discrete spaces related to the solutions of (1)"dsa" and (1)"rsa".

4 Strong coupling

To solve the heat equation, a classical approach is used : you discretizes time (implicit Euler) to obtain a sequence of steady problems to which the domain decomposition strategy is applied (see [8],[21]). Indeed, a waveform relaxation algorithm exists (see [16]), where the decomposition method is directly formulated for the original problem without time discretization, but this kind of algorithm is useful when the physical problems that are solved in the subdomains are not the same and require different numerical methods, and it is not the case here.

Of particular interest is the iterative strong coupling method, which is a domain decomposition strategy especially tailored to solve dimensionally heterogeneous problems. The method is explained in 4.1. From 4.2, schemes dedicated to the three kind of transmission conditions are defined. All schemes are analyzed from an algebraic point of view. The interface systems obtained by the strong coupling method have $2p$ unknowns. Through substitution, classical methods with p unknowns are found, such as the Schur complement system or the flux equation, adjusted to the specific context, preconditioned or not. On the same level, the Robin equation is introduced. The connection is finally made with classical domain decomposition methods such as Dirichlet-Neumann or Neumann-Dirichlet method in 4.5.

4.1 The method

The methodology proposed in [3],[4],[18],[19] and [32], is applied now to solve (1). Let us begin by writing (1) as a set of segregated schemes and by performing an Euler's implicate time-discretization (see the beginning of Section 3.2).

$$\left\{ \begin{array}{l} \frac{v_j^{n+1}(x^{e_j}) - v_j^n(x^{e_j})}{k} - \frac{\partial^2 v_j^{n+1}}{\partial x^{e_j 2}}(x^{e_j}) = f_j(x^{e_j}, t_{n+1}), \quad x^{e_j} \in (\delta, l_j), \\ j = 1, \dots, p \\ v_j^0(x^{e_j}) = 0, \quad x^{e_j} \in (\delta, l_j), j = 1, \dots, p \\ v_j^{n+1}(l_j) = 0 \\ v_j^{n+1}(\delta) = \alpha_j^{n+1}, j = 1, \dots, p \quad (a) \\ \frac{\partial v_j^{n+1}}{\partial x^{e_j}}(\delta) = \beta_j^{n+1}, j = 1, \dots, p \quad (b) \end{array} \right. \quad (26)$$

$$\left\{ \begin{array}{l} \frac{u^{n+1}(x, y) - u^n(x, y)}{k} - \Delta u^{n+1}(x, y) = f(x, y, t_{n+1}), \quad (x, y) \in \Omega(0) \\ u^0(x, y) = 0, \quad (x, y) \in \Omega(0) \\ \frac{\partial u^{n+1}}{\partial n}(x, y) = 0, \quad (x, y) \in \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \\ u^{n+1}(x, y) = \alpha_j^{n+1}, \quad (x, y) \in \gamma_j, j = 1, \dots, p \quad (a) \\ \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u^{n+1}}{\partial n} d\gamma = \beta_j^{n+1}, j = 1, \dots, p \quad (b) \end{array} \right. \quad (27)$$

The basic idea from the previous references is to consider the numerical resolution of the 2D problem (27) on one hand, and of the 1D problems (26) on the other hand, as black-boxes which receive the input data $(\alpha_j^{n+1}, j = 1, \dots, p)$ and give back $(\beta_j^{n+1}, j = 1, \dots, p)$ as output data (or conversely if the problem is well defined). For instance, if we choose that (26) receives the input $(\beta_j^{n+1}, j = 1, \dots, p)$ and (27) receives the input $(\alpha_j^{n+1}, j = 1, \dots, p)$, this can be expressed by

$$\left\{ \begin{array}{l} \alpha_j^{n+1} = \mathcal{F}_{1D,j}^{n+1}(\beta_j^{n+1}), j = 1, \dots, p \\ \beta_j^{n+1} = \mathcal{F}_{2D,j}^{n+1}(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}), j = 1, \dots, p \end{array} \right. \quad (28)$$

where $\mathcal{F}_{1D,j}^{n+1}$ gives the value of the solution of (26-option (b)) at the point δ on the axis j , and $\mathcal{F}_{2D,j}^{n+1}$ gives the mean value on the interface γ_j of the solution of (27-option (a)) at the time t_{n+1} , where (26-option (b)) is the system (26) with its penultimate equation (a) removed, and (27-option (a)) is the system (27) with its last equation (b) removed. The options (a) and (b) are exclusive, and this will always be the case below. Eventually, a system in the interface unknowns

$(\alpha_j^{n+1}, \beta_j^{n+1}, j = 1, \dots, p)$ is obtained, which is fully discretized and solved by an iterative method.

On the contrary, in the previous paper [24], account is taken of the equations (26-option (a)) and (27-option (a)) in addition to the other equations (26b) and (27b). They are related by (1-c) which is written below at the time t_{n+1} :

$$\frac{\partial v_j^{n+1}}{\partial x^{e_j}}(\delta) = \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u^{n+1}}{\partial n} d\gamma, j = 1, \dots, p \quad (29)$$

Therefore the 2D and 1D problems are not understood as separated black-boxes. The values $\beta_j^{n+1}, j = 1, \dots, p$, are no more unknowns and the only interface unknowns that are kept are $\alpha_j^{n+1}, j = 1, \dots, p$. The fully discretization leads to the monolithic scheme (13) which is solved by a direct method (full matrix method). Let us remark that $\alpha_j^{n+1} = v_{j,0}^{n+1}, j = 1, \dots, p$, in (13).

4.2 The numerical scheme

4.2.1 Discretization

The fully discrete scheme corresponding with (28) to solve (1) is the following :

$$\left\{ \begin{array}{l} h_i^{e_j} \frac{v_{j,i}^{n+1} - v_{j,i}^n}{k} + F_{i+1/2}^{j,n+1} - F_{i-1/2}^{j,n+1} = h_i^{e_j} f_i^{e_j,n}, i = 1, \dots, N_j, j = 1, \dots, p, \\ n \in \{0, \dots, N_k\} \\ F_{i+1/2}^{j,n+1} = -\frac{v_{j,i+1}^{n+1} - v_{j,i}^{n+1}}{h_{i+1/2}^{e_j}}, i = 1, \dots, N_j, v_{j,N_j+1}^{n+1} = 0, j = 1, \dots, p, \\ n \in \{0, \dots, N_k\} \\ v_{j,i}^0 = 0, i = 1, \dots, N_j, j = 1, \dots, p \\ F_{1/2}^{j,n+1} = -\beta_j^{n+1}, n \in \{0, \dots, N_k\} \quad (b) \\ f_i^{e_j,n} = \frac{1}{h_i^{e_j}} \int_{x_{i-1/2}^{e_j}}^{x_{i+1/2}^{e_j}} f_j(x_1, t_{n+1}) dx_1, i = 1, \dots, N_j, j = 1, \dots, p, \\ n \in \{0, \dots, N_k\} \\ \frac{v_{j,1}^{n+1} - \alpha_j^{n+1}}{h_{1/2}^{e_j}} = \beta_j^{n+1}, j = 1, \dots, p, n \in \{0, \dots, N_k\} \quad (a) \end{array} \right. \quad (30)$$

$$\left\{ \begin{array}{l} m(K) \frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = m(K) f_K^n, \quad \forall K \in \mathcal{T}, n \in \{0, \dots, N_k\} \\ F_{K,\sigma}^{n+1} = -\frac{m(\sigma)}{d_\sigma} (u_L^{n+1} - u_K^{n+1}), \quad \forall \sigma \in \mathcal{E}_{int}, \text{ if } \sigma = \sigma_{K/L}, n \in \{0, \dots, N_k\} \\ F_{K,\sigma}^{n+1} = \begin{cases} -\frac{m(\sigma)}{d_\sigma} (\alpha_j^{n+1} - u_K^{n+1}) & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \\ 0 & , \forall \sigma \subset \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \end{cases} \quad (a) \\ u_K^0 = 0, \quad \forall K \in \mathcal{T}, \\ f_K^n = \frac{1}{m(K)} \int_K f(x, t_{n+1}) dx, \quad \forall K \in \mathcal{T}, n \in \{0, \dots, N_k\} \\ \frac{1}{\theta_j \varepsilon} \sum_{\substack{\sigma \subset \gamma_j \\ \sigma \in \mathcal{E}_K}} \frac{m(\sigma)}{d_\sigma} (\alpha_j^{n+1} - u_K^{n+1}) = \beta_j^{n+1}, j = 1, \dots, p, n \in \{0, \dots, N_k\} \quad (b) \end{array} \right. \quad (31)$$

As above, let us denote (30-option (b)) (resp. (31-option (a))) the system (30) (resp. (31)) with its last equation (30a) (resp. (31b)) removed. The fully-discretized version of (28) is the system constituted by (30a) and (31b) (except that (30a) should be re-written $\alpha_j^{n+1} = \dots$). Of course, $v_{j,1}^{n+1}$ in (30a) is an implicate function of β_j^{n+1} since $(v_{j,i}^{n+1})_i$ is the solution of (30-option (b)). The same thing about the value of $(u_K^{n+1})_K$ on γ_j in (31b) is true since $(u_K^{n+1})_K$ is the solution of (31-option (a)), so a function of $(\alpha_j^{n+1})_{j=1,\dots,p}$.

The system constituted by (30a) and (31b) will be named the interface system. Interior unknowns are eliminated by the means of (30-option (b)) and (31-option (a)).

4.2.2 Formulation for the interface system with $2p$ unknowns

The 1D discrete operator in (30a) consists to solve the 1D Neumann problem (30-option (b)) and to give an approximate value of the solution at the point δ on the axis j . It is denoted by $S_{1D,j}^{ND,n+1}$ below, $j = 1, \dots, p$.

The 2D discrete operator in (31b) consists to solve the 2D Dirichlet problem (31-option (a)) and to give a discrete mean value of the normal derivative on $\gamma_j, j = 1, \dots, p$. It can be described by introducing general 2D problems, at the continuous level, with a Dirichlet or Neumann boundary value which is not compulsory constant. Below we give the semi-discrete version of the general Dirichlet problem (option (a)) and of the general Neumann problem (option (c))

$$\left\{ \begin{array}{l} \frac{u^{n+1}(x, y) - u^n(x, y)}{k} - \Delta u^{n+1}(x, y) = f(x, y, t_{n+1}), \quad (x, y) \in \Omega(0) \\ u^0(x, y) = 0, \quad (x, y) \in \Omega(0) \\ \frac{\partial u^{n+1}}{\partial n}(x, y) = 0, \quad (x, y) \in \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \\ u^{n+1}(x, y) = \tilde{\alpha}_j^{n+1}(x, y), \quad (x, y) \in \gamma_j, j = 1, \dots, p \quad (a) \\ \frac{\partial u^{n+1}}{\partial n}(x, y) = \tilde{\beta}_j^{n+1}(x, y), \quad (x, y) \in \gamma_j, j = 1, \dots, p \quad (c) \end{array} \right. \quad (32)$$

where, this time, the value of the solution $\tilde{\alpha}_j^{n+1}$ or of its normal derivative $\tilde{\beta}_j^{n+1}$ on γ_j may

depend on the space variables. Then a full space-time discretization of these general problems, with $u_K^0 = 0$, $\forall K \in \mathcal{T}$, and $n \in \{0, \dots, N_k\}$, is

$$\left\{ \begin{array}{l} m(K) \frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = m(K) f_K^n, \quad \forall K \in \mathcal{T} \\ F_{K,\sigma}^{n+1} = -\frac{m(\sigma)}{d_\sigma} (u_L^{n+1} - u_K^{n+1}), \quad \forall \sigma \in \mathcal{E}_{int}, \text{ if } \sigma = \sigma_{K/L} \\ F_{K,\sigma}^{n+1} = \begin{cases} -\frac{m(\sigma)}{d_\sigma} (\tilde{\alpha}_{j,\sigma}^{n+1} - u_K^{n+1}) & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \quad (a) \\ -m(\sigma) \tilde{\beta}_{j,\sigma}^{n+1} & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \quad (c) \\ 0 & , \forall \sigma \subset \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \end{cases} \end{array} \right. \quad (33)$$

where $\tilde{\alpha}_{j,\sigma}^{n+1}$ approaches the value of $\tilde{\alpha}_j^{n+1}$ on the edge σ (and the same for $\tilde{\beta}_{j,\sigma}^{n+1}$). As previously, the options (a) and (c) are exclusive. The equation (33c) gives the value on γ_j of a discrete normal derivative of the solution of the 2D Dirichlet problem (33-option (a)) that is to say $\tilde{\beta}_{j,\sigma}^{n+1} = \frac{\tilde{\alpha}_{j,\sigma}^{n+1} - u_K^{n+1}}{d_\sigma}$, $\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p$. This can be written

$$(\tilde{\beta}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} - S_{2D}^{DN, n+1} (\tilde{\alpha}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} = 0 \quad (34)$$

where $S_{2D}^{DN, n+1}$ is the discrete operator that solves the 2D Dirichlet problem (33-option (a)) and gives the value on γ_j of a discrete normal derivative of the solution. To make clear the linear part of the operator $S_{2D}^{DN, n+1}$, this can be written

$$(\tilde{\beta}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} - A_{2D}^{DN} (\tilde{\alpha}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} = B_{2D}^{DN, n+1} \quad (35)$$

where $B_{2D}^{DN, n+1}$ is a matrix $n_\sigma \times 1$ and A_{2D}^{DN} is a square real matrix $n_\sigma \times n_\sigma$.

Let us denote $S_{2D}^{ND, n+1}$, which is used later, the operator that gives the discrete value on γ_j of the solution of the 2D Neumann problem (33-option (c)). We have

$$(\tilde{\alpha}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} = S_{2D}^{ND, n+1} (\tilde{\beta}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} \quad (36)$$

and also

$$(\tilde{\alpha}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} = (A_{2D}^{DN})^{-1} ((\tilde{\beta}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} - B_{2D}^{DN, n+1}) \quad (37)$$

and still

$$S_{2D}^{DN, n+1} S_{2D}^{ND, n+1} = S_{2D}^{ND, n+1} S_{2D}^{DN, n+1} = I \quad (38)$$

where I stands for the identity operator in $\mathbb{R}^{n_\sigma}$.

Now, to completely describe (31b), let us introduce the two following linear discrete operators R_j (mean value operator) and E (trivial extension operator). For all w_j and $w_{j,\sigma}, j = 1, \dots, p$, we define

$$R_j(w_{i,\sigma})_{\sigma \subset \gamma_i, i=1, \dots, p} = \frac{1}{\theta_{j\mathcal{E}}} \sum_{\sigma \subset \gamma_j} m(\sigma) w_{j,\sigma} \quad (39)$$

$$E(w_j)_{j=1, \dots, p} = (w_{j,\sigma})_{\sigma \subset \gamma_j, j=1, \dots, p}, w_{j,\sigma} = w_j, \forall \sigma \subset \gamma_j, j = 1, \dots, p$$

We keep the same notations for the $1 \times n_\sigma$ matrix R_j and the $n_\sigma \times p$ matrix E . If $w_{j,\sigma} = w_j, \forall \sigma \subset \gamma_j$, then $R_j(w_{i,\sigma})_{\sigma \subset \gamma_i, i=1, \dots, p} = w_j$, and especially $R_j E(w_i)_{i=1, \dots, p} = w_j$.

Then the 2D discrete operator of (31b) can be written as a combination of $S_{2D}^{DN, n+1}, R_j, j = 1, \dots, p$ and E . The matrix R_j is a set of p blocks of size $1 \times n_{\sigma,j}, j = 1, \dots, p$, whose blocks are

nil except the j th. The matrix E is a set of $p \times p$ blocks, each block of size $n_{\sigma,j} \times 1, j = 1, \dots, p$, whose blocks are nil except the diagonal ones where all terms are equal to 1.

Eventually, the interface system can be written

$$\begin{cases} \alpha_j^{n+1} - S_{1D,j}^{ND,n+1}(\beta_j^{n+1}) = 0, j = 1, \dots, p \\ \beta_j^{n+1} - R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) = 0, j = 1, \dots, p \end{cases} \quad (40)$$

It gives rise to a linear system which can be written

$$\begin{cases} \alpha_j^{n+1} - a_j^{ND} \beta_j^{n+1} = b_j^{ND,n+1}, j = 1, \dots, p \\ \beta_j^{n+1} - R_j A_{2D}^{DN} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) = R_j B_{2D}^{DN,n+1}, j = 1, \dots, p \end{cases} \quad (41)$$

with $a_j^{ND}, b_j^{ND,n+1} \in \mathbb{R}$.

The first equations of (41) can be assembled to form

$$(\alpha_1^{n+1}, \dots, \alpha_p^{n+1})^T = A_{1D}^{ND} (\beta_1^{n+1}, \dots, \beta_p^{n+1})^T + B_{1D}^{ND,n+1} \quad (42)$$

where A_{1D}^{ND} is a diagonal matrix.

To be even shorter, (40) can be written $R^{n+1}(X^{n+1}) = 0$ with $X^T = (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p)$, and R^{n+1} an affine function. Such an interface system is solved in [18] by using the matrix-free GMRES method. Indeed, hybrid direct/iterative solvers are generally developed for domain decomposition methods (see for instance [11],[28]). The advantage of using Krylov type methods is that they only require matrix-vector products, which can be performed in parallel without computing the matrix of the interface system. In this paper, the GMRES method is used for all the numerical experiments. The scheme corresponding to (40) will be referred to as *DmNND*, by using the capital letters D and N which define the interface conditions, writing firstly the input/output values of the 2D problem, and secondly those of the 1D problems, and where "m" in front of "N" stands for "mean value of the Neumann values" that is to say the output values of the 2D problem are the mean values of the normal derivatives of the solution on the interfaces (see Table 4 to take a comprehensive view).

4.2.3 Formulation for the interface systems with p unknowns

By using direct elimination, more precisely by replacing in (40) the value of each $\beta_j^{n+1}, j = 1, \dots, p$, this yields

$$S_{1D,j}^{ND,n+1} R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) - \alpha_j^{n+1} = 0, j = 1, \dots, p \quad (43)$$

Since

$$S_{1D,j}^{ND,n+1} S_{1D,j}^{DN,n+1}(\alpha_j^{n+1}) = \alpha_j^{n+1}, j = 1, \dots, p \quad (44)$$

we see that (43) consists in solving the linear system

$$a_j^{ND} \left(R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) - S_{1D,j}^{DN,n+1}(\alpha_j^{n+1}) \right) = 0, j = 1, \dots, p \quad (45)$$

And so, (43) is a method to solve the so-called multi-scale Schur complement system

$$R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) - S_{1D,j}^{DN,n+1}(\alpha_j^{n+1}) = 0, j = 1, \dots, p \quad (46)$$

with the preconditioner A_{1D}^{ND} . Indeed, the Schur complement system is the algebraic counterpart of the Steklov-Poincaré equation. In the classical domain decomposition literature, the Steklov-Poincaré equation requires that the normal derivatives of the solutions of local Dirichlet, with Dirichlet data equal to α_j^{n+1} , $j = 1, \dots, p$, problems in the subdomains, are equal. The equation (46) is just an adaptation of this by using mean values.

By using again elimination, by replacing this time in (40) the value of each α_j^{n+1} , $j = 1, \dots, p$, this yields

$$R_j S_{2D}^{DN,n+1} E \left(S_{1D,1}^{ND,n+1}(\beta_1^{n+1}), \dots, S_{1D,p}^{ND,n+1}(\beta_p^{n+1}) \right) - \beta_j^{n+1} = 0, j = 1, \dots, p \quad (47)$$

Since

$$R_j S_{2D}^{DN,n+1} S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) = \beta_j^{n+1}, j = 1, \dots, p \quad (48)$$

we see that (47) consists in solving the linear system

$$R A_{2D}^{DN} \left(E \left(S_{1D,1}^{ND,n+1}(\beta_1^{n+1}), \dots, S_{1D,p}^{ND,n+1}(\beta_p^{n+1}) \right) - S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) \right) = 0 \quad (49)$$

where R is a block matrix with p lines, collection of R_j , $j = 1, \dots, p$.

In our opinion, we can consider that (49) is the flux equation that corresponds to the problem (1). In the classical domain decomposition literature, for instance in [29], the flux equation corresponds to solve local Neumann problems in the subdomains, with Neumann data equal (equal to β_j^{n+1} , $j = 1, \dots, p$, here), and to ensure both local solutions have the same value on the interface. Here, since the values on the interface are not equal, it is requested in a second step that the solving of the local Dirichlet problems with these different values gives ultimately the same mean value of the normal derivatives on the interface.

Remark 6 *The Schur complement system (46) is nothing else but (13c) or (14), which means that α_j^{n+1} (that is $v_{j,0}^{n+1}$ in the monolithic scheme) is a convex combination of the approximated values of the solution on each side of the interface γ_j , $j = 1, \dots, p$. It is interesting to note that (13c) played a key role in [24] where the system (13) was solved by a direct method (full matrix method).*

Remark 7 *To keep the frame of mind of the strong coupling strategy, that is to say to have $2 \times p$ unknowns, the interface system (40) to solve (1) should not be solved through substitution. Indeed, the Schur complement system (46) has just p unknowns. The same is true when considering the problems (1)"dsa" and (1)"rsa". The schemes obtained through substitution are nevertheless addressed for comparison purposes, and for this paper to be connected with the classical literature.*

4.3 Different choices of boundary condition

To provide meaning to the multi-dimensional interface conditions, it is usually assumed that the function (or the flux) is constant or the mean value on the interface is considered, and this might give a problem which is not well defined. Below, we consider several boundary conditions in (26) and (27) that give a problem different from (40) which remains well defined. It is determined to what extent the Schur complement system or the flux equation are found or not. Each time, for reasons of simplicity, only one kind of boundary conditions is chosen on the interfaces of the 2D domain, and another on those of the 1D domains.

4.3.1 Neumann boundary conditions for the 1D and the 2D problem

Firstly, let us consider again (26) and (27) and make another choice of options than the one made in (28). We choose the normal derivatives $(\beta_1^{n+1}, \dots, \beta_p^{n+1})$ as the unknown common input values on the interfaces, we solve the 1D and 2D Neumann problems (26-option (b)) and (32-option (c)) with $\tilde{\beta}_j^{n+1}(x, y) = \beta_j^{n+1}, \forall (x, y) \in \gamma_j$, to find $u^{n+1}, v_1^{n+1}, \dots, v_p^{n+1}$, and we write the interface equations (26a) and (27a) which means that the value of u^{n+1} on γ_j is the same as the value of $v_j^{n+1}, j = 1, \dots, p$. It is worth noting that we look at (32-option (c)) because (27-option (b)) is not well defined. The systems (27) and (32) are the same except the options. The interface equation implies that the value of the solution is constant on each interface in such way that the problem does not have any solution in general.

Remark 8 *The problem (27-option (b)) is not well defined as such, but assuming the solution is constant along the interface makes it well-posed. However this kind of non-standard 2D problems is not addressed in this paper, following [18] in which the authors choose to not consider non-standard problems because most available codes do not offer such boundary conditions as ready-to-use option.*

To provide meaning to the previous problem, it is possible to change the context by considering this time the problem (1)"dsa" by choosing interface conditions different from (27a). Let us write the interface conditions according to the relation (6d)

$$\frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u^{n+1} d\gamma = \alpha_j^{n+1} \quad (50)$$

which means that the mean value of u^{n+1} on γ_j is the value of $v_j^{n+1}(\delta), j = 1, \dots, p$.

The finite volume scheme, which was (30)(31) with the first choice of boundary conditions, becomes now (30)(51) given below.

$$\left\{ \begin{array}{l} m(K) \frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = m(K) f_K^n, \quad \forall K \in \mathcal{T} \\ F_{K,\sigma}^{n+1} = -\frac{m(\sigma)}{d_\sigma} (u_L^{n+1} - u_K^{n+1}), \quad \forall \sigma \in \mathcal{E}_{int}, \text{ if } \sigma = \sigma_{K/L} \\ F_{K,\sigma}^{n+1} = \begin{cases} -\frac{m(\sigma)}{d_\sigma} (\tilde{\alpha}_{j,\sigma}^{n+1} - u_K^{n+1}) & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \\ -m(\sigma) \beta_j^{n+1} & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \\ 0 & , \forall \sigma \subset \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \end{cases} \quad \begin{array}{l} (a) \\ (c) \end{array} \\ \frac{1}{\theta_j \varepsilon} \sum_{\substack{\sigma \subset \gamma_j \\ \sigma \in \mathcal{E}_K}} m(\sigma) \tilde{\alpha}_{j,\sigma}^{n+1} = \alpha_j^{n+1}, j = 1, \dots, p, n \in \{0, \dots, N_k\} \quad (d) \end{array} \right. \quad (51)$$

The interface system is (30a)(51a)(51d) which depends on the solutions of (30-option (b)) and (51-option (c)). It can be written

$$\begin{cases} \alpha_j^{n+1} - S_{1D,j}^{ND,n+1}(\beta_j^{n+1}) = 0, j = 1, \dots, p \\ \alpha_j^{n+1} - R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) = 0, j = 1, \dots, p \end{cases} \quad (52)$$

This leads to what the flux equation could be at the discrete level

$$S_{1D,j}^{ND,n+1}(\beta_j^{n+1}) - R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) = 0, j = 1, \dots, p \quad (53)$$

to solve (1)"dsa". This is in line with the classical one (except the use of mean values). The scheme corresponding to (52) will be referred to as *NmDND*, where "m" in front of "D" stands for "mean value of the Dirichlet values" that is to say the output values of the 2D problem are the mean values of the solution on the interfaces (see Table 4).

4.3.2 Dirichlet boundary conditions for the 1D and the 2D problems

Let us make one more time a new choice of options in (26) and (27) to solve (1). We choose $(\alpha_1^{n+1}, \dots, \alpha_p^{n+1})$ as the unknown common values on the interfaces, we solve the 1D and 2D Dirichlet problems (26-option (a)) and (27-option (a)), to find $u^{n+1}, v_1^{n+1}, \dots, v_p^{n+1}$, and we write (26b) and (27b) which means that the mean value of the normal derivative of u^{n+1} on γ_j is the same as the derivative of $v_j^{n+1}, j = 1, \dots, p$ at the point δ .

The finite volume scheme is no more (30)(31). The 1D part (30) is modified by replacing (30b) by

$$F_{1/2}^{j,n+1} = -\frac{v_{j,1}^{n+1} - \alpha_j^{n+1}}{h_{1/2}^{e_j}} \quad (54)$$

which discretizes the option (a) in (26) and allows to describe the operator $S_{1D,j}^{DN,n+1}$ completely. The description of the interface system is (30a)(31b), depending on the solutions of (30) with the option (54) instead of (30b) and (31-option (a)). The discrete interface system is

$$\begin{cases} \beta_j^{n+1} - S_{1D,j}^{DN,n+1}(\alpha_j^{n+1}) = 0, j = 1, \dots, p \\ \beta_j^{n+1} - R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) = 0, j = 1, \dots, p \end{cases} \quad (55)$$

The scheme corresponding to (55) will be referred to as *DmNDN* (see Table 4). It leads to (46), the Schur complement system for the problem (1).

4.3.3 Dirichlet boundary conditions for the 1D problems and Neumann boundary conditions for the 2D problem

Eventually, we choose to solve the 1D Dirichlet and 2D Neumann problems (26-option (a)) and (32-option (c)) with $\tilde{\beta}_j^{n+1}(x, y) = \beta_j^{n+1}, \forall (x, y) \in \gamma_j$. If we write the interface system (26b) and (27a), this involves that the value of u^{n+1} on γ_j is constant. Indeed, we have again as previously a problem that does not have any solution in general.

However, as above, a Neumann boundary condition for the 2D problem can be considered if (5) is replaced with (6) in (1), that is to say if the problem (1)"dsa" is considered. The finite volume scheme consists to solve (30) with the option (54) supplemented with (51-option (c)), and the interface system is (30a)(51a)(51d). It can be written

$$\begin{cases} \beta_j^{n+1} - S_{1D,j}^{DN,n+1}(\alpha_j^{n+1}) = 0, j = 1, \dots, p \\ \alpha_j^{n+1} - R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) = 0, j = 1, \dots, p \end{cases} \quad (56)$$

which gives the solution of (1)"dsa". The scheme corresponding to (56) will be referred to as *NmDDN* (see Table 4).

Let us now look at the resulting p unknowns schemes. If the value of each $\alpha_j^{n+1}, j = 1, \dots, p$, is replaced in (40), this yields

$$S_{1D,j}^{DN,n+1}(R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1})) - \beta_j^{n+1} = 0, j = 1, \dots, p \quad (57)$$

Since

$$S_{1D,j}^{DN,n+1} S_{1D,j}^{ND,n+1}(\beta_j^{n+1}) = \beta_j^{n+1}, j = 1, \dots, p \quad (58)$$

we see that (57) consists in solving the linear system

$$A_{1D}^{DN} \left((S_{1D,1}^{ND,n+1}(\beta_1^{n+1}), \dots, S_{1D,p}^{ND,n+1}(\beta_p^{n+1})) - R S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) \right) = 0 \quad (59)$$

We get the flux equation (53) with the preconditioner A_{1D}^{DN} .

Solving (56) through substitution by replacing this time the value of each $\beta_j^{n+1}, j = 1, \dots, p$, yields

$$\alpha_j^{n+1} - R_j S_{2D}^{ND,n+1} E \left(S_{1D,1}^{DN,n+1}(\alpha_1^{n+1}), \dots, S_{1D,p}^{DN,n+1}(\alpha_p^{n+1}) \right) = 0, j = 1, \dots, p \quad (60)$$

Since

$$R_j S_{2D}^{ND,n+1} S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) = \alpha_j^{n+1}, j = 1, \dots, p \quad (61)$$

the system (60) can be rewritten to give

$$R A_{2D}^{ND} \left(S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) - E \left(S_{1D,1}^{DN,n+1}(\alpha_1^{n+1}), \dots, S_{1D,p}^{DN,n+1}(\alpha_p^{n+1}) \right) \right) = 0, j = 1, \dots, p \quad (62)$$

We get an equation in the variable $\alpha_j^{n+1}, j = 1, \dots, p$, which may be considered as the Schur complement system for the problem (1)"dsa". First, local Dirichlet problems are solved which do not give solutions with the same normal derivative on the interface. Then it is required in a second step that the solving of the local Neumann problems gives ultimately the same mean value of the solutions on the interface.

4.4 Robin boundary conditions

Robin boundary conditions allow us to define several kind of schemes within the framework of the iterative strong coupling method. This leads to look at now the problem (1)"rsa", which is considered in both forms of (7) and (8).

We consider the continuous problem, just time-discretized, that is to say (26) and (27), where new options are offered in addition, for instance

$$\frac{\partial v_j^{n+1}}{\partial x^{e_j}}(\delta) + q_j v_j^{n+1}(\delta) = \omega_j^{n+1}, j = 1, \dots, p \quad (63)$$

and

$$\frac{\partial u^{n+1}}{\partial n}(x, y) + q_j u^{n+1}(x, y) = \omega_j^{n+1}, (x, y) \in \gamma_j, j = 1, \dots, p. \quad (64)$$

where $\omega_j^{n+1}, j = 1, \dots, p$, are constant.

Below, the relation (64) is addressed, first as an option that gives input values and later as output values (in the latter case, adaptation is necessary). When this option corresponds to input values, the problem (1)"rsa" is well defined. When (64) gives output values, then the 2D

problem is solved with Dirichlet or Neumann input boundary values, which brings us back to solve (1) or (1)"dsa".

In the same way as the flux equation or the Schur complement system are defined in the classical domain decomposition literature or in the multi-scale case, we define here "the Robin equation" by writing that the output values given by some local Robin problems on both sides of the interfaces are equal. This relates to Schur dual methods.

The table below summarizes the interface conditions which are used, and gives the associated problem each time. The cases where Robin conditions are used (both as input or output values) are only considered .

2D	-	1D	-	problem
input	output	input	output	-
R	mD	R	D	(1)"rsa"
R	mN	R	N	(1)"rsa"
R	mD	D	R	(1)"rsa"
R	mN	N	R	(1)"rsa"
D	mdR	D	R	(1)
N	muR	N	R	(1)"dsa"
D	mdR	R	D	(1)
N	muR	R	N	(1)"dsa"

TABLE 1 – Interface conditions, one of them being Robin.

The notation "mN" or "mD" has already been encountered, it means "mean value of the normal derivatives of the solution on the interfaces" or "mean value of the solution" itself. The new notation "mdR" means "mean value of the normal derivatives in (64)", and "muR" means "mean value of the solution in (64)". Indeed (64) can not be used as such as output value.

Remark 9 *It is possible to choose Robin conditions as input values for the 2D problem to solve (1) and (1)"dsa". To keep the length of this paper within reasonable limits, the corresponding options will be considered in another paper.*

4.4.1 Robin boundary conditions for the 1D problems and Robin boundary conditions for the 2D problem

Here, we look at the problem (1)"rsa" in the form of (7). We assume that the problem (26) with the option (63) (instead of options (a) and (b)) and the problem (27) with the option (64) (instead of options (a) and (b)) receive Robin input data $(\omega_j^{n+1}, j = 1, \dots, p)$ and give back $(\alpha_j^{n+1}, j = 1, \dots, p)$ as Dirichlet output data. The interface system is (26a) and (50).

As above it is useful to introduce the general Robin problem (27) with the option

$$\frac{\partial u^{n+1}}{\partial n}(x, y) + q_j u^{n+1}(x, y) = \tilde{\omega}_j^{n+1}(x, y), (x, y) \in \gamma_j, j = 1, \dots, p. \quad (65)$$

where, this time, $\tilde{\omega}_j^{n+1}, j = 1, \dots, p$, may depend on the space variables.

The finite volume scheme with the general option (65) is given below

$$\left\{ \begin{array}{l} m(K) \frac{u_K^{n+1} - u_K^n}{k} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = m(K) f_K^n, \quad \forall K \in \mathcal{T} \\ F_{K,\sigma}^{n+1} = -\frac{m(\sigma)}{d_\sigma} (u_L^{n+1} - u_K^{n+1}), \quad \forall \sigma \in \mathcal{E}_{int}, \text{ if } \sigma = \sigma_{K/L} \\ F_{K,\sigma}^{n+1} = \begin{cases} -\frac{m(\sigma)}{d_\sigma} (\tilde{\alpha}_{j,\sigma}^{n+1} - u_K^{n+1}) & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \\ -m(\sigma) \tilde{\beta}_j^{n+1} & , \forall \sigma \subset \gamma_j, \sigma \in \mathcal{E}_K, j = 1, \dots, p \\ 0 & , \forall \sigma \subset \partial\Omega(0) \setminus (\cup_{j=1}^p \gamma_j) \end{cases} \quad \begin{array}{l} (a) \\ (c) \end{array} \\ \frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} m(\sigma) \tilde{\alpha}_{j,\sigma}^{n+1} = \alpha_j^{n+1}, j = 1, \dots, p, \quad (d) \\ \frac{\tilde{\alpha}_{j,\sigma}^{n+1} - u_K^{n+1}}{d_\sigma} + q_j \tilde{\alpha}_{j,\sigma}^{n+1} = \tilde{\omega}_{j,\sigma}^{n+1}, \forall \sigma \subset \gamma_j, j = 1, \dots, p, \quad (r) \end{array} \right. \quad (66)$$

where $\tilde{\omega}_{j,\sigma}^{n+1}$ approaches the value of $\tilde{\omega}_j^{n+1}$ on the edge σ . It is the same scheme as (33) with additional options.

The 1D discrete operator that is used consists to solve (30) with the option (54) and

$$\frac{v_{j,1}^{n+1} - \alpha_j^{n+1}}{h_{1/2}^{e_j}} + q_j \alpha_j^{n+1} = \omega_j^{n+1} \quad (67)$$

which is the discretization of (63), and then to give the value of α_j^{n+1} , it is called $S_{1D,j}^{RD,n+1}$.

The 2D discrete operator that solves the general 2D Robin problem (66-option (a) and (r)) and gives the discrete values of the solution on the interfaces is called $S_{2D}^{RD,n+1}$. This can be written

$$(\tilde{\alpha}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} - S_{2D}^{RD,n+1} (\tilde{\omega}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} = 0 \quad (68)$$

To make clear the linear part of the operator $S_{2D}^{RD,n+1}$, this can be written

$$(\tilde{\alpha}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} - A_{2D}^{RD} (\tilde{\omega}_{j,\sigma}^{n+1})_{\sigma \in \gamma_j, j=1, \dots, p} = B_{2D}^{RN,n+1} \quad (69)$$

where $B_{2D}^{RD,n+1}$ is a matrix $n_\sigma \times 1$ and A_{2D}^{RD} is a square real matrix $n_\sigma \times n_\sigma$.

Finally, the discrete interface system, which is (67) and (66d), where $\tilde{\omega}_j^{n+1}(x, y) = \omega_j^{n+1}$, $\forall (x, y) \in \gamma_j$, can be written

$$\begin{cases} \alpha_j^{n+1} = S_{1D,j}^{RD,n+1}(\omega_j^{n+1}), j = 1, \dots, p \\ \alpha_j^{n+1} = R_j S_{2D}^{RD,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}), j = 1, \dots, p \end{cases} \quad (70)$$

The scheme corresponding to (70) will be referred to as *RmDRD* (see Table 7 to take a comprehensive view). That gives a relation with $(\omega_j^{n+1}, j = 1, \dots, p)$. We call this relation the Robin equation for the problem (1)"rsa", which can be put on the same level as the well known flux equation or Schur complement system :

$$R_j S_{2D}^{RD,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) = S_{1D,j}^{RD,n+1}(\omega_j^{n+1}), j = 1, \dots, p \quad (71)$$

Now let us look at the problem (1)"rsa" in the form of (8). Again, we solve the problem (26) with the option (63) and the problem (27) with the option (64). The interface system is (26b) and (27b). At the discrete level, we need the operators $S_{1D,j}^{RN,n+1}$ and $S_{2D}^{RN,n+1}$ to describe the problem. They are defined just as $S_{1D,j}^{RD,n+1}$ and $S_{2D}^{RD,n+1}$ above, by changing the output values, more precisely by considering Neumann values instead of Dirichlet values on the interfaces. We introduce

$$(1 - q_j h_{1/2}^{e_j})\beta_j^{n+1} + q_j v_{j,1}^{n+1} = \omega_j^{n+1} \quad (72)$$

and

$$(1 + q_j d_\sigma)\tilde{\beta}_{j,\sigma}^{n+1} + q_j u_K^{n+1} = \tilde{\omega}_{j,\sigma}^{n+1} \quad (73)$$

which are just another way to write (67) and (66r). The new operators solve the systems (30-option (b)) supplemented by (72), and (66-option (c)) supplemented by (73). Then they give the values of the normal derivative of the solution on the interfaces. To completely describe (8), the mean value of the normal derivative remains to be defined :

$$\frac{1}{\theta_j \varepsilon} \sum_{\sigma \subset \gamma_j, \sigma \in \mathcal{E}_K} m(\sigma) \tilde{\beta}_{j,\sigma}^{n+1} = \beta_j^{n+1}, j = 1, \dots, p, \quad (74)$$

Finally, the discrete interface system is (72) and (74). This can be written

$$\begin{cases} \beta_j^{n+1} = S_{1D,j}^{RN,n+1}(\omega_j^{n+1}), j = 1, \dots, p \\ \beta_j^{n+1} = R_j S_{2D}^{RN,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}), j = 1, \dots, p \end{cases} \quad (75)$$

The scheme corresponding to (75) will be referred to as *RmNRN* (see Table 7). That gives

$$R_j S_{2D}^{RN,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) = S_{1D,j}^{RN,n+1}(\omega_j^{n+1}), j = 1, \dots, p \quad (76)$$

Let us prove that this relation is equivalent to the Robin equation (71). Indeed, as far as the 1D discrete operators are concerned, we conclude from (30), (54) and (67) that for all $\omega_j^{n+1}, j = 1, \dots, p$,

$$S_{1D,j}^{RN,n+1}(\omega_j^{n+1}) + q_j S_{1D,j}^{RD,n+1}(\omega_j^{n+1}) = \omega_j^{n+1}, j = 1, \dots, p \quad (77)$$

A similar relation exists for the 2D discrete operators. If I stands for the identity operator (in a convenient set which depends simply on the context) then we have

$$S_{2D}^{RN,n+1} + Q S_{2D}^{RD,n+1} = I \quad (78)$$

where the matrix Q (the same notation as the operator is used) of size $n_\sigma \times n_\sigma$, is block diagonal, and the diagonal blocks of size $n_{\sigma,j} \times n_{\sigma,j}, j = 1, \dots, p$, are equal to $q_j I$. In the following, the diagonal matrix $\hat{Q}$ of size $p \times p$ whose diagonal coefficients are just $q_j, j = 1, \dots, p$, is also useful.

Now, it is easy to see that, by replacing $S_{1D,j}^{RN,n+1}$ and $S_{2D}^{RN,n+1}$ with (77) and (78) in (76), we obtain

$$R_j (I - Q S_{2D}^{RD,n+1}) E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) = \omega_j^{n+1} - q_j S_{1D,j}^{RD,n+1}(\omega_j^{n+1}), j = 1, \dots, p \quad (79)$$

Since $R_j E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) = \omega_j^{n+1}$ and $R_j Q = q_j R_j$, we find (71). Both relations are equivalent, as (7) and (8).

4.4.2 Dirichlet or Neumann boundary conditions for the 1D problems and Robin boundary conditions for the 2D problem

Here and in the following sub-sections, the review of the different possibilities continue. The notations for the discrete operators are absolutely similar to the previous ones. Their definition is straightforward and it is not given expansively for the sake of brevity.

If the 1D problems (26-option (a)) receive Dirichlet input values, and give Robin output values by means of (63); and the 2D problem (27) with the option (64) receive Robin input values and gives Dirichlet output values as in the previous sub-section by using (50); then the (discrete) interface system for the problem (1)"rsa" in the form of (7) is

$$\begin{cases} \omega_j^{n+1} = S_{1D,j}^{DR,n+1}(\alpha_j^{n+1}), j = 1, \dots, p \\ \alpha_j^{n+1} = R_j S_{2D}^{RD,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}), j = 1, \dots, p \end{cases} \quad (80)$$

The first equation is given by (67), with the scheme (30) with the option (54), and the second one is already described above. The scheme corresponding to (80) will be referred to as *RmDDR* (see Table 7).

Now, if the value of each $\alpha_j^{n+1}, j = 1, \dots, p$, is replaced in (80) this yields

$$A_{1D}^{DR} \left((S_{1D,1}^{RD,n+1}(\omega_1^{n+1}), \dots, S_{1D,p}^{RD,n+1}(\omega_p^{n+1})) - R S_{2D}^{RD,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) \right) = 0 \quad (81)$$

by swapping N and R in (58). We get the Robin equation (71) with the preconditioner A_{1D}^{DR} (linear part of the associated operator).

By replacing this time the value of each $\omega_j^{n+1}, j = 1, \dots, p$, in (80), this leads to (82) below by using the operator $S_{2D}^{DR,n+1}$. It corresponds to the 2D Dirichlet problem (32-option (a)) whose discrete writing is (66-option (a)). The Robin output values are obtained by using (65), which means (66r) at the discrete level. Now, swapping N and R in (61) yields

$$R A_{2D}^{RD} \left(S_{2D}^{DR,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) - E \left(S_{1D,1}^{DR,n+1}(\alpha_1^{n+1}), \dots, S_{1D,p}^{DR,n+1}(\alpha_p^{n+1}) \right) \right) = 0, j = 1, \dots, p \quad (82)$$

An equation in the variable $\alpha_j^{n+1}, j = 1, \dots, p$, is obtained, which can be considered as the Schur complement system for the problem (1)"rsa". As seen above, an adaptation of the Schur complement system to the multi-scale case consists thus to successively solve local Dirichlet problems and local Robin problems.

The problem (1)"rsa" in the form of (8) is now addressed. We solve the problems (26-option (b)) and (27) with the option (64). The interface system is (63) and (27b). At the discrete level, the interface system for the problem (1)"rsa" is written as

$$\begin{cases} \omega_j^{n+1} = S_{1D,j}^{NR,n+1}(\beta_j^{n+1}), j = 1, \dots, p \\ \beta_j^{n+1} = R_j S_{2D}^{RN,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}), j = 1, \dots, p \end{cases} \quad (83)$$

The first equation is given by (72), with the scheme (30-option (b)), and the second one is already described just in the previous sub-section. The scheme corresponding to (83) will be

referred to as *RmNNR* (see Table 7).

Now, if the value of each $\beta_j^{n+1}, j = 1, \dots, p$, is replaced in (83), while swapping D and R in (44), this yields

$$A_{1D}^{NR} \left((S_{1D,1}^{RN,n+1}(\omega_1^{n+1}), \dots, S_{1D,p}^{RN,n+1}(\omega_p^{n+1})) - R S_{2D}^{RN,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) \right) = 0 \quad (84)$$

We get the equation (76) with the preconditioner A_{1D}^{NR} (linear part of the associated operator). It is reminded that (76) is equivalent to (71), which is called Robin equation in the paper.

By replacing this time the value of each $\omega_j^{n+1}, j = 1, \dots, p$, in (83), the operator $S_{2D}^{NR,n+1}$ is introduced to obtain a factorization. It corresponds to the resolution of (32-option (c)). The Robin output values are obtained by using (65). The corresponding scheme is (66-option (c)), with (73) to obtain the Robin values. Then, D and R are swapped in (48) to obtain

$$R A_{2D}^{RN} \left(S_{2D}^{NR,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) - E \left(S_{1D,1}^{NR,n+1}(\beta_1^{n+1}), \dots, S_{1D,p}^{NR,n+1}(\beta_p^{n+1}) \right) \right) = 0, j = 1, \dots, p \quad (85)$$

We get this time an equation in the variable $\beta_j^{n+1}, j = 1, \dots, p$, which can be considered as a multi-scale flux equation for the problem (1) "rsa".

4.4.3 Dirichlet or Neumann boundary conditions for the 1D problems and for the 2D problem

First, let us look at the Dirichlet problem. We solve (26-option (a)). For the 2D problem to be well defined, the option (a) of (27) must be selected. Thus, the scheme that is proposed below can be used only to solve the problem (1). The interface system is (63) and

$$\frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \left(\frac{\partial u^{n+1}}{\partial n} + q_j u^{n+1} \right) d\gamma = \omega_j^{n+1}, j = 1, \dots, p. \quad (86)$$

The discretization of the Robin output values is defined above and this gives the (discrete) interface system

$$\begin{cases} \omega_j^{n+1} = S_{1D,j}^{DR,n+1}(\alpha_j^{n+1}), j = 1, \dots, p \\ \omega_j^{n+1} = R_j S_{2D}^{DR,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}), j = 1, \dots, p \end{cases} \quad (87)$$

It is relevant to note that

$$S_{2D}^{DR,n+1} = S_{2D}^{DN,n+1} + Q \quad (88)$$

which in turn means that $R_j S_{2D}^{DR,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) = R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) + q_j \alpha_j^{n+1}$ since $R_j Q = q_j R_j$ and $R_j E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) = \alpha_j^{n+1}$. Actually, the mean value of the normal derivative is only computed since u^{n+1} is constant on $\gamma_j, j = 1, \dots, p$. It is as if (86) is replaced by

$$\frac{1}{\theta_j \varepsilon} \int_{\gamma_j} \frac{\partial u^{n+1}}{\partial n} d\gamma + q_j u^{n+1}(x, y) = \omega_j^{n+1}, (x, y) \in \gamma_j, j = 1, \dots, p. \quad (89)$$

The scheme corresponding to (87) will be referred to as *DmdRDR*, where "md" in front of "R" stands for "mean value of the Robin values", complemented by "when the mean values of the

normal derivatives just need to be computed and the solution is constant on the interfaces" (see Table 4 to take a comprehensive view).

The system (87) leads to the Schur complement system (46) for the problem (1). This result is clear by using (88) and the relations below for all $\alpha_j^{n+1}, j = 1, \dots, p$,

$$S_{1D,j}^{DR,n+1}(\alpha_j^{n+1}) = S_{1D,j}^{DN,n+1}(\alpha_j^{n+1}) + q_j \alpha_j^{n+1}, j = 1, \dots, p. \quad (90)$$

However, the schemes *DmNDN* and *DmdRDR* are not equivalent because the unknowns of the interface systems are not the same.

Secondly, the Neumann problem is considered. We solve (26-option (b)) and (32-option (c)) with $\tilde{\beta}_j^{n+1}(x, y) = \beta_j^{n+1}, \forall (x, y) \in \gamma_j$. Therefore, the scheme below can be used only to solve the problem (1)"dsa". The Robin output values are obtained by using (63) and (86). The (discrete) interface system is

$$\begin{cases} \omega_j^{n+1} = S_{1D,j}^{NR,n+1}(\beta_j^{n+1}), j = 1, \dots, p \\ \omega_j^{n+1} = R_j S_{2D}^{NR,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}), j = 1, \dots, p \end{cases} \quad (91)$$

where $S_{2D}^{NR,n+1}$ solves the general 2D Neumann problem (33-option (c)) and gives the Robin values (73).

As above it is worth noting that

$$S_{2D}^{NR,n+1} = I + Q S_{2D}^{ND,n+1} \quad (92)$$

and then $R_j S_{2D}^{NR,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) = \beta_j^{n+1} + q_j R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1})$ since $R_j Q = q_j R_j$ and $R_j E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) = \beta_j^{n+1}$. Actually, the mean value of u^{n+1} is only computed since the normal derivative is constant on $\gamma_j, j = 1, \dots, p$. It is as if (86) is replaced by

$$\frac{\partial u^{n+1}}{\partial n}(x, y) + q_j \frac{1}{\theta_j \varepsilon} \int_{\gamma_j} u^{n+1} d\gamma = \omega_j^{n+1}, (x, y) \in \gamma_j, j = 1, \dots, p. \quad (93)$$

The scheme corresponding to (91) will be referred to as *NmuRNR*, where "mu" in front of "R" stands for "mean value of the Robin values", complemented by "when the mean values of the solution just need to be computed and the normal derivative of the solution is constant on the interfaces" (see Table 4).

One can easily see why the system (91) leads to the flux equation (53) for the problem (1)"dsa", as we have the relations for all $\beta_j^{n+1}, j = 1, \dots, p$,

$$S_{1D,j}^{NR,n+1}(\beta_j^{n+1}) = \beta_j^{n+1} + q_j S_{1D,j}^{ND,n+1}(\beta_j^{n+1}), j = 1, \dots, p \quad (94)$$

and the relation (92) for the 2D discrete operator. Then the interface system (91) can be written in the form of (53) multiplied by a factor equal to q_j , which is just like preconditioning by the matrix $\hat{Q}$.

4.4.4 Robin boundary conditions for the 1D problems and Dirichlet or Neumann boundary conditions for the 2D problem

If the 1D problems (26) receive Robin input values with the option (63), and give Dirichlet output values; and the 2D problem (27-option (a)) receive Dirichlet input values and gives

Robin output values by using (86) or in an equivalent manner (89) since the solution is constant on the interface ; then the (discrete) interface system for the problem (1) is

$$\begin{cases} \alpha_j^{n+1} = S_{1D,j}^{RD,n+1}(\omega_j^{n+1}), j = 1, \dots, p \\ \omega_j^{n+1} = R_j S_{2D}^{DR,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}), j = 1, \dots, p \end{cases} \quad (95)$$

The scheme will be referred to as *DmdRRD*.

Now, if the value of each $\omega_j^{n+1}, j = 1, \dots, p$, is replaced in (95), this yields

$$A_{1D}^{RD} \left((S_{1D,1}^{DR,n+1}(\alpha_1^{n+1}), \dots, S_{1D,p}^{DR,n+1}(\alpha_p^{n+1})) - R S_{2D}^{DR,n+1} E(\alpha_1^{n+1}, \dots, \alpha_p^{n+1}) \right) = 0 \quad (96)$$

by using (44) where N and R are swapped. As seen with (87), the Schur complement system (46) for the problem (1) with the preconditioner A_{1D}^{RD} (linear part of the associated operator) is involved here.

Replacing this time the value of each $\alpha_j^{n+1}, j = 1, \dots, p$, in (95) yields

$$R A_{2D}^{DR} \left(S_{2D}^{RD,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) - E \left(S_{1D,1}^{RD,n+1}(\omega_1^{n+1}), \dots, S_{1D,p}^{RD,n+1}(\omega_p^{n+1}) \right) \right) = 0, j = 1, \dots, p \quad (97)$$

if N and D are swapped in (48). We get an equation in the variable $\omega_j^{n+1}, j = 1, \dots, p$, which can be considered as a multi-scale Robin equation for the problem (1).

Now, the 1D problems (26) with the option (63), which give Neumann output values ; and the 2D Neumann problem (32-option (c)) with $\tilde{\beta}_j^{n+1}(x, y) = \beta_j^{n+1}, \forall (x, y) \in \gamma_j$ which gives Robin output values by using (86) or in an equivalent manner (93) since the normal derivative of the solution is constant on the interface ; are addressed. They solve the problem (1)"dsa". The (discrete) interface system is

$$\begin{cases} \beta_j^{n+1} = S_{1D,j}^{RN,n+1}(\omega_j^{n+1}), j = 1, \dots, p \\ \omega_j^{n+1} = R_j S_{2D}^{NR,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}), j = 1, \dots, p \end{cases} \quad (98)$$

The scheme will be referred to as *NmuRRN*.

Now, if the value of each $\omega_j^{n+1}, j = 1, \dots, p$, is replaced in (98), and (58) is used after swapping D and R , this yields

$$A_{1D}^{RN} \left((S_{1D,1}^{NR,n+1}(\beta_1^{n+1}), \dots, S_{1D,p}^{NR,n+1}(\beta_p^{n+1})) - R S_{2D}^{NR,n+1} E(\beta_1^{n+1}, \dots, \beta_p^{n+1}) \right) = 0 \quad (99)$$

We recognize the version of (91) with two unknowns, which is the flux equation (53) for the problem (1)"dsa" with the preconditioner A_{1D}^{RN} (linear part of the associated operator).

Replacing this time the value of each $\beta_j^{n+1}, j = 1, \dots, p$, in (98) yields

$$R A_{2D}^{NR} \left(S_{2D}^{RN,n+1} E(\omega_1^{n+1}, \dots, \omega_p^{n+1}) - E \left(S_{1D,1}^{RN,n+1}(\omega_1^{n+1}), \dots, S_{1D,p}^{RN,n+1}(\omega_p^{n+1}) \right) \right) = 0, j = 1, \dots, p \quad (100)$$

after swapping D and R in (61). We obtain a multi-scale Robin equation for the problem (1)"dsa".

4.5 Some iterative substructuring methods

Here we make an exploration of some classical iterative substructuring methods, for instance the Neumann-Dirichlet method and the Dirichlet-Neumann method. The schemes are adapted to the multi-scale case. The connection is made with the Schur complement system and the flux equation.

4.5.1 The Neumann-Dirichlet and Dirichlet-Neumann methods

If the block Gauss-Seidel method was used instead of GMRES to solve (40), we would get the following system

$$\begin{cases} \alpha_j^{n+1(l+1)} - S_{1D,j}^{ND,n+1}(\beta_j^{n+1(l)}) = 0, j = 1, \dots, p \\ \beta_j^{n+1(l+1)} - R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1(l+1)}, \dots, \alpha_p^{n+1(l+1)}) = 0, j = 1, \dots, p \end{cases} \quad (101)$$

and the resolution would be slow down. The scheme (101) is a geometrical multi-scale version of the classical iterative substructuring method of Neumann-Dirichlet to solve (1). Thus, the strategy that consists to solve (40) by GMRES can be read as an accelerated version of the Neumann-Dirichlet method. It is a well-known strategy (see [11],[29]).

Remark 10 *The most general Neumann-Dirichlet method depends on a parameter η . The generalization of the previous geometrical multi-scale version can be written :*

$$\beta_j^{n+1(l+1)} = \eta \frac{1}{\theta_j \varepsilon} \sum_{\substack{\sigma \subset \gamma_j \\ \sigma \in \mathcal{E}_K}} \frac{m(\sigma)}{d_\sigma} (\alpha_j^{n+1(l+1)} - u_K^{n+1(l+1)}) + (1 - \eta) \beta_j^{n+1(l)}.$$

In order to simplify the presentation, η has been taken equal to 1 above.

Let us solve again (40), by changing the order of the equations, and then by using the Gauss-Seidel method. We obtain

$$\begin{cases} \beta_j^{n+1(l+1)} - R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1(l)}, \dots, \alpha_p^{n+1(l)}) = 0, j = 1, \dots, p \\ \alpha_j^{n+1(l+1)} - S_{1D,j}^{ND,n+1}(\beta_j^{n+1(l+1)}) = 0, j = 1, \dots, p \end{cases} \quad (102)$$

which is a geometrical multi-scale version of the classical iterative substructuring method of Dirichlet-Neumann. The link between the Gauss-Seidel algorithm and the methods of Dirichlet-Neumann or Neumann-Dirichlet is already mentioned for instance in [18] and [28].

Now, using the Gauss-Seidel method to solve (56) gives

$$\begin{cases} \alpha_j^{n+1(l+1)} - R_j S_{2D}^{DN,n+1} E(\beta_1^{n+1(l)}, \dots, \beta_p^{n+1(l)}) = 0, j = 1, \dots, p \\ \beta_j^{n+1(l+1)} - S_{1D,j}^{ND,n+1}(\alpha_j^{n+1(l+1)}) = 0, j = 1, \dots, p \end{cases} \quad (103)$$

which is another geometrical multi-scale writing of the Neumann-Dirichlet method to solve this time (1)"dsa".

Then a version of the Dirichlet-Neumann method is proposed to solve (1)"dsa" :

$$\begin{cases} \beta_j^{n+1(l+1)} - S_{1D,j}^{DN,n+1}(\alpha_j^{n+1(l)}) = 0, j = 1, \dots, p \\ \alpha_j^{n+1(l+1)} - R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1(l+1)}, \dots, \beta_p^{n+1(l+1)}) = 0, j = 1, \dots, p \end{cases} \quad (104)$$

It is also possible to write a geometric multi-scale version of the Agoshkov and Lebedev method (defined in [1] and [26]) by taking suitable values of the parameters, which would allow us to find all the previous interface systems depending on a Robin interface condition, including those that solve (1)"rsa".

Eventually, the writing of these classical iterative substructuring methods in the multi-scale framework is of little interest, since these methods come down to the solving of systems that we have already obtained in the previous subsections, and since the solving by GMRES is more efficient.

4.5.2 Richardson procedures

In the literature, the classical iterative Neumann-Dirichlet method is interpreted as a preconditioned Richardson procedure for the flux equation. It is also true in some ways in the multi-scale framework. Indeed, by solving (101) through substitution, this yields

$$\beta_j^{n+1(l+1)} = R_j S_{2D}^{DN,n+1} E \left(S_{1D,1}^{ND,n+1}(\beta_1^{n+1(l)}), \dots, S_{1D,p}^{ND,n+1}(\beta_p^{n+1(l)}) \right), j = 1, \dots, p \quad (105)$$

that is to say a Richardson method to solve (47). We see above that (47) leads to (49) which is what we called the flux equation for the problem (1). However, there is not any preconditioning here.

By solving (103) through substitution, this yields

$$\beta_j^{n+1(l+1)} = S_{1D,j}^{DN,n+1} (R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1(l)}, \dots, \beta_p^{n+1(l)})), j = 1, \dots, p \quad (106)$$

and a Richardson method to solve (57) is obtained, which is exactly this time a preconditioned version of the flux equation for (1)"dsa".

By solving (102) through substitution, this yields

$$\alpha_j^{n+1(l+1)} = S_{1D,j}^{ND,n+1} (R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1(l)}, \dots, \alpha_p^{n+1(l)})), j = 1, \dots, p \quad (107)$$

that is to say a Richardson method to solve (43). So, the Dirichlet-Neumann method is related to the Schur complement system, not only in the general context (see [26], [29]), but also in this multi-scale case since (43) is a method to solve the Schur complement system for (1) with the operator A_{1D}^{ND} as a preconditioner.

By solving (104) through substitution, this yields

$$\alpha_j^{n+1(l+1)} = R_j S_{2D}^{ND,n+1} E(S_{1D,1}^{DN,n+1}(\alpha_1^{n+1(l)}), \dots, S_{1D,p}^{DN,n+1}(\alpha_p^{n+1(l)})), j = 1, \dots, p \quad (108)$$

thus, we obtain a Richardson method to solve (60), that is to say the multi-scale Schur complement system (62) for (1)"dsa". As above, the operator that solve a 2D Neumann problem can in no way be considered as a preconditioner. It is in this sens that the multi-scale case is different here from the classical case.

4.5.3 A parallel algorithm.

Let us solve (40) this time by using the Jacobi method

$$\begin{cases} \beta_j^{n+1(l+1)} - R_j S_{2D}^{DN,n+1} E(\alpha_1^{n+1(l)}, \dots, \alpha_p^{n+1(l)}) = 0, j = 1, \dots, p \\ \alpha_j^{n+1(l+1)} - S_{1D,j}^{ND,n+1}(\beta_j^{n+1(l)}) = 0, j = 1, \dots, p \end{cases} \quad (109)$$

We get a parallel algorithm, which is the parallel version of the sequential algorithms (multi-scale) Neumann-Dirichlet or Dirichlet-Neumann above (this is the same for both methods) to solve (1). This process is in tune with the iterative strong coupling method which keeps twice as many unknowns than classical methods. However, the resolution of the interface system by a Krylov method is more efficient.

Remark 11 *In a similar manner as above the interface system (56) for the problem (1)"dsa" can be solved by using the Jacobi method. This gives*

$$\begin{cases} \alpha_j^{n+1(l+1)} - R_j S_{2D}^{ND,n+1} E(\beta_1^{n+1(l)}, \dots, \beta_p^{n+1(l)}) = 0, j = 1, \dots, p \\ \beta_j^{n+1(l+1)} - S_{1D,j}^{DN,n+1}(\alpha_j^{n+1(l)}) = 0, j = 1, \dots, p \end{cases} \quad (110)$$

5 Numerical experiments

FIGURE 2 – a) The domain Ω_ε . b) The domain D_ε .

The first test that is considered consists to approach the solution of the heat equation (3) with a specific right hand side such that the exact solution u_ε is known and smooth (see [24]), in a domain Ω_ε with one node and two branches. The domain is represented at Figure 2a, with $\varepsilon = 0.2$, $\theta_1 = \theta_2 = 1$, $l_1 = l_2 = 1$. The exact solution is given below.

$$u_\varepsilon(x, y, t) = \begin{cases} 10^7 (t - 0.5)^5 (x_1^{(e_1)} - 0.5)^3 (x_1^{(e_1)} - 0.9)^3 & \text{if } x_1^{(e_1)} \in]0.5; 0.9[\text{ and } t > 0.5 \\ 10^7 (t - 0.5)^5 (x_1^{(e_2)} - 0.4)^3 (x_1^{(e_2)} - 0.8)^3 & \text{if } x_1^{(e_2)} \in]0.4; 0.8[\text{ and } t > 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (111)$$

The differences between u_ε and the approximate solutions of (1), (1)"dsa", or (1)"rsa", obtained by using the monolithic schemes (monolithic solutions) are estimated. The geometrical

multi-scale domain D_ε in which the model problems are defined is represented at Figure 2b. The tests have been done with $k = 0.002$ and $h_0 = h_i^{e_j} = h = 0.02$ for all i and j , which corresponds to a mesh $\mathcal{T}$ with 1684 control volumes. Two different values of δ are considered.

Then, we study the differences between u_ε and the monolithic solutions as a function of the size of the mesh h .

In this paper, the influence of ε is not studied. The error is studied as a function of ε in [24] for the problem (1).

The numerical experiments are carried out with a continuous solution, thus the three monolithic solutions should be close. On the contrary, if we considered a phenomena such that the solution was discontinuous across the interfaces, then the three monolithic solutions would be clearly distinct. For this reason, it is interesting to see the accuracy with which each of them can be approached, by using the iterative strong coupling strategy. The schemes obtained through substitution with a reduced number of unknowns, as the Schur complement system, the flux equation or the Robin equation, are addressed for comparative purposes. Almost thirty options using Dirichlet, Neumann or Robin interface conditions are compared. The code that is used is sequential, it cannot benefit from the parallel structure of the method. On the other hand, parallelization is relevant in the case of partitions using many subdomains. All the schemes deriving from a partitioned approach lead to similar computation times. Our attention focuses on accuracy.

5.1 Comparison between the three monolithic solutions

The monolithic finite volume schemes which are used here are (13) to solve (1), and (13) modified according (16), (17) and (21), to solve (1)"dsa" and (1)"rsa". The monolithic solutions are compared with u_ε at $T = 1$. The domain Ω_ε is truncated with $\delta = 0.3$.

In the 2D part of the domain, the exact solution is nil. In the 1D part of the domain, on the vertical axis, the curve of the exact solution decreases sharply before it shifts up from $y = 0.6$ forming a hollow, as shown in Figure 3b. We do not represent the curve of the exact solution on the horizontal axis : it is very similar to the one on the vertical axis with this time the hollow at $x = 0.7$. The approximate solutions of (1) and (1)"dsa" are very close together, the two plots cannot be distinguished from Figure 3a (2D part) and Figure 3b (1D part). The difference between these first two approximations can be seen from Figure 4a (in the order of 10^{-3}), and Figure 5b (in the order of 10^{-6}). The difference between the exact solution and its approximations is maximum at the interfaces in the 2D part of the domain (see Figure 3a), whereas the maximum deviation is reached at the level of the hollow in the 1D part of the domain, as evident from Figure 5a in the case of the vertical axis ($y = 0.6$ is the location of the hollow).

The hybrid solution (1)"rsa" offers a wide range of approximate solutions. When the parameter q (for the sake of simplicity $q = q_1 = q_2$) in the Robin interface condition is equal to zero, (1)"rsa" with the condition (7) is brought back to (1)"dsa" and (1)"rsa" with the condition (8) is not well posed, and when q tends to infinity (1)"rsa" with the condition (8) is brought back to (1) and (1)"rsa" with the condition (7) is not well posed. Playing with the value of q , a range of approximate solutions can be constructed between the solution of (1) and the one of (1)"dsa". As already presented in Figure 4a, the difference between the approximate solutions of (1) and (1)"rsa" with the condition (7) is in the order of 10^{-3} if $q = 0$, whereas with q equal to 10^3 , 10^4 , 10^8 , the order of magnitude of the difference is 10^{-5} , 10^{-6} , and 10^{-10} respectively.

Figure 4b and Figure 5b refer to $q = 50$ (with the condition (7)). Schemes with the condition (8) have not been represented, because the solutions are almost the same as those obtained with schemes with the condition (7) : the order of magnitude of the difference between the conditions (7) and (8) is 10^{-15} , 10^{-14} , and 10^{-10} respectively with q equal to 1, 10^4 , 10^8 . From 10^8 , when q increases, the difference increases too since (7) gives a problem which is no longer well posed, but such great values of q are actually not used.

FIGURE 3 – a) The approximate solutions of (1) and (1)"dsa". b) The exact solution (continuous line), the approximate solutions of (1) ($\square$), (1)"dsa" (+), and of (1)"rsa" ($\circ$).

FIGURE 4 – a) Difference between the approximate solutions of (1) and (1)"dsa". b) Difference between the approximate solutions of (1) and (1)"rsa" with $q = 50$.

Now, we may wonder what would happen if the value of δ changed. For instance, let us choose $\delta = 0.5$ this time. And thus, on one hand, the interface is closer to the area where the solution drops, on the other hand, the 2D accuracy is kept on a larger area.

We begin by the convergence of the approximate solutions of (1), of (1)"dsa" and of (1)"rsa", to u_ε when the size of the mesh h tends to zero. Let us compare $\delta = 0.3$ and $\delta = 0.5$.

Based on the error estimate reminded in Remark 2, since the right-hand side satisfies all the prerequisites, the order one of convergence with respect to h is predicted for the problem (1) provided that the term $\frac{h}{\sqrt{\varepsilon}}$ dominates, with respect to k ($k = 0.002$) as well as ε^J . We take values for h that ranges from 0.02 to 0.175 with $\delta = 0.3$, and from 0.02 to 0.125 with $\delta = 0.5$. The convergence of the three monolithic schemes ($\delta = 0.3$ and $\delta = 0.5$) in L^2 norm can be seen

FIGURE 5 – a) Difference between the exact solution and the approximate solution of (1) ($\square$), of (1)"dsa" (+), and of (1)"rsa" with $q = 50$ ($\circ$). b) Difference between the approximate solutions of (1) and (1)"dsa" (+) and between the approximate solutions of (1) and (1)"rsa" with $q = 50$ ($\circ$).

at the bottom of Figure 7c (the three curves cannot be distinguished). The convergence order is approximately the same with $\delta = 0.3$ and $\delta = 0.5$. The slope of the line of least squares (logarithmic scale) gives an average value of the order of convergence in terms of h , equal to about 2 in either case. This has been already noticed in [24] for the problem (1), and it can be seen here that the error estimate in Remark 2 seems to be valid too for the problems (1)"dsa" and (1)"rsa".

With $\delta = 0.5$, the curve have more reliable points (the threshold effect is very inconvenient with $\delta = 0.3$, preventing us from taking into account the points associated with small values of h) and looks like more regular. For sake of completeness, the H^1 norm is calculated in the case $\delta = 0.5$, and we see at the top of Figure 7c that the general form of the curve is the same as the one below with the L^2 norm. A very slight difference is only observed between the approximations of (1) and (1)"dsa" in H^1 norm when $h = 0.02$, the "rsa" approximation is not represented because the corresponding points are between the two others.

The difference between the approximations of (1) and (1)"dsa" with $\delta = 0.3$ are reflected in Figure 4a : in the order of 10^{-3} in the 2D part of the domain ($h = 0.02$). As presented in Figure 6a, the difference is in the order of 10^{-2} with $\delta = 0.5$. The same difference is observed between (1) and (1)"rsa" with $q = 50$ in Figures 4b and 6b. As shown in Figures 5b and 7b, in the 1D part of the domain, the difference stays in the order of 10^{-6} in both cases. As evident from Figures 5a and 7a, though, the difference between the approximations and u_ε is smaller with $\delta = 0.5$.

When $\delta = 0.5$, there is a 2D part of the domain close to the interface where the exact solution decreases sharply. Maybe this involves a much higher difference between the three monolithic solutions. The value $\delta = 0.5$ appears to have discriminatory power greater than $\delta = 0.3$. Rather, the expansion of the 2D part of the domain when $\delta = 0.5$ gives more accurate results in the 1D part of the domain and more regular error curves than when $\delta = 0.3$. For these reasons, we keep the value $\delta = 0.5$ in the following.

FIGURE 6 – a) Difference between the approximate solutions of (1) and (1)"dsa" (with $\delta = 0.5$). b) Difference between the approximate solutions of (1) and (1)"rsa" with $q = 50$ (with $\delta = 0.5$).

FIGURE 7 – a) Difference between the exact solution and the approximate solution of (1) ($\square$), of (1)"dsa" (+) , and of (1)"rsa" with $q = 50$ ($\circ$) (with $\delta = 0.5$). b) Difference between the approximate solutions of (1) and (1)"dsa" (+) and between the approximate solutions of (1) and (1)"rsa" with $q = 50$ ($\circ$) (with $\delta = 0.5$). c) Error between the exact solution and its approximations as a function of h . At the bottom with $\delta = 0.3$ (+) and $\delta = 0.5$ ($\square$) in L^2 norm (the three monolithic schemes give results that cannot be distinguished). At the top, with $\delta = 0.5$, (1) ($\square$) and (1)"dsa" (+) in H^1 norm.

5.2 Approximation of the monolithic solutions by using the iterative strong coupling strategy. First part.

5.2.1 The four unknowns schemes

We address several iterative schemes that converge to the approximate solutions given by the monolithic schemes (13) to solve (1), or (13) modified according (16) and (17) to solve (1)"dsa". These iterative schemes are defined by using the iterative strong coupling strategy previously described, and by choosing various interface conditions on the 2D part and on the 1D parts. The interface systems are solved by using the GMRES method. We give below the overview of all possibilities.

We take $\varepsilon = 0.2, \delta = 0.5$, with $h = 0.02, k = 0.002$. The difference between u_ε and its approximation by using any of the options discussed is always the same, it is also the value that we have obtained with each of the monolithic schemes. Therefore, the comparison of these schemes with u_ε do not allow us to distinguish between them. That is why we must more likely look at the difference between the approximate solutions which are obtained by using the options of the strong coupling method and by using the monolithic schemes. Each option is denominated

by the four corresponding letters, preceded by "m", "md" or "mu", where appropriate.

First, we look at the evolution of the solution over time. We use for instance the *DmNND* option to solve (1). Figure 8a refers to the L^2 and H^1 norms of the difference between the approximate solutions obtained with the *DmNND* option and with the monolithic scheme, for each of the four iterations of the GMRES algorithm (with tolerance 10^{-6} on the relative residual), during the 300th ($T=0.6$), 400th ($T=0.8$) and 500th ($T=1$) iteration in time. We use the basic GMRES algorithm, without the restarting option and without preconditioning techniques. On this example at iteration 300 in time, the GMRES method converges with 3 iterations instead of 4 (the solutions obtained at iterations 3 and 4 are the same, it can be seen from Figure 8a that the two corresponding points are lined up) to a solution with relative residual between 10^{-7} and 10^{-6} . This happens about 50 times, successively. From the 350th iteration in time, the GMRES method converges with 4 iterations to a solution with nil relative residual. Throughout the numerical experiments, we obtain a larger or smaller number of iterations in time where the relative residual of the GMRES method is close to the tolerance : we never observe high accuracy when this number is large. With regard to the *DmNND* option when $T=1$, the final iteration gives a L^2 norm equal to $4.8 \cdot 10^{-10}$ and a H^1 norm equal to $1.7 \cdot 10^{-9}$. It is clear that the L^2 norm and the H^1 norm behave in the same way. Therefore, in the following, all results are expressed only with the L^2 norm.

FIGURE 8 – a) Difference between the approximated *DmNND* solution and the approximated monolithic solution of (1) over time. b) Evolution of the error according to q for the *DmdRDR* to solve (1), and *RmDDR* and *RmDRD* schemes to solve (1)"rsa". c) Evolution of the error according to q for the *DmdRRD* options (four unknowns and two unknowns schemes) to solve (1).

Let us now take a closer look at the conditioning of the matrix of the interface system. The interface system used above, which corresponds to the *DmNND* option to solve (1), is given in (40) and is explained below (following (30a) and (31b)).

$$\begin{cases} \alpha_1^{n+1} - v_{1,1}^{n+1} + h_{1/2}^{e_1} \beta_1^{n+1} = 0 \\ \beta_1^{n+1} - \frac{1}{\theta_1 \varepsilon} \sum_{\sigma \in \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} \alpha_1^{n+1} + \frac{1}{\theta_1 \varepsilon} \sum_{\sigma \in \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} u_K^{n+1} = 0 \\ \alpha_2^{n+1} - v_{2,1}^{n+1} + h_{1/2}^{e_2} \beta_2^{n+1} = 0 \\ \beta_2^{n+1} - \frac{1}{\theta_2 \varepsilon} \sum_{\sigma \in \gamma_2, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} \alpha_2^{n+1} + \frac{1}{\theta_2 \varepsilon} \sum_{\sigma \in \gamma_2, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} u_K^{n+1} = 0 \end{cases} \quad (112)$$

The matrix of the linear system is the one obtained by keeping the equations in the same order and taking $(\alpha_1^{n+1}, \beta_1^{n+1}, \alpha_2^{n+1}, \beta_2^{n+1})$ as the unknown. In this way, the diagonal terms of the matrix are equal to 1, since $v_{j,1}^{n+1}$ only depends on $\beta_j^{n+1}, j = 1, 2$, and u_K^{n+1} only depends on α_1^{n+1} and α_2^{n+1} . The condition number of the matrix is equal to 245. To try to improve the condition number, what we propose here is to implement some variants by selecting in turn the coefficients of the variables equal to 1, acting as if the solution of the separated problems are nil (since the contributions from $v_{j,1}^{n+1}$ and u_K^{n+1} are not known or readily available). It is shown that it is a simple and efficient way to affect the condition number. The linear system (112) (results in Figure 8a) is numbered as variant 0. Keeping the lines 1 and 3 as such, dividing the line 2 by $-\frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma}$ and the line 4 by $-\frac{1}{\theta_2 \varepsilon} \sum_{\sigma \subset \gamma_2, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma}$, give the variant 1. Keeping the lines 2 and 4 as the variant 1, and dividing the line 1 by $h_{1/2}^{e_1}$ and the line 3 by $h_{1/2}^{e_2}$, give the variant 2. Eventually, the variant 3 is obtained by keeping the lines 1 and 3 as the variant 2, and the lines 2 and 4 as the variant 0. Each variant is determined by the choice of the variables whose coefficient is equal to 1 if $v_{j,1}^{n+1}$ and u_K^{n+1} are nil. As the operations are the same in the two first equations and in the two last equations, information can be condensed into the table below.

variant	coeff equal to 1 in eq 1	coeff equal to 1 in eq 2
0	α_1	β_1
1	α_1	α_1
2	β_1	α_1
3	β_1	β_1

TABLE 2 – The four variants.

The condition number of the interface matrix and the L^2 norm of the difference between the monolithic solution and the approximate solution obtained with each of the four variants of the *DmNND* option when $T = 1$ can be seen from Table 3.

-	<i>DmNND</i>	-	<i>DmNDN</i>	-
variant	condition number	L^2 norm	condition number	L^2 norm
0	245	$4.8 \cdot 10^{-10}$	1122	$9.2 \cdot 10^{-7}$
1	196	$2.4 \cdot 10^{-11}$	47	$1.7 \cdot 10^{-13}$
2	19474	0.017	4250	$2.0 \cdot 10^{-7}$
3	52	$1.9 \cdot 10^{-5}$	22	$2.4 \cdot 10^{-9}$

TABLE 3 – Condition numbers, error estimates.

It shows that the best accuracy is not obtained with the matrix that has the smallest condition number. This comment applies equally to *DmNND* and *DmNDN*. The linear system associated to the option *DmNDN* is (55) and can be explained as (112) too, but this time $v_{j,1}^{n+1}$ only depends on $\alpha_j^{n+1}, j = 1, 2$. This result is not so surprising, given that the accuracy depends also on the calculation method of $v_{j,1}^{n+1}$ and u_K^{n+1} , which in turn depends on the condition number of the matrix of the discretized separated problems : the 1D parts give a small condition number (equal to 21) whereas the condition number for the 2D part (equal to 236927) is very large. Table 3 indicates that the variant 1 gives the more accurate results. Quite the contrary, the

variant 2 give poor results. We can see here that the simple proposed operation is effective in improving the accuracy.

As regards the choice of the size of the mesh, we note that the accuracy is in the order of 10^{-6} for each variant of the *DmNND* option (except the variant 2, which is even worst) with $h = 0.01$, undoubtedly due to stability problems. In contrast, the accuracy is in the order of 10^{-13} or 10^{-14} for each variant with $h = 0.05$ or $h = 0.1$, only these values of h are too large for the results to be of particular interest. The condition number of the interface matrix varies very little with respect to h in the case of the variant 3, the one of the variant 2 is always the larger, the one of the variant 1 varies the most. If $h = 0.025$ or $h = 0.03125$, the condition number of the four variants are of the same order of magnitude as in Table 3, except that the one of the variant 1 is a bit larger and the best accuracy is obtained with the variant 0. So it is not always the same variant which gives the best accuracy when h changes. We choose to work with $h = 0.02$ in the following.

2D	-	1D	-	problem	reference	L^2 norm
input	output	input	output	-	-	-
D	mN	N	D	(1)	(40)	$2.4 \cdot 10^{-11}$
D	mN	D	N	(1)	(55)	$1.7 \cdot 10^{-13}$
D	mdR	D	R	(1)	(87)	$3.3 \cdot 10^{-14}$
D	mdR	R	D	(1)	(95)	$2.7 \cdot 10^{-14}$
N	mD	D	N	(1)"dsa"	(56)	$8.1 \cdot 10^{-14}$
N	mD	N	D	(1)"dsa"	(52)	$4.4 \cdot 10^{-14}$
N	muR	N	R	(1)"dsa"	(91)	$7.5 \cdot 10^{-14}$
N	muR	R	N	(1)"dsa"	(98)	$6.8 \cdot 10^{-12}$

TABLE 4 – Interface conditions, the corresponding problems which are solved, the interface systems, error estimates.

Now, we look at the results obtained by using the many other options when $T = 1$. The results of the four options to solve (1) and those of the four options to solve (1)"dsa" are resumed in Table 4.

It is worth mentioning that the parameter in Robin interface conditions is usually optimized to obtain the best convergence factor when an iterative method is used. However, in the framework of our linear geometrical multi-scale problem, the interface systems have only four unknowns and the GMRES method converges with four iterations. So, there is no need for optimization of iteration count, which is already very low. Nevertheless, we feel it is of interest to find the value of the parameter that gives the smallest difference (in L^2 norm) between the monolithic solutions and their approximations by using the iterative strong coupling strategy. Identification of an optimal parameter is done a posteriori, which shows what degree of accuracy can be achieved. The optimization is made each time the Robin condition is used as input or output value.

To solve (1), the *DmNND* and *DmNDN* options have just been discussed.

The *DmdRDR* option corresponds to the interface system (87) and is explained following (67) and (89). Several variants can be considered as above (according to Table 2 by changing β by ω). The results of the variant 2 have improved in comparison with *DmNDN* by choosing suitable values of q (we obtained 10^{-9} if $q = 50$), but however the variant that has given the best accuracy corresponds to the variant 1. Therefore, the variant that gives the more accurate

results when $q = 0$ is also the one that gives the more accurate results when $q \neq 0$. As a Robin condition corresponds to a Neumann condition when $q = 0$, the *DmdRRD* option generalizes the *DmdNDN* option and it seems natural that the optimal variants correspond to each other. The two first equations of the variant 1 are

$$\left\{ \begin{array}{l} \alpha_1^{n+1} + \frac{v_{1,1}^{n+1} - h_{1/2}^{e_1} \omega_1^{n+1}}{qh_{1/2}^{e_1} - 1} = 0 \\ \alpha_1^{n+1} - \frac{\frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} u_K^{n+1}}{\frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} + q} - \frac{\omega_1^{n+1}}{\frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_\sigma} + q} = 0 \end{array} \right. \quad (113)$$

They are (112) when $q = 0$. The third and fourth equations are similar to the first and the second one respectively, by correcting the subscripts. Again the matrix of the linear system is the one obtained by keeping the equations in the same order and taking $(\alpha_1^{n+1}, \beta_1^{n+1}, \alpha_2^{n+1}, \beta_2^{n+1})$ as the unknown. The separated problems have solutions such that $v_{j,1}^{n+1}$ only depends on α_j^{n+1} , $j = 1, 2$, and u_K^{n+1} only depends on α_1^{n+1} and α_2^{n+1} .

The linear system associated to *DmdRRD* can be explained as (113) too (the best accuracy is obtained with the same variant), but this time $v_{j,1}^{n+1}$ only depends on ω_j^{n+1} , $j = 1, 2$.

The *DmdRRD* and *DmdRDR* options are the ones that achieve the best accuracy to approach the monolithic solution associated with (1), they give a L^2 norm of the difference between the approximate solutions equal to $2.7 \cdot 10^{-14}$ ($q = 3$) and $3.3 \cdot 10^{-14}$ ($q = 11$) respectively.

The variation of the error obtained with the last two schemes as a function of q is not simple to understand. Although only positive values of q are taken into consideration in Table 4, it is interesting to note that the results can be further enhanced using negative values of q .

Looking first at *DmdRDR*, values of the error ranging from 10^{-14} to 10^{-10} are observed when $q > 0$, and from 10^{-15} to 10^{-8} when $q < 0$ ($3.7 \cdot 10^{-15}$ if $q = -21.5$). As shown in Figure 8b, numerous values of q give an error in the order of 10^{-13} , but not all of them, for instance the error is $5.7 \cdot 10^{-10}$ if $q = 120$, and even worse if negative values of q are authorized.

Doing the same thing with *DmdRRD*, looking at Figure 8c, values of the error ranging from 10^{-14} to 10^{-9} when $q > 0$, and from 10^{-15} to 10^{-7} when $q < 0$, are observed. Let us note that q determines both the matrices of the separated 1D problems and the matrix of the interface system. Large amplitude oscillations have been found for small values of q . For instance, we obtained $1.4 \cdot 10^{-9}$ as error with $q = 7$ and $4.1 \cdot 10^{-14}$ with $q = 8$; $7.7 \cdot 10^{-15}$ with $q = -19.5$ and $7.5 \cdot 10^{-7}$ with $q = -20$. It can be seen from Table 5 that the condition numbers of the matrix of the interface system when $q = -19.5$ and when $q = -20$ are approximately the same, the condition numbers of the matrix of the separated 1D problems as well. The condition number cannot therefore explain the difference between the two results. Moreover, how it is that the accuracy is only in the order of 10^{-7} with reasonable order of magnitude condition numbers? However, it is clear that very bad condition number of the matrices of the separated 1D problems leads to inaccurate results, as Table 5 notes when $q = 21.822$.

Just a few values, which are not uniformly located, are presented in Figure 8. The plotted values are more numerous in areas where the error seems to be more sensitive to variations in q , for instance between -50 and 50 . We do not consider it useful to make more computations because as of now it can be concluded that these schemes are not easy to use.

q	condition number interface	condition number 1D	L^2 norm
-20	354	21	$7.5 \cdot 10^{-7}$
-19.5	350	21	$7.7 \cdot 10^{-15}$
21	19	440	$8.0 \cdot 10^{-12}$
21.822	44564	1698119	$8.3 \cdot 10^{-8}$
22	55	2006	$8.5 \cdot 10^{-11}$

TABLE 5 – *DmdRDR* : condition numbers, error estimates.

Now the options that solve (1)"dsa" are reviewed. The *NmDDN* and *NmDND* options are explained following (30a) and (51a,b,c). The variant 3 (see Table 2) gives the best accuracy. The two first equations are given below (the others are similar).

$$\begin{cases} \frac{\alpha_1^{n+1} - v_{1,1}^{n+1}}{h_{1/2}^{e_1}} + \beta_1^{n+1} = 0 \\ \beta_1^{n+1} - \theta_1 \varepsilon \frac{\alpha_1^{n+1}}{\sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} m(\sigma) d_\sigma} + \frac{\sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} m(\sigma) u_K^{n+1}}{\sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} m(\sigma) d_\sigma} = 0 \end{cases} \quad (114)$$

The separated problems have solutions such that u_K^{n+1} only depends on β_1^{n+1} and β_2^{n+1} ; and $v_{j,1}^{n+1}$ only depends on $\alpha_j^{n+1}, j = 1, 2$ in the case of *NmDDN*, and $\beta_j^{n+1}, j = 1, 2$ in the case of *NmDND*. This variant of the schemes gives values of the error in the order of 10^{-14} (see Table 4), another variants give a comparatively low accuracy (in the order of 10^{-6} with the variant 2 for both schemes).

The *NmuRNR* option corresponds to the interface system (91) and is explained following (72) and (73). Several variants can be considered as above (according to Table 2 by changing α by ω). The variant that has given the best accuracy ($7.5 \cdot 10^{-14}$ with $q = 122$) corresponds to the variant 0. Unlike the schemes *DmdRDR* and *DmNDN*, the optimal variants for *NmuRNR* and *NmDND* do not correspond to each other. This can be due to the fact that a Robin condition corresponds to a Dirichlet condition (multiplied by a factor equal to q) when q tends to infinity, and this is not quantifiable. The two first equations of the variant 0 (the two last equations are similar) are

$$\begin{cases} \omega_1^{n+1} - (1 - q h_{1/2}^{e_1}) \beta_1^{n+1} - q v_{1,1}^{n+1} = 0 \\ \frac{-\omega_1^{n+1}}{q \frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} m(\sigma) d_\sigma + 1} + \beta_1^{n+1} + q \frac{\frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} m(\sigma) u_K^{n+1}}{q \frac{1}{\theta_1 \varepsilon} \sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} m(\sigma) d_\sigma + 1} = 0 \end{cases} \quad (115)$$

The separated problems have solutions such that $v_{j,1}^{n+1}$ only depends on $\beta_j^{n+1}, j = 1, 2$, and u_K^{n+1} only depends on β_1^{n+1} and β_2^{n+1} . Here, the use of a Robin condition does not represent a significant gain compared with *NmDND*, which gives already an accuracy in the order of 10^{-14} . The option *NmuRRN*, where this time $v_{j,1}^{n+1}$ only depends on $\omega_j^{n+1}, j = 1, 2$, does not give better result than the *NmDDN* option, whatever the variant chosen.

It can be concluded that, to solve (1)"dsa", *NmDDN* and *NmDND* are the simplest and most accurate options. If you are free to choose, (1)"dsa" is a better choice than (1) because

the solution of the corresponding monolithic scheme can be approached more accurately. The approximations provided by *NmDDN* and *NmDND* are quite satisfactory and make unnecessary the use of a Robin boundary condition and risky optimization of the parameter q .

5.2.2 The two unknowns schemes

Now, let us compare the previous results with the ones obtained with more classical methods as the Schur complement system. These rank among the two unknowns schemes, which are reviewed now, that arise from the four unknowns schemes by substitution. The results can be seen from Table 6.

It is recalled that after one variable per interface is eliminated, the *DmNDN* option leads to the

2D	-	1D	-	problem	L^2 norm				
input	output	input	output	-	Schur		Flux		Robin
D	mN	N	D	(1)	$2.7 \cdot 10^{-11}$	(45)	$5.8 \cdot 10^{-9}$	(49)	
D	mN	D	N	(1)	$3.7 \cdot 10^{-10}$	(46)			
D	mdR	D	R	(1)	$3.7 \cdot 10^{-10}$				
D	mdR	R	D	(1)	$8.5 \cdot 10^{-12}$	(96)			$3.9 \cdot 10^{-10}$ (97)
N	mD	D	N	(1)"dsa"	$8.9 \cdot 10^{-12}$	(62)	$1.8 \cdot 10^{-9}$	(59)	
N	mD	N	D	(1)"dsa"			$1.3 \cdot 10^{-10}$	(53)	
N	muR	N	R	(1)"dsa"			$1.3 \cdot 10^{-10}$		
N	muR	R	N	(1)"dsa"			$4.2 \cdot 10^{-11}$	(99)	$1.7 \cdot 10^{-11}$ (100)

TABLE 6 – Two unknowns schemes, the corresponding problems which are solved, error estimates (in bold if preconditioning) and references of the interface systems.

multi-scale Schur method (46). In this case, the interface system has only two unknowns and two equations. The GMRES method converges with two iterations. It gives a L^2 norm of the difference equal to $3.7 \cdot 10^{-10}$, and it does not give a more accurate result than the scheme with four unknowns. The multi-scale Schur method obtained from the *DmdRDR* option is equivalent to the previous one (see (90) and (88)), and gives exactly the same result.

Let us look at the multi-scale preconditioned Schur complement systems (45) and (96), deriving from the *DmNND* and *DmdRRD* options. Better accuracy is obtained : $2.7 \cdot 10^{-11}$ and $8.5 \cdot 10^{-12}$ ($q = 2$) respectively. The only problem by using (96) is the choice of the Robin parameter. However, it is not a major drawback because the results varie little with respect to the parameter q (see Figure 8c) with values ranging from 10^{-12} to 10^{-9} , with a majority of values in the 10^{-10} . Now, looking at the flux equation that we have defined to solve (1), the results are disappointing.

The flux equation (53) to solve the problem (1)"dsa" derives from the *NmDND* option after the elimination of one variable. The L^2 norm of the difference is equal to $1.3 \cdot 10^{-10}$, and it gives a less accurate result than the *NmDND* option with four unknowns. As predicted (by using (94) and (92)), after the elimination of an unknown, the *NmuRNR* option gives the same result as the flux equation. The multi-scale preconditioned flux equation (99) gives $4.2 \cdot 10^{-11}$ ($q = 2$) with values from 10^{-9} to 10^{-11} , whereas (59) isn't as good as the flux equation without any preconditioning. Rather, it is the Schur complement system that we have defined to solve (1)"dsa" that gives the greatest accuracy $8.9 \cdot 10^{-12}$.

The Robin equations that we have defined to solve (1) and (1)"dsa" give intermediate values between those of the flux equation and the Schur complement system. Both schemes give regular results as a function of q . For instance, for the Robin equation to solve (1), the values range from 10^{-9} to 10^{-10} between $q = 50$ and $q = 1000$; they remain around 10^{-11} for the Robin equation to solve (1)"dsa".

When comparing Tables 4 and 6, it can be concluded that the schemes with two unknowns don't reach the same accuracy as the ones with four unknowns. It is to be noted however that the two unknowns schemes are constructed by substituting one variable. To do this, one interface equation has to be given in writing as $\omega_1 = \dots$, which corresponds to the variants 0 or 3, for example to define the Schur method from the *DmdRRD* option, and then the latter expression is used in the other equation. It is found that the preconditioned Schur method (result in the order of 10^{-12}) is more accurate than these corresponding variants (results in the order of 10^{-10}). This holds for all the preconditioned Schur or flux equations. If the problem at hand was one-dimensional, with two subdomains (left and right), it is well known that $A_{1D,l}^{DN} - A_{1D,r}^{DN}$ would be an efficient preconditioner for $A_{1D,l}^{ND} - A_{1D,r}^{ND}$, the condition number of the resulting matrices being independent of h . The similarity between the 1D-1D composition and the 1D-2D composition is thus noted regarding the positive effect on accuracy. With regard to conditioning, it is interesting to note that the condition number of the interface system is equal to 1 at 10^{-2} , for all the p unknowns schemes, with or without preconditioning, that is to say for instance even (46) or (49).

Lastly, it is worth observing that the two unknowns schemes that depend on a Robin parameter are much more stable (with respect to this parameter) than the corresponding four unknowns schemes (*DmdRRD* or *NmuRRN*). The two unknowns (Schur) and four unknowns schemes corresponding to the *DmdRRD* option can be seen from Figure 8c.

5.3 Approximation of the monolithic solutions by using the iterative strong coupling strategy. Second part.

Now, we address several iterative schemes that converge to the approximate solution given by the monolithic scheme (13) modified according (16) and (21) to solve (1)"rsa". Unlike the problems (1) and (1)"dsa", the problem (1)"rsa" depends on the Robin parameter q . Each value of the parameter q corresponds to a specific monolithic scheme and the same value of q is chosen to write the Robin boundary conditions in the strong coupling method framework. Existence and uniqueness of the solution can be ensured only if $q \geq 0$.

We take $\varepsilon = 0.2, \delta = 0.5$, with $h = 0.02, k = 0.002$. The difference (L^2 norm) between the approximate solutions given by the options of the strong coupling method and by the monolithic scheme is examined. The results can be seen from Table 7.

The best accuracy is obtained with the options *RmDRD* or *RmDDR* by using the variant 3 according to Table 4 by changing β by ω , the two first equations are

$$\begin{cases} \omega_1^{n+1} + \left(\frac{1}{h_{1/2}^{e_1}} - q\right)\alpha_1^{n+1} - \frac{v_{1,1}^{n+1}}{h_{1/2}^{e_1}} = 0 \\ \omega_1^{n+1} - \frac{\theta_1 \varepsilon \alpha_1^{n+1}}{\sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)d_\sigma}{q d_\sigma + 1}} + \frac{\sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)u_K^{n+1}}{q d_\sigma + 1}}{\sum_{\sigma \subset \gamma_1, \sigma \in \mathcal{E}_K} \frac{m(\sigma)d_\sigma}{q d_\sigma + 1}} = 0 \end{cases} \quad (116)$$

2D	-	1D	-	ref	L^2 norm				variant
input	output	input	output	-	$q = 1$	$q = 10$	$q = 50$	$q = 120$	-
R	mD	R	D	(70)	$5.5 \cdot 10^{-14}$	$4.9 \cdot 10^{-14}$	$5.0 \cdot 10^{-13}$	$8.4 \cdot 10^{-11}$	3
R	mD	D	R	(80)	$8.1 \cdot 10^{-14}$	$1.1 \cdot 10^{-13}$	$7.9 \cdot 10^{-14}$	$2.5 \cdot 10^{-13}$	3
R	mN	R	N	(75)	$1.1 \cdot 10^{-12}$	$2.4 \cdot 10^{-8}$	$5.1 \cdot 10^{-11}$	$2.5 \cdot 10^{-12}$	1
					$3.1 \cdot 10^{-7}$	$2.4 \cdot 10^{-11}$	$1.5 \cdot 10^{-10}$	$1.3 \cdot 10^{-11}$	2
R	mN	N	R	(83)	$3.0 \cdot 10^{-12}$	$8.4 \cdot 10^{-11}$	$2.1 \cdot 10^{-10}$	$4.1 \cdot 10^{-13}$	1
					$2.1 \cdot 10^{-7}$	$1.9 \cdot 10^{-11}$	$1.5 \cdot 10^{-13}$	$4.8 \cdot 10^{-6}$	2
R	mD	R	D	(71)	$4.9 \cdot 10^{-11}$	$7.9 \cdot 10^{-9}$	$5.3 \cdot 10^{-9}$	$7.7 \cdot 10^{-10}$	Robin
R	mD	D	R	(81)	$1.7 \cdot 10^{-9}$	$1.1 \cdot 10^{-8}$	$3.1 \cdot 10^{-9}$	$7.7 \cdot 10^{-5}$	Robin
				(82)	$4.6 \cdot 10^{-12}$	$4.1 \cdot 10^{-10}$	$1.2 \cdot 10^{-9}$	$4.3 \cdot 10^{-10}$	Schur
R	mN	R	N	(76)	$4.9 \cdot 10^{-11}$	$7.9 \cdot 10^{-9}$	$5.3 \cdot 10^{-9}$	$7.7 \cdot 10^{-10}$	Robin
R	mN	N	R	(84)	$1.3 \cdot 10^{-10}$	$7.9 \cdot 10^{-10}$	$2.2 \cdot 10^{-10}$	$4.6 \cdot 10^{-11}$	Robin
				(85)	$6.3 \cdot 10^{-11}$	$2.9 \cdot 10^{-6}$	$6.2 \cdot 10^{-6}$	$2.3 \cdot 10^{-8}$	Flux

TABLE 7 – Four and two unknowns schemes. Interface conditions, references of the interface systems, error estimates (in bold if preconditioning).

We remind that the monolithic scheme (1)"rsa" is brought back to (1)"dsa" if $q = 0$ and that the iterative scheme $RmDRD$ is brought back to $NmDND$. On the contrary, when q is very large, the problem is ill-posed and the iterative method stagnates. The behaviour of the $RmDDR$ option is similar, $RmDDR$ is brought back to $NmDDN$ if $q = 0$. The options $RmDRD$ or $RmDDR$ generalize the options $NmDND$ and $NmDDN$ that are already very successful. Let us remark again that the variant that gives the best results is the same for the four schemes. Accuracy of $4.4 \cdot 10^{-14}$ is observed with $q = 6$ with the scheme $RmDRD$, and $3.1 \cdot 10^{-14}$ with $q = 8$ with the scheme $RmDDR$. It seems that $RmDRD$ and $RmDDR$ behave as well as $NmDND$ and $NmDDN$, at least for values of q that are not too large. Oscillations like the ones obtained with the $DmdRDR$ and $DmdRRD$ options to solve (1) are not observed here. From Figure 8b, we see that all values from the $RmDDR$ option are ranging from 10^{-14} to 10^{-13} and all values from the $RmDRD$ option from 10^{-14} to 10^{-11} . In view of the accuracy and stability of the $RmDDR$ option, the solution of the problem (1)"rsa" is of great interest and gives us much greater flexibility, as there is no required continuity of the solution or its normal derivative on the interfaces.

Conversely, the options $RmNNR$ and $RmNRN$ are ill-posed if $q = 0$. The variants are defined in Table 4 by changing α by ω . However, none of them particularly stands out. The variants 0 and 3 give accuracy between 10^{-5} and 10^{-10} . The variants 1 and 2, while better, are rather uneven. Some results are shown in Table 7.

Now, let us compare these results with those of the ensuing two unknowns schemes. First the Robin equation (71) arises from the $RmDRD$ or $RmNRN$ scheme (both schemes are equivalent, see (78) and (79)). It is shown from Table 7 that the Robin equation does not lead to optimal results. Nor the preconditioned Robin equations obtained from $RmDDR$ and $RmNNR$. The Schur and flux equations that we have defined to solve (1)"rsa" do not provide a great accuracy. Regarding the four unknowns schemes, different variants are considered and one of them is particularly efficient to improve the $RmDRD$ and $RmDDR$ options. The two unknowns schemes do not offer this possibility and their accuracy cannot be increased in this way.

6 Conclusion

In this paper, the solving of the heat equation in a geometrical multi-scale domain is considered. Three kinds of transmission conditions are examined : continuity of the solution and of the mean value of its normal derivative ; continuity of the normal derivative of the solution and of the mean value of the solution itself ; and Robin condition where there is no required continuity of the solution or its normal derivative on the interfaces. They correspond to problems referred to as (1), (1)"dsa" and (1)"rsa" respectively. They lead to three different monolithic schemes. Existence and uniqueness results are proved. These schemes are compared to three different families of schemes deriving from the strong coupling method.

The strong coupling strategy requires to choose interface conditions. They are carefully written to be consistent with each of the transmission conditions. The original schemes have $2p$ interface unknowns. By elimination, we define p interface unknowns schemes that are a counterpart to the Schur complement system or the flux equation. The Robin equation is defined on the same level. The numerical experiments ($p = 2$) show that some versions of the schemes provide more accuracy and have better stability properties than others. The difference between the approximations obtained by the monolithic scheme to solve (1)"dsa" and by the interface system deriving from the strong coupling method, is the smallest in L^2 norm when a Neumann constant boundary condition is used as input value for the 2D separated problem, and a mean value of the Dirichlet boundary condition is used as output value. For (1)"rsa", the option with Robin constant boundary condition as input value for the 2D separated problem, and a mean value of the Dirichlet boundary condition as output value, and conversely for the 1D problems, is the more accurate. This option is particularly stable with respect to the Robin parameter. Both problems can be approximated in that way at 10^{-14} . Using Dirichlet constant boundary condition as input value for the 2D separated problem, and a mean value of the Neumann boundary condition as output value, and the same choice for the 1D problems, is the more accurate to solve (1) at 10^{-13} . The two unknowns schemes are less accurate. However, it is noticed that the two unknowns schemes depending on a Robin condition are more stable with respect to the Robin parameter than the corresponding four unknowns schemes. Outside the framework of the geometrical multi-scale problems, it is well known that the composition of the operators Dirichlet-to-Neumann and Neumann-to-Dirichlet gives an efficient preconditioning. The numerical experiments show that it is true for geometrical multi-scale problems too, regardless of whether Dirichlet, Neumann or Robin boundary conditions are used.

Although the results make clear the interest of Robin conditions and the flexibility of the problem (1)"rsa", we do not propose in the present paper, methods to find an optimal parameter. Regarding this issue, the paper shows at least what degree of accuracy can be obtained and points out schemes that are stable against the Robin parameter.

Lastly, it is noticed that Robin conditions can be written to be used as input values for the 2D separated problem to solve (1) and (1)"dsa". A more natural writing of the Robin equation can then be devised in these cases. This is related to the Schwarz method (see [20]). This study will be carried out in another paper.

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