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Regular Switching Components

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Abstract

We consider a problem of Discrete Tomography which consists of reconstructing a finite lattice set $S \subset \mathbb{Z}^2$ from given horizontal and vertical X-rays (i.e. with prescribed numbers of points in each row and column). Without additional requirements, the problem can be solved in polynomial time. Many variants require the solution to be in a chosen class \mathcal{A} . For instance, the problem is NP-complete for the class $\mathcal{A} = \mathcal{H} \cap \mathcal{V}$ of \mathcal{HV} -convex lattice sets and it is polynomial for the class $\mathcal{A} = \mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$ of \mathcal{HV} -convex polyominoes. Twenty years after these results, the problem's complexity remains unknown for $\mathcal{A} = \mathcal{C}$, the class of \mathcal{C} -convex lattice sets (i.e. two-dimensional lattice polytopes).

The main difficulty to solve this problem comes from combinatorial structures called *switching components*. Switching components are closed path with horizontal and vertical edges such that the solutions S are composed of either the elements of even or of odd indices. This binary choice can be encoded by a Boolean variable associated with the switching component and the convexity constraints are simply encoded by SAT clauses (2-clauses for \mathcal{HV} -convexity and 3-clauses for \mathcal{C} -convexity).

The purpose of the paper is to investigate the properties of the switching components and the consequences of the convexity requirements. We divide the switching components in two classes: *regular* if their turning angle is constant, *irregular* otherwise. We prove that adjacent regular switching components have the same Boolean values. This property allows us to merge them into extended switching components. We prove that if all switching components are regular, then the extended switching components are all in-

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dependent (then the number of solutions with the considered feet is 2^n , where n is the number of extended switching components). Finally, we prove that they are geometrically ordered.

Keywords: Discrete Tomography, X-rays, HV-convex lattice sets

1. Introduction

A word about some friends of Maurice Nivat

The scientific community involved in the field of Discrete Tomography started their work about 25 years ago. Maurice Nivat promoted the field by co-organizing or participating in some weeks of workshops in Dagstuhl (1997), Thionville (1999), Sienna (2000), and Oberwolfach (2000). There, he presented the field to younger researchers who became his students and friends. This tribute to Maurice Nivat is also an opportunity to honor the memory of three of his friends who unfortunately died much too early: Alain Daurat, the last PhD student of Maurice (1973-2011), Alberto Del Lungo (1965-2003) [1], and Attila Kuba (1953-2006).

1.1. Open problems in Discrete Tomography

The field of Discrete Tomography [2, 3, 4] started in the mid 90s when the classical algorithms of Computerized Tomography failed to reconstruct lattice sets as requested for instance in Electron Microscopy for the investigation of crystals [5, 6]. Due to the devices providing the measurements and the complexity of the considered problems, special attention has been given to dimension 2. The state of the art around this question was already rich in complexity results, the most fundamental due to D. Gale and R.J. Ryser in 1957 [7, 8]. They proved independently that a finite lattice set with prescribed number of points in each row and column can be computed in polynomial time. This central problem has been extended in several directions including the following:

- Increasing the dimension of the lattice: The problem becomes NP-hard from dimension 3 [9]. The question can also be related to timetables and multi-commodity flow problems [10, 11].
- Increasing the number of X-ray directions: An *X-ray* is the vector which counts the number of points of a lattice set in the sequence of

consecutive parallel lines of given direction. If 3 or more directions are used, then the problem becomes NP-hard [12].

- Reconstructing several disjoint sets: The problem is NP-hard when the objective is to reconstruct two or more disjoint sets from the X-rays [13, 14].
- Adding geometrical or topological constraints as we are going to precise next.

Instead of searching for any lattice set with the prescribed X-rays as in the initial problem investigated by D. Gale and H.G Ryser [7, 8], we can search for a solution satisfying additional constraints. The constraints which are added can be topological, for instance by searching for 4-connected solutions (4-connected finite subsets of \mathbb{Z}^2 are called *polyominoes*), or geometrical, as in convex solutions. We introduce for that purpose different classes of lattice sets. The class of the 4-connected lattice sets or in other words polyominoes is denoted \mathcal{P} . The lattice sets having consecutive points on each horizontal line are horizontally convex or equivalently \mathcal{H} -convex. Their class is denoted \mathcal{H} . In the same way, \mathcal{V} is the class of the vertically convex namely \mathcal{V} -convex lattice sets. By intersection, $\mathcal{H} \cap \mathcal{V}$ is the class of the \mathcal{HV} -convex lattice sets (Fig. 1). We end these notations with the class \mathcal{C} of the \mathcal{C} -convex lattice sets in the usual meaning of convexity. A lattice set is \mathcal{C} -convex if it is equal to the intersection of its real convex hull with the lattice (Fig. 2). In other words \mathcal{C} is the class of the lattice polytopes. Both \mathcal{C} -convex and \mathcal{HV} -convex lattice sets are not necessarily 4 or even 8-connected. The class of the \mathcal{HV} -convex polyominoes i.e 4-connected \mathcal{HV} -convex lattice sets is denoted $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$.

The reconstruction of lattice sets with prescribed X-rays and belonging to a chosen class has been started by two seminal papers:

- The first one is due to Richard Gardner and Peter Grizmann[15]. They considered the reconstruction of \mathcal{C} -convex lattice sets with different number of directions of X-rays. They characterized the sets of n directions for which any \mathcal{C} -convex set is uniquely determined by its X-rays. For 2 or 3 directions, there always exist ambiguous pairs or triplets of X-rays. For $n \geq 7$ directions, all \mathcal{C} -convex lattice sets are uniquely determined by their X-rays. For $3 < n < 7$, the so-called cross-ratios of the directions provide a characterization of the sets of directions with either the uniqueness property, or ambiguous X-rays [15]. When the

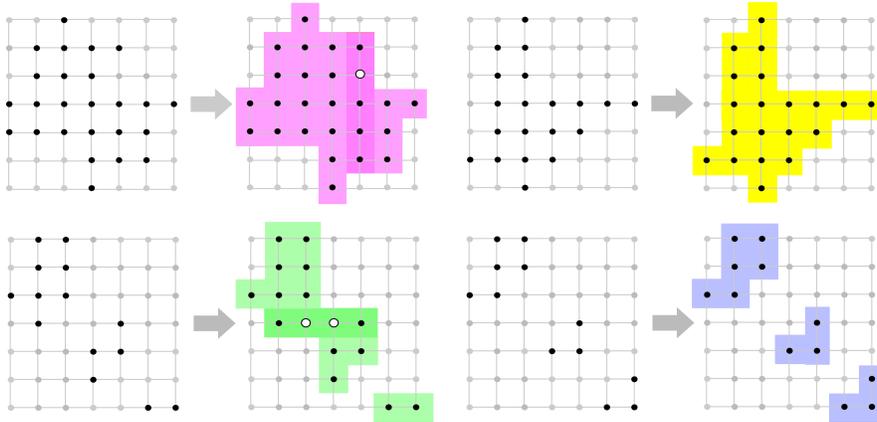


Figure 1: \mathcal{HV} -convexity. The two left lattice sets are not \mathcal{HV} -convex since their intersection with some vertical or horizontal line has holes. The two right lattice sets are \mathcal{HV} -convex.

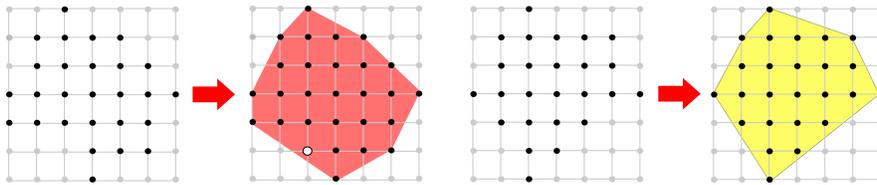


Figure 2: \mathcal{C} -convexity. The left lattice set is not \mathcal{C} -convex since its convex hull contains a lattice exterior point (in white). The right lattice set is \mathcal{C} -convex.

directions of X-rays provide uniqueness, these results have been completed by a polynomial time reconstruction algorithm by Sara Brunetti and Alain Daurat [16].

- The second paper on which we focus our attention is the reconstruction of \mathcal{HV} -convex polyominoes of \mathbb{Z}^2 from their horizontal and vertical X-rays. Elena Barcucci, Alberto Del Lungo, Renzo Pinzani, and Maurice Nivat proved in 1996 that this problem can be solved in polynomial time [17].

The polynomial time algorithm reconstructing a solution in the class $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$ from their horizontal and vertical X-rays is all the more noteworthy that almost all other related problems are NP-hard. Deciding whether there

exists a solution in the classes \mathcal{H} , \mathcal{V} , $\mathcal{H} \cap \mathcal{V}$, \mathcal{P} (or more generally with neighborhood constraints) is NP-complete in these cases [18, 17, 19]. The main remaining open question with orthogonal X-rays is the complexity of the reconstruction of lattice polytopes or in other words \mathcal{C} -convex lattice sets [20]. It is a challenging problem whose adjacency with the reconstruction of \mathcal{HV} -convex polyominoes has drawn a new attention on the original algorithm of [17]. Experts have noticed for twenty years that the 2-CNF formula built at the last step of the algorithm is always satisfiable (oral communications). However, this proposition remains a conjecture. Maurice Nivat and his co-authors left part of the combinatorial structure of the problem unsolved, although these questions are of major interest to tackle the remaining open problems of Discrete Tomography.

1.2. Results

In this paper, we provide new results in the framework of the reconstruction of lattice sets from their horizontal and vertical X-rays with convexity constraints. The reconstruction can be done by a generic algorithm denoted **ConvexTomo** whose strategy has been first described in [17] and also used in [21, 16]. **ConvexTomo** computes a partial solution and expresses the ambiguities about the remaining part in combinatorial structures called *switching components*. These structures are closed path with vertices on the lattice and only horizontal and vertical edges. Any switching component P provides two different ways to complete the partial solution, either by adding the vertices with odd indices or the ones with even indices. This binary choice is encoded by a Boolean variable denotes $P(S)$, where the argument $S \subset \mathbb{Z}^2$ is the considered lattice set.

We divide the switching components in two classes according to the variation of their turning angle: They might turn sometimes to the left (counterclockwise) and sometimes to the right (clockwise). If the path always turns to the same direction, the switching component is *regular*, and *irregular* otherwise. A larger part of the paper is devoted to the regular case.

The main lemma of this work (Lemma 1) is that adjacent regular switching components are equivalent in the sense that the convexity constraints enforce their Boolean variables to be equal. It follows that in the case where all the switching components are regular, any pair of switching component is either equivalent or independent (Theorem 1). This property allows us to merge the equivalent switching components into *extended switching components*. These objects are all independent. It provides the following structural

property:

Given a chosen position of the feet (the points of a solution with extremal coordinates), if the switching components can be computed and if they are all regular, then the number of solutions (with the chosen feet) is 2^n where n is the number of extended switching components (Property 1).

This property does not hold in general: with irregular switching components, the number of solutions is not necessarily a power of 2 (an example with 3 solutions is presented in Fig.3).

As the condition of regularity of all the switching components may seem a bit restrictive, we investigate the different cases in which it might occur. It leads us to prove new properties of the switching components and to provide some non-intuitive counter-examples.

- We prove that each switching component visits necessarily the four areas (South West, North West, North East and South East) of the set of undetermined points (Property 2).
- Except in one configuration of the feet (configuration e) of Figs. 20 and 21), the switching components are either all regular, or all irregular. In configuration e), we might have both regular and irregular switching components (this counter-intuitive case is illustrated in Fig.23).
- We prove that if all switching components are regular, then the extended switching components obtained by merging the adjacent switching components are well ordered (Property 3).

These properties might be helpful to reconstruct in polynomial time \mathcal{C} -convex lattice sets from their horizontal and vertical X-rays at least in the case of regular switching components [22].

In Section 2, we state the problem $DT_{\mathcal{A}}$, present the classical approach `ConvexTomo` and introduce the switching components. In Section 3, we prove the main result about the equivalence of the adjacent regular switching components. At last, Section 4 is devoted to the additional results (Properties 2 and 3).

2. Problem statement and generic algorithm `ConvexTomo`

First we introduce the notion of X-rays in order to state the problem $DT_{\mathcal{A}}$. Secondly, we present the classical generic algorithm `ConvexTomo`. This

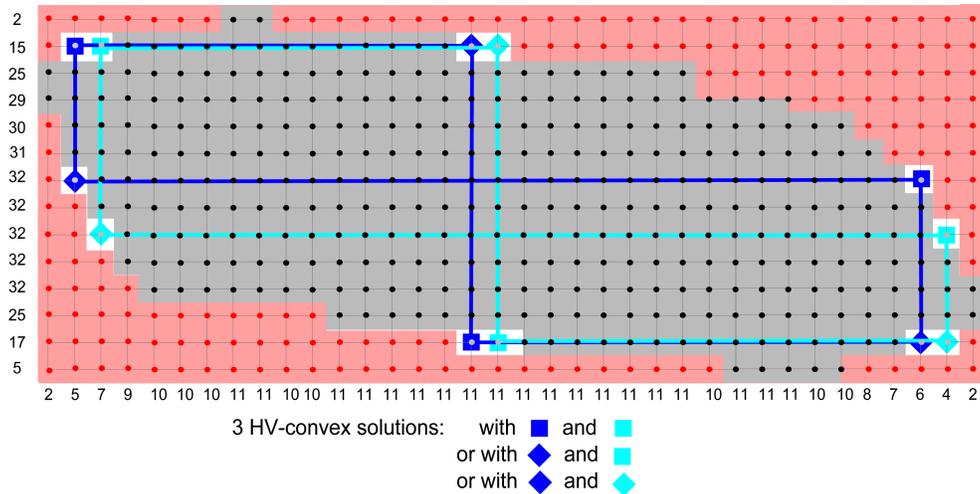


Figure 3: **Counter-example with irregular switching components.** Given prescribed X-rays $h = (5, 17, 25, \dots)$ and $v = (2, 5, 7, 9, \dots)$, we are searching for an \mathcal{HV} -convex or \mathcal{C} -convex lattice set with the prescribed X-rays. A partial solution can be computed. Some lattice points are inside the solution (black points, green cells), some lattice points are excluded (red points, pink cells) while some points remain undetermined (grey points, white cells covered by blue and cyan diamonds and squares). The blue path joining alternatively the blue squares and diamonds by horizontal and vertical edges is a switching component, and the same for the cyan path. These two switching components are both irregular since their path turn sometimes clockwise and sometimes counterclockwise. With these switching components, we can complete the partial solution by adding either the blue squares or the blue diamonds and either the cyan squares or the cyan diamonds (but not both). These two binary choices are encoded by two Boolean variables. The constraints of convexity make them dependant. If we add the blue squares, then we have to add cyan squares (if we add the cyan diamonds, then we have to add the blue diamonds). Due to this relation, we have here exactly 3 solutions. We prove in the following that with regular switching components, the number of solutions is necessarily a power of 2 (Property 1). This example shows that there is no hope to extend this property to the irregular case.

algorithm requires the class \mathcal{A} to be included in $\mathcal{H} \cap \mathcal{V}$ and to satisfy the partition property described later. These two assumptions allow us to define switching components.

2.1. Horizontal and vertical X-rays

An *X-ray* is the sequence of the cardinalities of the intersection between a given lattice set and the consecutive diophantine lines in a given direction.

In the case of the vertical and horizontal directions, it leads to the following definition.

Definition 1. Given a finite lattice set $S \subset \mathbb{Z}^2$ included in the rectangle $[1, m] \times [1, n]$, its vertical X-ray is the vector $V(S) \subset \mathbb{Z}^m$ whose coordinate $v_i(S)$ is the number of points of S in the vertical line $x = i$ (we have $V_i(S) = |\{(x, y) \in S | x = i\}|$). The horizontal X-ray of S is the vector $H(S) \subset \mathbb{Z}^n$ counting the number of points of S in each horizontal line $y = j$ (we have $H_j(S) = |\{(x, y) \in S | y = j\}|$) (Fig. 4).

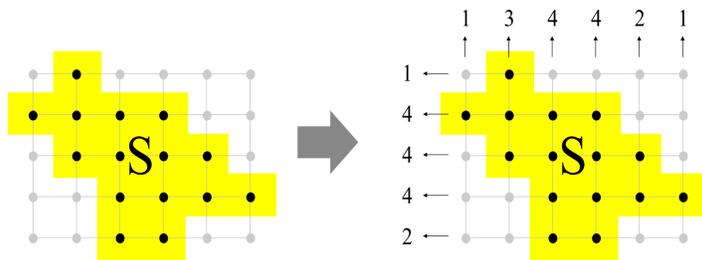


Figure 4: **The horizontal and vertical X-rays** of a given lattice set S .

2.2. Problem statement

We introduce the problem of deciding whether there exists a finite lattice set in a given class \mathcal{A} with prescribed X-rays:

Problem 1 ($\text{DT}_{\mathcal{A}}(h, v)$).

Given a class \mathcal{A} of lattice sets

Input: Two vectors $v \in \mathbb{Z}^m$, $h \in \mathbb{Z}^n$

Output: Does there exist a lattice set $S \in \mathcal{C}$ with $V(S) = v$ and $H(S) = h$?

The problems $\text{DT}_{\mathcal{A}}$ are known to be polynomial for the classes \mathcal{A} of the whole lattice sets [7, 8] and for the class $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$ of the \mathcal{HV} -convex polyominoes [17]. Its complexity is unknown for the class \mathcal{C} of the \mathcal{C} -convex polytopes and is NP-complete for many other classes (\mathcal{H} , \mathcal{V} , $\mathcal{H} \cap \mathcal{V}$, \mathcal{P} or lattice sets with neighborhood relations) [18, 17, 20].

2.3. The feet

We define the feet of a lattice set S as the subsets of its points with minimal or maximal coordinates.

Definition 2. Given a finite lattice set $S \subset \mathbb{Z}^2$, we consider its South, East, North and West feet as the sets

- $\text{South}(S) = S \cap \{y = y_{\min}(S)\}$ where $y_{\min}(S)$ is the min ordinate of the points of S .
- $\text{West}(S) = S \cap \{x = x_{\min}(S)\}$ where $x_{\min}(S)$ is the min abscissa of the points of S .
- $\text{North}(S) = S \cap \{y = y_{\max}(S)\}$ where $y_{\max}(S)$ is the max ordinate of the points of S .
- $\text{East}(S) = S \cap \{x = x_{\max}(S)\}$ where $x_{\max}(S)$ is the max abscissa of the points of S .

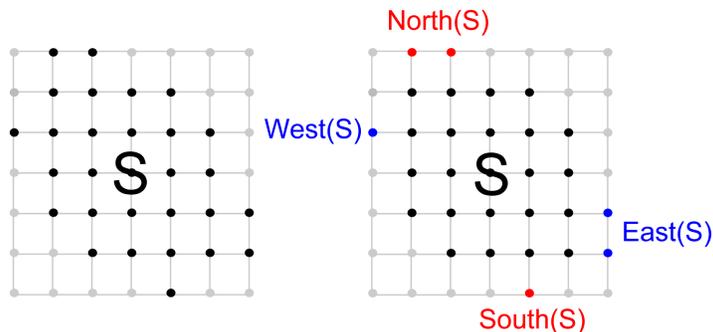


Figure 5: **The feet.** The four feet of a finite lattice set.

2.4. Generic algorithm *ConvexTomo*

The original algorithm of [17] has been developed for the reconstruction of \mathcal{HV} -convex polyominoes ($\text{DT}_{\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}}$), but it can be applied to solve $\text{DT}_{\mathcal{A}}$ with other classes \mathcal{A} than $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$. It requires two assumptions on \mathcal{A} . The first one is that the class \mathcal{A} is included in the class $\mathcal{H} \cap \mathcal{V}$ of the \mathcal{HV} -convex lattice sets. The second requirement is less direct. We present it after the introduction of the filling operations.

The generic algorithm `ConvexTomo` issued from [17] and used to solve $DT_{\mathcal{A}}$ can be decomposed in four steps:

1. Fix the feet (Fig.6).
2. Perform the filling operations: Take the X-rays and the geometrical constraints into account in order to compute a partial solution. Proceed until the filling operations do not allow to add or exclude any undetermined point.
3. If it remains undetermined points, connect them in path called switching components and represent the binary choice of adding either the points with odd or even indices by a Boolean variable.
4. Express the condition that the lattice set S is in the given class \mathcal{A} by SAT clauses and solve the SAT instance.

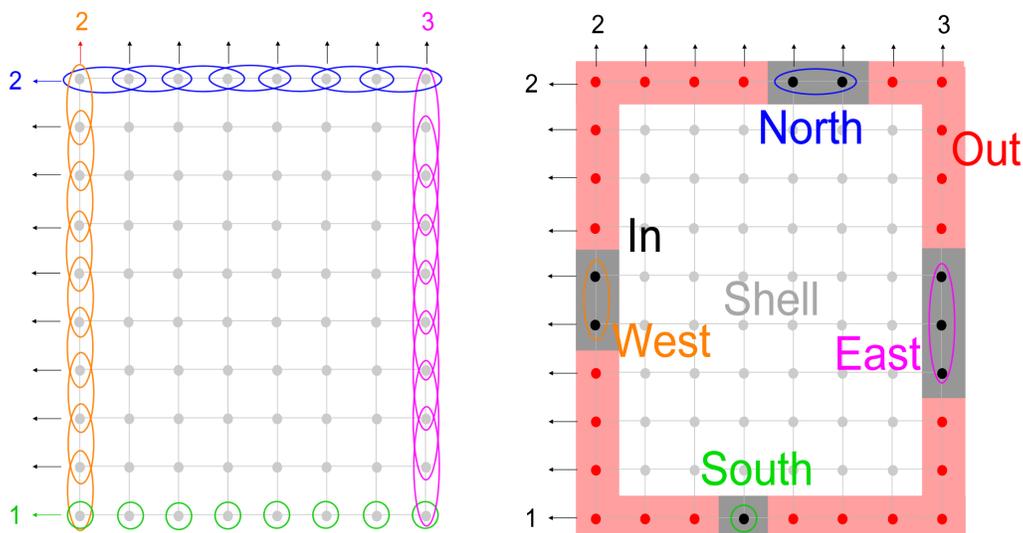


Figure 6: **Step 1.** On the left, the different possible positions of the feet. On the right, a position of the feet is chosen. The points are added in In (this set is represented by the black points/grey cells) while the other extreme points which have not been chosen are added in Out (red points/pink cells). The shell (grey points/white cells) is the set of the points which are not yet determined.

2.5. Filling operations

Given the chosen feet, the filling operations used for solving $DT_{\mathcal{A}}$ work with a partition of the lattice into three sets:

- The set In contains the points which are known to belong to all solutions.
- The set Out contains the points which are known to be excluded from all solutions.
- The set Shell is the set of the undetermined points.

The task of the filling operations is to take the constraints into account in order to include or exclude as many points as possible and thus decrease the cardinality of the shell (some filling operations are sketched in Fig.7). If a point of In has to be added to Out or conversely, this contradiction leads to the conclusion that the considered position of the feet admits no solution. The next step starts when the filling operations did not find any contradiction and are no longer able to decrease the shell size.

There exist different kinds of filling operations depending on the chosen class \mathcal{A} . The property of \mathcal{HV} -convexity and the prescribed X-rays provide the main operations. The first operations are uniquely related to \mathcal{HV} -convexity. The intermediary lattice points between two points of In on the same row or on the same column can be added to In. If there are two points $p \in \text{In}$ and $p' \in \text{Out}$ on the same row or column, then the points p'' with $p' \in [pp'']$ can be added to Out. There are also filling operations which take the prescribed X-rays into account with \mathcal{HV} -convexity (Fig.7). We refer to [21, 16] for a complete presentation of the operations.

Some of the filling operations can be much more complex. It is the case of the operations based on the 4-connectivity used for reconstructing \mathcal{HV} -convex polyominoes [17].

2.6. Second assumption: partition property

The algorithm `ConvexTomo` works under the assumption that the set of undetermined points can be decomposed in switching components. It requires first to provide a partition of the shell in four subsets that we denote NW, NE, SE, and SW for North-West, North-East, South-East and South-West according to their relative location towards the partial solution In (Fig. 8). We call *partition property* the property that after the end of the filling operations, the position of the solutions are sufficiently determined to decompose the shell in these four regions NW, NE, SE, and SW. Therefore there are only two assumptions to use the generic algorithm `ConvexTomo` for $\text{DT}_{\mathcal{A}}$:

1. The class \mathcal{A} of the considered lattice sets should be included in $\mathcal{H} \cap \mathcal{V}$.

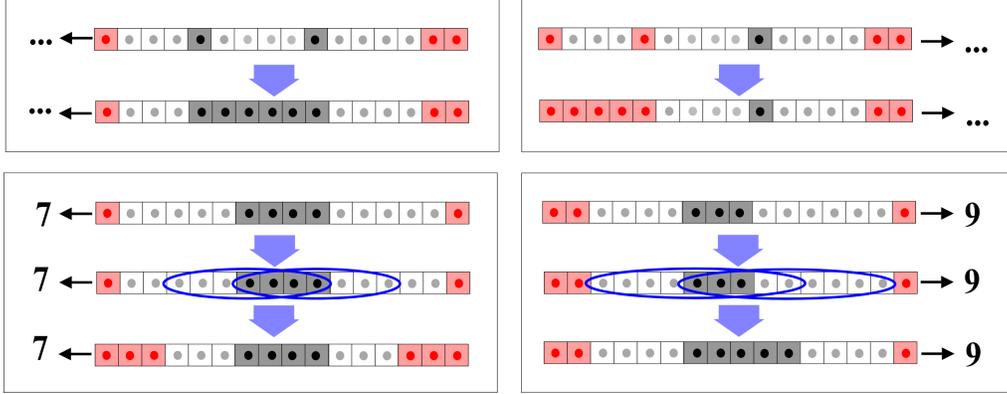


Figure 7: **Four filling operations.** Each rectangle presents a configuration of the set In (black points) and of the set Out (the excluded points are in red). Due to \mathcal{HV} -convexity, the two upper configurations lead to include or exclude new points. Below, the values of the horizontal X-ray allow again to increase In or Out.

2. The partition property namely the guarantee that the shell provided by the filling operations associated with \mathcal{A} can always be partitioned in four parts NW, NE, SE, and SW according to their locations.

The classes $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$ and \mathcal{C} both verify these requirements. They are both included in $\mathcal{H} \cap \mathcal{V}$. For $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$, the filling operations based on 4-connectivity provide a 4-connected set In [17]. Then, any 4-path joining the four feet provides the partition of the set of the undetermined points. It provides the partition property. In the case of the \mathcal{C} -convex lattice sets, the convex hull of the feet in \mathbb{R}^2 provides directly a partition of the shell in four areas (Fig. 8).

Then the algorithm `ConvexTomo` can be applied to solve $\text{DT}_{\mathcal{A}}(h, v)$ for $\mathcal{A} = \mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$ or $\mathcal{A} = \mathcal{C}$ but in order to remain as general as possible, we just assume in the following that the condition $\mathcal{A} \subset \mathcal{HV}$ and the partition property are satisfied. The reader should keep in mind that the whole paper is devoted to the decision problem $\text{DT}_{\mathcal{A}}$ with \mathcal{A} satisfying these two assumptions.

2.7. Corresponding points

We define vertical correspondences between the South and the North points and horizontal correspondences between the West and the East points.

Definition 3. We consider an instance $\text{DT}_{\mathcal{A}}(h, v)$ with X-rays h and v and a class $\mathcal{A} \subset \mathcal{HV}$ satisfying the partition property. Then the filling operations

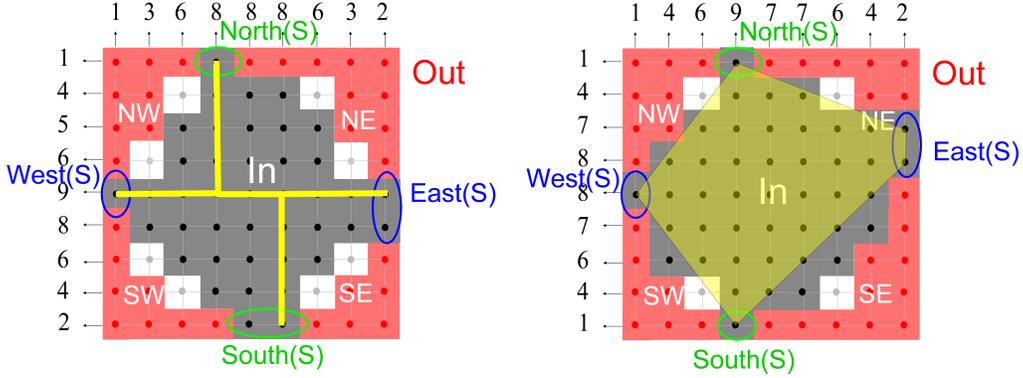


Figure 8: **The 4 areas of the shell.** On the left, we show an instance of $DT_{\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}}(h, v)$. After the filling operations associated with $\mathcal{H} \cap \mathcal{V} \cap \mathcal{P}$, the set In is 4-connected. A 4-path connecting the feet is drawn in yellow. It provides the partition of the grey undetermined points in South-East, North-East, North-West, and South-West areas respectively denoted SE, NE, NW, and SW. On the right, we consider an instance of $DT_C(h, v)$. The partition of the shell in SE, NE, NW, and SW is given directly by the convex hull of the feet.

provide a shell which can be decomposed in four areas SE, NE, NW, and SW.

The vertical correspondent of a point $\underline{p} = (i, j)$ of the South areas $SE \cup SW$ is the point $\bar{p} = (i, j + v_j)$.

The vertical correspondent of a point $\bar{p} = (i, j)$ of the North areas $NE \cup NW$ is the point $\underline{p} = (i, j - v_j)$.

The horizontal correspondent of a point $|p = (i, j)$ of the West areas $SW \cup NW$ is the point $p| = (i + h_j, j)$.

The horizontal correspondent of a point $p| = (i, j)$ of the East areas $SE \cup NE$ is the point $|p = (i - h_j, j)$.

The vertical and horizontal correspondents have several properties.

1. The horizontal and vertical correspondents of an undetermined point p are also undetermined (otherwise, the filling operations allow us to assign p to In or Out).
2. Correspondences are symmetric relations.
3. If p is in a solution, then its correspondents are outside.
4. A point p and its correspondent p' cannot be both outside a solution because the segment of the solution is necessarily in between and there are not enough points.

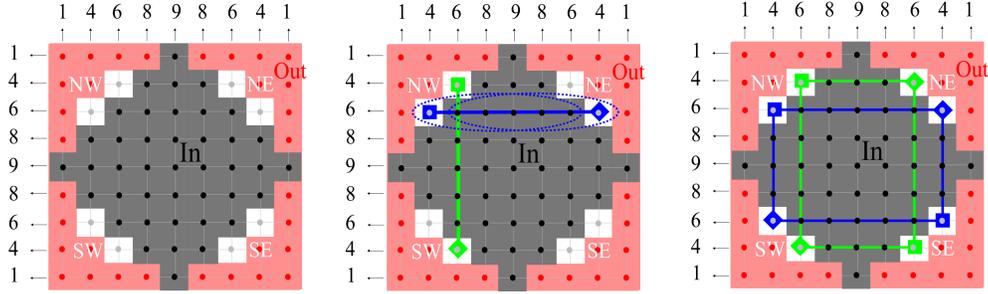


Figure 9: **Corresponding points.** On the left, a configuration with the sets In (black points), Out (red points), and Shell (white cells, grey points) of the undetermined points. In the middle, a pair of vertical correspondents (green) and a pair of horizontal correspondents (blue). We represent the points alternatively with squares or diamonds. Notice that the segment represented by the dotted ellipse has only two possible positions. Due to its length, if it contains the square, then it does not contain the diamond and conversely. On the right, the correspondences define closed paths called switching components which provide a partition of the undetermined points. For each switching component, either the squares or the diamonds belong to a solution.

It follows that if p and p' are correspondents, then p is in a solution S if and only if p' is not in S .

2.8. Switching components

The correspondences between the points define paths. The undetermined point p_1 has a horizontal correspondent p_2 which has a vertical correspondent p_3 which has an horizontal correspondent p_4 and so on until we come back to p_1 as the vertical correspondent (since the shell is finite, the path is necessarily closed). This path is referred as a *switching component* (Figs. 9 and 10).

Definition 4. We consider an instance $\text{DT}_{\mathcal{A}}(h, v)$ with a class $\mathcal{A} \subset \mathcal{H} \cap \mathcal{V}$ satisfying the partition property. The shell provided by the filling operations is decomposed in SE, NE, NW, and SW.

A switching component P is a closed path of corresponding points p_r with indices $r \in \mathbb{Z}/2l\mathbb{Z}$ (the length of the sequence is $2l$) so that p_{2k} and p_{2k+1} are vertical correspondents, while p_{2k-1} and p_{2k} are horizontal correspondents (Fig. 10). We denote by $P[0]$ the set of the vertices with even indices and by $P[1]$ the set of the vertices with odd indices.

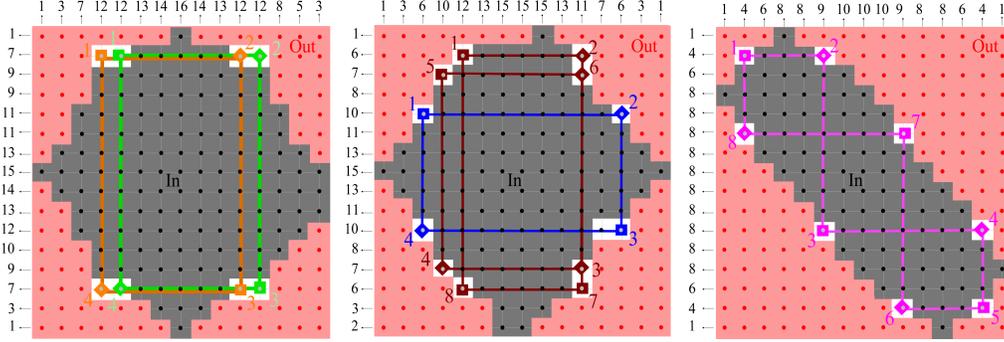


Figure 10: **Switching components.** On the left, two switching components P^1 and P^2 . The squares represent their points with even indices (the set $P[0]$) and the diamonds represent the points with odd indices (the set $P[1]$). Notice that there are only 2 \mathcal{C} -convex solutions since the \mathcal{C} -convexity forces adding $P^1[0] \cup P^2[0]$ or $P^1[1] \cup P^2[1]$ to the solution while the sets including $P^1[0] \cup P^2[1]$ or $P^1[1] \cup P^2[0]$ also have the prescribed X-rays but are not \mathcal{C} -convex. In the middle, regular switching components are not restricted to only one turn (the brown switching component makes 2 turns). On the right, an irregular switching component. It turns clockwise at point 2 and counterclockwise at point 3. On the contrary, the switching components of the two left images are regular (they always turn clockwise).

By construction, the two sets $P[0]$ and $P[1]$ have the same horizontal and vertical X-rays, since each point has a unique vertical and horizontal correspondent in the other set (they are respectively represented with diamonds and squares in all figures). In the same way as the corresponding points, any solution S of $\text{DT}_{\mathcal{A}}(h, v)$ with the considered feet falls in one of the two possible cases:

- either $P[0] \subset S$ and $P[1] \cap S = \emptyset$,
- or $P[1] \subset S$ and $P[0] \cap S = \emptyset$.

In other words, considering a switching component P , a solution contains either the points of the switching component with odd indices or the ones with even indices.

2.9. Boolean variables for encoding the state of each switching component

The two possible states of a switching components P in a solution S are encoded by a Boolean variable denoted $P(S)$. Its value is null if $P[0]$ is in

S and $P[1]$ is excluded while $P(S) = 1$ if $P[1]$ is in S and $P[0]$ is excluded. If we consider the whole sets of switching components P^q for the indices q going from $q = 1$ to the number Q of switching components, then the values of the Q Boolean variables $P^q(S)$ provide a complete characterization of S . Moreover, the 2^Q possible assignments of the Q Boolean variables all provide a set S with the prescribed horizontal and vertical X-rays. The problem is to determine whether there exists an assignment which provides a solution S in the chosen class \mathcal{A} .

2.10. Encoding the convexity with clauses

The search for an assignment of the Boolean variables which provides a solution S in the chosen class \mathcal{A} is done by expressing the constraints issued from \mathcal{A} on the Boolean variables associated with the switching components. With \mathcal{HV} -convexity, E. Barcucci et al. have proved that the \mathcal{HV} -convexity constraints can be expressed by a 2-CNF formula i.e a conjunction of SAT clauses with at most 2 literals per clause (Fig.11) [17].

The same approach holds for encoding \mathcal{C} -convexity with SAT clauses (Fig.11). The number of literals is however no more bounded by 2. Clauses with 3 literals might be necessary with the difficulty that 3-SAT is no more polynomial but NP-complete. It explains why, as far as we know, `ConvexTomo` is not polynomial for solving $\text{DT}_{\mathcal{C}}$. A better understanding of the relations between the switching components might be of interest in order to develop an alternative approach.

3. Main result

We prove that the Boolean variables associated with adjacent regular switching components are necessarily equal. Then we consider that the switching components are equivalent. This main proposition allows to state several properties.

3.1. Regular switching components

Before investigating the relationships between the switching components, we divide them in two groups according to the regularity of their turning angle.

Definition 5. A switching component is regular if the turning angle at each vertex has a constant orientation (in other words, the switching component always turns clockwise, or always counterclockwise (Fig.10)).

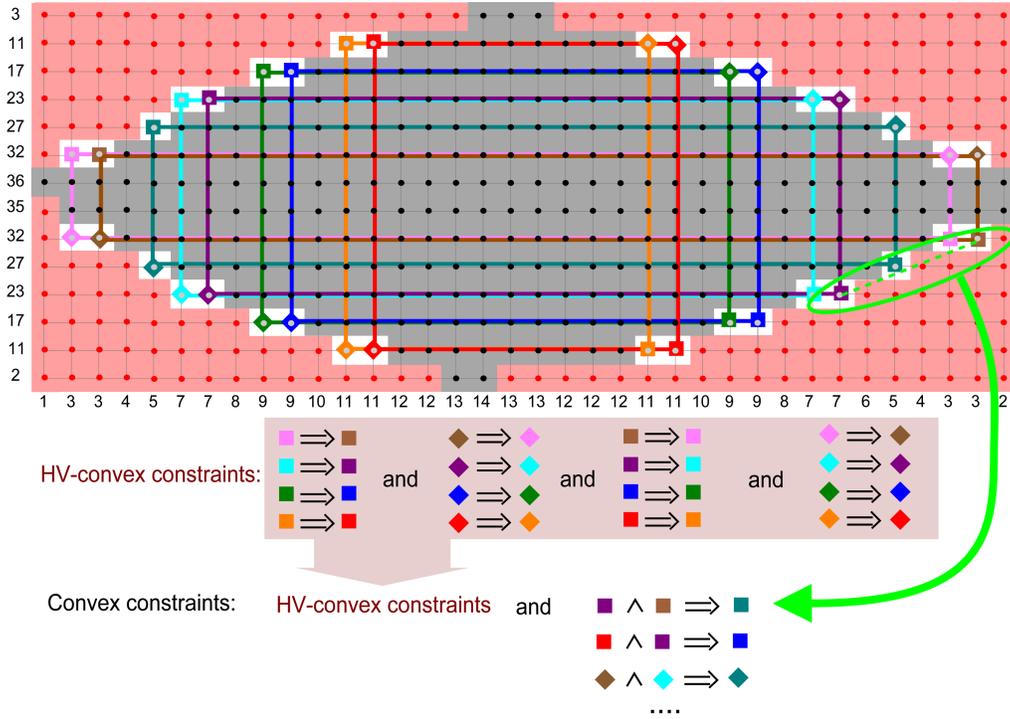


Figure 11: **2-SAT and 3-SAT clauses encoding convexity.** The switching components being encoded by Boolean variables (represented here geometrically with colored squares and diamonds), \mathcal{HV} -convexity is expressed by a conjunction of 2-SAT clauses. Consequently, the search of an \mathcal{HV} -convex solution is reduced to a 2-SAT instance that can be solved in linear time [23]. For expressing the \mathcal{C} -convexity in the same manner, we need clauses with 3 literals. It leads to the difficulty that 3-SAT is NP-complete.

The switching components which have both clockwise and anticlockwise turning angles are simply said *irregular* (Fig.10).

3.2. Adjacent switching components

We define an adjacency relation between switching components. Two switching components P and P' are *adjacent* if they contain respectively a point p and a point p' which are 4-adjacent.

3.3. Adjacent regular switching components are equivalent

We provide a first explanation about the property that the 2-CNF formula expressing the \mathcal{HV} -convexity of a solution of $\text{DT}_{\mathcal{A}}(h, v)$ can be often reduced

to a set of equalities.

Lemma 1. *We consider an instance of $\text{DT}_{\mathcal{A}}(h, v)$ with a class $\mathcal{A} \subset \mathcal{H} \cap \mathcal{V}$ satisfying the partition property (Section 2.6). The four feet being chosen and the filling operations performed, the Boolean variables of any pair of adjacent regular switching components are equal.*

This lemma can be understood as follows. If P and P' are two switching components with an adjacent pair of points p_1 and p'_1 , then the \mathcal{HV} -convexity constraint provides a first relation of dependency: if p_1 is in S , then p'_1 is in S . It can be formulated by the 2-clause $\overline{P}(S) \vee P'(S)$. According to Lemma 1, at some other places, some other points of the two switching components provide necessarily the converse clause $P(S) \vee \overline{P}'(S)$ with the consequence that $P(S)$ and $P'(S)$ are necessarily equal.

PROOF. We assume w.l.g that the points $p_1 = (x_1, y_1)$ and $p'_1 = (x'_1, y'_1)$ both belong to the North West area NW and that p'_1 is the right 4-neighbor of p_1 (Fig.12).

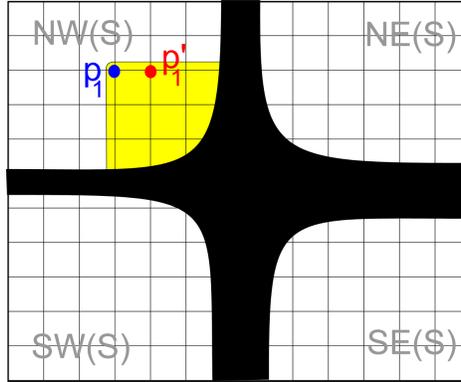


Figure 12: **Assumption of Lemma 1.** We consider two adjacent switching components. By symmetries, we can assume w.l.g that p'_1 is the right 4-neighbor of p_1 . It follows by H -convexity that if p_1 is in the solution S (the odd part $P[1]$ of the switching component P is in S namely $P(S) = 1$), then p'_1 is also in S (the odd part $P'[1]$ of the switching component P' is also in S namely $P'(S) = 1$). The black zone is a sketch of the set In . It underlines the property that with regular switching components horizontal correspondences are only between NW and NE or SW and SE while vertical correspondences are only between NW and SW or NE and SE.

We consider the positions of the points $p_0 = (x_0, y_0)$ (vertical correspondent of p_1), p_2 and $p_3 = (x_3, y_3)$. We distinguish three cases according to the comparison of y_0 and y_3 .

- Case 1. $y_0 < y_3$
- Case 2. $y_0 = y_3$
- Case 3. $y_0 > y_3$

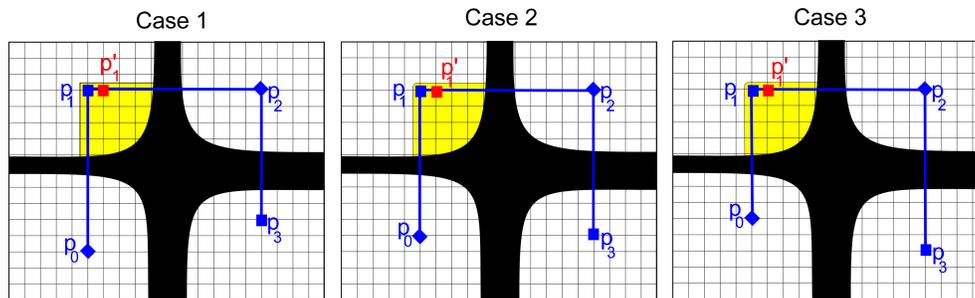


Figure 13: **The three possible configurations used for proving Lemma 1.** We develop the three cases in the following.

The three cases are developed in Figs. 14, 15, and 16. We follow the path of the second switching component P' by starting from p'_1 . This starting point is in the region denoted A in Figs. 14, 15, and 16. We compute the oriented graph of correspondences between the regions reached by P' . This graph provides a description of all the possible paths and thus of all the possible configurations starting from the region A .

According to the relative position of p'_1 towards p_1 , the \mathcal{HV} -convexity implies the clause $\overline{P}(S) \vee P'(S)$. Some regions provide the converse relation $P(S) \vee \overline{P'}(S)$ which provides the equality $P(S) = P'(S)$. They are colored in grey in all the figures. Therefore the task of the proof is to show that any closed path starting from p'_1 passes necessarily through a grey region. We prove it by computing the graphs of the correspondences of the regions as drawn in Figs.14 and 15.

In the first and second cases (Figs.14 and 15), either the path passes through a grey region and provides the equality, or it cannot come back to the initial region A which is excluded since the path is closed. It provides the equality.

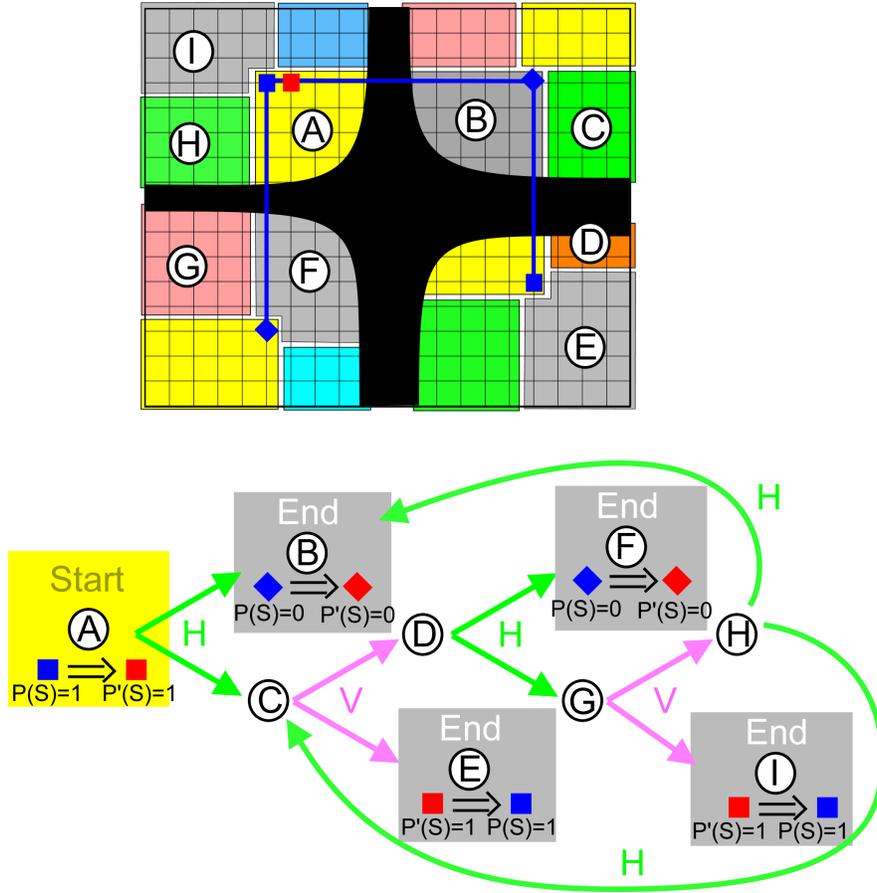


Figure 14: **Case 1.** In the first case, we decompose the four areas in different regions and follow the path of the switching component P' . It starts with a point p'_1 in the region A . The path ends also necessarily in A . Due to the relative position of p'_1 towards p_1 , we have $P(S) = 1 \implies P'(S) = 1$. The horizontal correspondents of the points in region A are either in region B , or in region C . The graph of the correspondences between the regions that we can reach from region A is drawn below. If the switching component P' passes through one of the grey regions B or F , the \mathcal{HV} -convexity constraint leads to $P(S) = 0 \implies P'(S) = 0$. If it passes through the grey regions E or I , then \mathcal{HV} -convexity leads to $P'(S) = 1 \implies P(S) = 1$. Therefore, if the switching component passes through one of the four grey regions, then we have the equality $P(S) = P'(S)$. We notice at last that the only way to avoid the grey regions is to follow the path $(CDGH)^*$ but it is excluded because it would not be possible to end the path in region A .

In the third case, we proceed in two steps. In the first step, we consider

path P passes through one of these regions, then the equality $P(S) = P'(S)$ is guaranteed by \mathcal{HV} -convexity. These grey regions that the path P should avoid in order to have $P(S) \neq P'(S)$ are drawn in the third row of Fig. 16. If the switching component P avoids the four grey regions, then the figure 16 shows that the path P after p_3 is restricted to a green region which does not allow to come back to the initial point. This case can be excluded, it leads to the following partial conclusion (i): If the switching component P' has two horizontal correspondents in the regions C and D , the equality of P and P' is guaranteed.

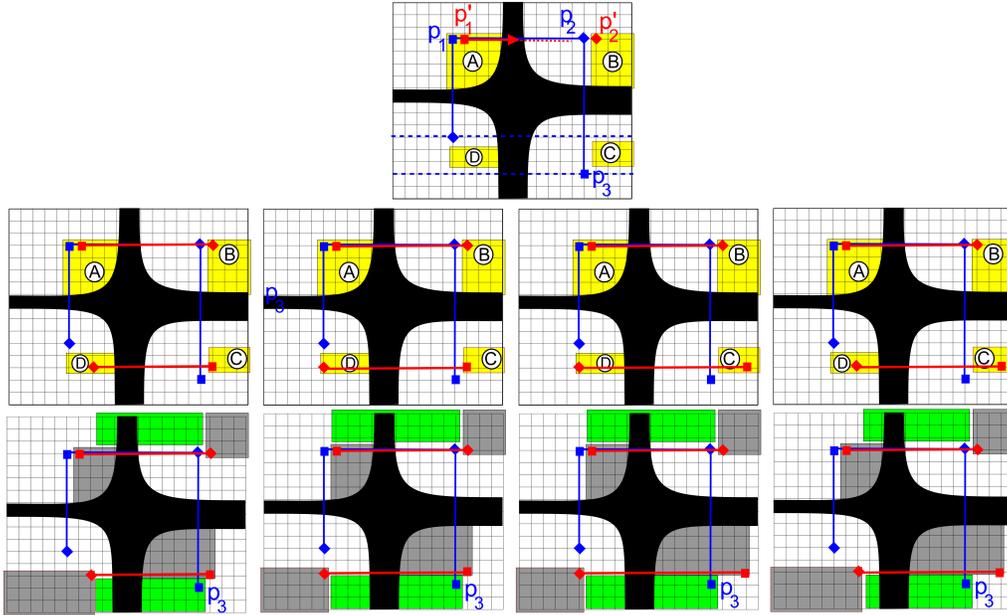


Figure 16: **Case 3. Preliminary analysis.** We introduce four regions A , B , C and D . The point p_1 and p_2 have coordinates noticed (x_1, y_1) and (x_2, y_2) . The points $p'_1 = (x_1 + 1, y_1)$ and $p'_2 = (x_2 + 1, y_2)$ (in red) are respectively in A and B . In the second row, we assume that there is a pair of horizontal correspondents of P' in the regions C and D . We denote their abscissa u and v with $u < v$. Four sub-cases are possible: $u > x_1$ and $v = x_2 + 1$, $u = x_1$ and $v = x_2 + 1$, $u = x_1$ and $v > x_2 + 1$, $u > x_1$ and $v > x_2 + 1$. In the third row, we consider the path of the switching component P after p_3 (in blue). Either it passes through one of the grey regions induced by P' and which provide the equality between P and P' , or it is maintained in the green area which does not allow to come back to p_1 . This last case is excluded with the consequence that if P' has two horizontal correspondents in B and C , we have the wanted equality $P(S) = P'(S)$.

We come back to the path followed by the switching component P' as drawn in Fig.17. We partition the undetermined points in regions according to their positions relatively to p_0, p_1, p_2 and p_3 and investigate again the regions crossed by the switching component P' . The grey regions drawn in Fig.17 are the ones which, due to \mathcal{HV} -convexity, provide the equality of the switching component $P(S) = P'(S)$. The diagram of the correspondences between the regions shows that either P' passes through a grey region, or passes through the edge BC for which the conclusion (i) provides already the equality $P = P'$, or is not closed which is in contradiction with its definition. It follows that in all valid cases, we have the equality $P(S) = P'(S)$. \square

3.4. Extended switching components are independent

We consider the case where all the switching components are regular. Lemma 1 provides the following corollary:

Theorem 1. *We consider an instance of $\text{DT}_{\mathcal{A}}(h, v)$ with a class $\mathcal{A} \subset \mathcal{H} \cap \mathcal{V}$ satisfying the partition property. The filling operations allow us to compute the switching components (Fig.11). If they are all regular, then any pair of switching components P and P' is either equivalent (for any solution S , $P(S) = P'(S)$ with the convention the first point of any switching component is in NW) or independent.*

Proof. We assume that the \mathcal{HV} -convexity constraint provides a dependency between two switching components P and P' . It means that they are in a configuration where if a point p_k of P is in a solution S , then the point $p'_{k'}$ of P' is also in S . By symmetries, we assume $p_k = (x_k, y_k)$ is in the North West area. It follows that $p'_{k'}$ is its South East quadrant $\{(x, y) \in \text{NW} | x \geq x_k, y \leq y_k\}$ (Fig.18). Thus there is a 4-connected path of undetermined points from p_k to $p'_{k'}$. By induction, Lemma 1 proves that their switching components are all equivalent and thus, provides the equality $P(S) = P'(S)$. \square

Theorem 1 states that we have either equivalent or independant switching components It leads to merge the equivalent switching components in one single *extended switching components*.

Definition 6. We consider an instance of $\text{DT}_{\mathcal{A}}(h, v)$ with a class $\mathcal{A} \subset \mathcal{H} \cap \mathcal{V}$ satisfying the partition property.

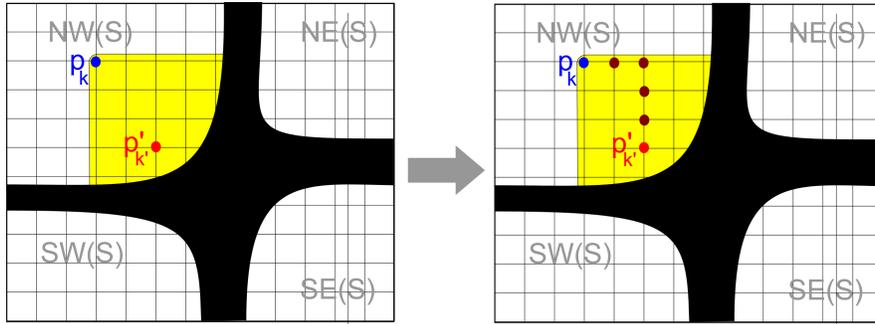


Figure 18: **A 4-connected path.** \mathcal{HV} -convex dependencies between switching components arise for instance in the case where the point $p'_{k'}$ of the switching component P' is in the South East quadrant of p_k (in yellow). If p_k is in a solution S , then by \mathcal{HV} -convexity, $p'_{k'}$ is also necessarily in S . In any case, we can connect p_k and $p'_{k'}$ by a 4-connected path of undetermined points (in brown). As their switching components are assumed to be all regular, according to Lemma 1, all the switching components of these brown points are equivalent. It provides the equivalence between the switching components of the blue and red points.

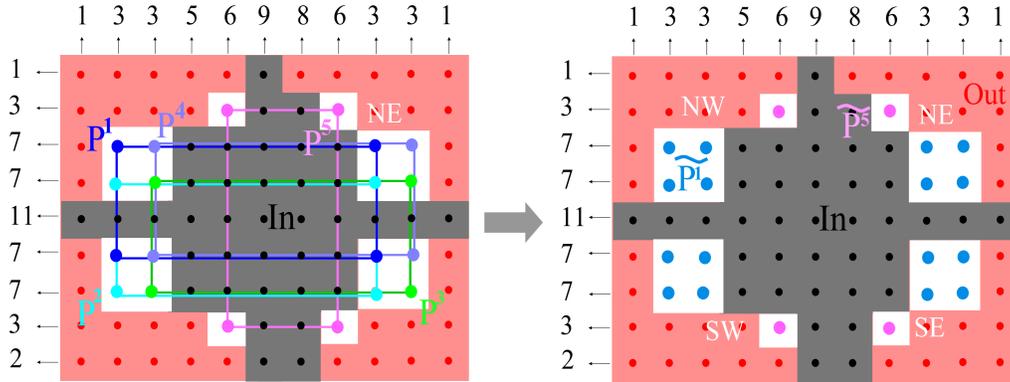


Figure 19: **Extended switching components.** On the left, due to \mathcal{HV} -convexity, the switching components P^1, P^2, P^3, P^4 are necessarily equal while P^5 is independent. On the right, we merge P^1, P^2, P^3, P^4 in the extended switching component \tilde{P}^1 and P^5 becomes \tilde{P}^5 . Extended switching components are less structured, they are just sets of points.

definition, for any solution S of the instance $\text{DT}_{\mathcal{A}}(h, v)$, either $\tilde{P}_{\text{NW}} \cup \tilde{P}_{\text{SE}}$ is in S and $(\tilde{P}_{\text{SW}} \cup \tilde{P}_{\text{NE}}) \cap S = \emptyset$ or $\tilde{P}_{\text{SW}} \cup \tilde{P}_{\text{NE}}$ is in S and $(\tilde{P}_{\text{NW}} \cup \tilde{P}_{\text{SE}}) \cap S =$

\emptyset . Theorem 1 means that if all the switching components are regular, all extended switching components are independent. An obvious corollary of Theorem 1 is therefore the following property:

Property 1. *We consider an instance of $\text{DT}_{\mathcal{A}}(h, v)$ with a class $\mathcal{A} \subset \mathcal{H} \cap \mathcal{V}$ satisfying the partition property. Given a position of the feet, if all the switching components are regular, then the number of solutions (with the chosen feet) is 2^n where n is the number of extended switching components.*

Property 1 does not hold with irregular switching components. A case with 3 solutions is shown in Fig.3. We end this section with a last remark about extended switching components in the regular case. Under the assumption that all the switching components are regular, distinct switching components pass necessarily through distinct rows and columns. Otherwise they would have been merged. This property is used for ordering the extended switching components.

4. Complementary results

We come back to the general switching components and provide several properties. We are interested in the characterization of the configurations where all the switching components are regular.

4.1. Any switching component visits the 4 areas

Due to their constant turning angle, the regular switching components have a cyclic property. With an initial point p_1 chosen in NW, we have $p_{1+4k} \in \text{NW}$, $p_{2+4k} \in \text{NE}$, $p_{3+4k} \in \text{SE}$, $p_{4k} \in \text{SW}$. This property does not hold with irregular switching components. We can even ask whether an irregular switching component can avoid one of the four areas NE, NW, SW, SE.

Property 2. *Any switching component passes through the four areas NE, NW, SW, SE.*

Proof. We assume w.l.g that there exists a switching component P which does not pass through the area NE. We start from a point $p_k \in \text{NW}$ and follow the path of the switching component. The horizontal correspondent p_{k+1} of p_k is necessarily in SE. The vertical correspondent p_{k+2} of p_{k+1} is therefore in NW. By induction, for any positive integer n , we have $p_{k+2n} \in \text{NW}$ and $p_{k+2n+1} \in \text{SE}$. The turning angles are alternatively $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$. Such a path cannot come back to the point p_k . It leads to a contradiction. \square

4.2. Structure of the switching components according to the positions of the feet

We investigate the relations between the positions of the feet and the structure of the switching components. In order to determine them, we start with an initial position of the South and North feet, and consider all the possible cases by adding first the West foot and secondly the East foot. This investigation is described step by step in Fig.20.

We can assume without loss of generality that the South foot is on the left of the North foot. We notice that the vertical X-rays passing through the South and North feet determine completely these columns (and we will have the same property with the rows of the East and West feet). Then, two cases are possible: either the points of In determined by the South and North feet have a common row or they don't share any row. In both cases, we introduce the possible positions of the West foot. The points of In on the rows of this foot might have different configurations. Many of them are not \mathcal{HV} -convex. After three feet, it remains only three possible cases. By adding the last East foot, we obtain 8 configurations which can be reduced to 6 by removing the symmetric cases. We denote them with letters from a) to f) (Fig.20). Each of them has specific possible switching components (Fig.21).

- In case a), different structures of switching components are possible but they are necessarily irregular (Fig.22).
- In case b), the structure of the switching components is constrained. They are all irregular (Fig.22). The points p_k and p_{k+8} are in the same area of the shell. Moreover, starting from an initial point p_1 in the South West area, there is no area containing simultaneously points of switching components with odd and even indices. It follows that the constant assignment $P[S] = 0$ for all switching components, or the constant assignment $P[S] = 1$ are two trivial solutions of the instance. In other words, there are at least two solutions.
- In case c), we can have switching components with different structures, but necessarily irregular (Fig.22).
- In case d), the structure of the switching components (all irregular according to Fig.22) is constrained. The point p_{k+6} is in the same area of the shell as p_k . As in b) with points $p_1 \in SW$, there are at least

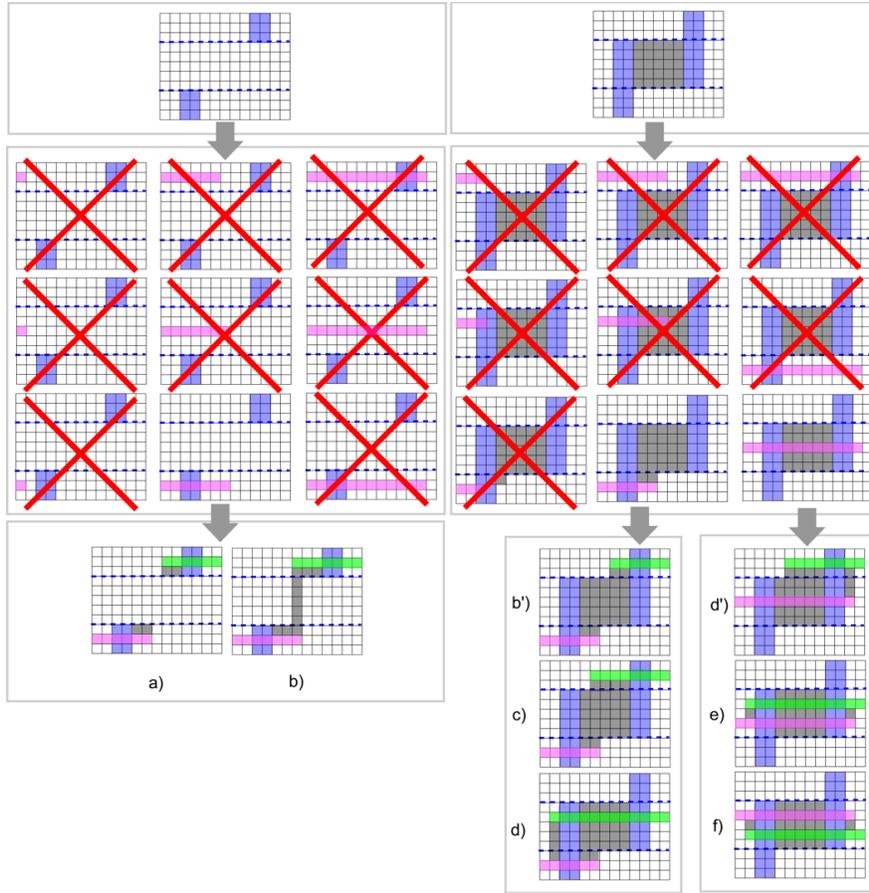


Figure 20: **The possible configurations of the rows and columns of the feet.** For the clarity of the figure, the lattice points are represented by the square cell which surrounds them. We start with the position of the South and North feet and consider all the compatible cases with \mathcal{HV} -convexity by adding the West and East feet. It leads at the end to only 6 possible configurations denoted with letters from a) to f) (the configurations b) and b') as well as d) and d') are symmetric).

two trivial solutions obtained with constant assignments of the Boolean variables.

- The case e) is the unique configuration which admits simultaneously irregular and regular switching components. Such a case is illustrated in Fig.21. This example goes against intuition.

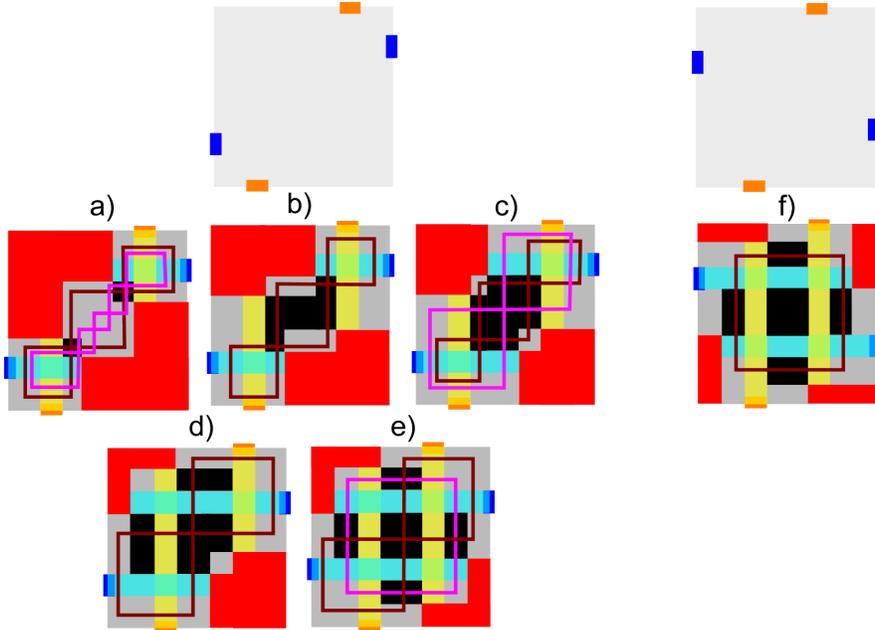


Figure 21: **The switching components in the 6 possible configurations.** Above the positions of the feet (blue and orange) and below, the six different possible configurations obtained in Fig. 20. The set Out of the cells of the excluded points is colored in red and the region of the possibly undetermined points is in grey. In each configuration, we have different structures of switching components. Regular switching components are only possible in the cases e) and f). Irregular switching components are possible in the cases a), b), c), d) and e).

- In the case f), all the switching components are regular.

It remains a lot of open questions about irregular switching components. The case e) is the only one where irregular and regular switching components might exist together (Fig.23). In the cases b), d), and f), the structures of the switching components guarantee that the two constant assignment of the Boolean variables ($P(S) = 1$ for all switching components or $P(S) = 0$ for all switching components) are trivial solution. In the configuration a), c) and e), we still don't know if it is possible to have an expression of the \mathcal{HV} -convexity constraints in 2-clauses which is not feasible.

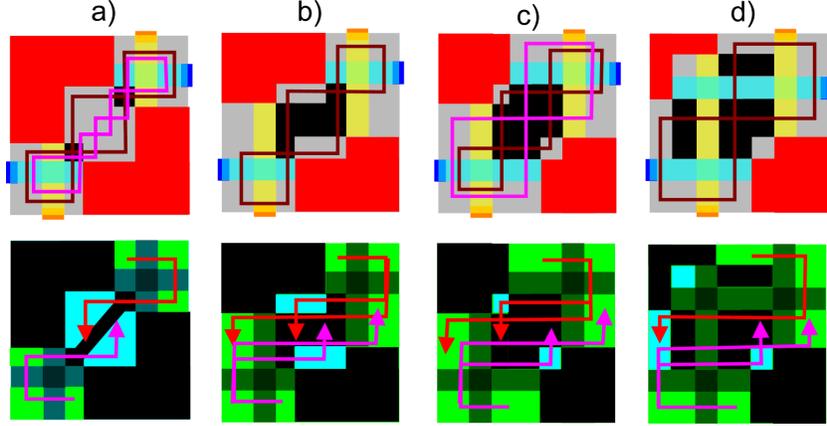


Figure 22: **The switching components in configurations a), b), c) and d)** cannot have regular switching components. It can be easily proved by following the path of any switching component in the shell (blue and green). After a few steps, any path has a turning angle different from the first one.

4.3. Extended regular switching components can be ordered

We introduce an order relation on the points of the shell. For a pair of points $p(x, y)$ and $p'(x', y')$, we denote $p < p'$ if $x < x'$ or if $x = x'$ and $y < y'$ for the points of the West areas of the shell and $p < p'$ if $x > x'$ or $x = x'$ and $y < y'$ for the East areas.

An order relation $<$ defined on an arbitrary set of points X induces an order relation $A < B$ between a pair of subsets $A \subset X$ and $B \subset X$ if for any pair $a \in A$ and $b \in B$, we have $a < b$. The constraint to order sets is that their points are well-ordered. Can the switching components be ordered? Unfortunately not, the points of switching components are not necessarily well ordered, even in the regular case (Fig.24). But it becomes true with extended switching components in the case where the switching components are all regular.

Property 3. *If all switching components are regular, the extended switching components can be ordered.*

Proof. We assume that switching components have been defined and that they are all regular. As already noticed, distinct extended switching components pass necessarily through distinct rows and columns. They can share neither a row nor a column. They are also necessarily 4-connected with the

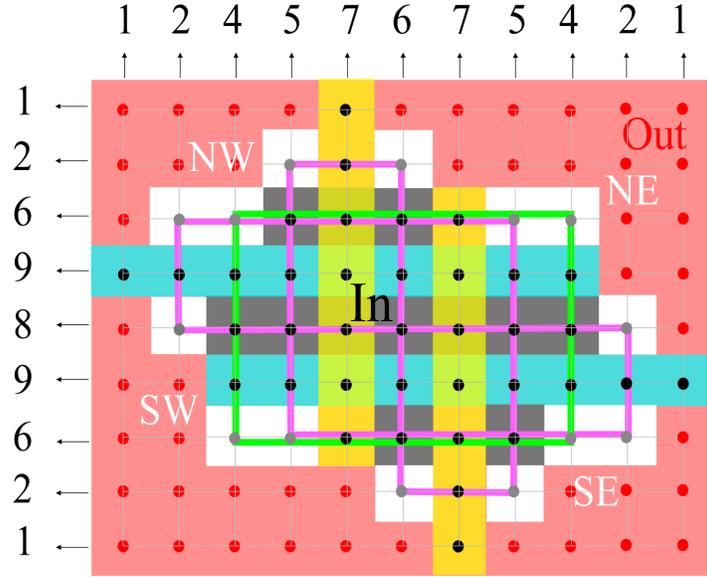


Figure 23: **An instance with regular and irregular switching components.** The only configuration which might provide both regular and irregular switching components is e). In all the other cases, the switching components are either all regular, or all irregular.

set In. On each area of undetermined points, they follow the boundary of In (Fig.25). Then the only case which can prevent them to be ordered is when one extended switching component is surrounded by another. We assume that such a configuration exists and show that it leads to a contradiction (Fig.26).

We consider 4-connected components of extended switching components that we call $4CC$. We assume that a $4CC$ denoted \widetilde{P}_1 of the switching component \widetilde{P} is surrounded by two $4CC$ \widetilde{P}'_1 and \widetilde{P}'_2 of another extended switching component \widetilde{P}' . It means $\widetilde{P}'_1 < \widetilde{P}_1 < \widetilde{P}'_2$. We can assume w.l.g that \widetilde{P}'_2 is upper than all the $4CC$ of \widetilde{P} (otherwise we invert \widetilde{P} and \widetilde{P}'). By definition, there exists a path of horizontal correspondences connecting \widetilde{P}'_1 and \widetilde{P}'_2 . With the same number of turns around In, the surrounded $4CC$ \widetilde{P}_1 arrives on a $4CC$ of P that we denote \widetilde{P}_2 . We notice therefore that horizontal and vertical correspondences preserve the order of the $4CC$ s (since they don't share rows or columns). Then by turning around In, it follows from the initial assumption $\widetilde{P}'_1 < \widetilde{P}_1$ that we have also $\widetilde{P}'_2 < \widetilde{P}_2$. It contradicts the initial assumption

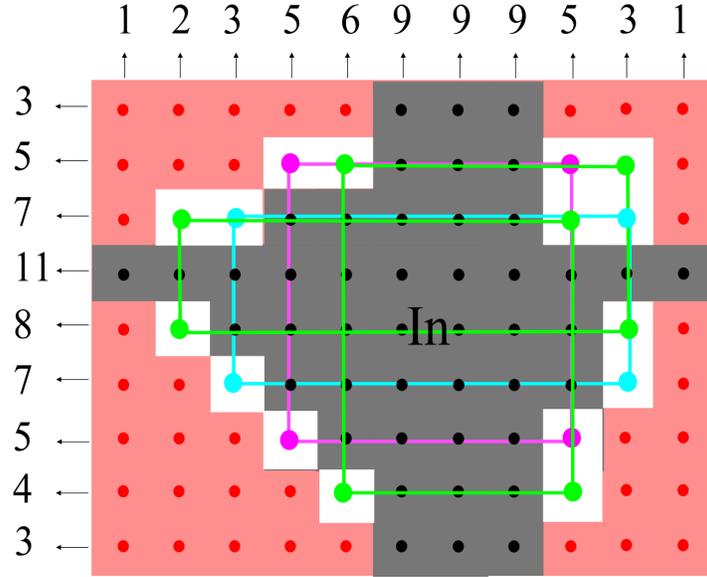


Figure 24: **An instance with regular switching components which can not be ordered directly.** The green switching components surrounds the two others.

that \widetilde{P}'_2 is upper than any 4CC of \widetilde{P} .

□

It follows from all these complementary properties that the extended switching components have a simple structure along the boundary of the set In: they are independent and well ordered. This property is a step forward in the direction of the reconstruction of \mathcal{C} -convex lattice sets in the case where all the switching components are regular.

5. Conclusion

Although the switching components have been widely used for reconstructing \mathcal{HV} -convex lattice sets from their horizontal and vertical X-rays, their combinatorial properties are still poorly known. We started the investigation of the regular case. It occurs when the feet are not placed in opposite corners of the rectangle (case f) of Fig.21). Then the switching components are all regular. According to Lemma 1, adjacent switching components are

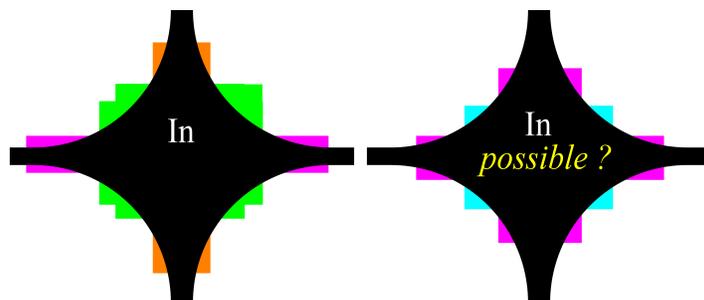


Figure 25: **Ordering the extended switching components.** The extended switching components follow the boundary of the set In (in black). They can be ordered if and only if the right case, with an extended switching component surrounding another one can be excluded.

equivalent. We merge them in extended switching components. While regular switching components might still have complex configurations, these new objects can be easily ordered along the boundary of the set In.

The proofs of the results that we stated in the paper have required new approaches based on the investigation of oriented graphs of regions. The regular switching components have revealed a part of their structure. It would be of interest to investigate the irregular switching components similarly but their structure is more complex. A better understanding of their relations could be useful to determine the status of the conjecture: the 2-CNF formula expressing the \mathcal{HV} -convexity of the solution is always satisfiable. We hope at last that it can help to determine the complexity of $DT_{\mathcal{A}}(h, v)$ for the class $\mathcal{A} = \mathcal{C}$ of the \mathcal{C} -convex lattice sets.

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