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ABSTRACT. This paper is devoted to a new family of reverse Hardy–Littlewood–Sobolev inequalities which involve a power law kernel with positive exponent. We investigate the range of the admissible parameters and the properties of the optimal functions. A striking open question is the possibility of concentration which is analyzed and related with free energy functionals and nonlinear diffusion equations involving mean field drifts.

RÉSUMÉ. Cet article est consacré à une nouvelle famille d'inégalités de Hardy–Littlewood–Sobolev inversées correspondant à un noyau en loi de puissances avec un exposant positif. Nous étudions le domaine des paramètres admissibles et les propriétés des fonctions optimales. Une question ouverte remarquable est la possibilité d'un phénomène de concentration, qui est analysé est relié à des fonctionnelles d'énergie libre et à des équations de diffusion non-linéaires avec termes de dérive donnés par un champ moyen.

1. INTRODUCTION

We are concerned with the following minimization problem. For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy\,.$$

For $0 < q < 1$ we consider

$$\mathscr{C}_{N,\lambda,q} := \inf\left\{\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(2-\alpha)/q}}\,:\,0\leq\rho\in\mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)\,,\ \rho\not\equiv 0\right\}\,,$$

where

$$\alpha := \frac{2N-q\,(2N+\lambda)}{N\,(1-q)}\,.$$

By convention, for any $p > 0$ we use the notation $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N}|\rho(x)|^p\,dx$ is finite. Note that α is determined by scaling and homogeneity: for given values of λ and q, the value of α is the only one for which there is a chance that the infimum is positive. We are asking whether $\mathscr{C}_{N,\lambda,q}$ is equal to zero or positive and, in the latter case, whether there is a unique minimizer. As we will see, there are three regimes $q < 2N/(2N+\lambda)$, $q = 2N/(2N+\lambda)$ and $q > 2N/(2N+\lambda)$, which respectively correspond to $\alpha > 0$, $\alpha = 0$ and

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$\alpha < 0$. The case $q = 2N/(2N+\lambda)$, in which there is an additional *conformal symmetry*, has already been dealt with in [19] by J. Dou and M. Zhu, in [2, Theorem 18] by W. Beckner, and in [37] by Q.A. Ngô and V.H. Nguyen, who have explicitly computed $\mathscr{C}_{N,\lambda,q}$ and characterized all solutions of the corresponding Euler–Lagrange equation. Here we will mostly concentrate on the other cases. Our main result is the following.

Theorem 1. *Let $N \geq 1$, $\lambda > 0$, $q \in (0,1)$ and define α as above. Then the inequality*

$$I_\lambda[\rho] \geq \mathscr{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho(x)\, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\, dx \right)^{(2-\alpha)/q} \tag{1}$$

holds for any nonnegative function $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$, for some positive constant $\mathscr{C}_{N,\lambda,q}$, if and only if $q > N/(N+\lambda)$. In this range, if either $N = 1$, 2 or if $N \geq 3$ and $q \geq \min\{1 - 2/N, 2N/(2N+\lambda)\}$, there is a radial positive, nonincreasing, bounded function $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$ which achieves the equality case.

This theorem provides a necessary and sufficient condition for the validity of the inequality, namely $q > N/(N+\lambda)$ or equivalently $\alpha < 1$. Concerning the existence of an optimizer, the theorem completely answers this question in dimensions $N = 1$ and $N = 2$. In dimensions $N \geq 3$ we obtain a sufficient condition for the existence of an optimizer, namely, $q \geq \min\{1-2/N, 2N/(2N+\lambda)\}$. This is not a necessary condition and, in fact, in Proposition 17 we prove existence in a slightly larger, but less explicit region.

In the whole region $q > N/(N+\lambda)$ we are able to prove the existence of an optimizer for the *relaxed inequality*

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\, dx \geq \mathscr{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho(x)\, dx + M \right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\, dx \right)^{(2-\alpha)/q} \tag{2}$$

with the same optimal constant $\mathscr{C}_{N,\lambda,q}$. Here ρ is an arbitrary nonnegative function in $\mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$ and M an arbitrary nonnegative real number. If $M = 0$, inequality (2) is reduced to inequality (1). It is straightforward to see that (2) can be interpreted as the extension of (1) to measures with an absolutely continous part ρ and an additional Dirac mass at the origin. Therefore the question about existence of an optimizer in Theorem 1 is reduced to the problem of whether the optimizer for this relaxed problem in fact has a Dirac mass. Fig. 1 summarizes these considerations.

The optimizers have been explicitly characterized in the conformally invariant case $q = q(\lambda) := 2N/(2N+\lambda)$ in [19, 2, 37] and are given, up to translations, dilations and multiplications by constants, by

$$\rho(x) = \left(1 + |x|^2\right)^{-N/q} \quad \forall\, x \in \mathbb{R}^N.$$

This result determines the value of the optimal constant in (1) as

$$\mathscr{C}_{N,\lambda,q(\lambda)} = \frac{1}{\pi^{\frac{\lambda}{2}}} \frac{\Gamma\left(\frac{N}{2} + \frac{\lambda}{2}\right)}{\Gamma\left(N + \frac{\lambda}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)} \right)^{1+\frac{\lambda}{N}}.$$

By a simple argument that will be exposed in Section 2, we can also find the optimizers in the special case $\lambda = 2$: if $N/(N+2) < q < 1$, then the optimizers for (1) are given by translations, dilations and constant multiples of

$$\rho(x) = \left(1+|x|^2\right)^{-\frac{1}{1-q}} .$$

In this case we obtain that

$$\mathscr{C}_{N,2,q} = \frac{N\,(1-q)}{\pi\,q}\left(\frac{(N+2)\,q-N}{2\,q}\right)^{\frac{2-N\,(1-q)}{N\,(1-q)}}\left(\frac{\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{1-q}-\frac{N}{2}\right)}\right)^{\frac{2}{N}} .$$

Returning to the general case (that is, $q \neq 2\,N/(2\,N+\lambda)$ and $\lambda \neq 2$), no explicit form of the optimizers is known, but we can at least prove a uniqueness result in some cases, see also Fig. 2.

Theorem 2. *Assume that $N/(N+\lambda) < q < 1$ and either $q \ge 1-1/N$ and $\lambda \ge 1$, or $2 \le \lambda \le 4$. Then the optimizer for (2) exists and is unique up to translation, dilation and multiplication by a positive constant.*

We refer to (1) as a *reverse Hardy–Littlewood–Sobolev inequality* as λ is positive. The Hardy–Littlewood–Sobolev (HLS) inequality corresponds to negative values of λ and is named after G. Hardy and J.E. Littlewood, see [23, 24], and S.L. Sobolev, see [39, 40]; also see [25] for an early discussion of rearrangement methods applied to these inequalities. In 1983, E.H. Lieb in [31] proved the existence of optimal functions for negative values of λ and established optimal constants. His proof requires an analysis of the invariances which has been systematized under the name of *competing symmetries*, see [11] and [32, 8] for a comprehensive introduction. Notice that rearrangement free proofs, which in some cases rely on the duality between Sobolev and HLS inequalities, have also been established more recently in various cases: see for instance [20, 21, 28]. Standard HLS inequalities, which correspond to negative values of λ in $I_\lambda[\rho]$, have many consequences in the theory of functional inequalities, particularly for identifying optimal constants.

Relatively few results are known in the case $\lambda > 0$. The conformally invariant case, *i.e.*, $q = 2\,N/(2\,N+\lambda)$, appears in [19] and is motivated by some earlier results on the sphere (see references therein). Further results have been obtained in [2, 37], still in the conformally invariant case. Another range of exponents, which has no intersection with the one considered in the present paper, was studied earlier in [41, Theorem G]. Here we focus on a non-conformally invariant family of interpolation inequalities corresponding to a given $\mathrm{L}^1(\mathbb{R}^N)$ norm. In a sense, these inequalities play for HLS inequalities a role analogous to Gagliardo-Nirenberg inequalities compared to Sobolev's conformally invariant inequality.

The study of (1) is motivated by the analysis of nonnegative solutions to the evolution equation

$$\partial_t \rho = \Delta \rho^q + \nabla \cdot \left(\rho\,\nabla W_\lambda * \rho\right) , \tag{3}$$

where the kernel is given by $W_\lambda(x) := \frac{1}{\lambda}|x|^\lambda$. Eq. (3) is a special case of a larger family of Keller-Segel type equations, which covers the cases $q = 1$ (linear diffusions), $q > 1$ (diffusions of porous medium type) in addition to $0 < q < 1$ (fast diffusions), and also the range of exponents $\lambda < 0$. Of particular interest is the original parabolic–elliptic Keller–Segel system which corresponds in dimension $N = 2$ to a limit case as $\lambda \to 0$, in which the kernel is $W_0(x) = \frac{1}{2\pi}\log|x|$ and the diffusion exponent is $q = 1$. The reader is invited to refer to [27] for a global overview of this class of problems and for a detailed list of references and applications.

According to [1, 38], (3) has a gradient flow structure in the Wasserstein-2 metric. The corresponding *free energy* functional is given by

$$\mathscr{F}[\rho] := -\frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx + \frac{1}{2\lambda}\,I_\lambda[\rho] \quad \forall\,\rho\in \mathrm{L}^1_+(\mathbb{R}^N)\,,$$

where $\mathrm{L}^1_+(\mathbb{R}^N)$ denotes the positive functions in $\mathrm{L}^1(\mathbb{R}^N)$. As will be detailed later, optimal functions for (1) are energy minimizers for $\mathscr{F}$ under a *mass* constraint. Smooth solutions $\rho(t,\cdot)$ of (3) with sufficient decay properties as $|x| \to +\infty$ conserve mass and center of mass over time while the free energy decays according to

$$\frac{d}{dt}\mathscr{F}[\rho(t,\cdot)] = -\int_{\mathbb{R}^N}\rho\left|\tfrac{q}{1-q}\nabla\rho^{q-1} - \nabla W_\lambda * \rho\right|^2 dx\,.$$

This identity allows us to identify the smooth stationary solutions as the solutions of

$$\rho_s = \big(C + (W_\lambda * \rho_s)\big)^{-\frac{1}{1-q}}$$

where C is a constant which has to be determined by the mass constraint. Thanks to the gradient flow structure, minimizers of the free energy $\mathscr{F}$ are stationary states of Eq. (3). When dealing with solutions of (3) or with minimizers of the free energy, without loss of generality we can normalize the mass to 1 in order to work in the space of probability measures $\mathscr{P}(\mathbb{R}^N)$. The general case of a bounded measure with an arbitrary mass can be recovered by an appropriate change of variables. Considering the *lower semicontinuous extension of the free energy to* $\mathscr{P}(\mathbb{R}^N)$ denoted by $\mathscr{F}^\Gamma$, we obtain counterparts to Theorems 1 and 2 in terms of $\mathscr{F}^\Gamma$.

Theorem 3. *The free energy $\mathscr{F}^\Gamma$ is bounded from below on $\mathscr{P}(\mathbb{R}^N)$ if and only if $N/(N+\lambda) < q < 1$. If $q > N/(N+\lambda)$, then there exists a global minimizer $\mu_* \in \mathscr{P}(\mathbb{R}^N)$ and, modulo translations, it has the form*

$$\mu_* = \rho_* + M_*\,\delta_0$$

for some $M_* \in [0,1)$. Moreover $\rho_* \in \mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)$ is radially symmetric, non-increasing and supported on $\mathbb{R}^N$.

If $M_* = 0$, then ρ_* is an optimizer of (1). Conversely, if $\rho \in \mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)$ is an optimizer of (1) with mass $M > 0$, then ρ/M is a global minimizer of $\mathscr{F}^\Gamma$ on $\mathscr{P}(\mathbb{R}^N)$.

Finally, if $N/(N+\lambda) < q < 1$ and either $q \ge 1 - 1/N$ and $\lambda \ge 1$, or $2 \le \lambda \le 4$, then the global minimizer μ_* of $\mathscr{F}^\Gamma$ on $\mathscr{P}(\mathbb{R}^N)$ is unique *up to translation.*

In the region of the parameters of Theorem 1 for which (1) is achieved by a radial function, this optimizer is also a minimizer of $\mathscr{F}$. If the minimizer μ_* of $\mathscr{F}^\Gamma$ has a singular part, then the constant $\mathscr{C}_{N,\lambda,q}$ is also achieved by μ_* in (2), up to a translation. Hence the results of Theorem 3 are equivalent to the results of Theorems 1 and 2.

The use of free energies to understand the long-time asymptotics of gradient flow equations like (3) and various related models with other interaction potentials than W_λ or more general pressure variables than ρ^{q-1} has already been studied in some cases: see for instance [1, 15, 16, 43]. The connection to Hardy–Littlewood–Sobolev type functional inequalities [10, 5, 9] is well-known for the range $\lambda \in (-N, 0]$. However, the case of W_λ with $\lambda > 0$ is as far as we know entirely new.

This paper results from the merging of two earlier preprints, [18] and [13], corresponding to two research projects that were investigated independently.

Section 2 is devoted to the proof of the reverse HLS inequality (1) and also of the optimal constant in the case $\lambda = 2$. In Section 3 we study the existence of optimizers of the reverse HLS inequality via the relaxed variational problem associated with (2). The regularity properties of these optimizers are analysed in Section 4, with the goal of providing some additional results of no-concentration. Section 5 is devoted to the equivalence of the reverse HLS inequalities and the existence of a lower bound of $\mathscr{F}^\Gamma$ on $\mathscr{P}(\mathbb{R}^N)$. The relative compactness of minimizing sequences of probability measures is also established as well as the uniqueness of the measure valued minimizers of $\mathscr{F}^\Gamma$, in the same range of the parameters as in Theorem 2. We conclude this paper by an Appendix A on a toy model for concentration which sheds some light on the threshold value $q = 1 - 2/N$ and by another Appendix B devoted to the simpler case $q \geq 1$, in order to complete the picture. From here on (except in Appendix B), we shall assume that $q < 1$ without further notice.

2. Reverse HLS inequality

The following proposition gives a necessary and sufficient condition for inequality (1).

Proposition 4. *Let* $\lambda > 0$.

(1) *If* $0 < q \leq N/(N+\lambda)$, *then* $\mathscr{C}_{N,\lambda,q} = 0$.
(2) *If* $N/(N+\lambda) < q < 1$, *then* $\mathscr{C}_{N,\lambda,q} > 0$.

The result for $q < N/(N+\lambda)$ was obtained in [14] using a different method. The result for $q = N/(N+\lambda)$, as well as the result for $2N/(2N+\lambda) \neq q > N/(N+\lambda)$, are new.

Proof of Proposition 4. Part (1). Let $\rho \geq 0$ be bounded with compact support and let $\sigma \geq 0$ be a smooth function with $\int_{\mathbb{R}^N} \sigma(x)\, dx = 1$. With another parameter $M > 0$ we consider

$$\rho_\varepsilon(x) = \rho(x) + M\,\varepsilon^{-N}\,\sigma(x/\varepsilon)\,,$$

where $\varepsilon > 0$ is a small parameter. Then $\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,dx = \int_{\mathbb{R}^N} \rho(x)\,dx + M$ and, by simple estimates,

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)^q\,dx \to \int_{\mathbb{R}^N} \rho(x)^q\,dx \quad \text{as} \quad \varepsilon \to 0_+ \tag{4}$$

and

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx \quad \text{as} \quad \varepsilon \to 0_+\,.$$

Thus, taking ρ_ε as a trial function,

$$\mathscr{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}} =: \mathscr{Q}[\rho, M]\,. \tag{5}$$

This inequality is valid for any M and therefore we can let $M \to +\infty$. If $\alpha > 1$, which is the same as $q < N/(N+\lambda)$, we immediately obtain $\mathscr{C}_{N,\lambda,q} = 0$ by letting $M \to +\infty$. If $\alpha = 1$, *i.e.*, $q = N/(N+\lambda)$, by taking the limit as $M \to +\infty$, we obtain

$$\mathscr{C}_{N,\lambda,q} \le \frac{2\int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{1/q}}\,.$$

Let us show that by a suitable choice of ρ the right side can be made arbitrarily small. For any $R > 1$, we take

$$\rho_R(x) := |x|^{-(N+\lambda)}\, \mathbb{1}_{1 \le |x| \le R}(x)\,.$$

Then

$$\int_{\mathbb{R}^N} |x|^\lambda \rho_R\,dx = \int_{\mathbb{R}^N} \rho_R^q\,dx = \left|\mathbb{S}^{N-1}\right| \log R$$

and, as a consequence,

$$\frac{\int_{\mathbb{R}^N} |x|^\lambda \rho_R(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho_R^{N/(N+\lambda)}\,dx\right)^{(N+\lambda)/N}} = \left(\left|\mathbb{S}^{N-1}\right| \log R\right)^{-\lambda/N} \to 0 \quad \text{as} \quad R \to \infty.$$

This proves that $\mathscr{C}_{N,\lambda,q} = 0$ for $q = N/(N+\lambda)$. □

In order to prove that $\mathscr{C}_{N,\lambda,q} > 0$ in the remaining cases, we need the following simple bound, which is known as a *Carlson type inequality* in the literature after [12] and whose sharp form has been established in [30] by V. Levin. Various proofs can be found in the literature and we insist on the fact that they are not limited to the case $q < 1$: see for instance [4, Ineq. 2(a)], [36, Chap. VII, Ineq. (8.1)] or [34, Section 4]. For completeness, we give a statement and a proof for the case we are interested in.

Lemma 5 (Carlson-Levin inequality)**.** *Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. Then there is a constant $c_{N,\lambda,q} > 0$ such that for all $\rho \ge 0$,*

$$\left(\int_{\mathbb{R}^N} \rho\,dx\right)^{1-\frac{N(1-q)}{\lambda q}} \left(\int_{\mathbb{R}^N} |x|^\lambda \rho(x)\,dx\right)^{\frac{N(1-q)}{\lambda q}} \ge c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{1/q}.$$

Equality is achieved if and only if

$$\rho(x) = \left(1 + |x|^\lambda\right)^{-\frac{1}{1-q}}$$

up to translations, dilations and constant multiples, and one has

$$c_{N,\lambda,q} = \frac{1}{\lambda}\left(\frac{(N+\lambda)\,q-N}{q}\right)^{\frac{1}{q}}\left(\frac{N\,(1-q)}{(N+\lambda)\,q-N}\right)^{\frac{N}{\lambda}\,\frac{1-q}{q}}\left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{\lambda}\right)\Gamma\left(\frac{N}{\lambda}\right)}\right)^{\frac{1-q}{q}}.$$

Proof. Let $R>0$. Using Hölder's inequality in two different ways, we obtain

$$\int_{\{|x|<R\}}\rho^q\,dx \le \left(\int_{\mathbb{R}^N}\rho\,dx\right)^q |B_R|^{1-q} = C_1\left(\int_{\mathbb{R}^N}\rho\,dx\right)^q R^{N(1-q)}$$

and

$$\int_{\{|x|\ge R\}}\rho^q\,dx \le \left(\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx\right)^q\left(\int_{\{|x|\ge R\}}|x|^{-\frac{\lambda q}{1-q}}\,dx\right)^{1-q}$$
$$= C_2\left(\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx\right)^q R^{-\lambda q+N(1-q)}.$$

The fact that $C_2<\infty$ comes from the assumption $q>N/(N+\lambda)$, which is the same as $\lambda\,q/(1-q)>N$. To conclude, we add these two inequalities and optimize over R.

The existence of a radial monotone non-increasing optimal function follows by standard variational methods; the expression for the optimal functions is a consequence of the Euler-Lagrange equations. The expression of $c_{N,\lambda,q}$ is then straightforward. □

Proof of Proposition 4. Part (2). By rearrangement inequalities it suffices to prove the inequality for symmetric non-increasing ρ's. For such functions, by the simplest rearrangement inequality,

$$\int_{\mathbb{R}^N}|x-y|^\lambda\rho(y)\,dx \ge \int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx \quad\text{for all}\quad x\in\mathbb{R}^N.$$

Thus,

$$I_\lambda[\rho] \ge \int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx\int_{\mathbb{R}^N}\rho\,dx. \tag{6}$$

In the range $\frac{N}{N+\lambda}<q<1$ (for which $\alpha<1$), we recall that by Lemma 5, for any symmetric non-increasing function ρ, we have

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^\alpha} \ge \left(\int_{\mathbb{R}^N}\rho\,dx\,dx\right)^{1-\alpha}\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx \ge c_{N,\lambda,q}^{2-\alpha}\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{\frac{2-\alpha}{q}}$$

because $2-\alpha=\frac{\lambda q}{N(1-q)}$. As a consequence, we obtain that

$$\mathscr{C}_{N,\lambda,q} \ge c_{N,\lambda,q}^{2-\alpha} > 0.$$

□

Corollary 6. *Let* $\lambda=2$ *and* $N/(N+2)<q<1$. *Then the optimizers for* (1) *are given by translations, dilations and constant multiples of*

$$\rho(x) = \left(1+|x|^2\right)^{-\frac{1}{1-q}}$$

and the optimal constant is

$$\mathscr{C}_{N,2,q} = 2\, c_{N,2,q}^{\frac{2q}{N(1-q)}}\,.$$

Proof. By rearrangement inequalities it is enough to prove (1) for symmetric non-increasing ρ's, and so $\int_{\mathbb{R}^N} x\,\rho(x)\,dx = 0$. Therefore

$$I_2[\rho] = 2\int_{\mathbb{R}^N}\rho(x)\,dx\int_{\mathbb{R}^N}|x|^2\,\rho(x)\,dx$$

and the optimal function is the one of the Carlson type inequality of Lemma 5. □

By taking into account the fact that

$$c_{N,2,q} = \frac{1}{2}\left(\frac{(N+2)\,q-N}{q}\right)^{\frac{1}{q}}\left(\frac{N\,(1-q)}{(N+2)\,q-N}\right)^{\frac{N}{2}\frac{1-q}{q}}\left(\frac{\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{2}\right)}\right)^{\frac{1-q}{q}}\,,$$

we recover the expression of $\mathscr{C}_{N,2,q}$ given in the introduction.

Remark 7. We can now make a few observations on the reverse HLS inequality (1) and its optimal constant $\mathscr{C}_{N,\lambda,q}$.

(i) The computation in the proof of Proposition 4, Part (2) explains a surprising feature of (1): $I_\lambda[\rho]$ controls a product of two terms. However, in the range $N/(N+\lambda) < q < 2\,N/(2\,N+\lambda)$ which corresponds to $\alpha \in (0,1)$, the problem is actually reduced (with a non-optimal constant) to the interpolation of $\int_{\mathbb{R}^N}\rho^q\,dx$ between $\int_{\mathbb{R}^N}\rho\,dx$ and $\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx$, which has a more classical structure.

(ii) There is an alternative way to prove (1) in the range $2\,N/(2\,N+\lambda) < q < 1$ using the results from [19, 2, 37]. We can indeed rely on Hölder's inequality to get that

$$\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{1/q} \le \left(\int_{\mathbb{R}^N}\rho(x)^{\frac{2N}{2N+\lambda}}\,dx\right)^{\eta\,\frac{2N+\lambda}{2N}}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^{1-\eta}$$

with $\eta := \frac{2\,N\,(1-q)}{\lambda\,q}$. By applying the conformally invariant inequality

$$I_\lambda[\rho] \ge \mathscr{C}_{N,\lambda,\frac{2N}{2N+\lambda}}\left(\int_{\mathbb{R}^N}\rho(x)^{\frac{2N}{2N+\lambda}}\,dx\right)^{\frac{2N+\lambda}{N}}$$

shown in [19, 2, 37], we obtain that

$$\mathscr{C}_{N,\lambda,q} \ge \mathscr{C}_{N,\lambda,\frac{2N}{2N+\lambda}} = \pi^{-\frac{\lambda}{2}}\,\frac{\Gamma\left(\frac{N}{2}+\frac{\lambda}{2}\right)}{\Gamma\left(N+\frac{\lambda}{2}\right)}\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1+\frac{\lambda}{N}}\,.$$

We notice that $\alpha = -2\,(1-\eta)/\eta$ is negative in the range $2\,N/(2\,N+\lambda) < q < 1$.

(iii) We have

$$\lim_{q\to N/(N+\lambda)_+}\mathscr{C}_{N,\lambda,q} = 0$$

because the map $(\lambda, q) \mapsto \mathscr{C}_{N,\lambda,q}$ is *upper semi-continuous*. The proof of this last property goes as follows. Let us rewrite $\mathscr{Q}[\rho, 0]$ defined in (5) as

$$\mathsf{Q}_{q,\lambda}[\rho] := \frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}}\,. \tag{7}$$

In this expression of the energy quotient, we emphasize the dependence in q and λ. As before, the infimum of $\mathsf{Q}_{q,\lambda}$ over the set of nonnegative functions in $\mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$ is $\mathscr{C}_{N,\lambda,q}$. Let (q, λ) be a given point in $(0,1) \times (0,\infty)$ and let $(q_n, \lambda_n)_{n\in\mathbb{N}}$ be a sequence converging to (q, λ). Let $\varepsilon > 0$ and choose a ρ which is bounded, has compact support and is such that $\mathsf{Q}_{q,\lambda}[\rho] \le \mathscr{C}_{N,\lambda,q} + \varepsilon$. Then, by the definition as an infimum, $\mathscr{C}_{N,q_n,\lambda_n} \le \mathsf{Q}_{q_n,\lambda_n}[\rho]$. On the other hand, the assumptions on ρ imply that $\lim_{n\to\infty} \mathsf{Q}_{q_n,\lambda_n}[\rho] = \mathsf{Q}_{q,\lambda}[\rho]$. We conclude that $\limsup_{n\to\infty} \mathscr{C}_{N,q_n,\lambda_n} \le \mathscr{C}_{N,\lambda,q} + \varepsilon$. Since ε is arbitrary, we obtain the claimed upper semi-continuity property.

3. Existence of minimizers and relaxation

We now investigate whether there are nonnegative minimizers in $\mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$ for $\mathscr{C}_{N,\lambda,q}$ if $N/(N+\lambda) < q < 1$. As mentioned before, the conformally invariant case $q = 2N/(2N+\lambda)$ has been dealt with before and will be excluded from our considerations. We start with the simpler case $2N/(2N+\lambda) < q < 1$, which corresponds to $\alpha < 0$.

Proposition 8. *Let $\lambda > 0$ and $2N/(2N+\lambda) < q < 1$. Then there is a minimizer for $\mathscr{C}_{N,\lambda,q}$.*

Proof. Let $(\rho_j)_{j\in\mathbb{N}}$ be a minimizing sequence. By rearrangement inequalities we may assume that the ρ_j are symmetric non-increasing. By scaling and homogeneity, we may also assume that

$$\int_{\mathbb{R}^N} \rho_j(x)\,dx = \int_{\mathbb{R}^N} \rho_j(x)^q\,dx = 1 \quad \text{for all } j \in \mathbb{N}.$$

This together with the symmetric non-increasing character of ρ_j implies that

$$\rho_j(x) \le C \min\left\{|x|^{-N}, |x|^{-N/q}\right\}$$

with C independent of j. By Helly's selection theorem we may assume, after passing to a subsequence if necessary, that $\rho_j \to \rho$ almost everywhere. The function ρ is symmetric non-increasing and satisfies the same upper bound as ρ_j.

By Fatou's lemma we have

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] \quad \text{and} \quad 1 \ge \int_{\mathbb{R}^N} \rho(x)\,dx\,.$$

To complete the proof we need to show that $\int_{\mathbb{R}^N} \rho(x)^q\,dx = 1$ (which implies, in particular, that $\rho \not\equiv 0$) and then ρ will be an optimizer.

Modifying an idea from [3] we pick $p \in \big(N/(N+\lambda), q\big)$ and apply (1) with the same λ and $\alpha(p) = \big(2N - p(2N+\lambda)\big)/\big(N(1-p)\big)$ to get

$$I_\lambda[\rho_j] \ge \mathscr{C}_{N,\lambda,p} \left(\int_{\mathbb{R}^N} \rho_j^p\,dx\right)^{(2-\alpha(p))/p}\,.$$

Since the left side converges to a finite limit, namely $\mathscr{C}_{N,\lambda,q}$, we find that the ρ_j are uniformly bounded in $\mathrm{L}^p(\mathbb{R}^N)$ and therefore we have as before

$$\rho_j(x) \le C'\,|x|^{-N/p}\,.$$

Since $\min\big\{|x|^{-N}, |x|^{-N/p}\big\} \in \mathrm{L}^q(\mathbb{R}^N)$, we obtain by dominated convergence

$$\int_{\mathbb{R}^N} \rho_j^q\,dx \to \int_{\mathbb{R}^N} \rho^q\,dx\,,$$

which, in view of the normalization, implies that $\int_{\mathbb{R}^N} \rho(x)^q\,dx = 1$, as claimed. □

Next, we prove the existence of minimizers in the range $N/(N+\lambda) < q < 2N/(2N+\lambda)$ by considering the *minimization of the relaxed problem* (2). The idea behind this relaxation is to allow ρ to contain a Dirac function at the origin. The motivation comes from the proof of the first part of Proposition 4. The expression of $\mathscr{Q}[\rho, M]$ as defined in (5) arises precisely from a measurable function ρ together with a Dirac function of strength M at the origin. We have seen that in the regime $q \le N/(N+\lambda)$ (that is, $\alpha \ge 1$) it is advantageous to increase M to infinity. This is no longer so if $N/(N+\lambda) < q < 2N/(2N+\lambda)$. While it is certainly disadvantageous to move M to infinity, it has to be investigated whether the optimum M is 0 or a positive finite value.

Let

$$\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q} := \inf\Big\{\mathscr{Q}[\rho, M] : 0 \le \rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)\,,\ \rho \not\equiv 0\,,\ M \ge 0\Big\}$$

where $\mathscr{Q}[\rho, M]$ is defined by (5). We know that $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q} \le \mathscr{C}_{N,\lambda,q}$ by restricting the minimization to $M = 0$. On the other hand, (5) gives $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q} \ge \mathscr{C}_{N,\lambda,q}$. Therefore,

$$\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q} = \mathscr{C}_{N,\lambda,q}\,,$$

which justifies our interpretation of $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$ as a *relaxed minimization* problem. Let us start with a preliminary observation.

Lemma 9. *Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. If $\rho \ge 0$ is an optimal function for either $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$ (for an $M \ge 0$) or $\mathscr{C}_{N,\lambda,q}$ (with $M = 0$), then ρ is radial (up to a translation), monotone non-increasing and positive almost everywhere on $\mathbb{R}^N$.*

Proof. Since $\mathscr{C}_{N,\lambda,q}$ is positive, we observe that ρ is not identically 0. By rearrangement inequalities and up to a translation, we know that ρ is radial and monotone non-increasing. Assume by contradiction that ρ vanishes on a set $E \subset \mathbb{R}^N$ of finite, positive measure. Then

$$\mathscr{Q}\big[\rho, M + \varepsilon\,\mathbb{1}_E\big] = \mathscr{Q}[\rho, M]\left(1 - \frac{2-\alpha}{q}\,\frac{|E|}{\int_{\mathbb{R}^N} \rho(x)^q\,dx}\,\varepsilon^q + o(\varepsilon^q)\right)$$

as $\varepsilon \to 0_+$, a contradiction to the minimality for sufficiently small $\varepsilon > 0$. □

Varying $\mathscr{Q}[\rho, M]$ with respect to ρ, we obtain *the Euler–Lagrange equation* on $\mathbb{R}^N$ for any minimizer (ρ_*, M_*) for $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$:

$$2\,\frac{\int_{\mathbb{R}^N} |x-y|^\lambda \rho_*(y)\,dy + M_* |x|^\lambda}{I_\lambda[\rho_*] + 2M_* \int_{\mathbb{R}^N} |y|^\lambda \rho_*(y)\,dy} - \frac{\alpha}{\int_{\mathbb{R}^N} \rho_*\,dy + M_*} - (2-\alpha)\,\frac{\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N} \rho_*(y)^q\,dy} = 0\,. \tag{8}$$

This equation follows from the fact that ρ_* is positive almost everywhere according to Lemma 9.

Proposition 10. *Let $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$. Then there is a minimizer for $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$.*

We will later show that for $N = 1$ and $N = 2$ there is a minimizer for the original problem $\mathscr{C}_{N,\lambda,q}$ in the full range of λ's and q's covered by Proposition 10. If $N \geq 3$, the same is true under additional restrictions.

Proof of Proposition 10. The beginning of the proof is similar to that of Proposition 8. Let $(\rho_j, M_j)_{j\in\mathbb{N}}$ be a minimizing sequence. By rearrangement inequalities we may assume that ρ_j is symmetric non-increasing. Moreover, by scaling and homogeneity, we may assume that

$$\int_{\mathbb{R}^N} \rho_j \, dx + M_j = \int_{\mathbb{R}^N} \rho_j^q = 1\,.$$

In a standard way this implies that

$$\rho_j(x) \leq C \min\left\{|x|^{-N}, |x|^{-N/q}\right\}$$

with C independent of j. By Helly's selection theorem we may assume, after passing to a subsequence if necessary, that $\rho_j \to \rho$ almost everywhere. The function ρ is symmetric non-increasing and satisfies the same upper bound as ρ_j. Passing to a further subsequence, we can also assume that $(M_j)_{j\in\mathbb{N}}$ and $\left(\int_{\mathbb{R}^N} \rho_j \, dx\right)_{j\in\mathbb{N}}$ converge and define $M := L + \lim_{j\to\infty} M_j$ where $L = \lim_{j\to\infty} \int_{\mathbb{R}^N} \rho_j \, dx - \int_{\mathbb{R}^N} \rho \, dx$, so that $\int_{\mathbb{R}^N} \rho \, dx + M = 1$. In the same way as before, we show that

$$\int_{\mathbb{R}^N} \rho(x)^q \, dx = 1\,.$$

We now turn our attention to the L^1-term. We cannot invoke Fatou's lemma because $\alpha \in (0,1)$ and therefore this term appears in $\mathscr{Q}$ with a positive exponent in the denominator. The problem with this term is that $|x|^{-N}$ is not integrable at the origin and we cannot get a better bound there. We have to argue via measures, so let $d\mu_j(x) := \rho_j(x) \, dx$. Because of the upper bound on ρ_j we have

$$\mu_j\left(\mathbb{R}^N \setminus B_R(0)\right) = \int_{\{|x|\geq R\}} \rho_j(x) \, dx \leq C \int_{\{|x|\geq R\}} \frac{dx}{|x|^{N/q}} = C' R^{-N(1-q)/q}\,.$$

This means that the measures are tight. After passing to a subsequence if necessary, we may assume that $\mu_j \to \mu$ weak * in the space of measures on $\mathbb{R}^N$. Tightness implies that

$$\mu(\mathbb{R}^N) = \lim_{j\to\infty} \int_{\mathbb{R}^N} \rho_j \, dx = L + \int_{\mathbb{R}^N} \rho \, dx\,.$$

Moreover, since the bound $C|x|^{-N/q}$ is integrable away from any neighborhood of the origin, we see that μ is absolutely continuous on $\mathbb{R}^N \setminus \{0\}$ and $d\mu/dx = \rho$. In other words,

$$d\mu = \rho \, dx + L\delta\,.$$

Using weak convergence in the space of measures one can show that

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \geq I_\lambda[\rho] + 2M\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx\,.$$

Finally, by Fatou's lemma,

$$\liminf_{j\to\infty}\int_{\mathbb{R}^N} |x|^\lambda\,\rho_j(x)\,dx \geq \int_{\mathbb{R}^N} |x|^\lambda\left(\rho(x)\,dx + L\delta\right) = \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx\,.$$

Hence

$$\liminf_{j\to\infty} \mathscr{Q}[\rho_j, M_j] \geq \mathscr{Q}[\rho, M]\,.$$

By definition of $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ the right side is bounded from below by $\mathscr{C}^{\rm rel}_{N,\lambda,q}$. On the other hand, by choice of ρ_j and M_j the left side is equal to $\mathscr{C}^{\rm rel}_{N,\lambda,q}$. This proves that (ρ, M) is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$. □

Next, we show that under certain assumptions a minimizer (ρ_*, M_*) for the relaxed problem must, in fact, have $M_* = 0$ and is therefore a minimizer of the original problem.

Proposition 11. *Let $N \geq 1$, $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$. If $N \geq 3$ and $\lambda > 2N/(N-2)$, then assume in addition that $q \geq 1-2/N$. If (ρ_*, M_*) is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$, then $M_* = 0$. In particular, there is a minimizer for $\mathscr{C}_{N,\lambda,q}$.*

Note that for $N \geq 3$, we are implicitly assuming $\lambda < 4N/(N-2)$ since otherwise the two assumptions $q < 2N/(2N+\lambda)$ and $q \geq 1-2/N$ cannot be simultaneously satisfied. For the proof of Proposition 11 we need the following lemma which identifies the sub-leading term in (4).

Lemma 12. *Let $0 < q < p$, let $f \in \mathrm{L}^p \cap \mathrm{L}^q(\mathbb{R}^N)$ be a symmetric non-increasing function and let $g \in \mathrm{L}^q(\mathbb{R}^N)$. Then, for any $\tau > 0$, as $\varepsilon \to 0_+$,*

$$\int_{\mathbb{R}^N} \left|f(x) + \varepsilon^{-N/p}\,\tau\, g(x/\varepsilon)\right|^q dx = \int_{\mathbb{R}^N} f^q\,dx + \varepsilon^{N(1-q/p)}\,\tau^q \int_{\mathbb{R}^N} |g|^q\,dx + o\left(\varepsilon^{N(1-q/p)}\,\tau^q\right).$$

Proof of Lemma 12. We first note that

$$f(x) = o\left(|x|^{-N/p}\right) \quad \text{as} \quad x \to 0 \tag{9}$$

in the sense that for any $c > 0$ there is an $r > 0$ such that for all $x \in \mathbb{R}^N$ with $|x| \leq r$ one has $f(x) \leq c\,|x|^{-N/p}$. To see this, we note that, since f is symmetric non-increasing,

$$f(x)^p \leq \frac{1}{\left|\left\{y \in \mathbb{R}^N :\ |y| \leq |x|\right\}\right|}\int_{|y|\leq|x|} f(y)^p\,dy\,.$$

The bound (9) now follows by dominated convergence.

It follows from (9) that, as $\varepsilon \to 0_+$,

$$\varepsilon^{N/p} f(\varepsilon x) \to 0 \quad \text{for any} \quad x \in \mathbb{R}^N,$$

and therefore, in particular, $\tau\, g(x) + \varepsilon^{N/p} f(\varepsilon x) \to \tau\, g(x)$ for any $x \in \mathbb{R}^N$. From the Brézis–Lieb lemma (see [7]) we know that

$$\int_{\mathbb{R}^N} \left|\tau\, g(x) + \varepsilon^{N/p} f(\varepsilon x)\right|^q dx = \tau^q \int_{\mathbb{R}^N} |g(x)|^q\, dx + \int_{\mathbb{R}^N} \left(\varepsilon^{N/p} f(\varepsilon x)\right)^q dx + o(1)\,.$$

By scaling this is equivalent to the assertion of the lemma. □

Proof of Proposition 11. We argue by contradiction and assume that $M_* > 0$. Let $0 \le \sigma \in \left(\mathrm{L}^1 \cap \mathrm{L}^q\left(\mathbb{R}^N\right)\right) \cap \mathrm{L}^1\left(\mathbb{R}^N, |x|^\lambda\, dx\right)$ with $\int_{\mathbb{R}^N} \sigma\, dx = 1$. We compute the value of $\mathcal{Q}[\rho, M]$ for the family $(\rho, M) = \left(\rho_* + \varepsilon^{-N} \tau\, \sigma(\cdot/\varepsilon), M_* - \tau\right)$ with a parameter $\tau < M_*$.
1) We have

$$I_\lambda\left[\rho_* + \varepsilon^{-N} \tau\, \sigma(\cdot/\varepsilon)\right] + 2\,(M_* - \tau) \int_{\mathbb{R}^N} |x|^\lambda \left(\rho_*(x) + \varepsilon^{-N} \tau\, \sigma(x/\varepsilon)\right) dx$$
$$= I_\lambda[\rho_*] + 2\, M_* \int_{\mathbb{R}^N} |x|^\lambda\, \rho_*(x)\, dx + R_1$$

with

$$R_1 = 2\,\tau \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_*(x) \left(|x-y|^\lambda - |x|^\lambda\right) \varepsilon^{-N} \sigma(y/\varepsilon)\, dx\, dy$$
$$+ \varepsilon^\lambda\, \tau^2\, I_\lambda[\sigma] + 2\,(M_* - \tau)\, \tau\, \varepsilon^\lambda \int_{\mathbb{R}^N} |x|^\lambda\, \sigma(x)\, dx\,.$$

Let us show that $R_1 = O\left(\varepsilon^\beta\, \tau\right)$ with $\beta := \min\{2, \lambda\}$. This is clear for the last two terms in the definition of R_1, so it remains to consider the double integral. If $\lambda \le 1$ we use the simple inequality $|x-y|^\lambda - |x|^\lambda \le |y|^\lambda$ to conclude that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_*(x) \left(|x-y|^\lambda - |x|^\lambda\right) \varepsilon^{-N} \sigma(y/\varepsilon)\, dx\, dy \le \varepsilon^\lambda \int_{\mathbb{R}^N} |x|^\lambda\, \sigma(x)\, dx \int_{\mathbb{R}^N} \rho_*\, dx\,.$$

If $\lambda > 1$ we use the fact that, with a constant C depending only on λ,

$$|x-y|^\lambda - |x|^\lambda \le -\lambda |x|^{\lambda-2}\, x \cdot y + C\left(|x|^{(2-\lambda)_+} |y|^\beta + |y|^\lambda\right). \tag{10}$$

Since ρ_* is radial, we obtain

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_*(x) \left(|x-y|^\lambda - |x|^\lambda\right) \varepsilon^{-N} \sigma(y/\varepsilon)\, dx\, dy$$
$$\le C\left(\varepsilon^\beta \int_{\mathbb{R}^N} |x|^{(2-\lambda)_+} \rho_*(x)\, dx \int_{\mathbb{R}^N} |y|^\beta \sigma(y)\, dy + \varepsilon^\lambda \int_{\mathbb{R}^N} |x|^\lambda\, \sigma(x)\, dx \int_{\mathbb{R}^N} \rho_*(x)\, dx\right).$$

Using the fact that $\rho_*, \sigma \in \mathrm{L}^1\left(\mathbb{R}^N\right) \cap \mathrm{L}^1\left(\mathbb{R}^N, |x|^\lambda\, dx\right)$ it is easy to see that the integrals on the right side are finite, so indeed $R_1 = O\left(\varepsilon^\beta\, \tau\right)$.
2) For the terms in the denominator of $\mathcal{Q}[\rho, M]$ we note that

$$\int_{\mathbb{R}^N} \left(\rho_*(x) + \varepsilon^{-N} \tau\, \sigma(x/\varepsilon)\right) dx + (M_* - \tau) = \int_{\mathbb{R}^N} \rho_*\, dx + M_*$$

and, by Lemma 12 applied with $p = 1$,

$$\int_{\mathbb{R}^N} \left(\rho_*(x) + \varepsilon^{-N} \tau\, \sigma(x/\varepsilon)\right)^q dx = \int_{\mathbb{R}^N} \rho_*^q\, dx + \varepsilon^{N(1-q)} \tau^q \int_{\mathbb{R}^N} \sigma^q\, dx + o\left(\varepsilon^{N(1-q)} \tau^q\right).$$

Thus,

$$\left(\int_{\mathbb{R}^N}\left(\rho_*(x)+\varepsilon^{-N}\tau\,\sigma(x/\varepsilon)\right)^q\,dx\right)^{-\frac{2-\alpha}{q}} = \left(\int_{\mathbb{R}^N}\rho_*^q\,dx\right)^{-\frac{2-\alpha}{q}}\left(1-\frac{2-\alpha}{q}\,\varepsilon^{N(1-q)}\tau^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx}+R_2\right)$$

with $R_2=o\left(\varepsilon^{N(1-q)}\tau^q\right)$.

Now we collect the estimates. Since (ρ_*,M_*) is a minimizer, we obtain that

$$\mathcal{Q}\left[\rho_*+\varepsilon^{-N}\tau\,\sigma(\cdot/\varepsilon),M_*-\tau\right]=\mathcal{C}_{N,\lambda,q}\left(1-\frac{2-\alpha}{q}\,\varepsilon^{N(1-q)}\tau^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx}+R_2\right) + R_1\left(\int_{\mathbb{R}^N}\rho_*\,dx+M_*\right)^{-\alpha}\left(\int_{\mathbb{R}^N}\left(\rho_*(x)+\varepsilon^{-N}\tau\,\sigma(x/\varepsilon)\right)^q\,dx\right)^{-\frac{2-\alpha}{q}}.$$

If $\beta=\min\{2,\lambda\}>N(1-q)$, we can choose τ to be a fixed number in $(0,M_*)$, so that $R_1=o\left(\varepsilon^{N(1-q)}\right)$ and therefore

$$\mathcal{Q}\left[\rho_*+\varepsilon^{-N}\tau\,\sigma(\cdot/\varepsilon),M_*-\tau\right]\le\mathcal{C}_{N,\lambda,q}\left(1-\frac{2-\alpha}{q}\,\varepsilon^{N(1-q)}\tau^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx}+o\left(\varepsilon^{N(1-q)}\right)\right).$$

Since $\alpha<2$, this is strictly less than $\mathcal{C}_{N,\lambda,q}$ for $\varepsilon>0$ small enough, contradicting the definition of $\mathcal{C}_{N,\lambda,q}$ as an infimum. Thus, $M_*=0$.

Note that if either $N=1$, 2 or if $N\ge3$ and $\lambda\le2N/(N-2)$, then the assumption $q>N/(N+\lambda)$ implies that $\beta>N(1-q)$. If $N\ge3$ and $\lambda>2N/(N-2)$, then $\beta=2\ge N(1-q)$ by assumption. Thus, it remains to deal with the case where $N\ge3$, $\lambda>2N/(N-2)$ and $2=N(1-q)$. In this case we have $R_1=O\left(\varepsilon^2\,\tau\right)$ and therefore

$$\mathcal{Q}\left[\rho_*+\varepsilon^{-N}\tau\,\sigma(\cdot/\varepsilon),M_*-\tau\right]\le\mathcal{C}_{N,\lambda,q}\left(1-\frac{2-\alpha}{q}\,\varepsilon^2\,\tau^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx}+O\left(\varepsilon^2\,\tau\right)\right).$$

By choosing τ small (but independent of ε) we obtain a contradiction as before. This completes the proof of the proposition. □

Remark 13. The extra assumption $q\ge1-2/N$ for $N\ge3$ and $\lambda>2N/(N-2)$ is dictated by the ε^2 bound on R_1. We claim that for any $\lambda\ge2$, this bound is optimal. Namely, one has

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho_*(x)\left(|x-y|^\lambda-|x|^\lambda\right)\varepsilon^{-N}\sigma(y/\varepsilon)\,dx\,dy = \varepsilon^2\,\frac{\lambda}{2}\left(1+\frac{\lambda-2}{N}\right)\int_{\mathbb{R}^N}|x|^{\lambda-2}\rho_*(x)\,dx\int_{\mathbb{R}^N}|y|^2\sigma(y)\,dy+o\left(\varepsilon^2\right)$$

for $\lambda\ge2$. This follows from the fact that, for any given $x\ne0$,

$$|x-y|^\lambda-|x|^\lambda=-\lambda\,|x|^{\lambda-2}x\cdot y+\frac{\lambda}{2}\,|x|^{\lambda-2}\left(|y|^2+(\lambda-2)\,\frac{(x\cdot y)^2}{|x|^2}\right)+O\left(|y|^{\min\{3,\lambda\}}+|y|^\lambda\right).$$

4. FURTHER RESULTS OF REGULARITY

In this section we discuss the existence of a minimizer for $\mathscr{C}_{N,\lambda,q}$ in the regime that is not covered by Proposition 11. In particular, we will establish a connection between the regularity of minimizers for the relaxed problem $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ and the presence or absence of a Dirac delta. This will allow us to establish existence of minimizers for $\mathscr{C}_{N,\lambda,q}$ in certain parameter regions which are not covered by Proposition 11.

Proposition 14. *Let $N\ge 3$, $\lambda>2N/(N-2)$ and $N/(N+\lambda)<q<\min\{1-2/N,2N/(2N+\lambda)\}$. If (ρ_*,M_*) is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ such that $(\rho_*,M_*)\in \mathrm{L}^{N(1-q)/2}(\mathbb{R}^N)\times[0,+\infty)$, then $M_*=0$.*

The condition that the minimizer (ρ_*,M_*) of $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ belongs to $\mathrm{L}^{N(1-q)/2}(\mathbb{R}^N)\times[0,+\infty)$ has to be understood as a *regularity* condition on ρ_*.

Proof. We argue by contradiction assuming that $M_*>0$ and consider a test function $\big(\rho_*+\varepsilon^{-N}\tau_\varepsilon\,\sigma(\cdot/\varepsilon),M_*-\tau_\varepsilon\big)$ such that $\int_{\mathbb{R}^N}\sigma\,dx=1$. We choose $\tau_\varepsilon=\tau_1\,\varepsilon^{N-2/(1-q)}$ with a constant τ_1 to be determined below. As in the proof of Proposition 11, one has

$$I_\lambda\big[\rho_*+\varepsilon^{-N}\tau_\varepsilon\,\sigma(\cdot/\varepsilon)\big]+2(M_*-\tau_\varepsilon)\int_{\mathbb{R}^N}|x|^\lambda\left(\rho_*(x)+\varepsilon^{-N}\sigma(x/\varepsilon)\right)dx\\ =I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx+R_1$$

with $R_1=O\big(\varepsilon^2\tau_\varepsilon\big)$. Note here that we have $\lambda\ge 2$. For the terms in the denominator we note that

$$\int_{\mathbb{R}^N}\left(\rho_*(x)+\varepsilon^{-N}\tau_\varepsilon\,\sigma(x/\varepsilon)\right)dx+(M_*-\tau_\varepsilon)=\int_{\mathbb{R}^N}\rho_*\,dx+M_*$$

and, by Lemma 12 applied with $p=N(1-q)/2$ and $\tau=\tau_\varepsilon$, *i.e.*, $\varepsilon^{-N}\tau_\varepsilon=\varepsilon^{-N/p}\tau_1$, we have

$$\int_{\mathbb{R}^N}\left(\rho_*(x)+\varepsilon^{-N}\tau_\varepsilon\,\sigma(x/\varepsilon)\right)^q\,dx=\int_{\mathbb{R}^N}\rho_*^q\,dx+\varepsilon^{N(1-q)}\tau_\varepsilon^q\int_{\mathbb{R}^N}\sigma^q\,dx+o\big(\varepsilon^{N(1-q)}\tau_\varepsilon^q\big)\,.$$

Because of the choice of τ_ε we have

$$\varepsilon^{N(1-q)}\tau_\varepsilon^q=\varepsilon^\gamma\tau_1^q\quad\text{and}\quad\varepsilon^2\tau_\varepsilon=\varepsilon^\gamma\tau_1\quad\text{with}\quad\gamma:=\frac{N-q\,(N+2)}{1-q}>0$$

and thus

$$\mathscr{Q}\big[\rho_*+\varepsilon^{-N}\tau_\varepsilon\,\sigma(\cdot/\varepsilon),M_*-\tau_\varepsilon\big]\le\mathscr{C}_{N,\lambda,q}\left(1-\frac{2-\alpha}{q}\,\varepsilon^\gamma\tau_1^q\,\frac{\int_{\mathbb{R}^N}\sigma^q\,dx}{\int_{\mathbb{R}^N}\rho_*^q\,dx}+O\big(\varepsilon^\gamma\tau_1\big)\right).$$

By choosing τ_1 small (but independent of ε) we obtain a contradiction as before. □

Proposition 14 motivates the study of the regularity of the minimizer (ρ_*,M_*) of $\mathscr{C}^{\rm rel}_{N,\lambda,q}$. We are not able to prove the regularity required in Proposition 14, but we can state a dichotomy result which is interesting by itself, and allows to deduce the existence of minimizers for $\mathscr{C}_{N,\lambda,q}$ in parameter regions not covered in Proposition 11.

Proposition 15. *Let $N\ge 1$, $\lambda>0$ and $N/(N+\lambda)<q<2N/(2N+\lambda)$. Let (ρ_*,M_*) be a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$. Then the following holds:*

(1) *If* $\int_{\mathbb{R}^N}\rho_*\,dx > \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$, *then* $M_*=0$ *and* ρ_* *is bounded with*

$$\rho_*(0)=\left(\frac{(2-\alpha)\,I_\lambda[\rho_*]\int_{\mathbb{R}^N}\rho_*\,dx}{\left(\int_{\mathbb{R}^N}\rho_*^q\,dx\right)\left(2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx\int_{\mathbb{R}^N}\rho_*\,dx-\alpha\,I_\lambda[\rho_*]\right)}\right)^{1/(1-q)}.$$

(2) *If* $\int_{\mathbb{R}^N}\rho_*\,dx = \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$, *then* $M_*=0$ *and* ρ_* *is unbounded.*

(3) *If* $\int_{\mathbb{R}^N}\rho_*\,dx < \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$, *then* ρ_* *is unbounded and*

$$M_*=\frac{\alpha\,I_\lambda[\rho_*]-2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx\,\int_{\mathbb{R}^N}\rho_*\,dx}{2\,(1-\alpha)\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}>0\,.$$

To prove Proposition 15, let us begin with an elementary lemma.

Lemma 16. *For constants* $A,\,B>0$ *and* $0<\alpha<1$, *define*

$$f(M):=\frac{A+M}{(B+M)^\alpha}\quad\textit{for any}\quad M\ge 0\,.$$

Then f *attains its minimum on* $[0,\infty)$ *at* $M=0$ *if* $\alpha\,A\le B$ *and at* $M=(\alpha\,A-B)/(1-\alpha)>0$ *if* $\alpha\,A>B$.

Proof. We consider the function on the larger interval $(-B,\infty)$. Let us compute

$$f'(M)=\frac{(B+M)-\alpha\,(A+M)}{(B+M)^{\alpha+1}}=\frac{B-\alpha\,A+(1-\alpha)\,M}{(B+M)^{\alpha+1}}\,.$$

Note that the denominator of the right side vanishes exactly at $M=(\alpha\,A-B)/(1-\alpha)$, except possibly if this number coincides with $-B$.

We distinguish two cases. If $A\le B$, which is the same as $(\alpha\,A-B)/(1-\alpha)\le -B$, then f is increasing on $(-B,\infty)$ and then f indeed attains its minimum on $[0,\infty)$ at 0. Thus it remains to deal with the other case, $A>B$. Then f is decreasing on $\big(-B,(\alpha\,A-B)/(1-\alpha)\big]$ and increasing on $\big[(\alpha\,A-B)/(1-\alpha),\infty\big)$. Therefore, if $\alpha\,A-B\le 0$, then f is increasing on $[0,\infty)$ and again the minimum is attained at 0. On the other hand, if $\alpha\,A-B>0$, then f has a minimum at the positive number $M=(\alpha\,A-B)/(1-\alpha)$. □

Proof of Proposition 15. Step 1. We vary $\mathcal{Q}[\rho_*,M]$ with respect to M. By the minimizing property of M_* the function

$$M\mapsto\mathcal{Q}[\rho_*,M]=\frac{2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}{\left(\int_{\mathbb{R}^N}\rho_*^q\,dx\right)^{(2-\alpha)/q}}\,\frac{A+M}{(B+M)^\alpha}$$

with

$$A:=\frac{I_\lambda[\rho_*]}{2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}\quad\text{and}\quad B:=\int_{\mathbb{R}^N}\rho_*(x)\,dx$$

attains its minimum on $[0,\infty)$ at M_*. From Lemma 16 we infer that

$$M_*=0\quad\text{if and only if}\quad\frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}\le\int_{\mathbb{R}^N}\rho_*(x)\,dx\,,$$

and that $M_*=\frac{\alpha I_\lambda[\rho_*]-2\left(\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx\right)\left(\int_{\mathbb{R}^N}\rho_*(y)\,dy\right)}{2\,(1-\alpha)\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$ if $\frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}>\int_{\mathbb{R}^N}\rho_*(x)\,dx$.

Step 2. We vary $\mathscr{Q}[\rho, M_*]$ with respect to ρ. Letting $x \to 0$ in the Euler–Lagrange equation (8), we find that

$$2\,\frac{\int_{\mathbb{R}^N} |y|^\lambda\,\rho_*(y)\,dy}{I_\lambda[\rho_*] + 2M_*\int_{\mathbb{R}^N} |y|^\lambda\,\rho_*(y)\,dy} - \alpha\,\frac{1}{\int_{\mathbb{R}^N}\rho_*(y)\,dy + M_*} = (2-\alpha)\,\frac{\rho_*(0)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy} \ge 0\,,$$

with the convention that the last inequality is an equality if and only if ρ_* is unbounded. Consistently, we shall write that $\rho_*(0) = +\infty$ in that case. We can rewrite our inequality as

$$M_* \ge \frac{\alpha I_\lambda[\rho_*] - 2\left(\int_{\mathbb{R}^N} |y|^\lambda\,\rho_*(y)\,dy\right)\left(\int_{\mathbb{R}^N}\rho_*\,dy\right)}{2\,(1-\alpha)\int_{\mathbb{R}^N} |y|^\lambda\,\rho_*(y)\,dy}$$

with equality if and only if ρ_* is unbounded. This completes the proof of Proposition 15. □

Next, we focus on matching ranges of the parameters (N, λ, q) with the cases (1), (2) and (3) in Proposition 15. For any $\lambda \ge 1$ we deduce from

$$|x-y|^\lambda \le \big(|x| + |y|\big)^\lambda \le 2^{\lambda-1}\left(|x|^\lambda + |y|^\lambda\right) \tag{11}$$

that

$$I_\lambda[\rho] < 2^\lambda \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx \int_{\mathbb{R}^N}\rho(x)\,dx\,.$$

For all $\alpha \le 2^{-\lambda+1}$, which can be translated into

$$q \ge \frac{2\,N\left(1-2^{-\lambda}\right)}{2\,N\left(1-2^{-\lambda}\right)+\lambda}\,,$$

that is,

$$\int_{\mathbb{R}^N}\rho_*\,dx \ge \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*(x)\,dx}\,,$$

so that Cases (1) and (2) of Proposition 15 apply and we infer that $M_* = 0$. Note that this bound for q is in the range $\big(N/(N+\lambda),\, 2N/(2N+\lambda)\big)$ for all $\lambda \ge 1$. See Fig. 1.

A better range for which $M_* = 0$ can be obtained for $N \ge 3$ using the fact that superlevel sets of a symmetric non-increasing function are balls. From the layer cake representation we deduce that

$$I_\lambda[\rho] \le 2\,A_{N,\lambda}\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx \int_{\mathbb{R}^N}\rho(x)\,dx\,,\quad A_{N,\lambda} := \sup_{0\le R,S<\infty} F(R,S)\,,$$

where

$$F(R,S) := \frac{\iint_{B_R\times B_S} |x-y|^\lambda\,dx\,dy}{|B_R|\int_{B_S} |x|^\lambda\,dx + |B_S|\int_{B_R} |y|^\lambda\,dy}\,.$$

For any $\lambda \ge 1$, we have $2\,A_{N,\lambda} \le 2^\lambda$ by (11), and also $A_{N,\lambda} \ge 1/2$ because by (6) $I_\lambda[\mathbb{1}_{B_1}] \ge |B_1|\int_{B_1} |y|^\lambda\,dy$. Note that $A_{N,2} = 1$ since, for $\lambda = 2$ and for any R, $S > 0$, $F(R,S) = 1$ by expanding the square in the numerator. The bound $A_{N,\lambda} \ge 1/2$ can be improved to $A_{N,\lambda} > 1$ for any $\lambda > 2$ as follows. We know that

$$A_{N,\lambda} \ge F(1,1) = \frac{N\,(N+\lambda)}{2}\iint_{0\le r,\,s\le 1} r^{N-1}\,s^{N-1}\left(\int_0^\pi \left(r^2+s^2-2\,r\,s\cos\varphi\right)^{\lambda/2}\frac{(\sin\varphi)^{N-2}}{W_N}\,d\varphi\right)dr\,ds$$

with the Wallis integral $W_N := \int_0^\pi (\sin\varphi)^{N-2}\,d\varphi$. For any $\lambda > 2$, we can apply Jensen's inequality twice and obtain

$$\int_0^\pi \left(r^2+s^2-2\,r\,s\cos\varphi\right)^{\lambda/2} \frac{(\sin\varphi)^{N-2}\,d\varphi}{W_N} \ge \left(\int_0^\pi \left(r^2+s^2-2\,r\,s\cos\varphi\right) \frac{(\sin\varphi)^{N-2}\,d\varphi}{W_N}\right)^{\lambda/2} = \left(r^2+s^2\right)^{\lambda/2}$$

and

$$\iint_{0\le r,\,s\le 1} r^{N-1}\,s^{N-1}\left(r^2+s^2\right)^{\lambda/2} dr\,ds \ge \frac{1}{N^2}\left(\iint_{0\le r,\,s\le 1} r^{N-1}\,s^{N-1}\left(r^2+s^2\right) N^2\,dr\,ds\right)^{\lambda/2} = \frac{1}{N^2}\left(\frac{2\,N}{N+2}\right)^{\lambda/2}.$$

Hence

$$A_{N,\lambda} \ge \frac{N+\lambda}{2\,N}\left(\frac{2\,N}{N+2}\right)^{\lambda/2} =: B_{N,\lambda}$$

where $\lambda \mapsto B_{N,\lambda}$ is monotone increasing, so that $A_{N,\lambda} \ge B_{N,\lambda} > B_{N,2} = 1$ for any $\lambda > 2$. In this range we can therefore define

$$\bar q(\lambda,N) := \frac{2\,N\left(1-\frac{1}{2A_{N,\lambda}}\right)}{2\,N\left(1-\frac{1}{2A_{N,\lambda}}\right)+\lambda}. \tag{12}$$

Based on a numerical computation, the curve $\lambda \mapsto \bar q(\lambda,N)$ is shown on Fig. 1. Note that in the case $\lambda = 2$, the curve $\bar q(\lambda,N)$ coincides with $N/(N+\lambda)$. The next result summarizes our considerations above.

Proposition 17. *Assume that $N \ge 3$. Then $\bar q$ defined by (12) is such that*

$$\bar q(\lambda,N) \le \frac{2\,N\left(1-2^{-\lambda}\right)}{2\,N\left(1-2^{-\lambda}\right)+\lambda} < \frac{2\,N}{2\,N+\lambda} \quad \text{for} \quad \lambda \ge 1 \quad \text{and} \quad \bar q(\lambda,N) > \frac{N}{N+\lambda} \quad \text{for} \quad \lambda > 2.$$

If (ρ_*, M_*) is a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ and if $\max\left\{\bar q(\lambda,N), \frac{N}{N+\lambda}\right\} < q < \frac{N-2}{N}$, then $M_* = 0$ and ρ_* is bounded.

Notice that $\frac{N}{N+\lambda} < \frac{N-2}{N}$ means $\lambda > \frac{2\,N}{N-2}$. We recall that the case $q \ge \frac{N-2}{N}$ has been covered in Proposition 11.

Proof. We recall that $q > \bar q(\lambda,N)$ defined by (12) means that

$$\int_{\mathbb{R}^N} \rho_*\,dx > \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*(x)\,dx},$$

so that Case (1) of Proposition 15 applies. The estimates on $\bar q$ follow from elementary computations. □

Next we consider the singularity of ρ_* at the origin in the unbounded case in more detail, in the cases which are not already covered by Propositions 8, 11 and 17.

Lemma 18. *Let $N\ge 3$, $\lambda>2N/(N-2)$ and $N/(N+\lambda)<q<\min\{1-N/2,\bar q(\lambda,N)\}$. Let (ρ_*,M_*) be a minimizer for $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ and assume that it is unbounded. Then there is a constant $C>0$ such that*

$$\rho_*(x)=C\,|x|^{-2/(1-q)}\big(1+o(1)\big)\quad as\quad x\to 0\,.$$

Proof. Since $\rho_*(x)\to\infty$ as $x\to 0$ we can rewrite the Euler–Lagrange equation (8) as

$$2\,\frac{\int_{\mathbb{R}^N}\big(|x-y|^\lambda-|y|^\lambda\big)\,\rho_*(y)\,dy+M_*|x|^\lambda}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\,\rho_*(y)\,dy}-(2-\alpha)\,\frac{\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}=0\,.$$

By Taylor expanding we have

$$\int_{\mathbb{R}^N}\Big(|x-y|^\lambda-|y|^\lambda\Big)\,\rho_*(y)\,dy+M_*\,|x|^\lambda=C_1\,|x|^2\big(1+o(1)\big)\quad\text{as}\quad x\to 0$$

with $C_1=\frac12\,\lambda\,(\lambda-1)\int_{\mathbb{R}^N}|y|^{\lambda-2}\rho_*(y)\,dy$, which is finite according to (6). This gives the claimed behavior for ρ_* at the origin. □

The proof of Lemma 18 relies only on (8). For this reason, we can also state the following result.

Proposition 19. *Let $N\ge 1$, $\lambda>0$ and $N/(N+\lambda)<q<1$. If $N\ge 3$ and $\lambda>2N/(N-2)$ we assume in addition that $q\ge\min\{1-N/2,\bar q(\lambda,N)\}$. If $(\rho_*,M_*)\in{\rm L}^1\cap{\rm L}^q\big(\mathbb{R}^N\big)\cap{\rm L}^1\big(\mathbb{R}^N,|x|^\lambda\,dx\big)\times\mathbb{R}^+$ solves (8), then $M_*=0$ and ρ_* is bounded.*

As a consequence, under the assumptions of Proposition 19, we recover that any minimizer (ρ_*,M_*) of $\mathscr{C}^{\rm rel}_{N,\lambda,q}$ is such that $M_*=0$ and ρ_* is bounded. Notice that the range $\bar q(\lambda,N)<1-2/N$ is covered in Proposition 17 but not here.

Proof. Assume by contradiction that ρ_* is unbounded. If $\lambda\ge 2$, the proof of Lemma 18 applies and we know that $\rho_*(x)\sim|x|^{-2/(1-q)}$ as $x\to 0$. For any $\lambda\in(0,1]$ we have that $|x-y|^\lambda\le|x|^\lambda+|y|^\lambda$. If $\lambda\in(1,2)$, using inequality (10) with the roles of x and y interchanged, we find that $\int_{\mathbb{R}^N}\big(|x-y|^\lambda-|y|^\lambda\big)\,\rho_*(y)\,dy\le C\,|x|^\lambda$ for some $C>0$. Hence, for some $c>0$,

$$\rho_*(x)\ge c\,|x|^{-\min\{\lambda,2\}/(1-q)}$$

for any $x\in\mathbb{R}^N$ with $|x|>0$ small enough. We claim that $\min\{\lambda,2\}/(1-q)\ge N$, which contradicts $\int_{\mathbb{R}^N}\rho_*\,dx<\infty$. □

By recalling the results of [19] in the conformally invariant case $q=2N/(2N+\lambda)$, and the results of Propositions 4, 8, 11, 19 and Lemma 9, we have completed the proof of Theorem 1.

5. Free Energy

In this section, we discuss the relation between the reverse HLS inequalities (1) and the free energy functional

$$\mathscr{F}[\rho]:=-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx+\frac{1}{2\lambda}\,I_\lambda[\rho]\,.$$

We also extend the free energy functional to the set of probability measures and prove a uniqueness result in this framework.

5.1. **Relaxation and extension of the free energy functional.** The kernel $|x-y|^\lambda$ is positive and continuous, so there is no ambiguity with the extension of I_λ to $\mathscr{P}(\mathbb{R}^N)$, which is simply given by

$$I_\lambda[\mu] = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda \, d\mu(x)\, d\mu(y)\,.$$

In this section we use the notion of weak convergence in the sense of probability theory: if μ_n and μ are probability measures on $\mathbb{R}^N$ then $\mu_n \rightharpoonup \mu$ means $\int_{\mathbb{R}^N} \varphi\, d\mu_n \to \int_{\mathbb{R}^N} \varphi\, d\mu$ for all bounded continuous functions φ on $\mathbb{R}^N$. We define the *extension* of $\mathscr{F}$ to $\mathscr{P}(\mathbb{R}^N)$ by

$$\mathscr{F}^\Gamma[\mu] := \inf_{\substack{(\rho_n)_{n\in\mathbb{N}}\subset C_c^\infty\cap\mathscr{P}(\mathbb{R}^N)\\ \text{s.t. } \rho_n\rightharpoonup\mu}} \liminf_{n\to\infty} \mathscr{F}[\rho_n]\,.$$

We also define a *relaxed free energy* by

$$\mathscr{F}^{\rm rel}[\rho,M] := -\frac{1}{1-q}\int_{\mathbb{R}^N} \rho(x)^q\, dx + \frac{1}{2\lambda}\, I_\lambda[\rho] + \frac{M}{\lambda}\int_{\mathbb{R}^N} |x|^\lambda\, \rho(x)\, dx\,.$$

The functional $\mathscr{F}^{\rm rel}$ can be characterized as the restriction of $\mathscr{F}^\Gamma$ to the subset of probability measures whose singular part is a multiple of a δ at the origin.

5.2. **Equivalence of the optimization problems and consequences.** According to Proposition 4, we know that $\mathscr{C}_{N,\lambda,q} = 0$ if $0 < q \le N/(N+\lambda)$, so that one can find a sequence of test functions $\rho_n \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ such that

$$\|\rho_n\|_{\mathrm{L}^1(\mathbb{R}^N)} = I_\lambda[\rho_n] = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_n(x)^q\, dx \ge n \in \mathbb{N}\,.$$

As a consequence, $\lim_{n\to\infty}\mathscr{F}[\rho_n] = -\infty$.

Next, let us consider the case $N/(N+\lambda) < q < 1$. Assume that $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ is such that $I_\lambda[\rho]$ is finite. For any $\ell > 0$ we define $\rho_\ell(x) := \ell^{-N}\rho(x/\ell)/\|\rho\|_{\mathrm{L}^1(\mathbb{R}^N)}$ and compute

$$\mathscr{F}[\rho_\ell] = -\,\ell^{(1-q)N}\,\mathsf{A} + \ell^\lambda\,\mathsf{B}$$

where $\mathsf{A} = \frac{1}{1-q}\int_{\mathbb{R}^N}\rho(x)^q\,dx/\|\rho\|^q_{\mathrm{L}^1(\mathbb{R}^N)}$ and $\mathsf{B} = \frac{1}{2\lambda}\, I_\lambda[\rho]/\|\rho\|^2_{\mathrm{L}^1(\mathbb{R}^N)}$. The function $\ell \mapsto \mathscr{F}[\rho_\ell]$ has a minimum which is achieved at $\ell = \ell_\star$ where

$$\ell_\star := \left(\frac{N(1-q)\,\mathsf{A}}{\lambda\,\mathsf{B}}\right)^{\frac{1}{\lambda-N(1-q)}}$$

and, with $\mathsf{Q}_{q,\lambda}$ as defined in (7), we obtain that

$$\mathscr{F}[\rho] \ge \mathscr{F}[\rho_{\ell_\star}] = -\,\kappa_\star\left(\mathsf{Q}_{q,\lambda}[\rho]\right)^{-\frac{N(1-q)}{\lambda-N(1-q)}} \quad \text{where} \quad \kappa_\star := \frac{\lambda-N(1-q)}{(1-q)\,\lambda}\,(2N)^{\frac{N(1-q)}{\lambda-N(1-q)}}\,.$$

As a consequence, we have the following result.

Proposition 20. *With the notations of Section 5.1, for any $q \in (0,1)$ and $\lambda > 0$, we have*

$$F_{N,\lambda,q} := \inf_\rho \mathscr{F}[\rho] = \inf_{\rho,M} \mathscr{F}^{\rm rel}[\rho,M] = \inf_\mu \mathscr{F}^\Gamma[\mu]$$

where the infima are taken on $\mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)$, $\big(\mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)\big)\times[0,\infty)$ *and* $\mathscr{P}(\mathbb{R}^N)$ *in case of, respectively,* $\mathscr{F}$, $\mathscr{F}^{\mathrm{rel}}$ *and* $\mathscr{F}^{\Gamma}$. *Moreover* $F_{N,\lambda,q}>-\infty$ *if and only if* $\mathscr{C}_{N,\lambda,q}>0$, *that is, if* $N/(N+\lambda)<q<1$ *and, in this case,*

$$F_{N,\lambda,q}=-\kappa_\star\,\mathscr{C}_{N,\lambda,q}^{-\frac{N(1-q)}{\lambda-N(1-q)}}=\mathscr{F}^{\mathrm{rel}}[\rho_*,M_*]=\mathscr{F}^{\Gamma}[\mu_*]$$

for some $\mu_*=M_*\,\delta+\rho_*$, $(\rho_*,M_*)\in\big(\mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)\big)\times[0,1)$ *such that* $\int_{\mathbb{R}^N}\rho_*(x)\,dx+M_*=1$. *Additionally, we have that*

$$I_\lambda[\rho_*]+2\,M_*\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx=2\,N\int_{\mathbb{R}^N}\rho_*(x)^q\,dx\,.$$

Since (ρ_*,M_*) *is also a minimizer for* $\mathscr{C}^{\mathrm{rel}}_{N,\lambda,q}$, *it satisfies all properties of Lemma 9 and Propositions 11, 15 and 17.*

Proof. This result is a simple consequence of the definitions of $\mathscr{F}^{\mathrm{rel}}$ and $\mathscr{F}^{\Gamma}$. The existence of the minimizer is a consequence of Propositions 8 and 10. If $\rho\in\mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)$ is a minimizer for $F_{N,\lambda,q}$, then $I_\lambda[\rho]=2\,N\int_{\mathbb{R}^N}\rho(x)^q\,dx$ because $\ell_\star=1$, and ρ is also an optimizer for $\mathscr{C}_{N,\lambda,q}$. Conversely, if $\rho\in\mathrm{L}^1_+\cap\mathrm{L}^q(\mathbb{R}^N)$ is an optimizer for $\mathscr{C}_{N,\lambda,q}$, then there is an $\ell>0$ such that $\ell^{-N}\rho(\cdot/\ell)/\|\rho\|_{\mathrm{L}^1(\mathbb{R}^N)}$ is an optimizer for $F_{N,\lambda,q}$. □

The discussion of whether $M_*=0$ or not in the statement of Proposition 20 is the same as in the discussion of the reverse Hardy–Littlewood–Sobolev inequality in Section 3. Except for the question of uniqueness, this completes the proof of Theorem 3.

5.3. **Properties of the free energy extended to probability measures.** From now on, unless it is explicitly specified, we shall denote by ρ the absolutely continuous part of the measure $\mu\in\mathscr{P}(\mathbb{R}^N)$. On $\mathscr{P}(\mathbb{R}^N)$, let us define

$$\mathscr{G}[\mu]:=\frac{1}{2\lambda}\,I_\lambda[\mu]-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho(x)^q\,dx \tag{13}$$

if $I_\lambda[\mu]<+\infty$ and extend it with the convention that $\mathscr{G}[\mu]=+\infty$ if $I_\lambda[\mu]=+\infty$. Notice that $\int_{\mathbb{R}^N}\rho(x)^q\,dx$ is finite by Lemma 5 and Eq. (6) whenever $I_\lambda[\rho]\le I_\lambda[\mu]$ is finite. Let us start with some technical estimates. The following is a variation of [14, Lemma 2.7].

Lemma 21. *Let* $N\ge1$ *and* $\lambda>0$, *then for any* $a\in\mathbb{R}^N$, $r>0$ *and* $\mu\in\mathscr{P}(\mathbb{R}^N)$ *we have*

$$I_\lambda[\mu]\ge2^{1-(\lambda-1)_+}\,\mu\big(B_r(a)\big)\left(\int_{\mathbb{R}^N}|y-a|^\lambda\,d\mu(y)-2^{(\lambda-1)_+}\,r^\lambda\right).$$

As a consequence, if $I_\lambda[\mu]<\infty$, *then* $\int_{\mathbb{R}^N}|y-a|^\lambda\,d\mu(y)$ *is finite for any* $a\in\mathbb{R}^N$ *and the infimum with respect to* a *is achieved.*

Proof. If $x\in B_r(a)$ and $y\in B_r(a)^c$, then

$$|x-y|^\lambda\ge\big(|y-a|-|x-a|\big)^\lambda\ge\big(|y-a|-r\big)^\lambda\ge2^{-(\lambda-1)_+}|y-a|^\lambda-r^\lambda\,.$$

We can therefore bound $I_\lambda[\mu]$ from below by

$$
\begin{aligned}
2 & \iint_{B_r(a)\times B_r(a)^c} |x-y|^\lambda \, d\mu(x)\, d\mu(y) \\
& \ge 2\,\mu\big(B_r(a)\big)\left(2^{-(\lambda-1)_+}\int_{B_r(a)^c} |y-a|^\lambda\, d\mu(y) - r^\lambda\,\mu\big(B_r(a)^c\big)\right) \\
& \quad = 2^{1-(\lambda-1)_+}\,\mu\big(B_r(a)\big)\left(\int_{\mathbb{R}^N} |y-a|^\lambda\, d\mu(y) - \int_{B_r(a)} |y-a|^\lambda\, d\mu(y) - 2^{(\lambda-1)_+}\, r^\lambda\,\mu\big(B_r(a)^c\big)\right) \\
& \quad \ge 2^{1-(\lambda-1)_+}\,\mu\big(B_r(a)\big)\left(\int_{\mathbb{R}^N} |y-a|^\lambda\, d\mu(y) - r^\lambda\,\mu\big(B_r(a)\big) - 2^{(\lambda-1)_+}\, r^\lambda\,\mu\big(B_r(a)^c\big)\right) \\
& \quad \ge 2^{1-(\lambda-1)_+}\,\mu_n\big(B_r(a)\big)\left(\int_{\mathbb{R}^N} |y-a|^\lambda\, d\mu_n(y) - 2^{(\lambda-1)_+}\, r^\lambda\right).
\end{aligned}
$$

This proves the claimed inequality.

Let $R>0$ be such that $\mu\big(B_R(0)\big)\ge 1/2$ and consider $a\in B_R(0)^c$, so that $|y-a|>|a|-R$ for any $y\in B_R(0)$. From the estimate

$$\int_{\mathbb{R}^N} |y-a|^\lambda\, d\mu(y) \ge \int_{B_R(0)} |y-a|^\lambda\, d\mu(y) \ge \frac{1}{2}\big(|a|-R\big)^\lambda,$$

we deduce that in $\inf_{a\in\mathbb{R}^N}\int_{\mathbb{R}^N}|y-a|^\lambda\, d\mu(y)$, a can be restricted to a compact region of $\mathbb{R}^N$. Since the map $a\mapsto \int_{\mathbb{R}^N}|y-a|^\lambda\, d\mu(y)$ is lower semi-continuous, the infimum is achieved. □

Corollary 22. *Let $\lambda>0$ and $N/(N+\lambda)<q<1$. Then there is a constant $C>0$ such that*

$$\mathscr{G}[\mu] \ge \frac{I_\lambda[\mu]}{4\lambda} - C \ge \frac{1}{4\lambda}\inf_{a\in\mathbb{R}^N}\int_{\mathbb{R}^N} |x-a|^\lambda\, d\mu(x) - C \quad \forall\,\mu\in\mathscr{P}(\mathbb{R}^N).$$

Proof. Let $\mu\in\mathscr{P}(\mathbb{R}^N)$ and let ρ be its absolutely continuous part with respect to Lebesgue's measure. By Theorem 1, we know that

$$\int_{\mathbb{R}^N} \rho(x)^q\, dx \le \left(\frac{I_\lambda[\rho]}{\mathscr{C}_{N,\lambda,q}}\right)^{\frac{N(1-q)}{\lambda}}$$

because $\int_{\mathbb{R}^N}\rho\, dx\le\mu(\mathbb{R}^N)=1$. Hence we obtain that

$$\mathscr{G}[\mu] \ge \frac{I_\lambda[\mu]}{4\lambda} - C \quad \text{with} \quad C = \min\left\{\frac{X}{4\lambda} - \left(\frac{X}{\mathscr{C}_{N,\lambda,q}}\right)^{\frac{N(1-q)}{\lambda}} \;:\; X>0\right\}.$$

As μ is a probability measure, the proof is completed using the inequality

$$\inf_{a\in\mathbb{R}^N}\int_{\mathbb{R}^N} |x-a|^\lambda\, d\mu(x) \le \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-a|^\lambda\, d\mu(x)\, d\mu(a) = I_\lambda[\mu].$$

□

Lemma 23. *If $\lambda>0$ and $N/(N+\lambda)<q<1$, then $\mathscr{G}$ is lower semi-continuous.*

Proof. Let $(\mu_n)\subset\mathscr{P}(\mathbb{R}^N)$ with $\mu_n\rightharpoonup\mu$. We denote by ρ_n and ρ the absolutely continuous part of μ_n and μ, respectively. We have to prove that $\liminf_{n\to\infty}\mathscr{G}[\mu_n]\ge\mathscr{G}[\mu]$. Either

$\liminf_{n\to\infty}\mathscr{G}[\mu_n]=+\infty$, or it is finite and then, up to the extraction of a subsequence, we know from Corollary 22 that $\mathscr{K}:=\sup_{n\in\mathbb{N}}I_\lambda[\mu_n]$ is finite. According to [38, Proposition 7.2], we also know that

$$\liminf_{n\to\infty} I_\lambda[\mu_n]\ge I_\lambda[\mu]\,.$$

According to [38, Theorem 7.7] or [6, Theorem 4], for any $r>0$ we have

$$\liminf_{n\to\infty}\left(-\int_{\overline{B_r}}\rho_n(x)^q\,dx\right)\ge-\int_{\overline{B_r}}\rho(x)^q\,dx\,.$$

Notice that the absolutely continuous part of the limit of $\mu_n\llcorner\overline{B_r}$ coincides with the absolutely continuous part of $\mu\llcorner\overline{B_r}$ as the difference is supported on ∂B_r.

We choose $r_0>0$ to be a number such that $\mu(B_{r_0})\ge 1/2$ and find $n_0\in\mathbb{N}$ such that for any $n\ge n_0$ we have $\mu_n(B_{r_0})\ge 1/4$. By applying Lemma 21, we obtain that

$$\int_{\mathbb{R}^N}|x|^\lambda\,d\mu_n(x)\le 2^{(\lambda-1)_+}\left(r_0^\lambda+2\,I_\lambda[\mu_n]\right)\le 2^{(\lambda-1)_+}\left(r_0^\lambda+2\,\mathscr{K}\right)$$

for any $n\ge n_0$. We apply Lemma 5 to $\rho=\rho_n\,\mathbb{1}_{B_r^c}$

$$\int_{B_r^c}\rho_n(x)^q\,dx\le c_{N,\lambda,q}^{-q}\left(\int_{B_r^c}\rho_n\,dx\right)^{q-\frac{N(1-q)}{\lambda}}\left(\int_{B_r^c}|x|^\lambda\,\rho_n\,dx\right)^{\frac{N(1-q)}{\lambda}}$$

and conclude that

$$\liminf_{n\to\infty}\left(-\int_{B_r^c}\rho_n(x)^q\,dx\right)\ge-c_{N,\lambda,q}^{-q}\left(\mu\big(B_r^c\big)\right)^{q-\frac{N(1-q)}{\lambda}}\left(2^{(\lambda-1)_+}\left(r_0^\lambda+2\,\mathscr{K}\right)\right)^{\frac{N(1-q)}{\lambda}}\,.$$

The right hand side vanishes as $r\to\infty$, which proves the claimed lower semi-continuity. □

After these preliminaries, we can now prove that $\mathscr{G}$, defined in (13), is the lower-semi-continuous envelope of $\mathscr{F}$. The precise statement goes as follows.

Proposition 24. *Let $0<q<1$ and $\lambda>0$. Let $\mu\in\mathscr{P}(\mathbb{R}^N)$*

(1) *If $q\le N/(N+\lambda)$, then $\mathscr{F}^\Gamma[\mu]=-\infty$.*
(2) *If $q>N/(N+\lambda)$, then $\mathscr{F}^\Gamma[\mu]=\mathscr{G}[\mu]$.*

Proof. Assume that $q\le N/(N+\lambda)$. Using the function $v(x)=|x|^{-N-\lambda}\left(\log|x|\right)^{-1/q}$, let us construct an approximation of any measure in $\mu\in\mathscr{P}(\mathbb{R}^N)$ given by a sequence $(\rho_n)_{n\in\mathbb{N}}$ of functions in $C_c^\infty\cap\mathscr{P}(\mathbb{R}^N)$ such that $\lim_{n\to\infty}\mathscr{F}[\rho_n]=-\infty$.

Let $\eta\in C_c^\infty(B_1)$ be a positive mollifier with unit mass and $\zeta\in C_c^\infty(B_2)$ be a cutoff function such that $\mathbb{1}_{B_1}\le\zeta\le 1$. Given any natural numbers i, j and k, we define $\eta_i(y):=i^N\eta(i\,y)$, $\zeta_j(y):=\zeta(y/j)$ and

$$f_{i,j,k}:=\left(1-\tfrac{1}{k}\right)\left(\mu*\eta_i\right)\zeta_i+\tfrac{1}{k}\,C_{i,j,k}\,(1-\zeta_i)\,\zeta_j\,v$$

where $C_{i,j,k}$ is a positive constant that has been picked so that $f_{i,j,k}\in\mathscr{P}(\mathbb{R}^N)$. We choose $i=n$, $j=e^n$ and $k=k(n)$ such that

$$\lim_{n\to\infty}k(n)=+\infty\quad\text{and}\quad\lim_{n\to\infty}k(n)^{-N\,q}\log(n/\log n)=+\infty\,.$$

By construction, $\rho_n := f_{n,j(n),k(n)} \rightharpoonup \mu$ as $n \to \infty$ and $\lim_{n\to\infty} \mathscr{F}[\rho_n] = -\infty$, so $\mathscr{F}^\Gamma[\mu] = -\infty$.

Assume that $q > N/(N+\lambda)$ and consider a sequence of functions in $C_c^\infty \cap \mathscr{P}(\mathbb{R}^N)$ such that $\rho_n \rightharpoonup \mu$ and $\lim_{n\to\infty} \mathscr{F}[\rho_n] = \mathscr{F}^\Gamma[\mu]$. If $I_\lambda[\mu] = \infty$, by the lower-semicontinuity of I_λ (see for instance [38, Proposition 7.2]), we know that $\lim_{n\to\infty} I_\lambda[\rho_n] = \infty$ and deduce from Corollary 22 that $\frac{1}{4\lambda} I_\lambda[\rho_n] - C \le \mathscr{F}[\rho_n]$ diverges, so that $\mathscr{F}^\Gamma[\mu] = \infty = \mathscr{G}[\mu]$.

Next, we assume that $I_\lambda[\mu] < \infty$. According to Lemma 23, we deduce from the lower semi-continuity of $\mathscr{G}$ that

$$\mathscr{F}^\Gamma[\mu] = \lim_{n\to\infty} \mathscr{F}[\rho_n] = \lim_{n\to\infty} \mathscr{G}[\rho_n] \ge \mathscr{G}[\mu]\,.$$

It remains to show the inequality $\mathscr{F}^\Gamma[\mu] \le \mathscr{G}[\mu]$. Let $\mu_R := \mu(B_R)^{-1}\, \mu \llcorner B_R$. We have that $\mu_R \rightharpoonup \mu$ as $R \to \infty$ and, by monotone convergence,

$$\lim_{R\to\infty} \mathscr{G}[\mu_R] = \mathscr{G}[\mu]\,.$$

Let $\eta_\varepsilon(x) := \varepsilon^{-N} \eta(x/\varepsilon)$ for a sufficiently regular, compactly supported, nonnegative function η such that $\int_{\mathbb{R}^N} \eta\, dx = 1$. Then $\mu_R * \eta_\varepsilon \in C_c^\infty \cap \mathscr{P}(\mathbb{R}^N)$ and $\mu_R * \eta_\varepsilon \rightharpoonup \mu_R$ as $\varepsilon \to 0$. Here we are using implicitly the metrizability of weak convergence. Since $\mu_R * \eta_\varepsilon \to \rho_R$ almost everywhere, Fatou's lemma implies that

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} (\mu_R * \eta_\varepsilon)^q\, dx \ge \int_{\mathbb{R}^N} \rho_R^q\, dx\,.$$

Moreover, since μ_R has compact support, the support of $\mu_R * \eta_\varepsilon$ is contained in a bounded set independent of ε and therefore the interaction term is, in fact, continuous under weak convergence (see, e.g., [38, Proposition 7.2]), that is,

$$\liminf_{\varepsilon\to 0} I_\lambda[\mu_R * \eta_\varepsilon] = I_\lambda[\mu_R]\,.$$

Thus, we have shown that

$$\liminf_{\varepsilon\to 0} \mathscr{F}[\mu_R * \eta_\varepsilon] \le \mathscr{G}[\mu_R]\,.$$

Hence for any $R = n \in \mathbb{N}$, we can find an $\varepsilon_n > 0$, small enough, such that $\mu_n * \eta_{\varepsilon_n} \rightharpoonup \mu$ and finally obtain that

$$\mathscr{F}^\Gamma[\mu] \le \lim_{n\to\infty} \mathscr{F}[\mu_n * \eta_{\varepsilon_n}] \le \mathscr{G}[\mu]\,.$$

□

In Section 3, using symmetric decreasing rearrangements, we proved that there is a minimizing sequence which converges to a minimizer. Here we have a stronger property.

Proposition 25. *Let $N/(N+\lambda) < q < 1$. Then* any *minimizing sequence for $\mathscr{F}^\Gamma$ is relatively compact, up to translations, with respect to weak convergence. In particular, there is a minimizer for $\mathscr{F}^\Gamma$.*

Proof. Let $(\mu_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $\mathscr{F}^\Gamma$ in $\mathscr{P}(\mathbb{R}^N)$. After an n-dependent translation we may assume that for any $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} |x|^\lambda \, d\mu_n(x) = \inf_{a\in\mathbb{R}^N} \int_{\mathbb{R}^N} |x-a|^\lambda \, d\mu_n(x)$$

according to Lemma 21. Corollary 22 applies

$$\int_{\mathbb{R}^N} |x|^\lambda \, d\mu_n(x) \le 4\,\lambda \left(\sup_n \mathscr{F}^\Gamma[\mu_n] + C \right),$$

which implies that $(\mu_n)_{n\in\mathbb{N}}$ is tight. By Prokhorov's theorem and after passing to a subsequence if necessary, $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to some $\mu_* \in \mathscr{P}(\mathbb{R}^N)$. By the lower-semicontinuity property of Lemma 23, we obtain that

$$\inf_{\mu\in\mathscr{P}(\mathbb{R}^N)} \mathscr{F}^\Gamma[\mu] = \lim_{n\to\infty} \mathscr{F}^\Gamma[\mu_n] \ge \inf_{\mu\in\mathscr{P}(\mathbb{R}^N)} \mathscr{F}^\Gamma[\mu],$$

which concludes the proof. □

Remark 26. By symmetrization, Lemma 9 and Proposition 20, we learn that, up to translations, any minimizer μ of $\mathscr{F}^\Gamma$ is of the form $\mu = \rho + M\delta$, with $M \in [0,1)$ and $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$. Moreover, ρ is radially symmetric non-increasing and strictly positive. The minimizers of $\mathscr{F}^\Gamma$ satisfy the Euler-Lagrange conditions given by (8). This can be also shown by taking variations directly on $\mathscr{F}^\Gamma$ as in [13].

5.4. **Uniqueness.**

Theorem 27. *Let $N/(N+\lambda) < q < 1$ and assume either that $1-1/N \le q < 1$ and $\lambda \ge 1$, or $2 \le \lambda \le 4$. Then the minimizer of $\mathscr{F}^\Gamma$ on $\mathscr{P}(\mathbb{R}^N)$ is unique up to translation.*

Notice that Theorem 2 is a special case of Theorem 27. Theorem 3 is a direct consequence of Proposition 20 and Theorem 27.

Proof. The proof relies on the notion of displacement convexity by mass transport in the range $1-1/N \le q < 1$, $\lambda \ge 1$ and on a recent convexity result, [33, Theorem 2.4], of O. Lopes in the case $2 \le \lambda \le 4$. Since $N/(N+4) < 1-1/N$ for $N \ge 2$, there is a range of parameters q and λ such that $N/(N+\lambda) < q < 1-1/N$ and $2 \le \lambda \le 4$, which is not covered by mass transport. Ranges of the parameters are shown in Fig. 2.

• *Displacement convexity and mass transport.* We assume that $1-1/N \le q < 1$ and $\lambda \ge 1$. Under these hypothesis, [35, Theorem 2.2] and [1, Theorem 9.4.12, p. 224] imply that the functional $\mathscr{F}^\Gamma$ restricted to the set of absolutely continuous measures is strictly geodesically convex with respect to the Wasserstein-2 metric. As the minimizers might not be absolutely continuous, we cannot apply these results directly but we can adapt their proofs. We shall say that the mesurable map $T : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ pushes forward the measure μ onto ν, or that T *transports* μ *onto* ν, if and only if

$$\int_{\mathbb{R}^N} \varphi\big(T(x)\big)\, d\mu(x) = \int_{\mathbb{R}^N} \varphi(x)\, d\nu(x)$$

for all bounded and continuous functions φ on $\mathbb{R}^N$. This will be written as $\nu = T\#\mu$.

Let us argue by contradiction and assume that there are two distinct radial minimizers $\mu_0 = \rho_0 + M_0\,\delta$ and $\mu_1 = \rho_1 + M_1\,\delta$, with $M_1 \geq M_0$. We define

$$F(s) = \mu_0(B_s) \qquad \text{and} \qquad G(s) = \mu_1(B_s)$$

on $(0,\infty)$. Both functions are monotone increasing according to Lemma 9 and Proposition 20, so that they admit well defined inverses $F^{-1} : [0,1) \to [0,\infty)$ and $G^{-1} : [0,1) \to [0,\infty)$. Let $T : \mathbb{R}^N \to \mathbb{R}^N$ with

$$T(x) := G^{-1}\big(F(|x|)\big)\,\frac{x}{|x|}$$

be the optimal transport map pushing μ_0 forward onto μ_1 according, *e.g.*, [43], which is noted as $T\#\mu_0 = \mu_1$. With $s_* := F^{-1}(M_1 - M_0)$, we note that $G^{-1}\big(F(s)\big) = 0$ for any $s \leq s_*$ and $s \mapsto G^{-1}\big(F(s)\big)$ is strictly increasing on $(s_*, 1)$. This implies that $T : B^c_{s_*} \to \mathbb{R}^N \setminus \{0\}$ is invertible and ∇T is positive semi-definite. We consider the midpoint of the nonlinear interpolant which is given by

$$\mu_{1/2} = \tfrac{1}{2}\,(I + T)\#\mu_0$$

where $I(x) = x$ denotes the identity map. For any $\lambda \geq 1$, we have that

$$I_\lambda[\mu_{1/2}] = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \left|\tfrac{1}{2}\left(x + T(x)\right) - \tfrac{1}{2}\left(y + T(y)\right)\right|^\lambda d\mu_0(x)\,d\mu_0(y)$$

$$< \iint_{\mathbb{R}^N\times\mathbb{R}^N} \left(\tfrac{1}{2}\,|x-y|^\lambda + \tfrac{1}{2}\left|T(x) - T(y)\right|^\lambda\right) d\mu_0(x)\,d\mu_0(y) = \tfrac{1}{2}\left(I_\lambda[\mu_0] + I_\lambda[\mu_1]\right).$$

Let Id be the identity matrix. By the change of variable formula as in [35], we obtain that

$$-\frac{1}{1-q}\int_{\mathbb{R}^N} \rho_{1/2}(x)^q\,dx = -\frac{1}{1-q}\int_{\mathbb{R}^N}\left(\frac{\rho_0(x)}{\det\left(\frac{1}{2}\big(\mathrm{Id} + \nabla T(x)\big)\right)}\right)^q \det\left(\tfrac{1}{2}\big(\mathrm{Id} + \nabla T(x)\big)\right)dx\,.$$

Using $q \geq 1 - 1/N$, the fact that ∇T is positive semi-definite and the concavity of $s \mapsto \det\big((1-s)\,\mathrm{Id} + s\,\nabla T\big)^{1-q}$, we obtain that

$$-\det\left(\tfrac{1}{2}\left(\mathrm{Id} + \nabla T\right)\right)^{1-q} \leq -\tfrac{1}{2}\,\det(\mathrm{Id}) - \tfrac{1}{2}\,\det\left(\nabla T\right)^{1-q}.$$

Hence

$$-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho_{1/2}^q\,dx \leq \frac{1}{2}\left(-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho_0^q\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N}\left(\frac{\rho_0}{\det\left(\nabla T\right)}\right)^q \det\left(\nabla T\right)dx\right).$$

Since $T : B^c_{s_*} \to \mathbb{R}^N$ is invertible and $T\#\rho_0 \llcorner B^c_{s_*} = \rho_1$, we can undo the change of variables:

$$-\frac{1}{1-q}\int_{\mathbb{R}^N}\left(\frac{\rho_0}{\det\left(\nabla T\right)}\right)^q \det\left(\nabla T\right)dx = -\frac{1}{1-q}\int_{B^c_{s_*}}\left(\frac{\rho_0}{\det\left(\nabla T\right)}\right)^q \det\left(\nabla T\right)dx$$
$$= -\frac{1}{1-q}\int_{\mathbb{R}^N}\rho_1^q\,dx\,.$$

Altogether, we have shown that $\mathscr{F}^\Gamma[\mu_{1/2}] < \frac{1}{2}\left(\mathscr{F}^\Gamma[\mu_0] + \mathscr{F}^\Gamma[\mu_1]\right)$, which contradicts the assumption that μ_0 and μ_1 are two distinct minimizers. Notice that displacement convexity is shown only in the set of radially decreasing probability measures of the form $\mu = \rho + M\,\delta$.

• *Linear convexity of the functional* $\mathscr{F}^\Gamma$. We assume that $2 \le \lambda \le 4$. Let $\mu_0 = \rho_0 + M_0\,\delta$ and $\mu_1 = \rho_1 + M_1\,\delta$ be two radial minimizers and consider the function

$$[0,1] \ni t \mapsto \mathscr{F}^\Gamma\big[(1-t)\,\mu_0 + t\,\mu_1\big] =: f(t)\,.$$

We shall prove that f is strictly convex if $\mu_0 \not\equiv \mu_1$. In this case, since μ_0 is a minimizer, we have $f(t) \ge f(0)$ for all $0 \le t \le 1$ and therefore $f'(0) \ge 0$. Together with the strict convexity this implies $f(1) > f(0)$, which contradicts the fact that μ_1 is a minimizer. This is why we compute

$$f''(t) = \frac{1}{\lambda}\,I_\lambda[\mu_0 - \mu_1] + q\int_{\mathbb{R}^N}\big((1-t)\,\rho_0 + t\,\rho_1\big)^{q-2}(\rho_1 - \rho_0)^2\,dx\,.$$

According to [33, Theorem 2.4], we have that $I_\lambda[h] \ge 0$ under the assumption $2 \le \lambda \le 4$, for all h such that that $\int_{\mathbb{R}^N}\big(1 + |x|^\lambda\big)\,|h|\,dx < \infty$ with $\int_{\mathbb{R}^N} h\,dx = 0$ and $\int_{\mathbb{R}^N} x\,h\,dx = 0$. Applied with $h = \rho_0 - \rho_1$, this proves the strict convexity if $M_0 = M_1 = 0$. We have now to adapt the result of O. Lopes to the measure valued setting, *i.e.*, $(M_0, M_1) \ne (0,0)$.

Some care is needed with the second term as the power $q-2$ is negative, but since we know that the optimizers are positive a.e. in $\mathbb{R}^N$ the last term in the expression of $f''(t)$ is strictly positive if $\rho_1 \not\equiv \rho_0$, or eventually $+\infty$.

We have to show that $I_\lambda[\mu_0 - \mu_1] > 0$. If $\lambda = 2$ or $\lambda = 4$, the convexity follows by expanding $|x-y|^\lambda$, so we can restrict our study to $2 < \lambda < 4$. By Plancherel's identity we obtain that

$$I_\lambda[\mu_0 - \mu_1] = (2\pi)^{\frac{N}{2}}\,2^{\lambda + \frac{N}{2}}\,\frac{\Gamma\left(\frac{\lambda+N}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\left\langle H_{-(N+\lambda)},\, |\hat\mu_0 - \hat\mu_1|^2\right\rangle$$

where $H_{-(N+\lambda)} \in \mathscr{S}'(\mathbb{R}^N)$ is a radial tempered distribution of homogeneity $-(N+\lambda)$. In particular, for any $\varphi \in \mathscr{S}(\mathbb{R}^N)$ we have

$$\langle H_{-(N+\lambda)}, \varphi\rangle = \int_{\mathbb{R}^N}\frac{1}{|\xi|^{N+\lambda}}\left(\varphi(\xi) - \sum_{|\alpha| \le [\lambda]}\frac{\xi^\alpha}{\alpha!}\,\partial^\alpha\varphi(0)\right)d\xi$$

where $[\lambda]$ denotes the integer part of λ: see [33, 22]. These identities extend by continuity to all bounded functions $\varphi \in C^2(\mathbb{R}^N)$ if $\lambda < 3$ and $C^3(\mathbb{R}^N)$ if $\lambda < 4$.

By Lemma 21, we know that $\int_{\mathbb{R}^N}|x|^\lambda\,d\mu_i(x)$ is finite for $i = 0,\,1$, so that $\hat\mu_i$ is of class C^2 if $\lambda < 3$ and of class C^3 if $\lambda < 4$. Since $\mu_i(\mathbb{R}^N) = 1$ and $\int_{\mathbb{R}^N} x\,d\mu_i = 0$, we infer $\hat\mu_i(0) = 1$ and $\nabla\hat\mu_i(0) = 0$. This implies that $\partial^\alpha|\hat\mu_0 - \hat\mu_1|^2(0) = 0$ for $|\alpha| \le 2$ if $\lambda < 3$ and for $|\alpha| \le 3$ if $\lambda < 4$. We conclude that

$$I_\lambda[\mu_0 - \mu_1] \ge 0$$

with strict inequality unless $\mu_0 = \mu_1$. Thus, we have shown that $f''(t) > 0$ as claimed. □

Appendix A. Toy Model for Concentration

Eq. (3) is a *mean field-type* equation, in which the *drift term* is an average of a spring force $\nabla W_\lambda(x)$ for any $\lambda > 0$. The case $\lambda = 2$ corresponds to linear springs obeying Hooke's law, while large λ reflect a force which is small at small distances, but becomes very large for large values of $|x|$. In this sense, it is a *strongly confining* force term. By expanding the diffusion term as $\Delta\rho^q = q\,\rho^{q-1}\left(\Delta\rho + (q-1)\,\rho^{-1}\,|\nabla\rho|^2\right)$ and considering ρ^{q-1} as a diffusion coefficient, it is obvious that this *fast diffusion* coefficient is large for small values of ρ and has to be balanced by a very large drift term to avoid a *runaway* phenomenon in which no stationary solutions may exist in $\mathrm{L}^1(\mathbb{R}^N)$. In the case of a drift term with linear growth as $|x| \to +\infty$, it is well known that the threshold is given by the exponent $q = 1-2/N$ and it is also known according to, *e.g.*, [26] for the pure fast diffusion case (no drift) that $q = 1-2/N$ is the threshold for the global existence of nonnegative solutions in $\mathrm{L}^1(\mathbb{R}^N)$, with constant mass.

In the regime $q < 1-2/N$, a new phenomenon appears which is not present in linear diffusions. As emphasized in [42], the diffusion coefficient ρ^{q-1} becomes small for large values of ρ and does not prevent the appearance of singularities. Let us observe that W_λ is a convolution kernel which averages the solution and can be expected to give rise to a smooth effective potential term $V_\lambda = W_\lambda * \rho$ at $x = 0$ if we consider a radial function ρ. This is why we expect that $V_\lambda(x) = V_\lambda(0) + O\left(|x|^2\right)$ for $|x|$ small, at least for $\lambda \ge 1$. With these considerations at hand, let us illustrate some consequences with a simpler model involving only a given, external potential V. Assume that u solves the *fast diffusion with external drift* given by

$$\partial_t u = \Delta u^q + \nabla\cdot\left(u\nabla V\right).$$

To fix ideas, we shall take $V(x) = \frac{1}{2}\,|x|^2 + \frac{1}{\lambda}\,|x|^\lambda$, which is expected to capture the behavior of the potential $W_\lambda * \rho$ at $x = 0$ and as $|x| \to +\infty$ when $\lambda \ge 2$. Such an equation admits a free energy functional

$$u \mapsto \int_{\mathbb{R}^N} V\,u\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N} u^q\,dx\,,$$

whose bounded minimizers under a mass constraint on $\int_{\mathbb{R}^N} u\,dx$ are, if they exist, given by

$$u_h(x) = \left(h + \frac{1-q}{q}\,V(x)\right)^{-\frac{1}{1-q}} \quad \forall\,x\in\mathbb{R}^N.$$

A linear spring would simply correspond to a fast diffusion Fokker–Planck equation when $V(x) = |x|^2$, *i.e.*, $\lambda = 2$. One can for instance refer to [29] for a general account on this topic. In that case, it is straightforward to observe that the so-called *Barenblatt profile* u_h has finite mass if and only if $q > 1-2/N$. For a general parameter $\lambda \ge 2$, the corresponding integrability condition for u_h is $q > 1-\lambda/N$. But $q = 1-2/N$ is also a threshold value for the regularity. Let us assume that $\lambda > 2$ and $1-\lambda/N < q < 1-2/N$, and consider the stationary solution u_h, which depends on the parameter h. The mass of u_h can be computed for any $h \ge 0$ as

$$m_\lambda(h) := \int_{\mathbb{R}^N}\left(h + \frac{1-q}{q}\,V(x)\right)^{-\frac{1}{1-q}} dx \le m_\lambda(0) = \int_{\mathbb{R}^N}\left(\tfrac{1}{2}\,|x|^2 + \frac{1-q}{\lambda\,q}\,|x|^\lambda\right)^{-\frac{1}{1-q}} dx\,.$$

Now, if one tries to minimize the free energy under the mass contraint $\int_{\mathbb{R}^N} u\,dx = m$, it is left to the reader to check that the limit of a minimizing sequence is simply the measure $\left(m - m_\lambda(0)\right)\delta + u_0$ for any $m > m_\lambda(0)$. For the model described by Eq. (3), the situation is by far more complicated because the mean field potential $V_\lambda = W_\lambda * \rho$ depends on the regular part ρ and we have no simple estimate on a critical mass as in the case of an external potential V.

APPENDIX B. OTHER RELATED INEQUALITIES

It is natural to ask why q has been taken in the range $(0,1)$ and whether an inequality similar to (1) holds for $q \ge 1$. The free energy approach of Section 5 provides simple guidelines to distinguish a fast diffusion regime with $q<1$ from a *porous medium* regime with $q>1$ and a linear diffusion regime with $q=1$ exactly as in the case of the Gagliardo-Nirenberg inequalities associated with the classical fast diffusion or porous medium equations and studied in [17].

Theorem 28. *Let $N \ge 1$, $\lambda > 0$ and $q \in (1,+\infty)$. Then the inequality*

$$I_\lambda[\rho]\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(\alpha-2)/q} \ge \mathscr{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^{\alpha} \tag{14}$$

holds for any nonnegative function $\rho \in \mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)$, for some positive constant $\mathscr{C}_{N,\lambda,q}$. Moreover, a radial positive, non-increasing, bounded function $\rho \in \mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)$ with compact support achieves the equality case.

Compared to (1) with $2N/(2N+\lambda) < q < 1$, notice that, as in the case of Gagliardo-Nirenberg inequalities, the position of $\int_{\mathbb{R}^N}\rho\,dx$ and $\int_{\mathbb{R}^N}\rho^q\,dx$ have been interchanged in the inequality. As in the case $q<1$, the exponent α is given by

$$\alpha = \frac{q(2N+\lambda)-2N}{N(q-1)}$$

and takes values in $(2+\lambda/N,+\infty)$ in the range $q>1$.

Proof. For any nonnegative function $\rho \in \mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)$ we have

$$I_\lambda[\rho] \ge \mathscr{C}_*\left(\int_{\mathbb{R}^N}\rho(x)^{\frac{2N}{2N+\lambda}}\,dx\right)^{2+\frac{\lambda}{N}}$$

with $\mathscr{C}_* = \mathscr{C}_{N,\lambda,2N/(2N+\lambda)}$ by Theorem 1. By Hölder's inequality,

$$\left(\int_{\mathbb{R}^N}\rho(x)^{\frac{2N}{2N+\lambda}}\,dx\right)^{2+\frac{\lambda}{N}}\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(\alpha-2)/q} \ge \left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^{\alpha},$$

and so

$$I_\lambda[\rho]\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{(\alpha-2)/q} \ge \mathscr{C}_*\left(\int_{\mathbb{R}^N}\rho(x)\,dx\right)^{\alpha}. \tag{15}$$

This proves (14) for some constant $\mathscr{C}_{N,\lambda,q} \ge \mathscr{C}_*$. The existence of a radial non-increasing minimizer is an easy consequence of rearrangement inequalities, Helly's selection theorem and Lesgue's theorem of dominated convergence as in the proof of Proposition 8. We read from the Euler–Lagrange equation

$$2\,\frac{\int_{\mathbb{R}^N}|x-y|^\lambda\rho(y)\,dy}{I_\lambda[\rho]} + \frac{(\alpha-2)\,\rho(x)^{q-1}}{\int_{\mathbb{R}^N}\rho(y)^q\,dy} - \frac{\alpha}{\int_{\mathbb{R}^N}\rho(y)\,dy} = 0,$$

that ρ has compact support. Indeed, because of the constraint $\rho \geq 0$, the equation is restricted to the interior of the support of ρ, which is either a ball or $\mathbb{R}^N$. Then we can use the Euler-Lagrange equation to write

$$\rho(x) = \left(C_1 - C_2 \int_{\mathbb{R}^N} |x-y|^\lambda \, \rho(y)\, dy\right)_+^{1/(q-1)}$$

for some positive constants C_1 and C_2, and since $\int_{\mathbb{R}^N} |x-y|^\lambda \rho(y)\, dy \sim |x|^\lambda \int_{\mathbb{R}^N} \rho(y)\, dy$ as $|x| \to +\infty$, the support of ρ has to be a finite ball by integrability of ρ. □

As in Proposition 20, we notice that the free energy functional also defined in the case $q > 1$ by $\mathscr{F}[\rho] := \frac{1}{q-1} \int_{\mathbb{R}^N} \rho^q \, dx + \frac{1}{2\lambda} I_\lambda[\rho]$ is bounded from below by an optimal constant which can be computed in terms of the optimal constant $\mathscr{C}_{N,\lambda,q}$ in (14) by a simple scaling argument.

At the threshold of the porous medium and fast diffusion regimes, there is a *linear regime* corresponding to $q = 1$. If we consider the limit of $\mathscr{F}[\rho] - \frac{1}{q-1} \int_{\mathbb{R}^N} \rho \, dx$ as $q \to 1$, we see that the limiting free energy takes the standard form $\rho \mapsto \int_{\mathbb{R}^N} \rho \log \rho \, dx + \frac{1}{2\lambda} I_\lambda[\rho]$, which is bounded from below according to the following *logarithmic Sobolev* type inequality.

Theorem 29. *Let $N \geq 1$ and $\lambda > 0$. Then the inequality*

$$\int_{\mathbb{R}^N} \rho \log \rho \, dx + \frac{N}{\lambda} \log\left(\frac{I_\lambda[\rho]}{\mathscr{C}_{N,\lambda,1}}\right) \geq 0 \tag{16}$$

holds for any nonnegative function $\rho \in \mathrm{L}^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \rho(x)\, dx = 1$ and $\rho \log \rho \in \mathrm{L}^1(\mathbb{R}^N)$, for some positive constant $\mathscr{C}_{N,\lambda,1}$. Moreover, a radial positive, non-increasing, bounded function $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$ achieves the equality case.

Proof. With $\varepsilon = 1/\alpha$, taking the log on both sides of (15) and multiplying by ε yields

$$g(\varepsilon) := \varepsilon \log\left(\frac{I_\lambda[\rho]}{\mathscr{C}_*}\right) + \frac{1-2\varepsilon}{q} \log\left(\int_{\mathbb{R}^N} \rho^q \, dx\right) - \log\left(\int_{\mathbb{R}^N} \rho \, dx\right) \geq 0 \, .$$

Since $q(\varepsilon) = 1 + \frac{\lambda}{N}\varepsilon + O(\varepsilon^2)$ for small $\varepsilon > 0$, we obtain $g(0) = 0$ in the limit, and the first order term is nonnegative for small enough ε,

$$g'(0) = \log\left(\frac{I_\lambda[\rho]}{\mathscr{C}_*}\right) - \frac{2N+\lambda}{N} \log\left(\int_{\mathbb{R}^N} \rho \, dx\right) + \frac{\lambda}{N} \frac{\int_{\mathbb{R}^N} \rho \log \rho \, dx}{\int_{\mathbb{R}^N} \rho \, dx} \geq 0 \, .$$

Hence there exists an optimal constant $\mathscr{C}_{N,\lambda,1} \geq \mathscr{C}_*$ such that

$$\int_{\mathbb{R}^N} \rho \log \rho \, dx + \frac{N}{\lambda} \left(\int_{\mathbb{R}^N} \rho \, dx\right) \log\left(\frac{I_\lambda[\rho]}{\mathscr{C}_{N,\lambda,1} \left(\int_{\mathbb{R}^N} \rho(x)\, dx\right)^{\frac{2N+\lambda}{N}}}\right) \geq 0 \, .$$

and (16) follows by taking into account the normalization. □

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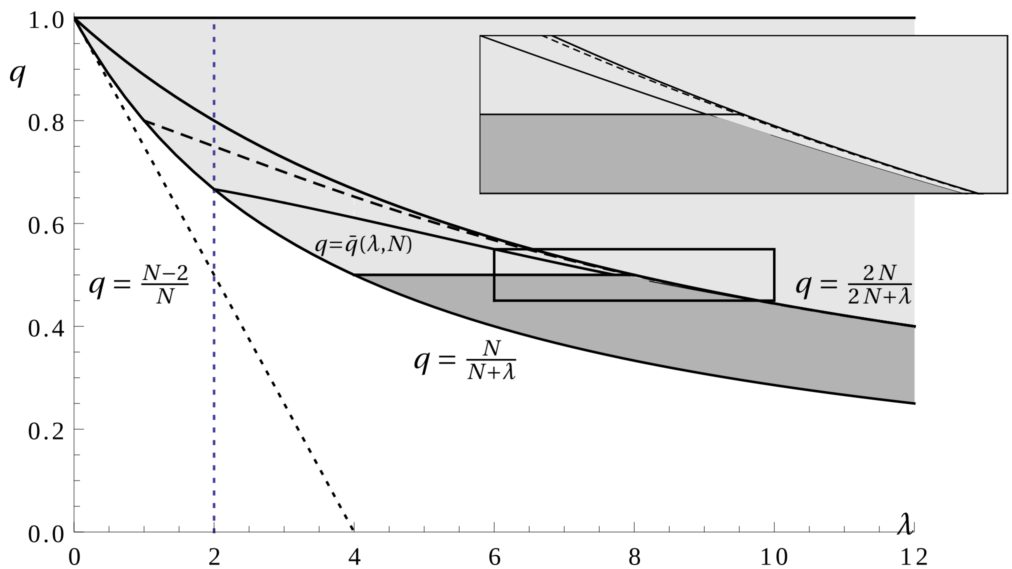

FIGURE 1. Main regions of the parameters (here $N = 4$), with an enlargement of the region inside the black rectangle. The case $q = 2N/(2N+\lambda)$ corresponding to $\alpha = 0$ has already been treated in [19, 2, 37]. Inequality (1) holds with a positive constant $\mathscr{C}_{N,\lambda,q}$ if $q > N/(N+\lambda)$, *i.e.*, $\alpha < 1$, which determines the admissible range corresponding to the grey area, and it is achieved by a function ρ (without any Dirac mass) in the light grey area. The dotted line is $q = 1-\lambda/N$: it is tangent to the admissible range of parameters at $(\lambda, q) = (0,1)$, and it is also the threshold line for integrable stationary solutions in the toy model in the Appendix. In the dark grey region, Dirac masses with $M_* > 0$ are not excluded. The dashed curve corresponds to the curve $q = 2N\big(1-2^{-\lambda}\big)/\big(2N\big(1-2^{-\lambda}\big)+\lambda\big)$ and can hardly be distinguished from $q = 2N/(2N+\lambda)$ when q is below $1-2/N$. The curve $q = \bar{q}(\lambda, N)$ of Corollary 17 is also represented. Above this curve, no Dirac mass appears when minimizing the relaxed problem corresponding to (1). Whether Dirac masses appear in the region which is not covered by Corollary 17 is an open question.

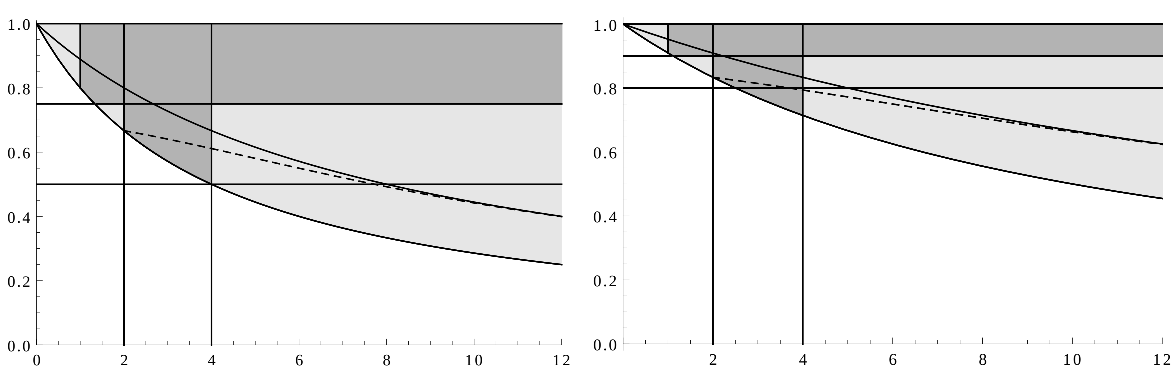

FIGURE 2. Darker grey areas correspond to regions of the parameters $(\lambda, q) \in (0, +\infty) \times [0,1)$ for which there is uniqueness of the measure-valued minimizer, with $N = 4$ (left) and $N = 10$ (right). The dashed curve is $q = \bar{q}(\lambda, N)$, above which minimizers are bounded, with no Dirac singularity. Horizontal lines correspond to $q = 0$, $1-2/N$, $1-1/N$ and 1.

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