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Immersions of transitive tournaments in digraphs with large minimum outdegree

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Abstract

We prove the existence of a function $h(k)$ such that every simple digraph with minimum outdegree greater than $h(k)$ contains an immersion of the transitive tournament on $k$ vertices. This solves a conjecture of Devos, McDonald, Mohar and Scheide.

In this note, all digraphs are without loops. Let $D$ be a digraph. We denote by $V(D)$ its vertex set and $A(D)$ its arc set. A digraph $D$ is simple if there is at most one arc from $x$ to $y$ for any $x, y \in V(D)$. Note that arcs in opposite directions are allowed. The multiplicity of a digraph $D$ is the maximum number of parallel arcs in the same direction in $D$. For an arc $a = (u, v)$ of a digraph $D$, we say that $u$ is the tail of $a$ and $v$ its head. The outdegree (resp. indegree) of a vertex $v$, denoted by $d^+(v)$ (resp. $d^-(v)$), is equal to the number of arcs $a$ of $D$ such that $v$ is the tail (resp. head) of $a$. The outneighbourhood (resp. inneighbourhood) of a vertex $v$, denoted by $N^+(v)$ (resp. $N^-(v)$), is the set of vertices $y$ such that $(v, y)$ (resp. $(y, v)$) is an arc of $D$. A directed path $P$ in a digraph $D$ is a set of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in A(D)$ for all $1 \leq i \leq k-1$. A directed cycle $C$ in a digraph $D$ is a set of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in A(D)$ for all $1 \leq i \leq k-1$ and $(x_k, x_1) \in A(D)$. A digraph $D$ is a tournament if exactly one of $(u, v)$ and $(v, u)$ is an arc of $D$ for all distinct $u, v \in V(D)$. The transitive tournament on $k$ vertices, denoted by $TT_k$, is the unique tournament on $k$ vertices without any directed cycle. The complete digraph on $k$ vertices is the simple digraph on $k$ vertices with every possible arc.

We say that a digraph $D$ contains an immersion of a digraph $H$ if there exists a mapping such that vertices of $H$ are mapped to distinct vertices of $D$, and the arcs of $H$ are mapped to directed paths joining the corresponding pairs of vertices of $D$, in such a way that these paths are pairwise arc-disjoint. If the directed paths are pairwise internally vertex-disjoint, we say that $D$ contains a subdivision of $H$.

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Understanding the necessary conditions for undirected graphs to contain a subdivision of a clique is a very natural and well-studied question. One of the most important examples is the following result by Mader [6]:

**Theorem 1** ([6]). For every $k \geq 1$, there exists an integer $f(k)$ such that every undirected graph with minimum degree greater than $f(k)$ contains a subdivision of $K_k$.

Bollobás and Thomason [1] as well as Komlós and Szemerédi [4] proved that $f(k) = O(k^2)$. In the case of digraphs, there exist examples of digraphs with large out- and indegree without a subdivision of the complete digraph on three vertices, as shown by Thomassen [7] (see also [3] for a simpler construction). However Mader [5] conjectured that an analogue should hold for transitive tournaments $TT_k$ in digraphs with large minimum outdegree.

**Conjecture 2** ([5]). For every $k \geq 1$, there exists an integer $g(k)$ such that every simple digraph with minimum outdegree greater than $g(k)$ contains a subdivision of $TT_k$.

The question turned out to be much more difficult than the undirected case, as the existence of $g(5)$ remains unknown. Weakening the statement, Devos, McDonald, Mohar and Scheide [3] made the following conjecture replacing subdivision with immersion and proved it for the case of Eulerian digraphs.

**Conjecture 3** ([3]). For every $k \geq 1$, there exists an integer $h(k)$ such that every simple digraph with minimum outdegree greater than $h(k)$ contains an immersion of $TT_k$.

As with subdivisions, Devos et al. showed in [3] the existence of digraphs with large out- and indegree without an immersion of the complete digraph on three vertices. Finding the right value for $h(k)$ in the case of undirected graphs is an interesting question on its own (see [2] for more details).

The goal of this note is to present a proof of this conjecture. Let $F(k,l)$ be the digraph consisting of $k$ vertices $x_1, \ldots, x_k$ and $l$ arcs from $x_i$ to $x_{i+1}$ for every $1 \leq i \leq k - 1$. It is clear that $F(k, \binom{k}{2})$ contains an immersion of $TT_k$, so the following theorem implies Conjecture 3.

**Theorem 4.** For every $k \geq 1$ and $l$, there exists a function $f(k,l)$ such that every digraph with minimum outdegree greater than $f(k,l)$ and multiplicity at most $kl$ contains an immersion of $F(k,l)$.

**Proof.** We prove the result for $f(k,l) = 2k^3l^2$ and $l \geq 2$. We proceed by induction on $k$. For $k = 1$ this is trivial because $F(1,l)$ is one vertex. Suppose now that the result holds for $k$ and assume for a contradiction that it does not hold for $k + 1$. Let $D$ be the digraph with the smallest number of arcs and vertices such that $D$ has multiplicity at most $(k+1)l$, all but at most $c_1 = k + (k+1)/l$ vertices have outdegree at least $f(k+1,l)$ and without an immersion of $F(k+1,l)$. By minimality of $D$, every vertex has outdegree exactly $f(k+1,l)$, except $c_1$ of them with outdegree 0. Call $T$ the set of vertices of outdegree 0. Suppose we want to remove arcs from $D$ such that the multiplicity of the remaining digraph is at most $kl$, while keeping the minimum outdegree as large as possible. For a vertex $v$, the worst case is when, for every vertex $y \in N^+(v)$, the multiplicity of $(v,y)$ is equal to $(k+1)l$. In this case we have to remove at most $l$ arcs for each of the $\frac{f(k+1,l)}{(k+1)l}$ vertices of $N^+(v)$. Therefore, removing $T$ and some of the parallel arcs, we obtain a digraph of outdegree greater than $d' = f(k+1,l) - c_1(k+1)l - \frac{f(k+1,l)}{(k+1)l}$ with multiplicity $kl$. Because $f(k+1,l) - f(k,l) = 2(3k^2 + 3k + 1)l^2$ and $c_1(k+1)l + \frac{f(k+1,l)}{(k+1)l} = k(k+1)l + 3(k+1)^2l^2$, we get that $d' \geq f(k,l)$ and by induction there
exists an immersion of $F(k, l)$ in $D - T$. Call $X = \{x_1, \ldots, x_k\}$ the set of vertices of the immersion and, numbering the paths arbitrarily, $P_{i,j}$ the $j$th directed path of this immersion from $x_i$ to $x_{i+1}$.

We can assume this immersion is of minimum size, so that every vertex in $P_{i,j}$ has exactly one outgoing arc in $P_{i,j}$. Let $D'$ be the digraph obtained from $D$ by removing all the arcs of the $P_{i,j}$ and the vertices $x_1, \ldots, x_k-1$. By the previous remark, the outdegree of each vertex in $D'$ is either 0 if this vertex belongs to $T$ or at least $f(k+1, l) - (k-1)l - (k-1)(k+1)l$.

For every vertex $y \in D' - x_k$, there do not exist $l$ arc-disjoint directed paths from $x_k$ to $y$ in $D'$, for otherwise there would be an immersion of $F(k+1, l)$ in $D$. Hence, by Menger’s Theorem there exists a set $E_y$ of less than $l$ arcs such that there is no directed path from $x_k$ to $y$ in $D' \setminus E_y$. Define $C_y$ for every vertex $y \in D' - x_k$ as the set of vertices which can reach $y$ in $D' \setminus E_y$. Now take $Y$ a minimal set such that $\cup_{y \in Y} C_y$ covers $D' - x_k$. We claim that $Y$ consists of at least $c_2 \geq \frac{f(k+1,l)-(k-1)l-(k-1)(k+1)l}{|l|} \geq 2c_1$ elements, as $\cup_{y \in Y} E_y$ must contain all the arcs of $D'$ with $x_k$ as tail.

For each $y \in Y$, define $S_y$ as the set of vertices which belong to $C_y$ and no other $C_{y'}$ for $y' \in Y$. Since $Y$ is minimal, every $S_y$ is non-empty. Note that for $u \in S_y$, if there exists $y' \in Y \setminus y$ and $v \in C_{y'}$ such that $uv \in A(D)$, then $uv \in E_{y'}$. Note that $T \subseteq Y$ as vertices in $T$ have outdegree 0 and if $y \in Y \setminus T$ then $S_y$ consists only of vertices of outdegree $f(k+1, l)$ in $D$.

Let $R$ be the digraph with vertex set $Y$ and arcs from $y$ to $y'$ if there is an arc from $S_y$ to $C_{y'}$. As noted before, $d_R^+(y) \leq |E_y| \leq l$. The average outdegree of the vertices of $Y \setminus T$ in $R$ is then at most $\frac{c_1l + (c_2 - c_1)l}{c_2 - c_1} \leq 2l$. Let $y$ be a vertex of $R \setminus T$ with outdegree at most this average. Let $H$ be the digraph induced on $D'$ by the vertices in $S_y$ to which we add $X$, all the arcs that existed in $D$ (with multiplicity) from vertices of $S_y$ to vertices of $X$ and the following arcs: For each $P_{i,j}$, let $z_1, z_2, \ldots, z_l = P_{i,j} \cap S_y$, where $z_i$ appears before $z_{i+1}$ on $P_{i,j}$ and add all the arcs $(z_i, z_{i+1})$ to $H$. Note that, if $(x, y)$ is an arc of $D'$, then by minimality of the immersion of $F(k, l)$, every time $x$ appears before $y$ on some $P_{i,j}$, then $P_{i,j}$ uses one of the arcs $(x, y)$. Thus for each pair of vertices $x$ and $y$ in $H$, either $(x, y) \in A(D)$ and the number of $(x, y)$ arcs in $H$ is equal to the one in $D$, or $(x, y) \notin A(D)$ and the number of $(x, y)$ arcs in $H$ is bounded by $(k-1)l$. This implies that $H$ has multiplicity at most $(k+1)l$.

**Claim 4.1.** $H$ is a digraph with multiplicity at most $(k+1)l$, such that all but at most $c_1$ vertices have outdegree greater than $f(k+1, l)$ and $H$ does not contain an immersion of $F(k+1, l)$.

*Proof of the claim.* Suppose $H$ contains an immersion of $F(k+1, l)$, then by replacing the new arcs by the corresponding directed paths along the $P_{i,j}$ we get an immersion of $F(k+1, l)$ in $D$. Moreover, we claim that the number of vertices in $H$ with outdegree smaller than $f(k+1, l)$ is at most $k + 2l + (k-1)l = c_1$. Indeed, the vertices of $H$ that can have outdegree smaller in $H$ than in $D$ are the $x_i$, or the vertices with outgoing arcs in $E_y$ for some $y' \in Y \setminus y$, or the vertices along the $P_{i,j}$. But with the additions of the new arcs, we know that there is at most one vertex per path $P_{i,j}$ that loses some outdegree in $H$.

However, since $H$ is strictly smaller than $D$, we reach a contradiction. 

\[ \Box \]

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