Topological rigidity as a monoidal equivalence

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Abstract

A topological commutative ring is said to be *rigid* when for every set X, the topological dual of the X-fold topological product of the ring is isomorphic to the free module over X. Examples are fields with a ring topology, discrete rings, and normed algebras. Rigidity translates into a dual equivalence between categories of free modules and of "topologically-free" modules and, with a suitable topological tensor product for the latter, one proves that it lifts to an equivalence between monoids in this category (some suitably generalized topological algebras) and coalgebras. In particular, we provide a description of its relationship with the standard duality between algebras and coalgebras, namely finite duality.

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1 Introduction

The main result of [14] states that given a (Hausdorff[1]) topological field $(\Bbbk, \tau)$, for every set X, the topological dual $((\Bbbk,\tau)^X)'$ of the X-fold topological product $(\Bbbk,\tau)^X$ is isomorphic to the vector space $\Bbbk^{(X)}$ of finitely-supported $\Bbbk$-valued maps defined on X (i.e., those maps $X \xrightarrow{f} \Bbbk$ such that for all but finitely many members x of X, $f(x) = 0$).

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[1] All topologies will be assumed separated.

Actually this topological property of *rigidity* is shared by more general topological (commutative unital) rings[2] than only topological fields (a fact not noticed in [14]). For instance any discrete ring is rigid in the above sense (see Lemma 18). And even if not all topological rings are rigid (see Section 4.3 for a counter-example), many of them still are (e.g., every real or complex normed commutative algebra).

It is our intention to study in more details some consequences of the property of rigidity for arbitrary commutative rings in particular for some of their topological algebras[3]. So far, for a topological ring (R, τ), rigidity reads as $((R, \tau)^X)' \simeq R^{(X)}$ (here, and everywhere else, R stands for the canonical left R-module structure on the underlying abelian group of R) for each set X. Suitably topologized (see Section 3.1), the algebraic dual $(R^{(X)})^*$ turns out to be isomorphic to $(R, \tau)^X$.

More appropriately the above correspondence may be upgraded into a dual equivalence of categories[4] between free and topologically-free modules, i.e., those topological modules isomorphic to some $(R, \tau)^X$ (Theorem 48) under the algebraic and topological dual functors. (This extends a similar interpretation from [14] to the realm of arbitrary commutative rigid rings.)

Under the rigidity assumption, the aforementioned dual equivalence enables to provide a topological tensor product $\circledast_{(\mathsf{R},\tau)}$ for topologically-free (R, τ)-modules by transporting the algebraic tensor product $\otimes_{\mathsf{R}}$ along the dual equivalence. It turns out that $\circledast_{(\mathsf{R},\tau)}$ is (coherently) associative, commutative and unital, i.e., makes monoidal the category of topologically-free modules (Proposition 60). Not too surprisingly the above dual equivalence remains well-behaved, i.e., monoidal, with respect to the (algebraic and topological) tensor products (Theorem 63). According to the theory of monoidal categories, this in turn provides a dual equivalence between monoids in the tensor category of topologically-free modules (some suitably generalized topological algebras) and coalgebras (Corollary 65). So there are two constructions: a *topological dual coalgebra* of a monoid (in the tensor category of topologically-free modules) and an *algebraic dual monoid* of a coalgebra, and these constructions are inverse one from the other (up to isomorphism).

There already exists a standard duality theory between algebras and coalgebras, over a field, known as *finite duality* but contrary to our "topological duality" it is merely an adjunction, not an equivalence. One discusses how these dualities interact (see Section 7) and in particular one proves that the

[2]In this contribution, every ring is assumed commutative and unital (see Section 2.1).

[3]The results of the present contribution also serve in a subsequent paper under preparation about topological semisimplicity of commutative topological algebras.

[4]A *dual* equivalence is an equivalence between a category and the opposite of another.

algebraic dual monoid of a coalgebra essentially corresponds to its finite dual (Section 7.2), that over a discrete field, the topological dual coalgebra of a monoid is a subcoalgebra of the finite dual coalgebra of its underlying algebra and furthermore that they are equal exactly when finite duality provides an equivalence of categories (Theorem 77).

2 Conventions, notations and basic definitions

2.1 Conventions

Except as otherwise stipulated, all topologies are Hausdorff, and every ring is assumed unital and commutative[5].

For a ring R, R denotes both its underlying set and the canonical left R-module structure on its underlying additive group. Likewise if A is an R-algebra, then A is both its underlying set and its underlying R-module. The unit of a ring R (resp., unital algebra A) is either denoted by 1_{R} (resp. 1_{A}). A ring map (or morphism of rings) is assumed to preserve the units.

A product of topological spaces always has the product topology. When for each $x \in X$, all (E_x, τ_x)'s are equal to the same topological space (E, τ), then the X-fold topological product $\prod_{x\in X}(E_x, \tau_x)$ is canonically identified with the set E^X of all maps from X to E equipped with the topology of simple convergence, and is denoted by $(E, \tau)^X$. Under this identification, the canonical projections $(E,\tau)^X \xrightarrow{\pi_x} (E,\tau)$ are given by $\pi_x(f) = f(x)$, $x \in X$, $f \in E^X$.

2.2 Basic definitions

1 Definition *Let* R *be a ring. A (Hausdorff, following our conventions) topology* τ *of (the carrier set of) the ring is called a* ring topology *when addition, multiplication and additive inversion of the ring are continuous. By* topological ring (R, τ) *is meant a ring together with a ring topology* τ *on it*[6]*. By a* field with a ring topology*, denoted* $(\Bbbk, \tau)$*, is meant a topological ring* $(\Bbbk, \tau)$ *with* $\Bbbk$ *a field.*

2 Example *A ring* R *with the discrete topology* d *is a topological ring.*

[5]To the contrary an algebra over a ring won't be assumed commutative, but associative and unital.

[6]In view of Section 2.1, the multiplication of a topological ring is jointly continuous.

Let (R, τ) be a topological ring. A pair (M, σ) consisting of a (left and unital[7]) R-module M and a topology σ on M which makes continuous the addition, opposite and scalar multiplication $R \times M \to M$, is called a *topological* (R, τ)*-module.* Such a topology is referred to as a (R, τ)*-module topology.* In particular, when R is a field $\Bbbk$, then this provides *topological* $(\Bbbk, \tau)$*-vector spaces.* Given topological (R, τ)-modules $(M, \sigma), (N, \gamma)$, a *continuous* (R, τ)*-linear map* $(M, \sigma) \xrightarrow{f} (N, \gamma)$ is a R-linear map $M \xrightarrow{f} N$ which is continuous. Topological (R, τ)-modules and these morphisms form a category[8] $\mathbf{TopMod}_{(\mathsf{R}, \tau)}$, which is denoted $\mathbf{TopVect}_{(\Bbbk, \tau)}$, when $\Bbbk$ is a field.

A pair (A, σ), with A a unital R-algebra, and σ a topology on A, is a *topological* (R, τ)*-algebra*, when σ is a module topology for the underlying R-module A, and the multiplication of A is a bilinear (jointly) continuous map. Given topological (R, τ)-algebras $(\mathsf{A}, \sigma), (\mathsf{B}, \gamma)$, a *continuous* (R, τ)*-algebra map* $(\mathsf{A}, \sigma) \xrightarrow{f} (\mathsf{B}, \gamma)$ is a unit-preserving R-algebra map $\mathsf{A} \xrightarrow{f} \mathsf{B}$ which is also continuous. Topological (R, τ)-algebras with these morphisms form a category ${}_1\mathbf{TopAlg}_{(\mathsf{R}, \tau)}$. One also has the full subcategory[9] ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R}, \tau)}$ of unital and commutative topological algebras.

3 Example *(R, τ) is a topological (R, τ)-module and the topological ring (R, τ) with the previous module structure, is a topological (R, τ)-algebra.*

4 Remark *A R-module (resp. R-algebra) with the discrete topology d is a topological (R, d)-module (resp. (R, d)-algebra).*

2.3 X-fold product and finitely-supported maps

The *opposite category* $\mathbf{C}^{\mathsf{op}}$ of $\mathbf{C}$ has the same objects and morphisms as $\mathbf{C}$ but with opposite composition. In other words one has $\mathbf{C}^{\mathsf{op}}(D, C) = \mathbf{C}(C, D)$. f^{op} denotes the $\mathbf{C}$-morphism f considered as a $\mathbf{C}^{\mathsf{op}}$-morphism. Let $\mathbf{C} \xrightarrow{F} \mathsf{D}$ be a functor. Then, $\mathbf{C}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathbf{D}^{\mathsf{op}}$ with for a $\mathbf{C}$-morphism f, $F^{\mathsf{op}}(f^{\mathsf{op}}) =$

[7]Unital means that the scalar action of the unit of R is the identity on the module.

[8]One does not worry about size issues and in this presentation "category" means a locally small category while "set" loosely means both small and large set. One assumes that the reader is familiar with basic notions from category theory among subcategories, (full, faithful) functors, natural transformations, natural isomorphisms, equivalence of categories, categorical product, terminal object, left/right adjoints, unit and counit of an adjunction. However some of them will be recalled in the text, essentially through footnotes. Of course [13] is a fundamental reference for this subject.

[9]By a *full subcategory* is meant a subcategory $\mathbf{D}$ of $\mathbf{C}$ such that $\mathbf{D}(C, D) = \mathbf{C}(C, D)$ for each $\mathbf{D}$-objects C, D, where as usually, $\mathbf{C}(C, D)$ denotes the *hom-set* of all $\mathbf{C}$-morphisms with domain C and codomain D.

$F(f)^{\mathsf{op}}$, is called the *opposite* of F. Each natural transformation[10] $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ has an *opposite natural transformation* $\alpha^{\mathsf{op}}\colon G^{\mathsf{op}} \Rightarrow F^{\mathsf{op}}\colon \mathbf{C}^{\mathsf{op}} \to \mathbf{D}^{\mathsf{op}}$ with $(\alpha^{\mathsf{op}})_C = \alpha_C^{\mathsf{op}}$ for each $\mathbf{C}$-object C.

Let R be a ring, and let X be a set. The R-module R^X of all maps from X to R, equivalently defined as the X-fold power of R in the category[11] $\mathbf{Mod}_{\mathsf{R}}$, is the object component of a functor[12] $\mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ whose action on maps is as follows: given $X \xrightarrow{f} Y$ and $g \in R^Y$, $P_{\mathsf{R}}(f)(g) = g \circ f$. R^X is merely not just a R-module but, under point-wise multiplication $R^X \times R^X \xrightarrow{M_X} R^X$, a commutative R-algebra, the usual *function algebra* on X, denoted $A_{\mathsf{R}}(X)$, with unit $1_{A_{\mathsf{R}}(X)} \coloneqq \sum_{x\in X} \delta_x^{\mathsf{R}}$, where δ_x^{R}, or simply δ_x, is the member of R^X with $\delta_x^{\mathsf{R}}(x) = 1_{\mathsf{R}}$, $x \in R$, and for $y \in X$, $y \neq x$, $\delta_x^{\mathsf{R}}(y) = 0$. This actually provides a functor[13] $\mathbf{Set}^{\mathsf{op}} \xrightarrow{A_{\mathsf{R}}} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$.

Let $f \in R^X$. The *support* of f is the set $supp(f) \coloneqq \{\, x \in X\colon f(x) \neq 0 \,\}$. Let $R^{(X)}$ be the sub-R-module of R^X consisting of all *finitely-supported maps* (or maps with *finite support*), i.e., the maps f such that $supp(f)$ is finite.

$R^{(X)}$ is actually the free R-module over X, and a basis is given by $\{\, \delta_x^{\mathsf{R}}\colon x \in X \,\}$. Observe that the map $X \xrightarrow{\delta_X^{\mathsf{R}}} R^X$, $x \mapsto \delta_x$, is one-to-one if, and only if, R is non-trivial.

5 Remark *Of course one has the* free module functor $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ *which is a left adjoint of the usual forgetful functor* $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$*;* $F_{\mathsf{R}}(X) \coloneqq R^{(X)}$*, and for* $X \xrightarrow{f} Y$*,* $p \in R^{(X)}$*,* $F_{\mathsf{R}}(f)(p) \coloneqq \sum_{y\in Y} \left(\sum_{x\in f^{-1}(\{\,y\,\})} p(x)\right) \delta_y^{\mathsf{R}}$*, i.e.,* $F_{\mathsf{R}}(f)(\delta_x^{\mathsf{R}}) = \delta_{f(x)}^{\mathsf{R}}$*,* $x \in X$*. The map* $X \xrightarrow{\delta_X^{\mathsf{R}}} |R^{(X)}|$ *is the component at* X *of the unit of the adjunction*[14] $F_{\mathsf{R}} \dashv |\cdot|\colon \mathbf{Set} \to \mathbf{Mod}_{\mathsf{R}}$.

Let (R, τ) be a topological ring and let X be a set. Since for a map

[10] The notation $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ means that α is a *natural transformation* between two functors $\mathbf{C} \xrightarrow{F,G} \mathbf{D}$. Given a $\mathbf{C}$-object C, $\alpha_C \in \mathbf{D}(F(C), G(C))$ denotes the *component at* C of α. Thus $\alpha = (\alpha_C)_C$. For each functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$, there is an *identity* at F, $id_F\colon F \Rightarrow F\colon \mathbf{C} \to \mathbf{D}$ with $(id_F)_C \coloneqq id_{F(C)}$, also denoted simply *id*.

[11] $\mathbf{Mod}_{\mathsf{R}}$ is the category of unital left-R-modules with R-linear maps. When R is a field $\Bbbk$ one uses $\mathbf{Vect}_{\Bbbk}$ instead.

[12] $\mathbf{Set}$ is the category of sets with set-theoretic maps.

[13] ${}_1\mathbf{Alg}_{\mathsf{R}}$ the category of (associative) unital R-algebras with unit-preserving algebra maps. (The multiplication m_{A} of an algebra A thus is a R-bilinear map $A \times A \xrightarrow{m_{\mathsf{A}}} A$.) ${}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ is the category of unital and commutative algebras.

[14] In this presentation, by $F \dashv G\colon \mathbf{C} \to \mathbf{D}$ is meant an adjunction with left adjoint $\mathbf{C} \xrightarrow{F} \mathbf{D}$ and right adjoint $\mathbf{D} \xrightarrow{G} \mathbf{C}$.

$X \xrightarrow{f} Y$, $\pi_x \circ P_{(\mathsf{R},\tau)}(f) = \pi_{f(x)}$, $x \in X$, $(R,\tau)^Y \xrightarrow{P_{\mathsf{R}}(f)} (R,\tau)^X$ is continuous, and thus one has a *topological power functor* $\mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$.

6 Lemma $(R,\tau)^X \times (R,\tau)^X \xrightarrow{M_X} (R,\tau)^X$ *is continuous.*

Proof: M_X is separately continuous because for each $x \in X$, $\pi_x \circ M_X = m_{\mathsf{R}} \circ (\pi_x \times \pi_x)$. Let $A \subseteq X$ be finite, and for each $x \in A$, let U_x be an open neighborhood zero in (R,τ). Continuity of m_{R} at zero implies the existence of neighborhoods V_x, W_x of zero such that $m_{\mathsf{R}}(V_x, W_x) \subseteq U_x$. $M_X(\bigcap_{x\in A} \pi_x^{-1}(V_x), \bigcap_{x\in A} \pi_x^{-1}(W_x)) \subseteq \bigcap_{x\in A} \pi_x^{-1}(U_x)$ ensures continuity at zero of M_X, and thus continuity by [19, Theorem 2.14, p. 16]. $\square$

$A_{(\mathsf{R},\tau)} : X \mapsto ((R,\tau)^X, M_X, 1_{A_{\mathsf{R}}(X)})$ is a functor too and the diagram below commutes, with the forgetful functors unnamed.

$$\begin{tikzcd}
{}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[rr] \arrow[dd] & & {}_{1,c}\mathbf{Alg}_{\mathsf{R}} \arrow[dd] \\
& \mathbf{Set}^{\mathsf{op}} \arrow[ul, "A_{(\mathsf{R},\tau)}"'] \arrow[ur, "A_{\mathsf{R}}"] \arrow[dl, "P_{(\mathsf{R},\tau)}"] \arrow[dr, "P_{\mathsf{R}}"'] & \\
\mathbf{TopMod}_{(\mathsf{R},\tau)} \arrow[rr] & & \mathbf{Mod}_{\mathsf{R}}
\end{tikzcd} \tag{1}$$

7 Notation *The underlying topological ring of $A_{(\mathsf{R},\tau)}(X)$ is denoted $(\mathsf{R},\tau)^X$ (of course, it is the X-fold product of (R,τ) in the category of rings).*

3 Recollection of results about algebraic and topological duals

3.1 Algebraic dual functor

Let R be a ring. Let M be a R-module. Let $M^* := \mathbf{Mod}_{\mathsf{R}}(M,R)$ be the *algebraic* (or *linear*) *dual* of M. This is readily a R-module on its own.

When (R,τ) is a topological ring, then M^* may be topologized with the initial topology ([6, 19]) $w^*_{(\mathsf{R},\tau)}$, called the *weak-$*$ topology*, induced by the family $(M^* \xrightarrow{\Lambda_M(v)} R)_{v\in M}$ of *evaluations* at some points, where $(\Lambda_M(v))(\ell) := \ell(v)$. This provides a structure of topological (R,τ)-module on M^*, which is even Hausdorff[15] (since if $\ell(v) = 0$ for all $v \in M$, then $\ell = 0$).

[15]The initial topology on X induced by a family F of maps all with domain X, is Hausdorff when F separates the points of X, i.e., when for each $x \neq y$ in X, there is a map $f \in F$ such that $f(x) \neq f(y)$.

Moreover given a linear map $M \xrightarrow{f} N$, $N^* \xrightarrow{f^*} M^*$, $\ell \mapsto f^*(\ell) \coloneqq \ell \circ f$, is continuous for the above topologies. Consequently, this provides a functor $\mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ called the *algebraic dual functor.*

8 Remark *M^* or f^* stand for $Alg_{(\mathsf{R},\tau)}(M)$ or $Alg_{(\mathsf{R},\tau)}(f)$.*

Up to isomorphism, one recovers the module of all R-valued maps on a set X, with its product topology, as the algebraic dual of the module of all finitely-supported maps duly topologized as above.

9 Lemma *For each set X, $(R,\tau)^X \simeq Alg_{(\mathsf{R},\tau)}(R^{(X)})$ (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$) under the map*

$$\rho_X : (R,\tau)^X \to ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$$

given by $(\rho_X(f)(p)) \coloneqq \sum_{x\in X} p(x)f(x)$, $f \in R^X$, $p \in R^{(X)}$.

Proof: Let $\ell \in (R^{(X)})^*$. Let us define $X \xrightarrow{\hat{\ell}} R$ by $\hat{\ell}(x) \coloneqq \ell(\delta_x)$, $x \in X$. That the two constructions are linear and inverse one from the other is clear.

It remains to make sure that there are also continuous. Let $\ell \in (R^{(X)})^*$, and let $x \in X$. Then, $\pi_x(\hat{\ell}) = \hat{\ell}(x) = \ell(\delta_x) = (\Lambda_{R^{(X)}}(\delta_x))(\ell)$, which ensures continuity of $((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{\rho_X^{-1}} (R,\tau)^X$. Let $f \in R^X$, and $p \in R^{(X)}$. As $(\Lambda_{R^{(X)}}(p))(\rho_X(f)) = (\rho_X(f))(p) = \sum_{x\in X} p(x)f(x) = \sum_{x\in X} \pi_x(p)f(x) = \sum_{x\in X} \pi_x(p)\pi_x(f)$, $\Lambda_{R^{(X)}}(p) \circ \rho_X$ is a finite linear combination of projections, whence is continuous for the product topology, so is ρ_X. $\square$

Let M be a free R-module. Let B be a basis of M. This defines a family of R-linear maps, the *coefficient maps* $(M \xrightarrow{b^*} R)_{b\in B}$ such that each $v \in M$ is uniquely represented as a finite linear combination $v = \sum_{b\in B} b^*(v)b$. One denotes $\mathbf{FreeMod}_{\mathsf{R}}$ the full subcategory of $\mathbf{Mod}_{\mathsf{R}}$ spanned[16] by the free modules. When $\Bbbk$ is a field, $\mathbf{FreeMod}_{\Bbbk}$ is just $\mathbf{Vect}_{\Bbbk}$ itself.

10 Example *For each set X, $p_x = \delta_x^*$, where $p_x \coloneqq R^{(X)} \hookrightarrow R^X \xrightarrow{\pi_x} R$, $x \in X$, with $R^{(X)} \hookrightarrow R^X$ the canonical inclusion.*

11 Remark *$b^*(d) = \delta_b(d)$, $b, d \in B$. So $B \xrightarrow{(-)^*} B^* \coloneqq \{\, b^* : b \in B \,\}$ is a bijection.*

[16] Each (even large) subset X of the objects of some category $\mathbf{C}$ determines uniquely a full subcategory of $\mathbf{C}$ called the *full subcategory of* $\mathbf{C}$ *spanned by* X, namely the subcategory $\mathbf{C}_X$ whose set of objects is X and $\mathbf{C}_X(C,D) = \mathbf{C}(C,D)$, $C, D \in X$.

Given a free R-module, any choice of a basis B provides the initial topology Π_B^τ on M^* induced by $(\Lambda_M(b))_{b\in B}$. (Of course, $\Pi_B^\tau \subseteq w^*_{(\mathsf{R},\tau)}$.)

12 Lemma *Let M be a free R-module. The topology Π_B^τ is independent of the choice of the basis B of M since it is equal to $w^*_{(\mathsf{R},\tau)}$. Moreover, for each basis B of M, $(M^*, w^*_{(\mathsf{R},\tau)}) \simeq (R,\tau)^B$ (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$).*

Proof: For $\ell \in M^*$, $(\Lambda_M(v))(\ell) = \sum_{b\in B} b^*(v)\ell(b) = \sum_{b\in B} b^*(v)(\Lambda_M(b))(\ell)$, $v \in M$, thus $\Lambda_M(v)$ is a finite linear combination of some $\Lambda_M(b)$'s, whence is continuous for Π_B^τ, and so $w^*_{(\mathsf{R},\tau)} \subseteq \Pi_B^\tau$. The last assertion is clear. $\square$

3.2 Topological dual functor

Let (R,τ) be a topological ring, and let (M,σ) be a topological (R,τ)-module. Let $(M,\sigma)' := \mathbf{TopMod}_{(\mathsf{R},\tau)}((M,\sigma),(R,\tau))$ be the *topological dual* of (M,σ), which is a R-submodule of M^*. Let $(M,\sigma) \xrightarrow{f} (N,\gamma)$ be a continuous (R,τ)-linear map between topological modules. Let $(N,\gamma)' \xrightarrow{f'} (M,\sigma)'$ be the R-linear map given by $f'(\ell) := \ell \circ f$. All of this evidently forms a functor $\mathbf{TopMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{Mod}_{\mathsf{R}}$.

Let (R,τ) be a topological ring, and let X be a set. Let $R^{(X)} \xrightarrow{\lambda_X} (R^X)^*$ be given by $(\lambda_X(p))(f) := \sum_{x\in X} p(x)f(x)$, $p \in R^{(X)}$, $f \in R^X$.

Let ρ_X be the map from Lemma 9. Then, for each $p \in R^{(X)}$, $\lambda_X(p) = \Lambda_{R^{(X)}}(p) \circ \rho_X$, which ensures continuity of $\lambda_X(p)$, i.e., $\lambda_X(p) \in ((R,\tau)^X)'$. Next lemma follows from the equality $p(x) = (\lambda_X(p))(\delta_x)$, $p \in R^{(X)}$, $x \in X$.

13 Lemma *$R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)'$ is one-to-one.*

4 Rigid rings: definitions and (counter-)examples

The notion of *rigidity*, recalled at the beginning of the Introduction, was originally but only implicitly introduced in [14, Theorem 5, p. 156] as the main result therein and the possibility that its conclusion could remain valid for more general topological rings than topological division rings was not noticed. Since a large part of this presentation is given for arbitrary rigid rings (Definition 15 below), one here provides a stock of basic examples.

As [14, Lemma 13, p. 158], one has the following fundamental lemma.

14 Lemma *Let* (R,τ) *be a topological ring, and let* X *be a set. For each* $f \in R^X$*,* $(f(x)\delta_x)_{x\in X}$ *is summable in* $(R,\tau)^X$ *with sum* f*.*

15 Definition *Let* (R,τ) *be a topological ring. It is said to be* rigid *when for each set* X*,* $R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)'$ *is an isomorphism in* $\mathbf{Mod}_R$*, i.e.,* λ_X *is onto. In this situation, one sometimes also called* rigid *a ring topology* τ *such that* (R,τ) *is rigid.*

16 Lemma *Let* $\ell \in ((R,\tau)^X)'$*.* $\ell \in im(\lambda_X)$ *if, and only if,* $\hat{\ell}: X \to R$ *given by* $\hat{\ell}(x) := \ell(\delta_x)$*, belongs to* $R^{(X)}$*. Moreover,* $im(\lambda_X) \xrightarrow{\hat{(-)}} R^{(X)}$*,* $\ell \mapsto \hat{\ell} = \sum_{x\in X} \ell(\delta_x)\delta_x$*, is the inverse of* λ_X*.*

4.1 Basic stock of examples

Of course, the trivial ring is rigid (under the (in)discrete topology!).

The first assertion of the following result is a slight generalization of the main theorem in [14], which is precisely the second assertion below, since the proof of [14, Theorem 5, p. 156] does not use continuity of the inversion.

17 Lemma *Let* $(\Bbbk,\tau)$ *be a field with a ring topology (see Definition 1). Then,* $(\Bbbk,\tau)$ *is rigid. In particular, any topological field*[17] *is rigid.*

18 Lemma *For each ring* R*, the discretely topologized ring* (R,d) *is rigid.*

Proof: Let $\ell \in ((R,\mathsf{d})^X)'$. As a consequence of Lemma 14, $(\ell(\delta_x))_x$ is summable in (R,d), with sum $\ell(1_{A_{\mathsf{R}}(X)})$. Since $\{0\}$ is an open neighborhood of zero in (R,d), $\ell(\delta_x) = 0$ for all but finitely many $x \in X$ ([19, Theorem 10.5, p. 73]). The conclusion follows by Lemma 16. □

Every normed, complex or real, commutative and unital algebra (e.g., Banach or C^*-algebra) is rigid.

19 Lemma *Let* $\Bbbk = \mathbb{R}, \mathbb{C}$*. Let* $(\mathsf{A}, \|\cdot\|)$ *be a commutative normed* $\Bbbk$*-algebra*[18] *with a unit. Then, as a topological ring under the topology induced by the norm, it is rigid.*

[17] A *topological field* is a field with a ring topology $(\Bbbk,\tau)$ such that the inversion $\alpha \mapsto \alpha^{-1}$ is continuous from $\Bbbk \setminus \{0\}$ to itself with the subspace topology.

[18] In a normed algebra $(\mathsf{A}, \|\cdot\|)$, unital or not, commutative or not, the norm is assumed *sub-multiplicative*, i.e., $\|xy\| \le \|x\|\|y\|$, which ensures that the multiplication of A is jointly continuous with respect to the topology induced by the norm.

Proof: Let $\tau_{\|-\|}$ be the topology on A induced by the norm of A, where A is the underlying $\Bbbk$-vector space of A. Let X be a set. Let $\ell \in ((A,\tau_{\|-\|})^X)'$. Let $f \in A^X$ be given by $f(x) = \frac{1}{\|\ell(\delta_x)\|}1_{\mathsf{A}}$ if $x \in supp(\hat{\ell})$ and $f(x) = 0$ for $x \notin supp(\hat{\ell})$. Since by Lemma 14, $(f(x)\delta_x)_{x\in X}$ is summable with sum f, $(f(x)\ell(\delta_x))_{x\in X}$ is summable in $(A,\tau_{\|-\|})$ with sum $\ell(f)$. So according to [19, Theorem 10.5, p. 73], for $1 > \epsilon > 0$, there exists a finite set $F_\epsilon \subseteq X$ such that $\|f(x)\ell(\delta_x)\| < \epsilon$ for all $x \in X \setminus F_\epsilon$. But $1 = \|f(x)\ell(\delta_x)\|$ for all $x \in supp(\hat{\ell})$ so that $supp(\hat{\ell})$ is finite, and λ_X is onto by Lemma 16. □

4.2 A supplementary example: von Neumann regular rings

A ring[19] is said to be *von Neumann regular* if for each $x \in R$, there exists $y \in R$ such that $x = xyx$ [10, Theorem 4.23, p. 65].

Let us assume that R is a (commutative) von Neumann regular ring. For each $x \in R$, there is a unique $x^\dagger \in R$, called the *weak inverse* of x, such that $x = xx^\dagger x$ and $x^\dagger = x^\dagger x x^\dagger$.[20]

20 Example *A field is a von Neumann regular with $x^\dagger := x^{-1}$, $x \neq 0$, and $0^\dagger = 0$. More generally, let $(\Bbbk_i)_{i\in I}$ be a family of fields. Let R be a ring, and let $\jmath\colon \mathsf{R} \hookrightarrow \prod_{i\in I}\Bbbk_i$ be a one-to-one ring map. Assume that for each $x \in R$, $\jmath(x)^\dagger \in im(\jmath)$, where for $(x_i)_{i\in I} \in \prod_{i\in I}\Bbbk_i$, $(x_i)^\dagger_{i\in I} := (x_i^\dagger)_{i\in I}$. Then, R is von Neumann regular.*

21 Remark *Let R be a von Neumann regular ring. For each $x \in R$, $x \neq 0$ if, and only if, $xx^\dagger \neq 0$. Moreover, $xx^\dagger$ belongs to the set $E(\mathsf{R})$ of all idempotents ($e^2 = e$) of R.*

22 Proposition *Let (R,τ) be a topological ring such that R is von Neumann regular. If $0 \notin \overline{E(\mathsf{R}) \setminus \{0\}}$, then (R,τ) is rigid. In particular, if $E(\mathsf{R})$ is finite, then (R,τ) is rigid.*

Proof: That the second assertion follows from the first is immediate. Let X be a set. Let us assume that $0 \notin \overline{E(\mathsf{R}) \setminus \{0\}}$. Let $V \in \mathfrak{V}_{(R,\tau)}(0)$[21] such that $V \cap (E(R) \setminus \{0\}) = \varnothing$. Let $\ell \in ((R,\tau)^X)'$. Let $f \in R^X$ be given by

[19] Assumed commutative and unital as in Section 2.1.

[20] Given $y \in R$ with $x = xyx$, then $z := yxy$ meets the requirements to be a "weak inverse" of x, and if y, z are two candidates, then one has $z = z^2x = z^2x^2y = (x^2z)zy = xzy = (x^2y)zy = (x^2z)y^2 = xy^2 = y$.

[21] Given a topological space (E,τ) and $x \in E$, $\mathfrak{V}_{(E,\tau)}(x)$ is the set of all neighborhoods of x.

$f(x) \coloneqq \ell(\delta_x)^\dagger$ for each $x \in X$. Since $(f(x)\ell(\delta_x))_{x\in X}$ is summable in (R,τ) with sum $\ell(f)$, by Cauchy's condition [19, Definition 10.3, p. 72], there exists a finite set $A_{f,V} \subseteq X$ such that for all $x \notin A_{f,V}$, $f(x)\ell(\delta_x) \in V$. But for $x \in X$, $f(x)\ell(\delta_x) = \ell(\delta_x)^\dagger\ell(\delta_x) \in E(\mathsf{R})$. Whence, in view of Remark 21, for all but finitely many x's, $f(x)\ell(\delta_x) = 0$, i.e., $\ell(\delta_x) = 0$. □

23 Remark *Lemma 17 is a consequence of Proposition 22 since for a field* $\Bbbk$*,* $E(\Bbbk) = \{0, 1_\Bbbk\}$*.*

Now, let $(E_i,\tau_i)_{i\in I}$ be a family of topological spaces. On $\prod_{i\in I} E_i$ is defined the *box topology* [9, p. 107] a basis of open sets of which is given by the "box" $\prod_{i\in I} V_i$, where each $V_i \in \tau_i$, $i \in I$. The product $\prod_{i\in I} E_i$ together with the box topology is denoted by $\sqcap_{i\in I}(E_i,\tau_i)$. (This topology is Hausdorff as soon as all the (E_i,τ_i)'s are.)

It is not difficult to see that given a family $(\mathsf{R}_i,\tau_i)_{i\in I}$ of topological rings, then $\sqcap_{i\in I}(\mathsf{R}_i,\tau_i)$ still is a topological ring (under component-wise operations).

24 Proposition *Let* $(\Bbbk_i)_{i\in I}$ *be a family of fields, and for each* $i \in I$*, let* τ_i *be a ring topology on* $\Bbbk_i$*. Let* R *be a ring with a one-to-one ring map* $\jmath\colon \mathsf{R} \hookrightarrow \prod_{i\in I}\Bbbk_i$*. Let us assume that for each* $x \in R$*,* $\jmath(x)^\dagger \in im(\jmath)$ *(*$(x_i)_i^\dagger$ *as in Example 20). Let* R *be topologized with the subspace topology* $\tau_\jmath$ *inherited from* $\sqcap_{i\in I}(\Bbbk_i,\tau_i)$*. Then,* $(\mathsf{R},\tau_\jmath)$ *is rigid.*

Proof: Naturally $(x_i)_{i\in I} \in E(\prod_{i\in I}\Bbbk_i)$ if, and only if, $x_i \in \{0, 1_{\Bbbk_i}\}$ for each $i \in I$. Now, for each $i \in I$, let U_i be an open neighborhood of zero in $(\Bbbk_i,\tau_i)$ such that $1_{\Bbbk_i} \notin U_i$. Then, $\prod_i U_i$ is an open neighborhood of zero in $\sqcap_{i\in I}(\Bbbk_i,\tau_i)$ whose only idempotent member is 0. Therefore, $0 \notin \overline{E(\prod_{i\in I}\Bbbk_i) \setminus \{0\}}$.

Under the assumptions of the statement, an application of Example 20 states that R is a (commutative) von Neumann regular ring. It is also of course a topological ring under $\tau_\jmath$ (since $\jmath$ is a one-to-one ring map). It is also clear that $E(R) \simeq E(\jmath(R)) \subseteq E(\prod_i \Bbbk_i)$. Furthermore, $\jmath(\overline{E(R)\setminus\{0\}}) = \overline{E(\jmath(R))\setminus\{0\}} \cap \jmath(R) \subseteq \overline{E(\prod_i\Bbbk_i)\setminus\{0\}}$, and thus $0 \notin \overline{E(R)\setminus\{0\}}$ according to the above discussion. Therefore, by Proposition 22, $(\mathsf{R},\tau_\jmath)$ is rigid. □

4.3 A counter-example

Let (R,τ) be a topological ring, and let us consider the topological $(\mathsf{R},\tau)^X$-module $((R,\tau)^X)^X$ for a given set X. To avoid confusion one denotes by $(R^X)^X \xrightarrow{\Pi_x} R^X$ the canonical projections.

Let us define a linear map $(R^X)^X \xrightarrow{\ell} (R,\tau)^X$ by setting $\ell(f)\colon x \mapsto (f(x))(x)$, $f \in (R^X)^X$. ℓ is continuous, and thus belongs to $(((R,\tau)^X)^X)'$, since for each $x \in X$, $\pi_x \circ \ell = \pi_x \circ \Pi_x$. Now, for each $x \in X$, $(\ell(\delta_x^{\mathsf{R}^X}))(x) = \delta_x^{\mathsf{R}^X}(x) = 1_{\mathsf{R}^X}$, so that $supp(\hat{\ell}) = X$. Consequently one obtains

25 Proposition *Let (R,τ) be topological ring, and let X be a set. If X is infinite, then $(\mathsf{R},\tau)^X$ is not rigid.*

However the above negative result may balanced by the following.

26 Proposition *Let (R,τ) be a rigid ring. If I is finite, then $(\mathsf{R},\tau)^I$ is rigid too.*

Proof: Let (R,τ) be a topological ring. For a set I, one recalls that $(\mathsf{R},\tau)^I$ is the underlying ring $(\mathsf{R},\tau)^I$ (Notation 7) of $A_{(\mathsf{R},\tau)}(I)$. Any topological $(\mathsf{R},\tau)^I$-module is also a topological (R,τ)-module under restriction of scalars[22] along the unit map $(\mathsf{R},\tau) \xrightarrow{\eta_I} (\mathsf{R},\tau)^I$, $\eta_I(1_{\mathsf{R}}) = 1_{\mathsf{R}^I}$, which of course is a ring map, and is continuous (because $\eta_I(\alpha) = m_{\mathsf{R}^I}(\eta_I(\alpha), 1_{\mathsf{R}^I})$, $\alpha \in R$.)

Let X be a set, and let $\ell \in (((R,\tau)^I)^X)'$, i.e., $((R,\tau)^I)^X \xrightarrow{\ell} (R,\tau)^I$ is continuous and $(\mathsf{R},\tau)^I$-linear, and by restriction of scalar along η_I it is also a continuous (R,τ)-linear. Therefore for each $i \in I$, $((R,\tau)^I)^X \xrightarrow{\ell} (R,\tau)^I \xrightarrow{\pi_i} (R,\tau)$ belongs to the topological dual space of $((R,\tau)^I)^X$ seen as a (R,τ)-module.

Let us assume that (R,τ) is rigid. Then, by Lemma 16, $supp(\widehat{\pi_i \circ \ell})$ is finite for each $i \in I$. One also has $supp(\hat{\ell}) = \bigcup_{i \in I} supp(\widehat{\pi_i \circ \ell})$, with $X \xrightarrow{\hat{\ell}} R^I$, $\hat{\ell}(x) \coloneqq \ell(\delta_x^{\mathsf{R}^I})$, $x \in X$. Whence if I is finite, then $supp(\hat{\ell})$ is finite too. $\square$

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[22] Let (R,τ) and (S,σ) be topological rings, and let $(\mathsf{R},\tau) \xrightarrow{f} (\mathsf{S},\sigma)$ be a continuous ring map. It may be used to transform a topological (S,σ)-module into a topological (R,τ)-module by restriction of scalars along f. In details, let (M,γ) be a topological (S,σ)-module. There is a scalar action of R on M given by $\alpha \cdot v \coloneqq f(\alpha)v$, $\alpha \in R$, $v \in M$ (where by juxtaposition is denoted the scalar action $\mathsf{S} \times M \to M$). Furthermore this action is again continuous (by composition of continuous maps). Let $f^*(M,\gamma)$ be the topological (R,τ)-module just obtained. At present let $(M,\gamma) \xrightarrow{g} (N,\pi)$ be a continuous (S,σ)-linear map. g is also R-linear because of $g(\alpha \cdot v) = g(f(\alpha)v) = f(\alpha)g(v) = \alpha \cdot g(v)$, and thus provides a continuous (R,τ)-linear map $f^*(M,\gamma) \xrightarrow{g} f^*(N,\pi)$. All this results in a functor $\mathbf{TopMod}_{(\mathsf{S},\sigma)} \xrightarrow{f^*} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ of *restriction of scalars along f*.

5 Rigidity as an equivalence of categories

The main result of this section is Theorem 48 which provides a translation of the rigidity condition on a topological ring into a dual equivalence between the category of free modules and that of topologically-free modules (see below), provided by the topological dual functor with equivalence inverse the (opposite of the) algebraic dual functor, with both functors conveniently co-restricted. The purpose of this section thus is to prove this result.

Topologically-free modules. Let (R,τ) be a topological ring. Let (M,σ) be a topological (R,τ)-module. It is said to be a *topologically-free* (R,τ)-*module* if $(M,\sigma)\simeq(R,\tau)^X$, in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$, for some set X. Such topological modules span the full subcategory $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ of $\mathbf{TopMod}_{(\mathsf{R},\tau)}$. For a field $(\Bbbk,\tau)$ with a ring topology, one defines correspondingly the category $\mathbf{TopFreeVect}_{(\Bbbk,\tau)}\hookrightarrow\mathbf{TopVect}_{(\Bbbk,\tau)}$ of *topologically-free* $(\Bbbk,\tau)$*-vector spaces.*

27 Remark *The topological power functor* $\mathbf{Set}^{\mathsf{op}}\xrightarrow{P_{(\mathsf{R},\tau)}}\mathbf{TopMod}_{(\mathsf{R},\tau)}$ *factors as indicated below (the co-restriction obtained is also called* $P_{(\mathsf{R},\tau)}$*).*

$$\begin{tikzcd}
\mathbf{Set}^{\mathsf{op}} \arrow[r, "P_{(\mathsf{R},\tau)}"] \arrow[dr, "P_{(\mathsf{R},\tau)}"'] & \mathbf{TopMod}_{(\mathsf{R},\tau)} \\
& \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \arrow[u, hook]
\end{tikzcd} \tag{2}$$

Topologically-free modules are characterized by the fact of possessing "topological bases" (see Corollary 31 below) which makes easier a number of calculations and proofs, once such a basis is chosen.

28 Definition *Let* (M,σ) *be a topological* (R,τ)*-module. Let* $B\subseteq M$*. It is said to be a* topological basis *of* (M,σ) *if the following hold.*

1. *For each* $v\in M$*, there exists a unique family* $(b'(v))_{b\in B}$*, with* $b'(v)\in R$ *for each* $b\in B$*, such that* $(b'(v)b)_b$ *is summable in* (M,σ) *with sum* v*.* $b'(v)$ *is referred to as the* coefficient *of* v *at* $b\in B$*.*
2. *For each family* $(\alpha_b)_{b\in B}$ *of elements of* R*, there is a member* v *of* M *such that* $b'(v)=\alpha_b$*,* $b\in B$*. (By the above point such* v *is unique.)*
3. σ *is equal to the initial topology induced by the* (topological) coefficient maps $(M\xrightarrow{b'}(R,\tau))_{b\in B}$*. (According to the two above points, each* b' *is* R*-linear.)*

29 Remark *It is an immediate consequence of the definition that for a topological basis B of some topological module, $0 \notin B$ and $b'(d) = \delta_b(d)$, $b, d \in B$ (since $\sum_{b\in B} \delta_b(d)b = d = \sum_{b\in B} b'(d)b$). In particular, $B \xrightarrow{(-)'} B' := \{ b' : b \in B \}$ is a bijection.*

30 Lemma *Let (M,σ) and (N,γ) be isomorphic topological (R,τ)-modules. Let $\Theta\colon (M,\sigma) \simeq (N,\gamma)$ be an isomorphism (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$). Let B be a topological basis of (M,σ). Then, $\Theta(B) = \{ \Theta(b) : b \in B \}$ is a topological basis of (N,γ).*

31 Corollary *Let (M,σ) be a (Hausdorff) topological (R,τ)-module. It admits a topological basis if, and only if, it is topologically-free.*

32 Example *Let (R,τ) be a topological ring. For each set X, $\{ \delta_x : x \in X \}$ is a topological basis of $(R,\tau)^X$. Moreover $\pi_x = \delta'_x$, $x \in X$.*

Let us now take the time to establish a certain number of quite useful properties of topological bases.

33 Lemma *Let (M,σ) be a topologically-free (R,τ)-module with topological basis B. Then, B is R-linearly independent and the linear span $\langle B\rangle$ of B is dense in (M,σ).*

Proof: Concerning the assertion of independence, it suffices to note that 0 may be written as $\sum_{b\in B} 0b$, and conclude by the uniqueness of the decomposition in a topological basis. Let $u \in M$ and let $V := \{ v \in M : b'(v) \in U_b,\ b \in A \} \in \mathfrak{V}_{(M,\sigma)}(0)$, where A is a finite subset of B and $U_b \in \mathfrak{V}_{(\mathsf{R},\tau)}(0)$, $b \in A$. Let $\alpha_b \in U_b$, $b \in A$, and $v := \sum_{b\in A} \alpha_b b - \sum_{b\in B\setminus A} b'(u)b \in V$. So $u + v \in \langle B\rangle$. Thus, $u + V$ meets $\langle B\rangle$ and $\langle B\rangle$ is dense in (M,σ). $\Box$

34 Corollary *Let (M,σ) be a topologically-free (R,τ)-module, and let (N,γ) be a topological (R,τ)-module. Let $(M,\sigma) \xrightarrow{f,g} (N,\gamma)$ be two continuous (R,τ)-linear maps. $f = g$ if, and only if, for any topological basis B of (M,σ), $f(b) = g(b)$ for each $b \in B$.*

Topologically-free modules allow for the definition of changes of topological bases (see Proposition 51 for a more general construction).

35 Lemma *Let (M,σ) and (N,γ) be two topologically-free (R,τ)-modules, and let B, D be respective topological bases. Let $f : B \to D$ be a bijection. Then, there is a unique isomorphism g in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$ such that $g(b) = f(b)$, $b \in B$.*

Proof: The question of uniqueness is settled by Corollary 34. If such an isomorphism g exists, then $g(v) = g\left(\sum_{b\in B} b'(v)b\right) = \sum_{b\in B} b'(v)g(b) = \sum_{b\in B} b'(v)f(b) = \sum_{d\in D}(f^{-1}(d))'(v)d$, $v \in M$. One observes that g as defined by the right hand-side of the last equality, is R-linear, and it is also continuous since for each $d \in D$, $d' \circ g = (f^{-1}(d))'$. □

36 Lemma *Let M be a free module with basis B. Then, $(M^*, w^*_{(\mathsf{R},\tau)})$ is a topologically-free module with topological basis $B^* := \{\, b^* : b \in B\,\}$ (see Remark 11).*

Proof: According to Lemma 12, $(M^*, w^*_{(\mathsf{R},\tau)})$ is a topologically-free module. Let $M \xrightarrow{\theta_B} R^{(B)}$ be the isomorphism given by $\theta_B(b) = \delta_b$, $b \in B$. Thus, $\theta^*_B : (R^{(B)})^* \simeq M^*$, and $\theta^*_B \circ \rho_B : R^B \simeq (R^{(B)})^* \simeq M^*$ is given by $\theta^*_B(\rho_B(\delta^{\mathsf{R}}_b)) = \rho_B(\delta^{\mathsf{R}}_b) \circ \theta_B = p_b \circ \theta_B = b^*$ for $b \in B$ (see Example 10 for the definition of p_b). Now, $\{\, \delta_b : b \in B\,\}$ being a topological basis of R^B, by Lemma 30, this shows that B^* is a topological basis of $(M^*, w^*_{(\mathsf{R},\tau)})$. □

37 Example *$\{\, p_x : x \in X\,\}$ is a topological basis of $(R^{(X)})^*$ (Example 10).*

38 Remark *If B is a basis of a free module M, then $B \simeq B^*$ under $b \mapsto b^*$, because for each $b, d \in B$, $b^*(d) = \delta_b(d)$.*

39 Corollary *Let (R,τ) be a topological ring. The algebraic dual functors $\mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ factors as illustrated in the diagram below[23]. Moreover the resulting co-restriction of $Alg_{(\mathsf{R},\tau)}$ (the bottom arrow of the*

[23]When $\Bbbk$ is a field with a ring topology τ, then one has the corresponding factorization of $\mathbf{Vect}^{\mathsf{op}}_{\Bbbk} \xrightarrow{Alg_{(\Bbbk,\tau)}} \mathbf{TopVect}_{(\Bbbk,\tau)}$.

$$\begin{tikzcd} \mathbf{Vect}^{\mathsf{op}}_{\Bbbk} \arrow[r, "Alg_{(\Bbbk,\tau)}"] \arrow[dr] & \mathbf{TopVect}_{(\Bbbk,\tau)} \\ & \mathbf{TopFreeVect}_{(\Bbbk,\tau)} \arrow[u, hook] \end{tikzcd} \tag{3}$$

diagram) is essentially surjective[24].

$$\begin{tikzcd} \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \arrow[r, "Alg_{(\mathsf{R},\tau)}"] & \mathbf{TopMod}_{(\mathsf{R},\tau)} \\ \mathbf{FreeMod}_{\mathsf{R}}^{\mathsf{op}} \arrow[r] \arrow[u, hook] & \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \arrow[u, hook] \end{tikzcd} \tag{4}$$

Proof: The first assertion is merely Lemma 36. Regarding the second assertion, let (M,σ) be a topologically-free module. So, for some set X, $(M,\sigma) \simeq (R,\tau)^X$. By Lemma 9, $(R,\tau)^X \simeq Alg_{(\mathsf{R},\tau)}(R^{(X)})$. □

40 Lemma *Let (R,τ) be a rigid ring. Let (M,σ) be a topologically-free (R,τ)-module with topological basis B. Then, $(M,\sigma)'$ is free with basis $B' := \{\, b' : b \in B \,\}$.*

Proof: Let $\Theta_B : (M,\sigma) \simeq (R,\tau)^B$ be given by $\Theta_B(b) = \delta_b$. Therefore $\Theta'_B : ((R,\tau)^B)' \simeq (M,\sigma)'$, and thus one has an isomorphism $\Theta'_B \circ \lambda_B : R^{(B)} \simeq (M,\sigma)'$. Since a module isomorphic to a free module is free, $(M,\sigma)'$ is free. The previous isomorphism acts as: $\Theta'_B(\lambda_B(\delta_b)) = \pi_b \circ \Theta_B = b'$ for $b \in B$. It follows from Lemma 30 that B' is a basis of $(M,\sigma)'$. □

41 Example *Let (R,τ) be a rigid ring. Let $(M,\sigma) = (R,\tau)^X$. By Example 32, $\{\, \delta'_x : x \in X \,\} = \{\, \pi_x : x \in X \,\}$ is a linear basis of $((R,\tau)^X)'$.*

42 Corollary *Let (R,τ) be a rigid ring. The functor $\mathbf{TopMod}_{(\mathsf{R},\tau)}^{\mathsf{op}} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{Mod}_{\mathsf{R}}$ factors*[25] *as indicated by the diagram below.*

$$\begin{tikzcd} \mathbf{TopMod}_{(\mathsf{R},\tau)}^{\mathsf{op}} \arrow[r, "Top_{(\mathsf{R},\tau)}"] & \mathbf{Mod}_{\mathsf{R}} \\ \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}^{\mathsf{op}} \arrow[r] \arrow[u, hook] & \mathbf{FreeMod}_{\mathsf{R}} \arrow[u, hook] \end{tikzcd} \tag{6}$$

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[24]A functor $\mathbf{C} \xrightarrow{F} \mathsf{D}$ is *essentially surjective* when each object D in $\mathbf{D}$ is isomorphic to an object of the form FC, for some object C in $\mathbf{C}$. Whence an equivalence of categories is a fully faithful and essentially surjective functor (see [13]).

[25]Correspondingly for a field $(\Bbbk,\tau)$ with a ring topology,

$$\begin{tikzcd} \mathbf{TopVect}_{(\Bbbk,\tau)}^{\mathsf{op}} \arrow[r, "Top_{(\Bbbk,\tau)}"] & \mathbf{Vect}_{\Bbbk} \\ \mathbf{TopFreeVect}_{(\Bbbk,\tau)}^{\mathsf{op}} \arrow[u, hook] \arrow[ur] & \end{tikzcd} \tag{5}$$

The topological dual of the algebraic dual of a free module. Let (R,τ) be a topological ring. Let M be a R-module, and let us consider as in Section 3.1, the R-linear map

$$M \xrightarrow{\Lambda_M} (M^*, w^*_{(\mathsf{R},\tau)})'$$

$(\Lambda_M(v))(\ell) = \ell(v)$, $v \in M$, $\ell \in M^*$.

43 Lemma *Let M be a projective R-module. Then, Λ_M is one-to-one. This holds in particular when M is a free R-module.*

Proof: Let us consider a dual basis for M, i.e., sets $B \subseteq M$ and $\{\ell_e : e \in B\} \subseteq M^*$, such that for all $v \in M$, $\ell_e(v) = 0$ for all but finitely many $\ell_e \in B^*$ and $v = \sum_{e \in B} \ell_e(v)e$ ([10, p. 23]). Let $v \in \ker \Lambda_M$, i.e., $(\Lambda_M(v))(\ell) = \ell(v) = 0$ for each $\ell \in M^*$. Then, in particular, $\Lambda_M(v)(\ell_e) = \ell_e(v) = 0$ for all $e \in B$, and thus $v = 0$. □

Let (R,τ) (resp. $(\Bbbk,\tau)$) be a rigid ring (resp. field). Let us still denote by $\mathbf{FreeMod}^{\mathsf{op}}_{\mathsf{R}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ (resp. $\mathbf{Vect}^{\mathsf{op}}_{\Bbbk} \xrightarrow{Alg_{(\Bbbk,\tau)}} \mathbf{TopFreeVect}_{(\Bbbk,\tau)}$) and by $\mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{FreeMod}_{\mathsf{R}}$ (resp. $\mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \xrightarrow{Top_{(\Bbbk,\tau)}} \mathbf{Vect}_{\Bbbk}$) the functors provided by Corollaries 39 and 42.

44 Proposition *Let us assume that (R,τ) is rigid. $\Lambda := (\Lambda_M)_M : id \Rightarrow Top_{(\mathsf{R},\tau)} \circ Alg^{\mathsf{op}}_{(\mathsf{R},\tau)} : \mathbf{FreeMod}_{\mathsf{R}} \to \mathbf{FreeMod}_{\mathsf{R}}$ is a natural isomorphism*[26].

Proof: Naturality is clear. Let (R,τ) be a topological ring. Let M be a free R-module. For each free basis X of M, the following diagram commutes in $\mathbf{Mod}_{\mathsf{R}}$, where $M \xrightarrow{\theta_X} R^{(X)}$ is the canonical isomorphism given by $\theta_X(x) = \delta^{\mathsf{R}}_x$, $x \in X$. Consequently, when (R,τ) is rigid, then for each free R-module M, $M \xrightarrow{\Lambda_M} (M^*, w^*_{(\mathsf{R},\tau)})'$ is an isomorphism.

$$\begin{tikzcd}
M \arrow[r, "\Lambda_M"] \arrow[dd, equal, "\theta_X"'] & (M^*, w^*_{(\mathsf{R},\tau)})' & \\
 & & ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})' \arrow[ul, "(\theta^*_X)'"'] \\
R^{(X)} \arrow[r, "\lambda_X"'] & ((R,\tau)^X)' \arrow[ur, "\rho'_X"'] &
\end{tikzcd} \tag{7}$$

[26] A *natural isomorphism* $\alpha : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ is a natural transformation the components of which are isomorphisms in $\mathbf{D}$. Two functors $F, G : \mathbf{C} \to \mathbf{D}$ are said to be *naturally isomorphic*, which is denoted $F \simeq G$, if there is a natural isomorphism $\alpha : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$.

$\square$

45 Corollary *Let us assume that* $(\Bbbk,\tau)$ *is a field with a ring topology. Then,* $\Lambda = (\Lambda_M)_M : id \Rightarrow Top_{(\Bbbk,\tau)} \circ Alg^{\mathsf{op}}_{(\Bbbk,\tau)} : \mathbf{Vect}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ *is a natural isomorphism.*

The algebraic dual of the topological dual of a topologically-free module. Let (M,σ) be a topological (R,τ)-module. Let us consider the R-linear map $M \xrightarrow{\Gamma_{(M,\sigma)}} ((M,\sigma)')^*$ by setting $(\Gamma_{(M,\sigma)}(v))(\ell) := \ell(v)$.

46 Proposition *Let us assume that* (R,τ) *is a rigid ring. Then,* $\Gamma : id \Rightarrow Alg_{(\mathsf{R},\tau)} \circ Top^{\mathsf{op}}_{(\mathsf{R},\tau)} : \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *is a natural isomorphism, with* $\Gamma := (\Gamma_{(M,\sigma)})_{(M,\sigma)}$.

Proof: Naturality is clear. Let $\Theta : (M,\sigma) \simeq (R,\tau)^X$ be an isomorphism (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$). Since (R,τ) is rigid, $\lambda_X : R^{(X)} \simeq ((R,\tau)^X)'$ is an isomorphism. Therefore $R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)' \xrightarrow{\Theta'} (M,\sigma)'$ is an isomorphism too in $\mathbf{Mod}_R$. In particular, $(M,\sigma)'$ is a free with basis $\{\, \Theta'(\lambda_X(\delta^{\mathsf{R}}_x)) : x \in X \,\}$. By Lemma 12, the weak-$*$ topology on $((M,\sigma)')^*$ is the same as the initial topology given by the maps $((M,\sigma)')^* \xrightarrow{\Lambda_{(M,\sigma)'}(\pi_x \circ \Theta)} (R,\tau)$, $x \in X$, because $\Theta'(\lambda_X(\delta^{\mathsf{R}}_x)) = \Theta'(\pi_x) = \pi_x \circ \Theta$. Therefore, $\Gamma_{(M,\sigma)}$ is continuous if, and only if, for each $x \in X$, $\Lambda_{(M,\sigma)'}(\pi_x \circ \Theta) \circ \Gamma_{(M,\sigma)} = \pi_x \circ \Theta$ is continuous. Continuity of $\Gamma_{(M,\sigma)}$ thus is proved.

That $\Gamma_{(M,\sigma)}$ is an isomorphism in $\mathbf{TopMod}_{(R,\tau)}$ follows from the commutativity of the diagram (in $\mathbf{TopMod}_{(R,\tau)}$) below (which may be checked by hand).

$$\begin{tikzcd}
 & (M,\sigma) \arrow[dl, "\Theta"'] \arrow[rr, "\Gamma_{(M,\sigma)}"] & & (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)}) \\
(R,\tau)^X \arrow[dr, "\rho_X"'] & & & \\
 & ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \arrow[rr, "(\lambda_X^{-1})^*"'] & & ((((R,\tau)^X)')^*, w^*_{(\mathsf{R},\tau)}) \arrow[uu, "((\Theta')^*)^{-1}"']
\end{tikzcd} \tag{8}$$

$\square$

47 Corollary *Let us assume that* $(\Bbbk,\tau)$ *is a field with a ring topology.* $\Gamma : id \Rightarrow Alg_{(\Bbbk,\tau)} \circ Top^{\mathsf{op}}_{(\Bbbk,\tau)} : \mathbf{TopFreeVect}_{(\Bbbk,\tau)} \to \mathbf{TopFreeVect}_{(\Bbbk,\tau)}$ *with* $\Gamma := (\Gamma_{(M,\sigma)})_{(M,\sigma)}$, *is a natural isomorphism.*

The equivalence and some of its immediate consequences. Collecting Proposition 44 and Lemma 46, one immediately gets the following.

48 Theorem *Let us assume that* (R,τ) *is rigid.* $\mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{FreeMod}_{\mathsf{R}}$ *is an equivalence of categories*[27]*, with equivalence inverse the functor* $\mathbf{FreeMod}_{\mathsf{R}} \xrightarrow{Alg^{\mathsf{op}}_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$.

49 Corollary $\mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \xrightarrow{Top_{(\Bbbk,\tau)}} \mathbf{Vect}_{\Bbbk}$ *is an equivalence of categories, and* $\mathbf{Vect}_{\Bbbk} \xrightarrow{Alg^{\mathsf{op}}_{(\Bbbk,\tau)}} \mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)}$ *is its equivalence inverse, whenever* $(\Bbbk,\tau)$ *is a field with a ring topology.*

Finite-dimensional vector spaces. Let $\Bbbk$ be a field. Let (M,σ) be a topologically-free $(\Bbbk,\mathsf{d})$-vector space with M finite-dimensional. Then, σ is the discrete topology on M. It follows that $(M,\sigma)' = M^*$, and the equivalence established in Corollary 49 coincides with the classical dual equivalence $\mathbf{FinDimVect}_{\Bbbk} \simeq \mathbf{FinDimVect}^{\mathsf{op}}_{\Bbbk}$ under the algebraic dual functor, where $\mathbf{FinDimVect}_{\Bbbk}$ is the category of finite-dimensional $\Bbbk$-vector spaces.

Linearly compact vector spaces. Let $\Bbbk$ be a field. A topological $(\Bbbk,\mathsf{d})$-vector space (M,σ) is said to be a *linearly compact* $\Bbbk$-vector space when $(M,\sigma) \simeq (\Bbbk,\mathsf{d})^X$ for some set X (see [4, Proposition 24.4, p. 105]). The full subcategory $\mathbf{LCpVect}_{\Bbbk}$ of $\mathbf{TopVect}_{(\Bbbk,\mathsf{d})}$ spanned by these spaces is equal to $\mathbf{TopFreeVect}_{(\Bbbk,\mathsf{d})}$.

50 Corollary *(of Theorem 48) Let* R *be a ring. For each rigid topologies* τ,σ *on* R*, the categories* $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *and* $\mathbf{TopFreeMod}_{(\mathsf{R},\sigma)}$ *are equivalent. Moreover, for each field* $(\Bbbk,\tau)$ *with a ring topology,* $\mathbf{TopFreeVect}_{(\Bbbk,\tau)}$ *is equivalent to* $\mathbf{LCpVect}_{\Bbbk}$*.*

In particular, one recovers the result of J. Dieudonné [8] that $\mathbf{Vect}^{\mathsf{op}}_{\Bbbk} \simeq \mathbf{LCpVect}_{\Bbbk}$.

[27] $\mathbf{C} \simeq \mathbf{D}$ means that the categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*, i.e., that there is an equivalence of categories $\mathbf{C} \xrightarrow{F} \mathbf{D}$. In this situation an *equivalent inverse* of F is a functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ such that there are two natural isomorphisms $\eta\colon id_{\mathbf{C}} \Rightarrow G\circ F\colon \mathbf{C} \to \mathbf{C}$ and $\epsilon\colon F\circ G \Rightarrow id_{\mathbf{D}}\colon \mathbf{D} \to \mathbf{D}$.

The universal property of $(R,\tau)^X$. For a ring R, the forgetful functor $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$ (see Remark 5) may be restricted as indicated in the following commutative diagram, and the restriction still is denoted $\mathbf{FreeMod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$.

$$\begin{tikzcd} \mathbf{Mod}_{\mathsf{R}} \arrow[r, "|\cdot|"] & \mathbf{Set} \\ \mathbf{FreeMod}_{\mathsf{R}} \arrow[u, hook] \arrow[ur] & \end{tikzcd} \tag{9}$$

Likewise $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ (see again Remark 5) may be co-restricted as indicated by the commutative diagram below, and the co-restriction is given the same name $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{FreeMod}_{\mathsf{R}}$.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "F_{\mathsf{R}}"] \arrow[dr] & \mathbf{Mod}_{\mathsf{R}} \\ & \mathbf{FreeMod}_{\mathsf{R}} \arrow[u, hook] \end{tikzcd} \tag{10}$$

(Of course, when R is a field $\Bbbk$, there is no need to consider the corresponding co-restrictions.)

The adjunction $F_{\mathsf{R}} \dashv |\cdot|\colon \mathbf{Set} \to \mathbf{Mod}_{\mathsf{R}}$ gives rise to a new one $F_{\mathsf{R}} \dashv |\cdot|\colon \mathbf{Set} \to \mathbf{FreeMod}_{\mathsf{R}}$ [13, p. 147], and by composition, for each rigid ring (R,τ), there is also the adjunction $Alg^{\mathsf{op}}_{(\mathsf{R},\tau)} \circ F_{\mathsf{R}} \dashv |\cdot| \circ Top_{(\mathsf{R},\tau)}\colon \mathbf{Set} \to \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$. Since $(R,\tau)^X \simeq ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$ (Lemma 9), this may be translated into a *universal property* of $(R,\tau)^X$, as explained below, which somehow legitimates the terminology *topologically-free*.

51 Proposition *Let us assume that (R,τ) is rigid. Let X be a set. For each topologically-free module (M,σ) and any map $X \xrightarrow{f} |(M,\sigma)'|$, there is a unique continuous (R,τ)-linear map $(M,\sigma) \xrightarrow{f^\sharp} (R,\tau)^X$ such that $|(f^\sharp)'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X = f$ (recall that $\delta^{\mathsf{R}}_X(x) = \delta^{\mathsf{R}}_x$, $x \in X$).*

Proof: There is a unique R-linear map $R^{(X)} \xrightarrow{\tilde{f}} (M,\sigma)'$ such that $|\tilde{f}| \circ \delta^{\mathsf{R}}_X = f$. Let us define the continuous linear map $(M,\sigma) \xrightarrow{f^\sharp} (R,\tau)^X \coloneqq (M,\sigma) \xrightarrow{\Gamma_{(M,\sigma)}} (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{(\tilde{f})^*} ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{\rho_X^{-1}} (R,\tau)^X$. One has

$$
\begin{array}{rcl}
|(f^\sharp)'|\circ|\lambda_X|\circ\delta_X^{\mathsf{R}} & = & |\Gamma'_{(M,\sigma)}|\circ|((\tilde{f})^*)'|\circ|(\rho_X^{-1})'|\circ|\lambda_X|\circ\delta_X^{\mathsf{R}}\\
& = & |\Gamma'_{(M,\sigma)}|\circ|((\tilde{f})^*)'|\circ|\Lambda_{R^{(X)}}|\circ\delta_X^{\mathsf{R}}\\
& & \text{(because } (\rho_X^{-1})'\circ\lambda_X=\Lambda_{R^{(X)}})\\
& = & |\Gamma'_{(M,\sigma)}|\circ|\Lambda_{(M,\sigma)'}|\circ|\tilde{f}|\circ\delta_X^{\mathsf{R}}\\
& & \text{(by naturality of } \Lambda)\\
& = & |\tilde{f}|\circ\delta_X^{\mathsf{R}}\\
& & \text{(triangular identities for an adjunction [13, p. 85])}\\
& = & f.
\end{array}
\tag{11}
$$

It remains to check uniqueness of $f^\sharp$. Let $(M,\sigma)\xrightarrow{g}(R,\tau)^X$ be a continuous linear map such that $|g'|\circ|\lambda_X|\circ\delta_X^{\mathsf{R}}=f$. Then, $g'\circ\lambda_X=\tilde{f}$. Thus, $\lambda_X^*\circ(g')^*=\tilde{f}^*=\rho_X\circ f^\sharp\circ\Gamma^{-1}_{(M,\sigma)}$. So $\rho_X\circ\Gamma^{-1}_{(R,\tau)^X}\circ(g')^*=\rho_X\circ f^\sharp\circ\Gamma^{-1}_{(M,\sigma)}$ because $\Gamma_{(R,\tau)^X}=(\lambda_X^{-1})^*\circ\rho_X$ (by direct inspection), and thus $\Gamma^{-1}_{(R,\tau)^X}\circ(g')^*=f^\sharp\circ\Gamma^{-1}_{(M,\sigma)}$. Then, by naturality of Γ^{-1}, $g\circ\Gamma^{-1}_{(M,\sigma)}=f^\sharp\circ\Gamma^{-1}_{(M,\sigma)}$. □

52 Corollary *Let* (R,τ) *be a rigid ring.* $\mathbf{Set}\xrightarrow{P^{\mathsf{op}}_{(\mathsf{R},\tau)}}\mathbf{TopFreeMod}^{\mathsf{op}}_{(R,\tau)}$ *is a left adjoint of* $\mathbf{TopFreeMod}^{\mathsf{op}}_{(R,\tau)}\xrightarrow{Top_{(\mathsf{R},\tau)}}\mathbf{FreeMod}_R\xrightarrow{|-|}\mathbf{Set}$*, and thus is naturally equivalent to* $\mathbf{Set}\xrightarrow{Alg^{\mathsf{op}}_{(\mathsf{R},\tau)}\circ F_{\mathsf{R}}}\mathbf{TopFreeMod}^{\mathsf{op}}_{(R,\tau)}$.

Proof: A quick calculation shows that $P_{(\mathsf{R},\tau)}(f)=(|\lambda_Y|\circ\delta_Y^{\mathsf{R}}\circ f)^\sharp$ for a set-theoretic map $X\xrightarrow{f}Y$. The relation $f\mapsto(\lambda_Y\circ\delta_Y^{\mathsf{R}}\circ f)^\sharp$ provides a functor from $\mathbf{Set}^{\mathsf{op}}$ to $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ whose opposite is, by construction, a left adjoint of $|-|\circ Top_{(\mathsf{R},\tau)}$ (this is basically the content of Proposition 51). □

6 Tensor product of topologically-free modules

In this section most of the ingredients so far introduced and developed fit together so as to lift the equivalence $\mathbf{FreeMod}\simeq\mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ to a duality between some (suitably generalized) topological algebras and coalgebras. Let us briefly describe how this goal is achieved:

1. A topological tensor product $\otimes_{(\mathsf{R},\tau)}$ for topologically-free modules (over a rigid ring (R,τ)) is provided by transport of the algebraic tensor product $\otimes_{\mathsf{R}}$ of R-modules along the dual equivalence from Section 5.

2. A topological basis (Definition 28) for $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ is described in terms of topological bases of $(M,\sigma),(N,\gamma)$.

3. The aforementioned equivalence is proved to be compatible with $\otimes_{\mathsf{R}}$ and $\circledast_{(\mathsf{R},\tau)}$ (i.e., it is a monoidal equivalence).

4. Accordingly, for category-theoretic reasons (see Appendix A), one obtains a dual equivalence between some topological algebras and coalgebras, still under the algebraic and topological dual functors.

6.1 Algebraic tensor product of modules

The results recalled below are well-known (cf. [5] for instance) but they give us the opportunity to introduce some notations used hereafter.

Let R be a ring. For each R-modules M,N, one denotes by $M\otimes_{\mathsf{R}}N$ their (algebraic) tensor product, and by $M\times N\xrightarrow{\otimes}M\otimes_{\mathsf{R}}N$ the *universal*[28] R-bilinear map. As usually the image of $(x,y)\in M\times N$ by $\otimes$ is denoted $x\otimes y$.

Furthermore, with the following isomorphisms $(M\otimes_{\mathsf{R}}N)\otimes_{\mathsf{R}}P\xrightarrow{\alpha_{M,N,P}}M\otimes_{\mathsf{R}}(N\otimes_{\mathsf{R}}P)$, $\alpha_{M,N,P}((x\otimes y)\otimes z)=x\otimes(y\otimes z)$, $M\otimes_{\mathsf{R}}R\xrightarrow{\rho_M}M$, $\rho_M(x\otimes 1_{\mathsf{R}})=x$, $R\otimes_{\mathsf{R}}N\xrightarrow{\lambda_N}N$, $\lambda_N(1_{\mathsf{R}}\otimes y)=y$, $M\otimes_{\mathsf{R}}N\xrightarrow{\sigma_{M,N}}N\otimes_{\mathsf{R}}M$, $\sigma_{M,N}(x\otimes y)=y\otimes x$, $\mathbb{M}\mathrm{od}_{\mathsf{R}}=(\mathbf{Mod}_{\mathsf{R}},\otimes_{\mathsf{R}},R)$ is a symmetric monoidal category ([13, p. 184]). Given two sets X,Y, $R^{(X)}\otimes_{\mathsf{R}}R^{(Y)}\simeq R^{(X\times Y)}$ under the unique R-linear map $R^{(X)}\otimes_{\mathsf{R}}R^{(Y)}\xrightarrow{\Phi_{X,Y}}R^{(X\times Y)}$ which maps $f\otimes g$ to $\sum_{(x,y)\in X\times Y}f(x)g(y)\delta^{\mathsf{R}}_{(x,y)}$, with inverse $R^{(X\times Y)}\xrightarrow{\Psi_{X,Y}}R^{(X)}\otimes_{\mathsf{R}}R^{(Y)}$ the unique R-linear whose values on the basis elements are given by $\Psi_{X,Y}(\delta^{\mathsf{R}}_{(x,y)})=\delta^{\mathsf{R}}_x\otimes_{\mathsf{R}}\delta^{\mathsf{R}}_y$.

By functoriality, it is clear that given free modules M,N, $M\otimes_{\mathsf{R}}N$ is free too. As a consequence, $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}=(\mathbf{FreeMod}_{\mathsf{R}},\otimes_{\mathsf{R}},R)$ is a (symmetric) monoidal subcategory[29] of $\mathbb{M}\mathrm{od}_{\mathsf{R}}$.

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[28] In other words, each R-bilinear map $M\times N\xrightarrow{f}P$ uniquely factors as

$$\begin{tikzcd} M\times N \arrow[r, "f"] \arrow[d, "\otimes"'] & P \\ M\otimes_{\mathsf{R}}P \arrow[ur, "\tilde f"'] & \end{tikzcd} \tag{12}$$

where $\tilde f$ is R-linear.

[29] By a *(symmetric) monoidal subcategory* of a (symmetric) monoidal category $\mathbb{C}=(\mathbf{C},-\otimes-,I)$ we mean a subcategory $\mathbf{C}'$ of $\mathbf{C}$, closed under tensor products, containing I, and the coherence constraints of $\mathbb{C}$ between $\mathbf{C}'$-objects. (The last condition is automatically fulfilled when $\mathbf{C}'$ is a full subcategory.) The embedding $E_{\mathbf{C}'}$ of $\mathbf{C}'$ into $\mathbf{C}$ then is a strict monoidal functor $\mathbb{E}_{\mathbf{C}'}$ (see e.g., [15, Def. 2, p. 4876].

6.2 Topological tensor product of topologically-free modules

We now wish to take advantage of the equivalence of categories $\mathbf{FreeMod}_{\mathsf{R}} \simeq \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ (Theorem 48) for a rigid ring (R,τ), to introduce a topological tensor product of topologically-free modules. **From here to the end of Section 6.2, (R,τ) denotes a rigid ring.**

The bifunctor $\circledast_{(\mathsf{R},\tau)}$. Let $(M,\sigma),(N,\gamma)$ be two topologically-free (R,τ)-modules. One defines their *topological tensor product over* (R,τ) as

$$(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma) \coloneqq Alg_{(\mathsf{R},\tau)}((M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'). \tag{13}$$

One immediately observes that $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ still is a topologically-free (R,τ)-module as $(M,\sigma)'$ and $(N,\gamma)'$ are free R-modules (Lemma 40), $(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'$ also is (Section 6.1), and the algebraic dual of a free module is topologically-free (Lemma 36).

Actually, this definition is just the object component of a bifunctor[30]

$$\mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \xrightarrow{-\circledast_{(\mathsf{R},\tau)}-} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$$

namely[31] $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \xrightarrow{Top^{\mathsf{op}}_{(\mathsf{R},\tau)} \times Top^{\mathsf{op}}_{(\mathsf{R},\tau)}} \mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}} \times \mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}} \xrightarrow{\otimes^{\mathsf{op}}_{\mathsf{R}}} \mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$.

53 Remark *Let X,Y be sets. One has*

$$\begin{array}{rcl} (R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^Y & = & ((((R,\tau)^X)' \otimes_{\mathsf{R}} ((R,\tau)^Y)')^*, w^*_{(\mathsf{R},\tau)}) \\ & \simeq & ((R^{(X)} \otimes_{\mathsf{R}} R^{(Y)})^*, w^*_{(\mathsf{R},\tau)}) \\ & & \textit{(under } (\lambda_X \otimes_{\mathsf{R}} \lambda_Y)^* \textit{)} \\ & \simeq & ((R^{(X\times Y)})^*, w^*_{(\mathsf{R},\tau)}) \\ & & \textit{(under } \Psi^*_{X,Y}\textit{; see Section 6.1)} \\ & \simeq & (R,\tau)^{X\times Y}. \\ & & \textit{(under } \rho^{-1}_{X\times Y}\textit{)} \end{array} \tag{14}$$

Given $f_i \in \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}((M_i,\sigma_i),(N_i,\gamma_i))$, $i=1,2$, then $f_1 \circledast_{(\mathsf{R},\tau)} f_2 \coloneqq (M_1,\sigma_1) \circledast_{(\mathsf{R},\tau)} (M_2,\sigma_2) \xrightarrow{(f_1'\otimes_{\mathsf{R}} f_2')^*} (N_1,\gamma_1) \circledast_{(\mathsf{R},\tau)} (N_2,\gamma_2)$.

---

[30]By a *bifunctor* is meant a functor with domain a product of two categories.
[31]For every categories $\mathbf{C},\mathbf{D}$, $(\mathbf{C}\times\mathbf{D})^{\mathsf{op}} = \mathbf{C}^{\mathsf{op}} \times \mathbf{D}^{\mathsf{op}}$.

In details, let $L \in ((M_1,\sigma_1)' \otimes_{\mathsf{R}} (M_2,\sigma_2)')^*$, $\ell_1 \in (N_1,\gamma_1)'$ and $\ell_2 \in (N_2,\gamma_2)'$. Then,

$$\begin{array}{rcl} ((f_1 \circledast_{(\mathsf{R},\tau)} f_2)(L))(\ell_1 \otimes \ell_2) & = & (((f' \otimes_{\mathsf{R}} g')^*)(\ell))(\ell_1 \otimes \ell_2) \\ & = & L((f_1' \otimes_{\mathsf{R}} f_2')(\ell_1 \otimes \ell_2)) \\ & = & L((\ell_1 \circ f_1) \otimes (\ell_2 \circ f_2)). \end{array} \tag{15}$$

A topological basis of $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$. Our next goal will be to explicitly describe a topological basis (Definition 28) of $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ in terms of topological bases of (M,σ) and (N,γ).

54 Definition *Given a ring S, for every S-modules M, N, one has a natural R-linear map $M^* \otimes_{\mathsf{S}} N^* \xrightarrow{\Theta_{M,N}} (M \otimes_{\mathsf{S}} M)^*$ given by $(\Theta_{M,N}(\ell_1 \otimes \ell_2))(u \otimes v) = \ell_1(u)\ell_2(v)$, $\ell_1 \in M^*$, $\ell_2 \in N^*$, $u \in M$ and $v \in N$.*

Let $(M,\sigma), (N,\gamma)$ be two topologically-free (R,τ)-modules. Let $u \in M$ and $v \in N$. Let us define

$$u \circledast v := \Theta_{(M,\sigma)',(N,\gamma)'}(\Gamma_{(M,\sigma)}(u) \otimes_{\mathsf{R}} \Gamma_{(N,\gamma)}(v)) \in (M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma). \tag{16}$$

In details, given $\ell_1 \in (M,\sigma)'$ and $\ell_2 \in (N,\gamma)'$, $(u \circledast v)(\ell_1 \otimes \ell_2) = \ell_1(u)\ell_2(v)$.

55 Lemma *Let (M,σ) and (N,γ) be both topologically-free (R,τ)-modules, with respective topological bases B, D. The map $B \times D \xrightarrow{-\circledast-} (M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ given by $(b,d) \mapsto b \circledast d$, is one-to-one.*

56 Lemma *Let (M,σ) and (N,γ) be both topologically-free (R,τ)-modules. The map $M \times N \xrightarrow{-\circledast-} (M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ is R-bilinear and separately continuous in both variable. Moreover, if $\tau = \mathsf{d}$, then $\circledast$ is even jointly continuous.*

Proof: R-bilinearity is clear. Since $(M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'$ is free on $\{\, x \otimes y : x \in X,\ y \in Y \,\}$, where X (resp. Y) is a basis of $(M,\sigma)'$ (resp. $(N,\gamma)'$), by Lemma 12, the topology $w^*_{(\mathsf{R},\tau)}$ on $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ is the initial topology induced by $((M,\sigma)' \otimes_R ((N,\mu)')^* \xrightarrow{\Lambda_{(M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'}(x \otimes y)} (R,\tau)$, $x \in X$, $y \in Y$.

Let $x \in X$, $y \in Y$, $u \in M$ and $v \in N$. Then, $\Lambda_{(M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'}(x \otimes y)(u \circledast v) = (u \circledast v)(x \otimes y) = x(u)y(v) = m_{\mathsf{R}}(x(u), x(v))$, and this automatically guarantees separate continuity in each variable of $\circledast$.

Let us assume that $\tau = \mathsf{d}$. According to the above general case, to see that $\circledast$ is continuous, by [19, Theorem 2.14, p. 17], it suffices to prove continuity at zero of $\circledast$. Let $A \subseteq X \times Y$ be a finite set, and for each $(x,y) \in A$, let $U_{(x,y)}$ be an open neighborhood of zero in (R, d). Let $A_1 \coloneqq \{ x \in X \colon \exists y \in Y,\ (x,y) \in A \}$ and $A_2 \coloneqq \{ y \in Y \colon \exists x \in X,\ (x,y) \in A \}$. A_1, A_2 are both finite and $A \subseteq A_1 \times A_2$. Let $u \in M$ such that $x(u) = 0$ for all $x \in A_1$, and $v \in N$ such that $y(v) = 0$ for all $y \in A_2$. Then, $(u \circledast v)(x \otimes y) = 0 \in U_{(x,y)}$ for all $(x,y) \in A_1 \times A_2$. □

57 Remark *Let X, Y be sets, and let $f \in R^X$, $g \in R^Y$. Since $\circledast$ is separately continuous by Lemma 56,*[32]

$$\begin{array}{rcl} f \circledast g & = & \left(\sum_{x\in X} f(x)\delta_x^{\mathsf{R}}\right) \circledast \left(\sum_{y \in Y} g(y)\delta_y^{\mathsf{R}}\right) \\ & = & \sum_{(x,y)\in X\times Y} f(x)g(y)\delta_x^{\mathsf{R}} \circledast \delta_y^{\mathsf{R}}. \\ & & \textit{(as a sum of a summable family)} \end{array} \tag{17}$$

For the same reason as above, if B is a topological basis of (M,σ) and D is a topological basis of (N,γ), then $u \circledast v = \sum_{(b,d)\in B\times D} b'(u)d'(v) b \circledast d$, $u \in M$, $v \in N$. In particular, one observes that $(b \circledast d)'(u \circledast v) = b'(u)d'(v)$, $b \in B$, $u \in M$, $d \in D$, and $v \in N$.

58 Proposition *Let (M,σ) and (N,γ) be topologically-free (R,τ)-modules, with respective topological bases B, D. Then, $(b \circledast d)_{(b,d)\in B\times D}$ is a topological basis of $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$.*

Proof: By virtue of Lemma 30 and Remark 53, $\{ (\delta_x^{\mathsf{R}} \circledast \delta_y^{\mathsf{R}})_{(x,y)\in X\times Y} \colon (x,y) \in X \times Y \}$ is a topological basis of $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^Y$ since one has $((\lambda_X^{-1} \otimes_{\mathsf{R}} \lambda_Y^{-1})^* (\Phi_{X,Y}^* (\rho_{X\times Y}(\delta_{(x,y)}^{\mathsf{R}})))) = \delta_x^{\mathsf{R}} \circledast \delta_y^{\mathsf{R}}$. Let us consider the isomorphism $\Theta_B \colon (M,\sigma) \simeq (R,\tau)^B$, $\Theta_B(b) = \delta_b$, $b \in B$ (resp. $\Theta_D \colon (N,\gamma) \simeq (R,\tau)^D$). By functoriality, $\Theta_B \circledast_{(\mathsf{R},\tau)} \Theta_D \colon (M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma) \simeq (R,\tau)^B \circledast_{(\mathsf{R},\tau)} (R,\tau)^D$, and since $(\theta_B \circledast_{(\mathsf{R},\tau)} \theta_D)(b \circledast d) = \delta_b \circledast \delta_d$, $(b,d) \in B \times D$, $\{ b \circledast d \colon (b,d) \in B \times D \}$ is a topological basis of $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$. □

59 Corollary *Let (M,σ) and (N,γ) be topologically-free (R,τ)-modules. Then, $\{ u \circledast v \colon u \in M,\ v \in N \}$ spans a dense subspace in $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$.*

Proof: It is clear in view of Lemma 33. □

Let (M_i, σ_i) and (N_i, γ_i), $i = 1, 2$, be topologically-free (R,τ)-modules. Let $(M_i,\sigma_i) \xrightarrow{f_i} (N_i, \gamma_i)$, $i = 1, 2$, be continuous (R,τ)-linear maps. Let $(u,v) \in M_1 \times M_2$. By Eq. (15) it is clear that

$$(f_1 \circledast_{(\mathsf{R},\tau)} f_2)(u \circledast v) = f_1(u) \circledast f_2(v). \tag{18}$$

[32] The second equality in Eq. (17) follows from the proof of [19, Theorem 10.15, p. 78] which, by inspection, shows that the cited result still is valid more generally after the replacement of a jointly continuous bilinear map by a separately continuous bilinear map.

If B and D are topological bases of (M_1,σ_1) and (M_2,σ_2) respectively, $(f_1 \circledast_{(\mathsf{R},\tau)} f_2)(u \circledast v) = \sum_{(b,d)\in B\times D} b'(u)d'(v)f_1(b) \circledast f_2(d)$ (in view of Remark 57).

6.3 Monoidality of $\circledast_{(\mathsf{R},\tau)}$ and its direct consequences

Since most of the proofs from this section mainly consist in rather tedious, but simple, inspections of commutativity of some diagrams, essentially by working with given topological or linear bases[33], and because they did not provide much understanding, they are not included in the presentation. The reader is kindly invited to consult Appendix A where are summarized some notions and notations about monoidal category theory.

60 Proposition *Let (R,τ) be a rigid ring.*

$$\mathbb{TopFreeMod}_{(\mathsf{R},\tau)} \coloneqq (\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}, \circledast_{(\mathsf{R},\tau)}, (R,\tau))$$

is a symmetric monoidal category[34].

61 Corollary *For each field $(\Bbbk,\tau)$ with a ring topology,*

$$\mathbb{TopFreeVect}_{(\Bbbk,\tau)} \coloneqq (\mathbf{TopFreeVect}_{(\Bbbk,\tau)}, \circledast_{(\Bbbk,\tau)}, (\Bbbk,\tau))$$

is a symmetric monoidal category.

62 Example *Let (R,τ) be a rigid ring. Let X be a set. Let us define a commutative monoid $M_{(\mathsf{R},\tau)}(X) \coloneqq ((R,\tau)^X, \mu_X, \eta_X)$ in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ by $\mu_X(f \circledast g) = \sum_{x\in X} f(x)g(x)\delta_x^{\mathsf{R}}$, $f,g \in R^X$ (under $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^X \simeq (R,\tau)^{X\times X}$ from Remark 53) and $\eta_X(1_{\mathsf{R}}) = \sum_{x\in X} \delta_x^{\mathsf{R}}$. This provides a functor $\mathbf{Set}^{\mathsf{op}} \xrightarrow{M_{(\mathsf{R},\tau)}} {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$.*

Let (R,τ) be a rigid ring. For each free R-modules M,N, let us define $\Phi_{M,N} \coloneqq (M^*, w^*_{(\mathsf{R},\tau)}) \circledast_{(\mathsf{R},\tau)} (N^*, w^*_{(\mathsf{R},\tau)}) = Alg_{(\mathsf{R},\tau)}((M^*, w^*_{(\mathsf{R},\tau)})' \otimes_{\mathsf{R}} (N^*, w^*_{(\mathsf{R},\tau)})') \xrightarrow{(\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*} ((M \otimes_{\mathsf{R}} N)^*, w^*_{(\mathsf{R},\tau)})$. According to Proposition 44, $\Phi_{M,N}$ is an isomorphism in $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$. Naturality in M,N is clear, so this provides a natural isomorphism

$$\begin{aligned}\Phi\colon & Alg_{(\mathsf{R},\tau)}(-) \circledast_{(\mathsf{R},\tau)} Alg_{(\mathsf{R},\tau)}(-) \Rightarrow Alg_{(\mathsf{R},\tau)}(- \otimes_{\mathsf{R}} -)\colon \\ & \mathbf{FreeMod}_{\mathsf{R}}^{\mathsf{op}} \times \mathbf{FreeMod}_{\mathsf{R}}^{\mathsf{op}} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}.\end{aligned} \tag{19}$$

[33] E.g., associativity of $\circledast_{(\mathsf{R},\tau)}$ is given by the isomorphism $(b \circledast d) \circledast e \mapsto b \circledast (d \circledast e)$ on basis elements (Lemma 35).

[34] See [13] for the definition of a symmetric monoidal category.

Furthermore, let us consider the isomorphism $(R,\tau) \xrightarrow{\phi} (R^*, w^*_{(\mathsf{R},\tau)})$ given by $\phi(1_{\mathsf{R}}) \coloneqq id_R$, with inverse $\phi^{-1}(\ell) = \ell(1_{\mathsf{R}})$.

Let $(M,\sigma),(N,\gamma)$ be topologically-free (R,τ)-modules. One defines the map $\Psi_{(M,\sigma),(N,\gamma)} \coloneqq (M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)' \xrightarrow{\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}} (Alg_{(\mathsf{R},\tau)}((M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'))' = ((M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma))'$. This gives rise to a natural isomorphism

$$\begin{aligned}&\Psi: Top_{(\mathsf{R},\tau)}(-) \otimes_{\mathsf{R}} Top_{(\mathsf{R},\tau)}(-) \Rightarrow Top_{(\mathsf{R},\tau)}(- \circledast_{(\mathsf{R},\tau)} -):\\ &\mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \to \mathbf{FreeMod}_{\mathsf{R}}.\end{aligned} \tag{20}$$

Let also $R \xrightarrow{\psi} (R,\tau)'$ be given by $\psi(1_{\mathsf{R}}) = id_R$ and $\psi^{-1}(\ell) = \ell(1_{\mathsf{R}})$.

63 Theorem *Let (R,τ) be a rigid ring.*

1. $\mathbb{Alg}_{(\mathsf{R},\tau)} = (Alg_{(\mathsf{R},\tau)}, \Phi, \phi): \mathbb{FreeMod}^{\mathsf{op}}_{\mathsf{R}} \to \mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *is a strong symmetric monoidal functor.*

2. $\mathbb{Top}_{(\mathsf{R},\tau)} = (Top_{(\mathsf{R},\tau)}, \Psi, \psi)$ *is a strong symmetric monoidal functor from* $\mathbb{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ *to* $\mathbb{FreeMod}_{\mathsf{R}}$*, so is* $\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}$ *(Remark 83, Appendix A.2) from* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *to* $\mathbb{FreeMod}^{\mathsf{op}}_{\mathsf{R}}$.

3. $\Lambda^{\mathsf{op}}: \mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)} \circ \mathbb{Alg}_{(\mathsf{R},\tau)} \Rightarrow \mathbb{id}: \mathbb{FreeMod}^{\mathsf{op}}_{\mathsf{R}} \to \mathbb{FreeMod}^{\mathsf{op}}_{\mathsf{R}}$ *is a monoidal isomorphism.*

4. $\Gamma: \mathbb{id} \Rightarrow \mathbb{Alg}_{(\mathsf{R},\tau)} \circ \mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}: \mathbb{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *is a monoidal isomorphism.*

In particular, $\mathbb{FreeMod}^{\mathsf{op}}_{\mathsf{R}}$ *and* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *are monoidally equivalent (Appendix A.2).*

64 Corollary *For each field $(\Bbbk,\tau)$ with a ring topology, the monoidal categories* $\mathbb{Vect}^{\mathsf{op}}_{\Bbbk}$ *and* $\mathbb{TopFreeVect}_{(\Bbbk,\tau)}$ *are monoidally equivalent.*

65 Corollary *For each rigid ring (R,τ), the induced natural transformations (see Remark 85 in Appendix A.2 and Example 79 in Appendix A.1)*

- $\widetilde{\Lambda^{\mathsf{op}}}: (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}}) \circ \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \Rightarrow id_{{}_\epsilon\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}}}: {}_\epsilon\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}} \to {}_\epsilon\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}}$,

- $\widetilde{\Lambda^{\mathsf{op}}}: (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}}) \circ \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \Rightarrow id_{{}_{\epsilon,coc}\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}}}: {}_{\epsilon,coc}\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}} \to {}_{\epsilon,coc}\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}}$,

- $\tilde{\Gamma}: id_{\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})} \Rightarrow \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}})$:
$\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \to \mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$,

- $\tilde{\Gamma}\colon id_{{}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})} \Rightarrow \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \circ (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})$: ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \to {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$.

are all natural isomorphisms.

Thus ${}_\epsilon\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}}$ *(resp.,* ${}_{\epsilon,coc}\mathbf{Coalg}^{\mathsf{op}}_{\mathsf{R}}$*) and* $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$ *(resp.,* ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$*) are equivalent for each rigid topology* τ *on* R.

In particular, $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \simeq \mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\sigma)})$ *(resp.,* ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \simeq {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\sigma)})$*), for each rigid topologies* τ,σ *on* R.

Proof: Follows from Theorem 63 together with Remarks 85 and 86 in Appendix A.2. □

66 Example *Let us make explicit the domain and codomain of the natural isomorphism* $\tilde{\Gamma}$ *from Corollary 65.*

Let $((M,\sigma),m,e)$ *be an object of* $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$. *Let its* topological dual coalgebra *be*

$$\begin{array}{rcl} ((M,\sigma)',\delta,\epsilon) & \coloneqq & (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})((M,\sigma),m,e) \\ & = & ((M,\sigma)',\Lambda^{-1}_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'} \circ m',\psi^{-1}\circ e'). \end{array} \tag{21}$$

In details, for $\ell \in (M,\sigma)'$, $\epsilon(\ell) = \psi^{-1}(e'(\ell)) = \psi^{-1}(\ell\circ e) = \ell(e(1_{\mathsf{R}}))$, *and* $\delta(\ell) = ((\Lambda^{-1}_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'} \circ m')(\ell)) = \sum_{i=1}^n \ell_i \otimes r_i$ *for some* $\ell_i, r_i \in (M,\sigma)'$. *Thus,* $\ell(m(u\circledast v)) = \sum_{i=1}^n \ell_i(u)r_i(v)$, $u,v\in M$.

Now, $(((M,\sigma)')^*,M,E) \coloneqq \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}((M,\sigma)',\delta,\epsilon)$ *is given by* $M \coloneqq \delta^*\circ(\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})^*$ *and* $E \coloneqq \epsilon^*\circ\phi$. *Thus* $E(1_{\mathsf{R}}) = \epsilon^*(\phi(1_{\mathsf{R}})) = \epsilon^*(id_R) = \epsilon$, *and given* $L_1, L_2 \in ((M,\sigma)')^*$, *and* $\ell\in(M,\sigma)'$,

$$\begin{array}{rcl} (M(L_1\circledast L_2))(\ell) & = & (\delta^*((\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})^*(L_1\circledast L_2)))(\ell) \\ & = & ((\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})^*(L_1\circledast L_2))(\delta(\ell)) \\ & = & (L_1\circledast L_2)((\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})(\delta(\ell))) \\ & = & (L_1\circledast L_2)(\sum_{i=1}^n \Lambda_{(M,\sigma)'}(\ell_i)\otimes\Lambda_{(M,\sigma)'}(r_i)) \\ & = & \sum_{i=1}^n \Lambda_{(M,\sigma)'}(\ell_i)(L_1)\Lambda_{(M,\sigma)'}(r_i)(L_2) \\ & = & \sum_{i=1}^n L_1(\ell_i)L_2(r_i). \end{array} \tag{22}$$

67 Corollary *The equivalence from Corollary 65 restricts to an equivalence between the category* ${}_\epsilon\mathbf{FinDimCoalg}_{\mathbb{k}}$ *(resp.* ${}_{\epsilon,coc}\mathbf{FinDimCoalg}_{\mathbb{k}}$*) of finite-dimensional (resp. cocommutative) coalgebras and* $\mathbf{Mon}(\mathbb{FinDimVect}_{\mathbb{k}})$ *(resp.* ${}_c\mathbf{Mon}(\mathbb{FinDimVect}_{\mathbb{k}})$*), where* $\mathbb{FinDimVect}_{\mathbb{k}} = (\mathbf{FinDimVect}_{\mathbb{k}},\otimes_{\mathbb{k}},\mathbb{k})$.

7 Relationship with finite duality

Over a field, there is a standard and well-known notion of duality between algebras and coalgebras, known as the finite duality [1, 7] and we have the intention to understand the relations if any, between the equivalence of categories from Corollary 65 and this finite duality.

Let $(-)^*\colon \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \to \mathbf{Mod}_{\mathsf{R}}$ be the usual algebraic dual functor. Then, $\mathbb{D}_* := ((-)^*, \Theta, \theta)$ is a lax symmetric monoidal functor from $\mathbb{Mod}_{\mathsf{R}}^{\mathsf{op}}$ to $\mathbb{Mod}_{\mathsf{R}}$ (where Θ is as in Definition 54, and $\theta\colon R \to R^*$ is the isomorphism $\theta(1_{\mathsf{R}}) = id_{\mathsf{R}}$ (and $\theta^{-1}(\ell) = \ell(1_{\mathsf{R}})$)). When $\Bbbk$ is a field, there is the *finite dual functor* $D_{fin}\colon \mathbf{Mon}(\mathbb{Vect}_{\Bbbk})^{\mathsf{op}} \to {}_{\epsilon}\mathbf{Coalg}_{\Bbbk}$ (denoted by $(-)^0$ in [1, 7]).The aforementioned finite duality is the adjunction $D_{fin}^{\mathsf{op}} \dashv \widetilde{\mathbb{D}_*}\colon {}_{\epsilon}\mathbf{Coalg}_{\Bbbk} \to \mathbf{Mon}(\mathbb{Vect}_{\Bbbk})$ (see e.g., [7, Theorem 1.5.22, p. 44], where $\widetilde{\mathbb{D}_*}$ is denoted by $(-)^*$).

7.1 The underlying algebra

Let (R, τ) be a rigid ring. Let $(M, \sigma), (N, \gamma)$ be topologically-free (R, τ)-modules. According to Lemma 56, $M \times N \xrightarrow{-\circledast-} (M, \sigma) \circledast_{(\mathsf{R},\tau)} (N, \gamma)$ is R-bilinear. Denoting by $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \xrightarrow{\|-\|} \mathbf{Mod}_{\mathsf{R}}$ the canonical forgetful functor, this means that there is a unique R-linear map $\|(M,\sigma)\| \otimes_{\mathsf{R}} \|(N,\gamma)\| \xrightarrow{\Xi_{(M,\sigma),(N,\gamma)}} \|(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)\|$ such that for each $u \in M$, $v \in N$, $\Xi_{(M,\sigma),(N,\gamma)}(u \otimes v) = u \circledast v$.

68 Lemma $\mathbb{A} := (\| - \|, (\Xi_{(M,\sigma),(N,\gamma)})_{(M,\sigma),(N,\gamma)}, id_R)$ *is a lax symmetric monoidal functor from* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *to* $\mathbb{Mod}_{\mathsf{R}}$.

Let $\tilde{\mathbb{A}}\colon \mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \to \mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}})$ be the functor induced by $\mathbb{A}$ as introduced in Appendix A.2. Using the functorial isomorphism $O\colon \mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}}) \simeq {}_1\mathbf{Alg}_{\mathsf{R}}$ (Example 79, Appendix A.1), to any monoid in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ is associated an ordinary algebra.

69 Definition *Let us define* $UA := \mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \xrightarrow{O\circ\tilde{\mathbb{A}}} {}_1\mathbf{Alg}_{\mathsf{R}}$. *Given a monoid* $((M,\sigma),\mu,\eta)$ *in* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$, *then* $UA((M,\sigma),\mu,\eta) = O(\tilde{\mathbb{A}}((M,\sigma),\mu,\eta))$ *is referred to as the* underlying (ordinary) algebra *of the monoid* $((M,\sigma),\mu,\eta)$. *In details,* $UA((M,\sigma),\mu,\eta) = (M, \mu_{bil}, \eta(1_{\mathsf{R}}))$ *with* $\mu_{bil}\colon M \times M \to M$ *given by* $\mu_{bil}(u,v) := \mu(u \circledast v)$.

70 Remark *Since by Lemma 68,* $\mathbb{A}$ *is symmetric, it induces a functor (see Remark 83, Appendix A.2)* ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \xrightarrow{\tilde{\mathbb{A}}} {}_c\mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}})$. *Be-*

cause one has the co-restriction ${}_c\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}}) \xrightarrow{O} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$*, one may consider the* underlying algebra *functor* $UA = {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \xrightarrow{O\circ\tilde{\mathbb{A}}} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$.

71 Example *(Continuation of Example 62)* $UA(M_{(\mathsf{R},\tau)}(X)) = A_{\mathsf{R}}X$.

7.2 Relations with $\widetilde{\mathbb{D}_*}$

Let (R,τ) be a rigid ring. Let $\mathbf{FreeMod}_{\mathsf{R}} \xrightarrow{E} \mathbf{Mod}_{\mathsf{R}}$ be the canonical embedding functor. Since $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ is a symmetric monoidal subcategory of $\mathbb{M}\mathrm{od}_{\mathsf{R}}$ it follows that $\mathbb{E} = (E, id, id)$ is a strict monoidal functor from $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ to $\mathbb{M}\mathrm{od}_{\mathsf{R}}$ (see Appendix A.2).

One claims that $\mathbb{D}_* \circ \mathbb{E}^{\mathrm{op}} = \mathbb{A} \circ \mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)}$. In particular, if $\Bbbk$ is a field (and τ is a ring topology on $\Bbbk$), then this reduces to $\mathbb{D}_* = \mathbb{A} \circ \mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}$.

That $\|-\| \circ Alg_{(\mathsf{R},\tau)} = (-)^* \circ E^{\mathrm{op}}$ is due to the very definition of $Alg_{(\mathsf{R},\tau)}$. Of course, $\|\phi\| = \theta$. That for each free modules M, N, $\|(\Lambda_M \otimes \Lambda_N)^*\| \circ \Xi_{M^*,N^*} = \Theta_{M,N}$ is easy to check. So $((-)^* \circ E^{\mathrm{op}}, \Theta, \theta) = (\|-\|, \Xi, id_R) \circ (Alg_{(\mathsf{R},\tau)}, \Phi, \phi) = (\|-\| \circ Alg_{(\mathsf{R},\tau)}, (\|\Phi_{M,N}\| \circ \Xi_{M^*,N^*})_{M,N}, \|\phi\|)$.

It follows that the finite dual monoid[35] $\widetilde{\mathbb{D}_*}(\mathsf{C})$ of a $\Bbbk$-coalgebra C, is equal to $\tilde{\mathbb{A}}(\widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}}(\mathsf{C}))$ whatever the ring topology τ on the field $\Bbbk$, and thus as ordinary algebras, $O(\widetilde{\mathbb{D}_*}(\mathsf{C})) = UA(\widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}}(\mathsf{C}))$.

72 Proposition *Let* $(\Bbbk,\tau)$ *be a field with a ring topology. The functor* $\mathbf{Mon}(\mathbb{F}\mathrm{ree}\mathbb{T}\mathrm{op}\mathbb{V}\mathrm{ect}_{(\Bbbk,\tau)}) \xrightarrow{O\circ\tilde{\mathbb{A}}} {}_1\mathbf{Alg}_{\Bbbk}$ *has a left adjoint, namely* $\widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}} \circ D_{fin}^{\mathrm{op}} \circ O^{-1}$.

Proof: One has $\widetilde{\mathbb{D}_*} = \tilde{\mathbb{A}} \circ \widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}}$, whence $\widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\Bbbk,\tau)}}) = \tilde{\mathbb{A}} \circ \widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}} \simeq \tilde{\mathbb{A}}$ (natural isomorphism) by Theorem 63. Since $\widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\tau)}} \circ D_{fin}^{\mathrm{op}}$ is a left adjoint of $\widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\Bbbk,\tau)}})$, it follows that[36] it is also the left adjoint of $\tilde{\mathbb{A}}$. $\square$

7.3 The underlying topological algebra

Let R be a ring. Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ be an object of $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\mathsf{d})})$. One knows that (A,σ) is an object of $\mathbf{TopMod}_{(\mathsf{R},\mathsf{d})}$ and $UA(\mathsf{A})$ is an ob-

[35] In $\mathbb{V}\mathrm{ect}_{\Bbbk}$.

[36] Because if F, G are two naturally isomorphic functors and L is a left adjoint of F, then it is also a left adjoint of G.

ject of ${}_1\mathbf{Alg}_{\mathsf{R}}$. Moreover, $(A,\sigma)\times(A,\sigma) \xrightarrow{\mu_{bil}} (A,\sigma)$ is continuous, since it is equal to the composition $(A,\sigma)\times(A,\sigma) \xrightarrow{-\circledast-} (A,\sigma)\circledast_{(\mathsf{R},\mathsf{d})}(A,\sigma) \xrightarrow{\mu} (A,\sigma)$ of continuous maps (see Lemma 56). Now, let $((A,\sigma),\mu,\eta) \xrightarrow{f} ((B,\gamma),\nu,\zeta)$ be a morphism in $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$. In particular, $(A,\sigma) \xrightarrow{f} (B,\gamma)$ is linear and continuous, and the following diagram commutes.

$$\begin{tikzcd}
(A,\sigma)\times(A,\sigma) \arrow[dddd, "f\times f"'] \arrow[dr, "-\circledast-"'] \arrow[drr, bend left, "\mu_{bil}"] & & \\
& (A,\sigma)\circledast_{(\mathsf{R},\mathsf{d})}(A,\sigma) \arrow[r, "\mu"] \arrow[d, "f\circledast_{(\mathsf{R},\mathsf{d})}f"'] & (A,\sigma) \arrow[d, "f"] \\
& (B,\gamma)\circledast_{(\mathsf{R},\mathsf{d})}(B,\gamma) \arrow[r, "\nu"'] & (B,\gamma) \\
& & \\
(B,\gamma)\times(B,\gamma) \arrow[uur, "-\circledast-"'] \arrow[uurr, bend right, "\nu_{bil}"'] & &
\end{tikzcd} \tag{23}$$

Since by assumption, one also has $f\circ\eta = \zeta$, it follows that $f(\eta(1_{\mathsf{R}})) = \zeta(1_{\mathsf{R}})$, and thus f is a continuous algebra map from $((A,\sigma),\mu_{bil},\eta(1_{\mathsf{R}}))$ to $((B,\nu_{bil},\zeta(1_{\mathsf{R}}))$.)

This provides a *topological algebra* functor $\mathbf{Mon}(\mathbb{TopFreeMod}_{(R,\mathsf{d})}) \xrightarrow{TA} {}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ and the following diagram commutes (the unnamed arrows are either the obvious forgetful functors or the evident embedding functor), so that TA is concrete[37] over $\mathbf{TopMod}_{(\mathsf{R},\mathsf{d})}$, whence faithful.

$$\begin{tikzcd}
& \mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}) \arrow[r, "TA"] \arrow[d] \arrow[dl, bend right, "\tilde{\mathbb{A}}"'] & {}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})} \arrow[d] \arrow[ddr, bend left] & \\
& \mathbf{TopFreeMod}_{(\mathsf{R},\mathsf{d})} \arrow[r, hook] \arrow[d, "\|\cdot\|"] & \mathbf{TopMod}_{(\mathsf{R},\mathsf{d})} \arrow[dl] & \\
\mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}}) \arrow[r] \arrow[rrr, bend right, "O"'] & \mathbf{Mod}_{\mathsf{R}} & & {}_1\mathbf{Alg}_{\mathsf{R}} \arrow[ll]
\end{tikzcd} \tag{25}$$

[37]A *concrete category* $\mathbf{C}$ over $\mathbf{D}$ is a pair $(\mathbf{C},\mathbf{C}\xrightarrow{U}\mathbf{D})$ with U a faithful functor. Given concrete categories $(\mathbf{C}_i,U_i)$, $i=1,2$, over $\mathbf{D}$, by a *concrete functor* $(\mathbf{C}_1,U_1)\xrightarrow{F}(\mathbf{C}_2,U_2)$ is meant an ordinary functor $\mathbf{C}_1\xrightarrow{F}\mathbf{C}_2$ such that the following diagram commutes. Such a functor is necessarily faithful.

$$\begin{tikzcd}
\mathbf{C}_1 \arrow[rr, "F"] \arrow[dr, "U_1"'] & & \mathbf{C}_2 \arrow[dl, "U_2"] \\
& \mathbf{D} &
\end{tikzcd} \tag{24}$$

73 Remark *When* A *is a commutative monoid in* $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$*, then* $TA(\mathsf{A})$ *is a commutative topological algebra.*

74 Example *For each set* X*,* $TA(M_{(\mathsf{R},\mathsf{d})}(X)) = A_{(\mathsf{R},\mathsf{d})}(X)$*.*

75 Proposition TA *is a full embedding (injective on objects and faithful).*

Proof: Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ and $\mathsf{B} = ((B,\gamma),\nu,\zeta)$ be two monoids in $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$. Let $TA(\mathsf{A}) \xrightarrow{g} TA(\mathsf{B})$ be a morphism in ${}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$. In particular, $g \in {}_1\mathbf{Alg}_{\mathsf{R}}(UA(\mathsf{A}),UA(\mathsf{B})) \cap \mathbf{Top}((|A|,\sigma),(|B|,\gamma))$. (Recall from Remark 5 that $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$ is the usual forgetful functor, and **Top** is the category of Hausdorff topological spaces.)

By assumption, for each $u,v \in A$, one has $g(\mu(u \otimes v)) = g(\mu_{bil}(u,v)) = \nu_{bil}(g(u),g(v)) = \nu(g(u)\otimes g(v))$. Thus, $g\circ\mu = \nu\circ(g\otimes_{(\mathsf{R},\mathsf{d})} g)$ on $\{\, u\otimes v: u,v \in A\,\}$. Since this set spans a dense subset of $(A,\sigma)\otimes_{(R,\tau)}(A,\sigma)$ (according to Corollary 59), by linearity and continuity, $g\circ\mu = \nu\circ(g\otimes_{(\mathsf{R},\mathsf{d})} g)$ on the whole of $(A,\sigma)\otimes_{(\mathsf{R},\mathsf{d})}(A,\sigma)$.

Moreover, $g(\eta(1_{\mathsf{R}})) = \zeta(1_{\mathsf{R}})$, then $g\circ\eta = \zeta$. Therefore, g may be seen as a morphism $\mathsf{A} \xrightarrow{f} \mathsf{B}$ in $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$ with $TA(f) = g$, i.e., TA is full.

Let $\mathsf{A} = ((A,\sigma),\mu,\eta), \mathsf{B} = ((B,\gamma),\nu,\zeta)$ be monoids in $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$ such that $TA(\mathsf{A}) = TA(\mathsf{B})$. In particular, $(A,\sigma) = (B,\gamma)$, and $\eta = \zeta$. By assumption $\mu_{bil} = \nu_{bil}$. Whence $\mu = \nu$ on $\{\, u\otimes v: u \in A,\ v \in B\,\}$, and by continuity they are equal on $(A,\sigma)\otimes_{(\mathsf{R},\mathsf{d})}(B,\gamma)$. So $\mathsf{A} = \mathsf{B}$, i.e., TA is injective on objects. □

As a consequence of Proposition 75, $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$ is isomorphic to a full subcategory of ${}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ ([2, Proposition 4.5, p. 49]). Accordingly a monoid in $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$ is essentially a topological algebra.

It is clear that ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}) \xrightarrow{TA} {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ of TA (see Remark 73) also is a full embedding functor.

7.4 Relations with D_{fin}

Let V be any vector space on a field $\Bbbk$. Then, V^* has a somewhat natural topology called the *V-topology* ([1]) or the *finite topology* ([7]), with a fundamental system of neighborhoods of zero consisting of spaces

$$W^\dagger := \{\, \ell \in V^*: \forall w \in W,\ \ell(w) = 0\,\} \tag{26}$$

where W runs over the finite-dimensional subspaces of V. This is manifestely the same topology as our $w^*_{(\mathbb{k},\mathsf{d})}$ (see Section 3.1). Accordingly this turns V^* into a linearly compact $\mathbb{k}$-vector space (p. 19). The closed subspace of $(V^*, w^*_{(\mathbb{k},\mathsf{d})})$ are exactly the subspaces of the form $W^\dagger$, where W is any subspace of V ([4, Proposition 24.4, p. 105]).

76 Lemma *Let W be a subspace of V. $\operatorname{codim}(W^\dagger)$ is finite if, and only if, $\dim(W)$ is finite. In this case, $\operatorname{codim}(W^\dagger) = \dim(W)$.*

Proof: One observes that $V^*/W^\dagger \simeq W^*$ because the map $V^* \xrightarrow{incl_W^*} W^*$ is onto, where $W \xrightarrow{incl_W} V$ is the canonical inclusion, and $\ker incl_W^* = W^\dagger$. Since $V^*/W^\dagger \simeq W^*$, it follows that $\operatorname{codim}(W^\dagger) = \dim W^*$. □

77 Theorem *Let $\mathbb{k}$ be a field. For each monoid A in $\mathbb{TopFreeVect}_{(\mathbb{k},\mathsf{d})}$, $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathbb{k},\mathsf{d})}})(\mathsf{A})$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. Furthermore, the assertions below are equivalent.*

1. *In $TA(\mathsf{A})$ every finite-codimensional ideal is closed.*
2. *$\tilde{\mathbb{A}}(A)$ is reflexive*[38]*.*
3. *The coalgebra $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathbb{k},\mathsf{d})}})(\mathsf{A})$ is coreflexive*[39]*.*
4. *$(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathbb{k},\mathsf{d})}})(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$.*

Proof: Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ be a monoid in $\mathbb{TopFreeMod}_{(\mathbb{k},\mathsf{d})}$. Whence its underlying topological vector space is a linearly compact vector space (p. 19). Let $\mathsf{C} := (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathbb{k},\mathsf{d})}})(\mathsf{A})$. Since $\tilde{\mathbb{A}} \circ \widetilde{\mathbb{Alg}_{(\mathbb{k},\mathsf{d})}} = \widetilde{\mathbb{D}_*}$ it follows that $\tilde{\mathbb{A}} \simeq \tilde{\mathbb{A}} \circ \widetilde{\mathbb{Alg}_{(\mathbb{k},\mathsf{d})}} \circ (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathbb{k},\mathsf{d})}}) \simeq \widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathbb{k},\mathsf{d})}})$ (naturally isomorphic). In particular, $\tilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}(\mathsf{C})$. By construction, the underlying topological vector space of A, namely (A,σ), is also the underlying topological vector space of $TA(\mathsf{A})$. Also $\mathsf{A}, TA(\mathsf{A})$ and $\tilde{\mathbb{A}}(\mathsf{A})$ share the same underlying vector space A, which is isomorphic to C^*, where C is the underlying vector space of the coalgebra C. Of course, $(A,\sigma) \simeq Alg_{(\mathbb{k},\mathsf{d})}(C) = (C^*, w^*_{(\mathbb{k},\mathsf{d})})$. Therefore, up to such an isomorphism, (A,σ) has a fundamental system of neighborhoods

[38] A monoid A in $\mathbb{Vect}_{\mathbb{k}}$ is *reflexive* when $\mathsf{A} \simeq \widetilde{\mathbb{D}_*}(D^{\mathsf{op}}_{fin}(\mathsf{A}))$ under the linear map $u \mapsto (\ell \mapsto \ell(u))$, which is the unit of the adjunction $D^{\mathsf{op}}_{fin} \dashv \widetilde{\mathbb{D}_*}$.

[39] A coalgebra C is *coreflexive*, when $\mathsf{C} \simeq D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{D}_*}(\mathsf{C}))$ under the natural inclusion $u \mapsto (\ell \mapsto \ell(u))$, which is the counit of $D^{\mathsf{op}}_{fin} \dashv \widetilde{\mathbb{D}_*}$.

of zero consisting of $V^\dagger = \{\,\ell \in A\colon \forall v \in V,\ \ell(v) = 0\,\}$ where V is a finite-dimensional subspace of C (see Eq. (26)).

Let $\ell \in (A,\sigma)'$. By continuity of ℓ, there exists a finite-dimensional subspace V of C such that $V^\dagger \subseteq \ker \ell$. Let B be a (finite) basis of V, and let D be the (necessarily finite-dimensional, by [11, Thm 1.3.2, p. 21]) subcoalgebra of C it generates. Then, $V \subseteq D$, which implies that $D^\dagger \subseteq V^\dagger \subseteq \ker \ell$. But $D^\dagger$ is a finite-codimensional ideal of $\tilde{\mathbb{A}}(\mathsf{A})$ (by Lemma 76 and [1, Theorem 2.3.1, p. 78]), whence $\ell \in A^0$.[40]

It remains to check that the above inclusion $incl_{(A,\sigma)'}$ is a coalgebra map from $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A})$ to $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, which would equivalently mean that $(A,\sigma)'$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. One thus needs to make explicit the two coalgebra structures so as to make possible a comparison. By construction the comultiplication of $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A})$ is given by the composition $\Lambda^{-1}_{(A,\sigma)'\otimes_\Bbbk (A,\sigma)'} \circ \mu'$. So for $\ell \in (A,\sigma)'$, $(\Lambda^{-1}_{(A,\sigma)'\otimes_\Bbbk (A,\sigma)'} \circ \mu')(\ell) = \sum_{i=1}^n \ell_i \otimes r_i$, for some $\ell_i, r_i \in (A,\sigma)'$. Therefore, given $\ell \in (A,\sigma)'$, $u, v \in A$, $\ell(\mu(u \circledast v)) = \sum_{i=1}^n \ell_i(u) r_i(v)$. The counit of $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A})$ is $(A,\sigma)' \xrightarrow{\eta'} (\Bbbk,\mathsf{d})' \xrightarrow{\psi^{-1}} \Bbbk$, i.e., $\ell \mapsto \ell(\eta(1_\Bbbk))$. It follows easily, from the explicit description of $D_{fin}(\mathsf{B})$ provided in [7, p. 35], for a monoid B in $\mathbb{Vect}_\Bbbk$, that the above comultiplication coincides with that of $D_{fin}(\tilde{\mathbb{A}}(A))$, and because it is patent that the counit of $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A})$ is the restriction of that of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, $(A,\sigma)'$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$.

It remains to prove the equivalence of the four assertions given in the statement. 2⇔3 since finite duality restricts to an equivalence of categories between the full categories of reflexive algebras and of coreflexive coalgebras (in a standard way; see e.g., [12, Proposition 4.2, p. 16]), and $\tilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A}))$.

The coalgebra $\mathsf{C} \coloneqq (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A})$ is coreflexive if, and only if, every finite-codimensional ideal of $\widetilde{\mathbb{D}_*}(\mathsf{C}) \simeq \tilde{\mathbb{A}}(\mathsf{A})$ is closed in the finite topology of C^* ([1, Lemma 2.2.15, p. 76]), which coincides with our topology $w^*_{(\Bbbk,\mathsf{d})}$, and thus it turns out that $(\widetilde{\mathbb{D}_*}(\mathsf{C}), w^*_{(\Bbbk,\mathsf{d})}) \simeq TA(\mathsf{A})$ (since C^* under the finite topology is equal to $Alg_{(\Bbbk,\mathsf{d})}(C) \simeq (A,\sigma)$ by functoriality). Whence 3⇔1.

Let us assume that in $TA(\mathsf{A})$ every finite-codimensional ideal is closed. Let $\ell \in D_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. By definition $\ker \ell$ contains a finite-codimensional ideal say I of $\tilde{\mathbb{A}}(\mathsf{A})$. Since I is closed, there exists a finite-dimensional subspace

[40] $A^0 \coloneqq \{\,\ell \in A^*\colon \ker \ell$ contains a finite-codimensional (two-sided) ideal of $UA(\mathsf{A})\,\}$ is the underlying vector space of the finite dual coalgebra $D_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$.

D of C such that $D^\dagger = I$ (since the closed subspaces are of the form $D^\dagger$ for a subspace D of C and by Lemma 76, $\mathrm{codim}(I) = \mathrm{codim}(D^\dagger) = \dim(D)$), whence I is open, which shows that $(A,\sigma) \xrightarrow{\ell} (\Bbbk, \mathsf{d})$ is continuous so 1⇒4.

Let $\mathsf{C} := (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, so that $\tilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}(\mathsf{C})$, as above. Whence $\mathsf{C} = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A})) \simeq D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{D}_*}(\mathsf{C}))$. This is not sufficient to ensure coreflexivity of C, since there is at this stage no guaranty that the above isomorphism corresponds to the counit of the adjunction $D^{\mathsf{op}}_{fin} \dashv \widetilde{\mathbb{D}_*}$ (see Footnote 39). One knows from the beginning of the proof that

$$\tilde{\mathbb{A}}(\tilde{\Gamma}_{\mathsf{A}}) \colon \mathbb{A}(\mathsf{A}) \simeq \tilde{\mathbb{A}}(\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A})))$$

which, in this case where $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, is the isomorphism $\|\Gamma_{(A,\sigma)}\| \colon A \simeq ((A,\sigma)')^* = (A^0)^*$, $u \mapsto (\ell \mapsto \ell(u))$. So $\tilde{\mathbb{A}}(\mathsf{A})$ is reflexive, and thus its finite dual coalgebra C is coreflexive. Thus 4⇒3. □

78 Example *Let R be a ring. Let $C_{\mathsf{R}}X = (R^{(X)}, d_X, e_X)$ be the group-like coalgebra on X, i.e., $d_X(\delta_x) = \delta_x \otimes \delta_x$, and $e_X(\delta_x) = 1_{\mathsf{R}}$, $x \in X$. The following diagram commutes (this may be checked by hand) for a rigid ring (R,τ).*

$$\begin{array}{ccc} ((R,\tau)^X)' & \xrightarrow{\quad \mu'_X \quad} & ((R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^X)' \\ \lambda_X \downarrow & & \uparrow \Lambda_{((R,\tau)^X)' \otimes_{\mathsf{R}} ((R,\tau)^X)'} \\ R^{(X)} \xrightarrow[d_X]{} R^{(X)} \otimes_{\mathsf{R}} R^{(X)} & \xrightarrow[\lambda_X^{-1} \otimes_{\mathsf{R}} \lambda_X^{-1}]{} & ((R,\tau)^X)' \otimes_{\mathsf{R}} ((R,\tau)^X)' \end{array} \tag{27}$$

Moreover $\underbrace{\eta'_X(\ell)} = \psi(e_X(\lambda_X(\ell)))$ for each $\ell \in ((R,\tau)^X)'$. All of this shows that $\lambda_X \colon (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})(M_{(\mathsf{R},\tau)}(X)) \simeq C_{\mathsf{R}}X$ is an isomorphism of coalgebras.

Let $\Bbbk$ be a field. It follows from Theorem 77 and Example 74 that in $A_{(\Bbbk,\mathsf{d})}(X)$ every finite-codimensional ideal is closed if, and only if $C_{\Bbbk}X$ is coreflexive if, and only if, $\{\pi_x : x \in X\} = {}_1\mathbf{Alg}_{\Bbbk}(A_{\Bbbk}(X), \Bbbk)$ ([17, Corollary 3.2, p. 528]). This holds in particular if $|X| \le |\Bbbk|$ (see [17, Corollary 3.6, p. 529]). If $\Bbbk$ is a finite field, then $C_{\Bbbk}(X)$ is coreflexive if, and only if, X is finite (see [17, Remark 3.7, p. 530]).

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A Monoidal categories and functors

This appendix contains basic facts about monoidal categories and monoidal functors, and a part of it is reprinted from [15]. See [13, Chap. VII] for more details.

Throughout $\mathbb{C} = (\mathbf{C}, - \otimes -, I, \alpha, \lambda, \rho) = (\mathbf{C}, - \otimes -, I)$ (resp. $\mathbb{C} = (\mathbf{C}, - \otimes -, I, \alpha, \lambda, \rho, \sigma)$) denotes a (resp. symmetric) monoidal category with α the associativity and λ and ρ the left and right unit constraints (resp., and σ the symmetry), referred to as *coherence constraints*. These constraints have to make commute some diagrams to ensure *coherence* of the (resp. symmetric) monoidal category (see [13, Chap. VII, p. 165]). If $\mathbb{C}$ is a (symmetric) monoidal category, then so is $\mathbb{C}^{\mathbb{op}} := (\mathbf{C}^{\mathsf{op}}, -\otimes^{\mathsf{op}}-, I, (\alpha^{-1})^{\mathsf{op}}, (\varrho^{-1})^{\mathsf{op}}, (\lambda^{-1})^{\mathsf{op}})$, called the *dual* monoidal category of $\mathbb{C}$.

A.1 Monoids and comonoids

A *monoid in* $\mathbb{C}$ is a triple $(C, C \otimes C \xrightarrow{m} C, I \xrightarrow{e} C)$ such that the diagrams

$$\begin{tikzcd}
(C \otimes C) \otimes C \arrow[rr, "m \otimes id_C"] \arrow[d, "\alpha_{C,C,C}"'] & & C \otimes C \arrow[dd, "m"] \\
C \otimes (C \otimes C) \arrow[d, "id_C \otimes m"'] & & \\
C \otimes C \arrow[rr, "m"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes I \arrow[r, "id_C \otimes e"] \arrow[dr, "\rho_C"'] & C \otimes C \arrow[d, "m"] & I \otimes C \arrow[l, "e \otimes id_C"'] \arrow[dl, "\lambda_C"] \\
& C &
\end{tikzcd}$$

commute, while a *monoid morphism* $(C, m, e) \longrightarrow (C', m', e')$ is any $f : C \to C'$ making the diagrams

$$\begin{tikzcd}
C \otimes C \arrow[r, "m"] \arrow[d, "f \otimes f"'] & C \arrow[d, "f"] \\
C' \otimes C' \arrow[r, "m'"'] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, "e'"'] & C \arrow[d, "f"] \\
& C'
\end{tikzcd}$$

commutative. This defines the category $\mathbf{Mon}\mathbb{C}$ of monoids in $\mathbb{C}$.
The category $\mathbf{Comon}\mathbb{C}$ of *comonoids* over $\mathbb{C}$ is defined to be $(\mathbf{Mon}\mathbb{C}^{\mathbb{op}})^{\mathsf{op}}$, the opposite of the category of monoids in $\mathbb{C}^{\mathbb{op}}$.

A monoid (C, m, e) is called *commutative* if, and only if, $m = m \circ \sigma_{C,C}$ with $\sigma_{C,C} : C \otimes C \to C \otimes C$ the symmetry; dually, a comonoid (C, μ, ϵ) is called *cocommutative*, provided that $\mu = \sigma_{C,C} \circ \mu$. By ${}_c\mathbf{Mon}\mathbb{C}$ and ${}_{coc}\mathbf{Comon}\mathbb{C}$ we denote the categories of commutative monoids and cocommutative comonoids respectively, with all (co)monoid morphisms as morphisms. Of course, one has ${}_{coc}\mathbf{Comon}\mathbb{C} = ({}_c\mathbf{Mon}\mathbb{C}^{\mathbb{op}})^{\mathsf{op}}$.

79 Example *1.* $\mathbf{Mon}(\mathbb{Set})$ *is (isomorphic to) the category of monoids, where* $\mathbb{Set} = (\mathbf{Set}, \times, 1)$ *(*$1 := \{\varnothing\}$*).*

2. Let $\mathbb{Mod}_{\mathsf{R}}$ *be the monoidal category of* R*-modules and* R*-linear maps for a commutative unital ring* R *with its usual tensor product* $\otimes_{\mathsf{R}}$*.*

(a) $\mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}})$ *is isomorphic to the category* ${}_1\mathbf{Alg}_{\mathsf{R}}$ *of "ordinary" unital* R*-algebras under the functor* O*, concrete over* $\mathbf{Mod}_{\mathsf{R}}$*, such that* $O(\mathsf{A}) := (A, m_{\mathsf{A}}, 1_{\mathsf{A}})$*, with* $m_{\mathsf{A}}(x, y) := \mu(x \otimes y)$*,* $x, y \in A$*, and* $1_{\mathsf{A}} := \eta(1_{\mathsf{R}})$*, where* $\mathsf{A} = (A, \mu_{\mathsf{A}}, \eta_{\mathsf{A}})$ *is a monoid in* $\mathbb{Mod}_{\mathsf{R}}$*.*

(b) Likewise ${}_c\mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}}) \simeq {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ *under the (co-)restriction of the above functor* O*.*

(c) $\mathbf{Comon}(\mathbb{Mod}_{\mathsf{R}})$ *is the category* ${}_\epsilon\mathbf{Coalg}_{\mathsf{R}}$ *of counital* R*-coalgebras ([1, 7]), and the category of cocommutative coalgebras* ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}$ *is equal to* ${}_{coc}\mathbf{Comon}(\mathbb{Mod}_{\mathsf{R}})$*.*

A.2 Monoidal functors and their induced functors

We briefly recall the following definitions and facts, too, which are fundamental for this note. See e.g. [3, 16, 18] for a more detailed treatment and for the missing proofs.

80 Definition *Let* $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ *and* $\mathbb{C}' = (\mathbf{C}', - \otimes' -, I')$ *be monoidal categories. A* (lax) monoidal functor from $\mathbb{C}$ to $\mathbb{C}'$ *is a triple* $\mathbb{F} := (F, \Phi, \phi)$*, where* $F : \mathbf{C} \to \mathbf{C}'$ *is a functor,* $\Phi_{C_1,C_2} : FC_1 \otimes' FC_2 \to F(C_1 \otimes C_2)$ *is a natural transformation and* $\phi : I' \to FI$ *is a* $\mathbf{C}$*-morphism, subject to certain coherence conditions. A lax monoidal functor is called* strong monoidal *(resp.* strict monoidal *), if* Φ *and* ϕ *are isomorphisms (resp. identities).* Φ, ϕ *are the* coherence constraints *of* $\mathbb{F}$*.*

Let $\mathbb{C}, \mathbb{C}'$ be symmetric. A monoidal functor $\mathbb{F}: \mathbb{C} \to \mathbb{C}'$ is said to be *symmetric* when furthermore it satisfies yet another coherence condition relative to the symmetry constraints.

81 Facts *Let* $\mathbb{C} := (\mathbf{C}, \otimes, I)$*,* $\mathbb{C}' := (\mathbf{C}', \otimes', I')$ *and* $\mathbb{C}'' := (\mathbf{C}'', \otimes'', I'')$ *be (symmetric) monoidal categories.*

1. $\mathbb{id}_{\mathbb{C}} := (id_{\mathbf{C}}, id_{-\otimes -}, id_I)$*, or simply* $\mathbb{id}$*, is a monoidal functor from* $\mathbb{C}$ *to itself, and serves as a unit for the composition of monoidal functors given below.*

2. *Given monoidal functors* $\mathbb{F} = (F, \Phi, \phi): \mathbb{C} \to \mathbb{D}$ *and* $\mathbb{G} = (G, \Psi, \psi): \mathbb{D} \to \mathbb{E}$*, one defines a monoidal functor* $\mathbb{G} \circ \mathbb{F} := \mathbb{H} = (H, \Theta, \theta): \mathbb{C} \to \mathbb{E}$ *with*

   (a) $H = G \circ F$.

   (b) Given objects C_1, C_2 *of* $\mathbf{C}$*,* Θ_{C_1,C_2} *is the composite* $\mathbf{C}''$*-morphism* $G(F(C_1)) \otimes'' GF(C_2) \xrightarrow{\Psi_{F(C_1),F(C_2)}} G(F(C_1) \otimes' F(C_2)) \xrightarrow{G(\Phi_{C_1,C_2})} G(F(C_1 \otimes C_2))$.

   (c) $\theta := I'' \xrightarrow{\psi} G(I') \xrightarrow{G(\phi)} G(F(I))$.

   H *is strong (resp. symmetric) when* F, G *so are.*

82 Proposition *Let* $\mathbb{F} = (F, \Phi, \phi): \mathbb{C} \to \mathbb{C}'$ *be a monoidal functor.*

$\tilde{\mathbb{F}}(M, m, e) = (FM, FM \otimes FM \xrightarrow{\Phi_{M,M}} F(M \otimes M) \xrightarrow{Fm} FM, I' \xrightarrow{\phi} FI \xrightarrow{Fe} FM)$ *and* $\tilde{\mathbb{F}} f = Ff$ *define an* induced functor $\tilde{\mathbb{F}}: \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$*, such that the diagram below commutes (with forgetful functors* U_m *and* U'_m*).*

$$\begin{tikzcd} \mathbf{Mon}\mathbb{C} \arrow[r, "\tilde{\mathbb{F}}"] \arrow[d, "U_m"'] & \mathbf{Mon}\mathbb{C}' \arrow[d, "U'_m"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{C}' \end{tikzcd} \tag{28}$$

83 Remark 1. *When* $\mathbb{F}$ *is symmetric, then* $\tilde{\mathbb{F}}$ *also provides a functor* ${}_c\mathbf{Mon}\mathbb{C}$ *to* ${}_c\mathbf{Mon}\mathbb{C}'$ *with similar properties as above.*

2. $\widetilde{\mathbb{id}_{\mathbb{C}}} = id_{\mathbf{Mon}(\mathbb{C})}$ *and* $\widetilde{\mathbb{G} \circ \mathbb{F}} = \tilde{\mathbb{G}} \circ \tilde{\mathbb{F}}$.

A strong monoidal functor $\mathbb{F} = (F, \Phi, \phi): \mathbb{C} \to \mathbb{C}'$ may be considered as a strong monoidal functor $\mathbb{F}^{\mathsf{d}} := (F^{\mathsf{op}}, (\Phi^{-1})^{\mathsf{op}}, (\phi^{-1})^{\mathsf{op}}): \mathbb{C}^{\mathbb{op}} \to (\mathbb{C}')^{\mathbb{op}}$, the *dual of* $\mathbb{F}$ ([16, Proposition 17, p. 639]).

84 Definition *Let $\mathbb{F} = (F, \Phi, \phi)$ and $\mathbb{G} = (G, \Psi, \psi)$ be monoidal functors from $\mathbb{C}$ to $\mathbb{C}'$. A natural transformation $\alpha : F \Rightarrow G : \mathbf{C} \to \mathbf{C}'$ is a* monoidal transformation $\alpha : \mathbb{F} \Rightarrow \mathbb{G} : \mathbb{C} \to \mathbb{C}'$ *when the following diagrams commute, for every $\mathbf{C}$-objects C_1, C_2.*

$$\begin{tikzcd} FC_1 \otimes' FC_2 \arrow[r, "\alpha_{C_1} \otimes' \alpha_{C_2}"] \arrow[d, "\Phi_{C_1,C_2}"'] & GC_1 \otimes' GC_2 \arrow[d, "\Psi_{C_1,C_2}"] \\ F(C_1 \otimes C_2) \arrow[r, "\alpha_{C_1 \otimes C_2}"'] & G(C_1 \otimes C_2) \end{tikzcd} \qquad \begin{tikzcd} I' \arrow[r, "\phi"] \arrow[dr, "\psi"'] & FI \arrow[d, "\alpha_I"] \\ & GI \end{tikzcd} \tag{29}$$

85 Remark 1. *Let $\alpha : \mathbb{F} \Rightarrow \mathbb{G} : \mathbb{C} \to \mathbb{C}'$ be a monoidal transformation. It induces $\tilde{\alpha} : \tilde{\mathbb{F}} \Rightarrow \tilde{\mathbb{G}} : \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$ with $\tilde{\alpha}_{(C,m,e)} := \alpha_C$.*

2. *When the monoidal functors and categories are symmetric, then α also induces $\tilde{\alpha} : \tilde{\mathbb{F}} \Rightarrow \tilde{\mathbb{G}} : {}_c\mathbf{Mon}\mathbb{C} \to {}_c\mathbf{Mon}\mathbb{C}'$ [3, Prop. 3.38].*

3. *Let $\alpha : \mathbb{F} \Rightarrow \mathbb{G} : \mathbb{C} \to \mathbb{C}'$ be a monoidal transformation between strong monoidal functors, then $\alpha^{\mathsf{op}} : G^{\mathsf{op}} \to F^{\mathsf{op}} : \mathbf{C}^{\mathsf{op}} \to \mathbf{C}'^{\mathsf{op}}$ also provides a monoidal transformation $\alpha^{\mathbb{d}} : \mathbb{G}^{\mathbb{d}} \to \mathbb{F}^{\mathbb{d}} : \mathbb{C}^{\mathbb{op}} \to \mathbb{C}'^{\mathbb{op}}$, called the* dual *of α.*

A *monoidal isomorphism* is a monoidal transformation which is a natural isomorphism.

86 Remark *$\alpha^{-1} : \mathbb{G} \Rightarrow \mathbb{F} : \mathbb{C} \to \mathbb{C}'$ is a monoidal isomorphism when so is α. Accordingly, the induced natural transformation $\tilde{\alpha} : \tilde{\mathbb{F}} \Rightarrow \tilde{\mathbb{G}} : \mathbf{Mon}(\mathbb{C}) \to \mathbf{Mon}(\mathbb{C}')$ is a natural isomorphism with inverse $(\tilde{\alpha})^{-1} = \widetilde{(\alpha^{-1})}$.*

A *monoidal equivalence* of monoidal categories is given by a monoidal functor $\mathbb{F} : \mathbb{C} \to \mathbb{C}'$ such that there are a monoidal functor $\mathbb{G}$ and monoidal isomorphisms $\eta : \mathbb{id} \Rightarrow \mathbb{G} \circ \mathbb{F}$ and $\epsilon : \mathbb{F} \circ \mathbb{G} \Rightarrow \mathbb{id}$. $\mathbb{C}, \mathbb{C}'$ are said *monoidally equivalent.*

87 Remark *If $\mathbb{F}$ is a monoidal equivalence, then $\tilde{\mathbb{F}}$ is an equivalence between the corresponding categories of monoids.*

88 Remark *Let $\mathbb{F}, \mathbb{G} : \mathbb{C}' \to \mathbb{C}$ be strong monoidal functors. Let $\eta : \mathbb{id} \Rightarrow \mathbb{G} \circ \mathbb{F}$ and $\epsilon : \mathbb{F} \circ \mathbb{G} \Rightarrow \mathbb{id}$ be monoidal isomorphisms. Whence $\eta^{\mathbb{d}} : \mathbb{F}^{\mathbb{d}} \circ \mathbb{G}^{\mathbb{d}} = (\mathbb{G} \circ \mathbb{F})^{\mathbb{d}} \Rightarrow \mathbb{id}^{\mathbb{d}}_{\mathbb{C}} = \mathbb{id}_{\mathbb{C}^{\mathbb{op}}}$ and $\epsilon^{\mathbb{d}} : \mathbb{id}_{\mathbb{C}'^{\mathbb{op}}} = (\mathbb{id}_{\mathbb{C}'})^{\mathbb{d}} \Rightarrow \mathbb{G}^{\mathbb{d}} \circ \mathbb{F}^{\mathbb{d}}$ are also monoidal isomorphisms, and this provides a monoidal equivalence between $\mathbb{C}^{\mathbb{op}}$ and $\mathbb{C}'^{\mathbb{op}}$.*