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Abstract

In the second section "Courant-Gelfand theorem" of his last published paper (Topological properties of eigenoscillations in mathematical physics, Proc. Steklov Institute Math. 273 (2011) 25–34), Arnold recounts Gelfand's strategy to prove that the zeros of any linear combination of the n first eigenfunctions of the Sturm-Liouville problem

$$- y''(s) + q(x)\, y(x) = \lambda\, y(x) \text{ in }]0,1[\,, \text{ with } y(0) = y(1) = 0\,,$$

divide the interval into at most n connected components, and concludes that "the lack of a published formal text with a rigorous proof ... is still distressing."
Inspired by Quantum mechanics, Gelfand's strategy consists in replacing the analysis of linear combinations of the n first eigenfunctions by that of their Slater determinant which is the first eigenfunction of the associated n particle operator acting on Fermions.
In the present paper, we implement Gelfand's strategy, and give a complete proof of the above assertion. As a matter of fact, we refine this strategy, and prove a stronger property taking the multiplicity of zeros into account, a result which actually goes back to Sturm (1836).

1 Introduction

On September 30, 1833, C. Sturm[1] presented a memoir on second order linear differential equations to the Paris Academy of Sciences. The main results are summarized in [19, 20], and were later published in the first volume of Liouville's journal, [21, 22]. We refer to [5] for more details. In this paper, we shall consider the following particular case.

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[1] Jacques Charles François Sturm (1803–1855)

Theorem 1.1 (Sturm (1836)). *Let q be a smooth real valued function defined is a neighborhood of the interval $[0,1]$. The Dirichlet eigenvalue problem*

$$\begin{cases} -y''(x) + q(x)\,y(x) = \lambda\,y(x) \ \textit{in} \ ]0,1[\,, \\ y(0) = y(1) = 0\,, \end{cases} \tag{1}$$

has the following properties.

1. *There exists an infinite sequence of (simple) eigenvalues*

$$\lambda_1 < \lambda_2 < \cdots \nearrow \infty,$$

   with an associated orthonormal family of eigenfunctions $\{h_j, j \geq 1\}$.

2. *For any $j \geq 1$, the eigenfunction h_j has exactly $(j-1)$ zeros in the interval $]0,1[$.*

3. *For any $1 \leq m \leq n$, let $U = \sum_{k=m}^{n} a_k\, h_k$ be any nontrivial real linear combination of eigenfunctions. Then,*

   (a) U has at most $(n-1)$ zeros in $]0,1[$, counted with multiplicities,

   (b) U changes sign at least $(m-1)$ times in $]0,1[$.

Sturm's motivations came from mathematical physics. He took a novel point of view, looking for qualitative behavior of solutions rather than for explicit solutions. To prove Assertions 1 and 2, he introduced the comparison and oscillation theorems which today bear his name. Assertion 3 came as a corollary of Sturm's investigation of the evolution of zeros of a solution $u(t,x)$ of the associated heat equation, with initial condition U, as times goes to infinity (in direct line with his motivations). We give a proof of Assertion 3 in Section 2.

Remarks 1.2. (i) In the framework of Fourier series, Assertion 3b is often referred to as the Sturm-Hurwitz theorem.
(ii) Sturm's theorem applies to more general operators, with more general boundary conditions; we refer to [5] for more details.

R. Courant[2] partly generalized Assertion 2, in Sturm's theorem, to higher dimensions.

Theorem 1.3. *Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \nearrow \infty$ be the Dirichlet eigenvalues of $-\Delta$ in a bounded domain of $\mathbb{R}^d$, listed in nondecreasing order, with multiplicities. Let u be any nontrivial eigenfunction associated with the eigenvalue λ_n, and let $\beta_0(u)$ denote the number of connected components of $\Omega \backslash \{0\}$ (*nodal domains*). Then,*

$$\beta_0(u) \leq n\,.$$

In a footnote of [10, p. 454], Courant makes the following statement.

Statement 1.4. *Any linear combination of the first n eigenfunctions divides the domain, by means of its nodes, into no more than n subdomains. See the Göttingen dissertation of H. Herrmann, Beiträge zur Theorie der Eigenwerten und Eigenfunktionen, 1932.*

[2] Richard Courant (1888–1972).

In the literature, Statement 1.4 is referred to as the "Courant-Herrmann theorem", "Courant-Herrmann conjecture", "Herrmann's theorem", or "Courant generalized theorem". In [6, 7], we call it the *Extended Courant property.*

Remarks 1.5. 1. It is easy to see that Courant's upper bound is not sharp. This is indeed the case whenever the eigenvalue λ_n is not simple. More generally, it can be shown that the number $\beta_0(u)$ is asymptotically smaller than $\gamma(n)\,n$ when n tends to infinity, where $\gamma(n) < 1$ is a constant which only depends on the dimension n. It is interesting to investigate the eigenvalues for which Courant's upper bound is sharp, see the review article [9]. For this research topic, we also refer to the surprising recent paper [12].

2. In dimension greater than or equal to 2, there is *no general lower bound* for $\beta_0(u)$, except the trivial ones (1 for λ_1, and 2 for $\lambda_k, k \geq 2$). Examples were first given by A. Stern in her 1924 Göttingen thesis, see [4].

In the early 1970's, V. Arnold[3] noticed that Statement 1.4, would provide a partial answer to one of the problems formulated by D. Hilbert[4].

> Citation from Arnold [3, p. 27].
>
> I immediately deduced from the generalized Courant theorem [Statement 1.4] new results in Hilbert's famous (16th) problem. ... And then it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. ... Hence, the statement of the generalized Courant theorem is not true (explicit counterexamples were soon produced by Viro). Courant died in 1972 and could not have known about this counterexample[5].

Arnold was very much intrigued by Statement 1.4, as is illustrated by [3], his last published paper, where he in particular relates a discussion with I. Gelfand[6], which we transcribe below, using Arnold's words, in the form of an imaginary dialog.

(Gelfand) *I thought that, except for me, nobody paid attention to Courant's remarkable assertion. But I was so surprised that I delved into it and found a proof.*

(Arnold is quite surprised, but does not have time to mention the counterexamples before Gelfand continues.)

However, I could prove this theorem of Courant only for oscillations of one-dimensional media, where $m = 1$.

(Arnold) *Where could I read it?*

(Gelfand) *I never write proofs. I just discover new interesting things. Finding proofs (and writing articles) is up to my students.*

Arnold then recounts Gelfand's strategy to prove Statement 1.4 in the *one-dimensional case.*

[3] Vladimir Igorevich Arnold (1937-2010).

[4] David Hilbert (1862–1943).

[5] [1], the first paper of Arnold on this subject, we are aware of, dates from 1973.

[6] Israel Moiseevich Gelfand (1913-2009).

> Quotations from [3, Section 2].
>
> Nevertheless, the one-dimensional version of Courant's theorem is apparently valid. ... Gelfand's idea was to replace the analysis of the system of n eigenfunctions of the one-particle quantum-mechanical problem by the analysis of the first eigenfunction of the n-particle problem (considering as particles, fermions rather than bosons). ...
>
> Unfortunately, [Gelfand's hints] do not yet provide a *proof* for this generalized theorem: many facts are still to be proved. ...
>
> Gelfand did not publish anything concerning this: he only told me that he hoped his students would correct this drawback of his theory. ...
>
> Viktor Borisovich Lidskii told me that "he knows how to prove all this". ...
>
> Although [Lidskii's] arguments look convincing, the lack of a published formal text with a proof of the Courant-Gelfand theorem is still distressing.

In [13], Kuznetsov refers to Statement 1.4 as *Herrmann's theorem*, and relates that Gelfand's approach *so attracted Arnold that he included Herrmann's theorem for eigenfunctions of problem [(1)] together with Gelfand's hint into the 3rd Russian edition of his Ordinary Differential Equations*, see Problem 9 in the "Supplementary problems" at the end of [2].

More precisely, Arnold's Problem 9 proposes to prove the following statement, which is the one-dimensional analogue of Statement 1.4.

Statement 1.6. *The zeros of any linear combination of the n first eigenfunctions of the Sturm-Liouville problem* (1) *divide the interval into at most n connected components.*

This statement is equivalent to saying that any linear combination of the n first eigenfunctions of (1) has at most $(n-1)$ zeros in the open interval. This is a weak form of Sturm's upper bound, Assertion 3a in Theorem 1.1. In the present paper, we implement Gelfand's strategy to prove Statement 1.6 (see also [7]), and we extend this strategy to take into account the multiplicities of zeros, and to prove Assertion 3a in Theorem 1.1. Inspired by Quantum mechanics, Gelfand's strategy consists in replacing the analysis of linear combinations of the n first eigenfunctions by that of their Slater determinant which is the first eigenfunction of the associated n particle operator acting on Fermions. We give more details in Section 5. Note that Assertion 3b is actually a consequence of Assertion 3a, see Section 2.

The paper is organized as follows. In Section 2, we give J. Liouville's[7] 1836 proof of Assertion 3 in Theorem 1.1. In Section 3, we introduce some notation. In Section 4, we give preliminary results on Vandermonde determinants, to be used later on. In Section 5, we explain Gelfand's strategy, and we apply it to a particular case, the harmonic oscillator. Section 6 is devoted to the proof of Assertion 3a in Theorem 1.1, in the general case, following Gelfand's strategy: in Subsection 6.2, we prove Sturm's weak upper bound on the number of zeros of a linear combination of eigenfunctions, Statement 1.6, thus solving Problem 9 in [2]; Sturm's strong upper bound is proved in Subsection 6.4.

[7] Joseph Liouville (1809–1882).

2 Liouville's proof of Sturm's theorem

Assertions 1 and 2 in Theorem 1.1 are well-known, and can be found in many textbooks. This is not the case for Assertion 3. In this section, we give a short proof, based on the arguments of Liouville [16], and Rayleigh[8] [18, § 142].

Proof of Assertion 3a. Write equation (1) for h_1 and for h_k, multiply the first one by h_k, the second by $(-h_1)$ and add to obtain the relation

$$\left(h_1\, h_k' - h_1'\, h_k\right)' = (\lambda_1 - \lambda_k)\, h_1\, h_k\,.$$

Multiply by a_k, and sum from $k = m$ to $k = n$ to obtain

$$\left(h_1\, U' - h_1'\, U\right)' = h_1\, U_1\,, \tag{2}$$

where $U_1 = \sum_{k=m}^{n}(\lambda_1 - \lambda_k)\, a_k\, h_k$. Integrating this relation from 0 to x, and using the Dirichlet boundary condition, gives

$$h_1(x)\, U'(x) - h_1'(x)\, U(x) = \int_0^x h_1(t)\, U_1(t)\, dt\,.$$

Note that the left hand side can be rewritten as $h_1^2(x)\, \frac{d}{dx}\frac{U}{h_1}(x)$ in $]0,1[$. Count zeros with multiplicities. Assume that U has N zeros in $]0,1[$. Then so does $\frac{U}{h_1}$, so that, by Rolle's theorem, $\frac{d}{dx}\frac{U}{h_1}$ has *a least* $(N-1)$ zeros in $]0,1[$. It follows that the function $x \mapsto \int_0^x h_1(t)\, U_1(t)\, dt$ has at least $(N-1)$ zeros in $]0,1[$. Note that it also vanishes at both 0 and 1 because the h_j form an orthonormal family. By Rolle's theorem again, we conclude that its derivative, $h_1\, U_1$, has at least N zeros in $]0,1[$. Because U and U_1 have the same form, we can repeat the argument, and conclude that, for any $\ell \geq 1$, the function $U_\ell = \sum_{k=m}^{n}(\lambda_1 - \lambda_k)^\ell\, a_k\, h_k$ has *at least* N zeros in $]0,1[$. Letting ℓ tend to infinity, using the fact that the eigenvalues λ_k are simple, and the fact that h_n has $(n-1)$ zeros in $]0,1[$, it follows that $N \leq (n-1)$.

Proof of Assertion 3b. Assume that U changes sign exactly M times at the points $z_1 < \cdots < z_M$ in the interval $]0,1[$, and that $M < (m-1)$, i.e., $M \leq (m-2)$. Consider the function,

$$V(x) := \begin{vmatrix} h_1(z_1) & \ldots & h_1(z_M) & h_1(x) \\ \vdots & & \vdots & \vdots \\ h_n(z_1) & \ldots & h_n(z_M) & h_n(x) \end{vmatrix}$$

It is easy to prove that the function V is not identically zero (see Lemma 6.1). It clearly vanishes at the points $z_j, 1 \leq j \leq M$, and it is a linear combination of the eigenfunctions $h_1, \ldots, h_M$ (develop the determinant with respect to the last column). According to Assertion 3a in Theorem 1.1, V does not have any other zero, and each z_j has order 1, so that V changes sign exactly at the points z_j. Since $M \leq (m-2)$, the functions U and V are orthogonal, and their product $U\, V$ does not change sign in $]0,1[$. It follows that $U\, V$ vanishes identically, a contradiction. $\square$

[8] John William Strutt, Lord Rayleigh (1842–1919).

Remark 2.1. With the above notation, we can rewrite (2) as

$$h_1\, U_1 = h_1\, U'' + (\lambda_1 - q)\, h_1\, U\,. \tag{3}$$

A similar relation holds between $U_{\ell+1}$ and U_ℓ. Using these relations, and letting ℓ tend to infinity as in the preceding proof, we obtain the following lemma which is interesting in itself.

Lemma 2.2. *The nonzero linear combination U cannot vanish at infinite order at any point in $[0,1]$. In particular, its zeros are isolated.*

3 Notation

Let n be an integer, $n \geq 1$, and $\mathrm{J} \subset \mathbb{R}$ an interval. Given n points $x_1, \ldots, x_n$ in J, we denote the corresponding vector by $\vec{x} = (x_1, \ldots, x_n) \in \mathrm{J}^n$. Generally speaking, we denote by $\vec{k} = (k_1, \cdots, k_n)$ a vector with positive integer entries.

We use the notation $\overset{\rightharpoonup}{c} = (c_1, \ldots, c_{n-1})$ for an $(n-1)$-vector with entries in J.

Given n real continuous functions $f_1, \ldots, f_n$ defined on J, we denote by $\vec{f}$ the vector-valued function $\big(f_1, \cdots, f_n\big)$, and we introduce the determinant

$$\left|\vec{f}(x_1) \ldots \vec{f}(x_n)\right| := \begin{vmatrix} f_1(x_1) & f_1(x_2) & \ldots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \ldots & f_2(x_n) \\ \vdots & \vdots & \ldots & \vdots \\ f_n(x_1) & f_n(x_2) & \ldots & f_n(x_n) \end{vmatrix}. \tag{4}$$

Given a vector $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we denote by

$$S_{\vec{b}}(x) = \sum_{j=1}^{n} b_j\, f_j(x)\,, \tag{5}$$

the linear combination of $f_1, \ldots, f_n$, with coefficients b_j's.

Let $\vec{c} \in \mathrm{J}^n$ be a vector of the form

$$\vec{c} = (\bar{c}_1, \ldots, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_2,\ \ldots\ , \bar{c}_p, \ldots, \bar{c}_p)\,, \tag{6}$$

with $\bar{c}_1$ repeated k_1 times, $\ldots$, $\bar{c}_p$ repeated k_p times, $1 \leq p \leq n$, $k_1 + \cdots + k_p = n$, and with $\bar{c}_1 < \bar{c}_2 < \cdots < \bar{c}_p$.

It will be convenient to relabel the variables $\vec{x} = (x_1, \ldots, x_n)$ according to the structure of $\vec{c}$, as follows,

$$\vec{x} = (x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, \ldots, x_{p,1}, \ldots, x_{p,k_p})\,, \tag{7}$$

so that,

$$\begin{cases} x_{1,1} = x_1,\ \ldots, x_{1,k_1} = x_{k_1} \text{ and, for } 2 \leq i \leq p\,, \\ x_{i,1} = x_{k_1+\cdots+k_{i-1}+1},\ \ldots, x_{i,k_i} = x_{k_1+\cdots+k_{i-1}+k_i}\,. \end{cases} \tag{8}$$

In this case, we will also write the vector $\vec{x}$ as

$$\vec{x} = \left(x^{(1)}, \dots, x^{(p)}\right), \tag{9}$$

with $x^{(i)} = (x_{i,1}, \dots, x_{i,k_i})$, for $1 \le i \le p$.

We shall usually use both ways of labeling inside a formula, there should not be any confusion.

We introduce the real polynomials

$$\begin{cases} Q_1(x_1) &= 1\,,\\ \text{and, for } n \ge 2, & \\ Q_n(x_1, \dots, x_n) &= \prod_{j=2}^{n} (x_1 - x_j)\,, \end{cases} \tag{10}$$

and

$$\begin{cases} P_1(x_1) &= 1\,,\\ \text{and, for } n \ge 2, & \\ P_n(x_1, \dots, x_n) &= \prod_{1 \le i < j \le n} (x_i - x_j) = \prod_{i=1}^{n-1} Q_{n+1-i}(x_i, \dots, x_n)\,. \end{cases} \tag{11}$$

4 Vandermonde determinants

Lemma 4.1. *The polynomial P_n, defined in (11), is the Vandermonde*[9] *determinant*

$$P_n(x_1, \dots, x_n) = \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \dots & x_n^{n-1} \end{vmatrix}. \tag{12}$$

Furthermore,

1. *P_n is anti-symmetric under the action of the group of permutations $\mathfrak{s}_n$, and homogenous of degree $\frac{n(n-1)}{2}$.*
2. *As a function of $x_1, \dots, x_n$, P_n is harmonic, $\Delta P_n = 0$, and satisfies*

$$\partial_{x_n}^{n-1}\, \partial_{x_{n-1}}^{n-2} \cdots \partial_{x_3}^{2}\, \partial_{x_2}\, P_n = (-1)^{\frac{n(n-1)}{2}} (n-1)!\,(n-2)! \,\dots\, 2!\,. \tag{13}$$

Proof. The identity (12) is well-known, and readily implies Assertion 1. The polynomial P_n being anti-symmetric, its Laplacian is also anti-symmetric, and hence, must be divisible by P_n. Being of degree less than P_n, ΔP_n must be zero. The identity (13) follows immediately from the multi-linearity of the determinant, or by induction on n. ☐

Notation. When $\vec{x} = (x_1, \dots, x_n)$, we will also write $P_n(\vec{x})$ for $P_n(x_1, \dots, x_n)$. We will denote by $D_n(\partial_{\vec{x}})$ the differential operator

$$D_n(\partial_{\vec{x}}) := \partial_{x_n}^{n-1}\, \partial_{x_{n-1}}^{n-2} \cdots \partial_{x_3}^{2}\, \partial_{x_2} \tag{14}$$

which appears in (13), so that

$$D_n(\partial_{\vec{x}}) P_n(\vec{x}\,) = (-1)^{\frac{n(n-1)}{2}} (n-1)!\,(n-2)! \,\dots\, 2!\,. \tag{15}$$

[9] Alexandre Théophile VANDERMONDE (1735–1796).

Lemma 4.2. *Given $\vec{x} = (\vec{y}, \vec{z}) \in \mathbb{R}^p \times \mathbb{R}^q$, the function*

$$\vec{x} \mapsto P_p(\vec{y})\, P_q(\vec{z})$$

is harmonic as a function on $\mathbb{R}^{p+q}$.

We shall now describe the local behaviour of the harmonic polynomial P_n near a point $\vec{c} \in \mathbb{R}_n$ at which it vanishes. We first treat two simple examples.

Example 4.3. Let $n = 5$, and $\vec{c} = (\bar{c}_1, \bar{c}_1, \bar{c}_2, \bar{c}_2, c_5)$, with $\bar{c}_1 < \bar{c}_2 < c_5$. Then, $P_5(\vec{c}) = 0$. Write $\vec{x} = \vec{c} + \vec{\xi}$. An easy computation gives,

$$P_5(\vec{c} + \vec{\xi}) = P_2(\xi_1, \xi_2)\, P_2(\xi_3, \xi_4) \left\{ \rho(\vec{c}) + \omega(\vec{c}, \vec{\xi}\,) \right\}, \tag{16}$$

where $\rho(\vec{c}) = (\bar{c}_1 - \bar{c}_2)^4\, (\bar{c}_1 - c_5)^2\, (\bar{c}_2 - c_5)^2$ is a nonzero constant, and where $\omega(\vec{c}, \vec{\xi}\,)$ denotes a polynomial in the $(\xi_i - \xi_j)$'s, with coefficients depending on $\vec{c}$, and without constant term.

Example 4.4. Let $n = 5$. Let $\vec{c} = (\bar{c}_1, \bar{c}_1, \bar{c}_1, c_4, c_5)$, with $\bar{c}_1 < c_4 < c_5$. Then, $P_5(\vec{c}) = 0$. Write $\vec{x} = \vec{c} + \vec{\xi}$. An easy computation gives,

$$P_5(\vec{c} + \vec{\xi}) = P_3(\xi_1, \xi_2, \xi_3) \left\{ \rho(\vec{c}) + \omega(\vec{c}, \vec{\xi}\,) \right\}, \tag{17}$$

where $\rho(\vec{c}) = (\bar{c}_1 - c_4)^3\, (\bar{c}_1 - c_5)^3\, (c_4 - c_5)$ is a nonzero constant, and where $\omega(\vec{c}, \vec{\xi}\,)$ denotes a polynomial in the $(\xi_i - \xi_j)$'s, with coefficients depending on $\vec{c}$, and without constant term.

Remark 4.5. In both examples, the leading term on the right hand side of $P_n(\vec{c} + \vec{\xi}\,)$ is a homogeneous harmonic polynomial is some of the variables ξ_j's, as we can expect from Bers's theorem, [8].

In the following lemma, we use both the standard coordinates names and their relabeling (7)–(9), for both variables $\vec{x}$ and $\vec{\xi}$.

Lemma 4.6. *Let p be an integer, $1 \leq p \leq n$, and $(k_1, \ldots, k_p)$ be a p-tuple of positive integers, such that $k_1 + \cdots + k_p = n$. Let $(\bar{c}_1, \ldots, \bar{c}_p)$ be a p-tuple, such that $\bar{c}_1 < \cdots < \bar{c}_p$. Let $\vec{c}$ be the n-vector*

$$\vec{c} = (c_1, \ldots, c_1, \ \ldots \ , c_p, \ldots, c_p)\,, \tag{18}$$

where each $\bar{c}_j$ is repeated k_j times, $1 \leq j \leq p$. Writing $\vec{x} = \vec{c} + \vec{\xi}$, and relabeling the coordinates of the vectors $\vec{x}$ and $\vec{\xi}$ as in (7)–(9), we have the following relation,

$$P_n(\vec{c} + \vec{\xi}\,) = \rho(\vec{c}\,)\, P_{k_1}(\xi_{1,1}, \ldots, \xi_{1,k_1}) \ \ldots \ P_{k_p}(\xi_{p,1}, \ldots, \xi_{p,k_p}) \left(1 + \omega(\vec{c}, \vec{\xi}\,)\right), \tag{19}$$

where $\rho(\vec{c}\,)$ is a nonzero constant depending only on $\vec{c}$, and where $\omega(\vec{c}, \vec{\xi}$ is a polynomial in the variables $(\xi_i - \xi_j)$'s, with coefficients depending on the c_j's, and without term of degree 0.

Proof. From the definition of P_n, and using the relabeling of the variables $\vec{x}$ and $\vec{\xi}$, as indicated in (7)–(9), we obtain the following relations.

$$P_n(\vec{c}+\vec{\xi}\,) = \left(\prod_{i=1}^{k_1} Q_{n+1-i}(c_i+\xi_i,\ldots,c_n+\xi_n)\right) \prod_{i=k_1+1}^{n} Q_{n+1-i}(c_i+\xi_i,\ldots,c_n+\xi_n)\,, \tag{20}$$

$$P_n(\vec{c}+\vec{\xi}\,) = \left(\prod_{i=1}^{k_1} Q_{n+1-i}(c_i+\xi_i,\ldots,c_n+\xi_n)\right) P_{n-k_1}(c_{2,1}+\xi_{2,1},\ldots,c_{p,1}+\xi_{p,k_p})\,, \tag{21}$$

Developing the factors Q_{n+1-i} for $i \le k_1$, we obtain,

$$\prod_{i=1}^{k_1} Q_{n+1-i}(c_i+\xi_i,\ldots,c_n+\xi_n) = \rho_1(\vec{c}\,)\, P_{k_1}(\xi_{1,1},\ldots,\xi_{1,k_1})\, \Big(1+\omega(\vec{c},\vec{\xi}\,)\Big)\,, \tag{22}$$

where

$$\rho_1(\vec{c}\,) = \left[Q_{n+1-k_1}(\bar{c}_1, c_{2,1},\ldots,c_{p,k_p})\right]^{k_1} \neq 0\,, \tag{23}$$

and where

$$\omega(\vec{c},\vec{\xi}\,) \tag{24}$$

is a polynomial in the variables $(\xi_i-\xi_j)$'s, with coefficients depending on the c_j's, and without term of degree 0. Finally, we have

$$P_n(\vec{c}+\vec{\xi}\,) = \rho_1(\vec{c}\,)\, P_{k_1}(\xi_{1,1},\ldots,\xi_{1,k_1})\, P_{n-k_1}(c_{2,1}+\xi_{2,1},\ldots,c_{p,1}+\xi_{p,k_p})\, \Big(1+\omega_1(\vec{c},\vec{\xi}\,)\Big)\,,$$

or, more concisely,

$$P_n(\vec{c}+\vec{\xi}\,) = \rho_1(\vec{c}\,)\, P_{k_1}\big(\xi^{(1)}\big)\, P_{n-k_1}\big(\xi^{(2)},\ldots,\xi^{(p)}\big)\, \Big(1+\omega_1(\vec{c},\vec{\xi}\,)\Big)\,. \tag{25}$$

We can then apply the same kind of computation to the factor P_{n-k_1}, and repeat the operation until we finally obtain the desired formula, with

$$\rho(\vec{c}\,) = \rho_1(\vec{c}\,)\,\cdots\,\rho_p(\vec{c}\,) \neq 0\,. \tag{26}$$

□

Notation. In the sequel, we shall use $\omega(\vec{c},\vec{\xi}\,)$ as a generic notation for a function in the variables $(\xi_i-\xi_j)$'s which tends to zero as $\vec{\xi}$ tends to zero.

5 Gelfand's strategy and the harmonic oscillator

In this section, we explain Gelfand's strategy to prove Statement 1.6, and how one can extend it to obtain a proof of Assertion 3a in Theorem 1.1, in the particular case of the harmonic oscillator.

Let $\mathfrak{H}^{(1)}$ denote the 1-particle *harmonic oscillator*

$$\mathfrak{H}^{(1)} := -\frac{d^2}{dx^2} + x^2 \tag{27}$$

on the line. The eigenvalues are given by $\{\lambda_n = 2n-1, n \geq 1\}$, they are simple, with associated orthonormal basis of eigenfunctions $\{h_n, n \geq 1\}$,

$$h_n(x) = \gamma_{n-1}\, H_{n-1}(x)\, \exp(-x^2/2)\,, \tag{28}$$

where H_m is the m-th Hermite polynomial, and γ_m a normalizing constant [14, Chap. 3]. The polynomial $H_m(x)$ has degree m, with leading coefficient 2^m, and satisfies the differential equation,

$$y''(x) - 2x\, y'(x) + 2m\, y(x) = 0 \tag{29}$$

on the line $\mathbb{R}$.

We consider the n-particle Hamiltonian in $\mathbb{R}^n$,

$$\mathfrak{H}^{(n)} := \sum_{j=1}^{n} \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) = -\Delta + |\vec{x}|^2\,. \tag{30}$$

Gelfand's strategy is to look at $\mathfrak{H}_F^{(n)}$, the operator $\mathfrak{H}^{(n)}$ restricted to *Fermions*, i.e., to functions which are anti-invariant under the action of the permutation group $\mathfrak{s}_n$ on $\mathbb{R}^n$,

$$L_F^2(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \mid f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) = \varepsilon(\sigma) f(x_1, \ldots, x_n),\ \forall \sigma \in \mathfrak{s}_n \right\}\,. \tag{31}$$

Equivalently, we consider the Dirichlet realization $\mathfrak{H}_F^{(n)}$ of $\mathfrak{H}^{(n)}$ in

$$\Omega_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 < x_2 \cdots < x_n\}\,. \tag{32}$$

Introduce the *Slater*[10] *determinant*

$$\mathfrak{S}_n(\vec{x}) = \begin{vmatrix} h_1(x_1) & \ldots & h_1(x_n) \\ \vdots & & \vdots \\ h_n(x_1) & \ldots & h_n(x_n) \end{vmatrix} = A_n\, \exp(-|\vec{x}\,|^2/2) \begin{vmatrix} H_0(x_1) & \ldots & H_0(x_n) \\ \vdots & & \vdots \\ H_{n-1}(x_1) & \ldots & H_{n-1}(x_n) \end{vmatrix}. \tag{33}$$

Using the properties of Hermite polynomials, we find that

$$\mathfrak{S}_n(\vec{x}) = B_n\, \exp(-|\vec{x}\,|^2/2)\, P_n(\vec{x})\,. \tag{34}$$

In the preceding equalities, A_n and B_n are nonzero constants depending only on n.

According to Arnold [3, Section 2], Gelfand noticed the following two facts.

> **A**. The (antisymmetric) eigenfunction $[\mathfrak{S}_n]$ of the operator $[\mathfrak{h}^{(n)}]$ is the first eigenfunction for this operator (on functions satisfying the Dirichlet condition in the fundamental domain $[\Omega_n]$).
>
> **B**. Choosing the locations $[(c_2, \ldots, c_n)]$ of the other electrons (except for the first one), one can obtain any linear combination of the first n eigenfunctions of the one-electron problem as a linear combination $[\mathfrak{S}_n(x, c_2, \ldots, c_n)]$ (up to multiplication by a nonzero constant).

[10] John Clark Slater (1900–1976).

Observe however that **B** is true only for linear combinations of the n first eigenfunctions which have $(n-1)$ distinct zeros.

In the case of the harmonic oscillator, the proof of facts **A** and **B** is easy. More precisely, we have the following proposition which implies Statement 1.6 in this particular case.

Proposition 5.1. *Recall the notation $\vec{h}(c) = \big(h_1(c), \ldots, h_n(c)\big)$.*

1. *The function $\mathfrak{S}_n(\vec{x}\,)$ is the first Dirichlet eigenfunction of $-\Delta + |\vec{x}\,|^2$ in Ω_n.*

2. *For any $\vec{c} = (c_1, \ldots, c_{n-1}) \in \Omega_{n-1}$, the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$, are linearly independent.*

3. *Given $\vec{b} \in \mathbb{R}^n \backslash \{0\}$, the linear combination*

$$S_{\vec{b}}(x) = \sum_{j=1}^{n} b_j h_j(x)$$

   has at most $(n-1)$ distinct zeros. Furthermore, if the function $\mathfrak{S}_n(\vec{x}\,)$ has exactly $(n-1)$ distinct zeros $c_1 < c_2 < \cdots < c_{n-1}$, then there exists a nonzero constant C such that

$$S_{\vec{b}}(x) = C\, \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x) \text{ for all } x \in \mathbb{R}\,.$$

4. *The function $\mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)$ vanishes at order 1 at each c_j, $1 \leq j \leq (n-1)$, and does not have any other zero.*

Proof. Assertion 1. It is clear that $\mathfrak{S}_n$ is an eigenfunction of $-\Delta + |x|^2$, and that it vanishes on $\partial\Omega_n$. From (12) and (34), we see that it does not vanish in Ω_n, so that $\mathfrak{S}_n$ must be the first Dirichlet eigenfunction for $-\Delta + |\vec{x}|^2$ in Ω_n.

Assertion 2. If the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$, were dependent, $\mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)$ would be identically zero. Developing this determinant with respect to the last column, we would have

$$\mathfrak{S}_{n-1}(c_1, \ldots, c_{n-1})\, h_n(x) + \cdots \equiv 0\,.$$

This is impossible because the h_j's are linearly independent and $\mathfrak{S}_{n-1}(c_1, \ldots, c_{n-1}) \neq 0\,.$

Assertion 3. Assume that $S_{\vec{b}}$ has at least n distinct zeros $c_1 < \cdots < c_n$. The n components $b_j, 1 \leq j \leq n$ would satisfy a system of n equations, whose determinant $\mathfrak{S}_n(c_1, \ldots, c_n)$ is positive. This would imply that $\vec{b} = \vec{0}$. Assume that $S_{\vec{b}}$ has exactly $(n-1)$ zeros, $c_1 < \cdots < c_{n-1}$. The function $x \mapsto \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)$ can be written as a linear combination $S_{\vec{s}(\vec{c})}(x)$, with coefficients $s_j(\vec{c}), 1 \leq j \leq n$ given by Slater like determinants. Both vectors $\vec{b}$ and $\vec{s}(\vec{c})$ would then be orthogonal to the $(n-1)$ independent vectors $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$. This implies that there exists a nonzero constant C such that $\vec{b} = C\,\vec{s}(\vec{c})$.

Assertion 4. It suffices to consider the case of c_1. Consider $\vec{c} = (c_1, c_1, c_2, \ldots, c_{n-1})$, and write

$$\mathfrak{S}_n(\vec{c} + \vec{\xi}\,) = B_n \exp(|\vec{c} + \vec{\xi}\,|^2/2)\, P_n(\vec{c} + \vec{\xi}\,)\,.$$

Using Lemma 4.6, we conclude that

$$\mathfrak{S}_n(\vec{c}+\vec{\xi}\,) = \alpha(\vec{c}\,)\,(\xi_1-\xi_2)\,\left(1+\omega(\vec{c},\vec{\xi}\,)\right)\,,$$

for some nonzero constant $\alpha(\vec{c}\,)$ depending on $\vec{c}$.

It follows that

$$\mathfrak{S}_n(c_1+\xi, c_1, \ldots, c_{n-1}) = \alpha(\vec{c}\,)\,\xi\,\left(1+\omega(\vec{c},\xi)\right)\,,$$

so that this function vanishes precisely at order 1 at c_1. ☐

Remark 5.2. It is standard in Quantum mechanics (except that the usual context for the one-particle Hamiltonian is a 3D-space) that the ground state energy is the sum of the n first eigenvalues of the one-particle Hamiltonian, a consequence of the Pauli[11] exclusion principle. This is for example the main motivation for considering this sum when analyzing the celebrated Lieb-Thirring's inequality in connection with the analysis of the *stability of matter* (see for example [15]).

The following lemma allows us to extend Gelfand's strategy, and to take care of the multiplicity of zeros to achieve a proof of Sturm's upper bound.

Lemma 5.3. *Let* $\vec{\bar{c}} = (\bar{c}_1, \ldots, \bar{c}_1, \ldots, \bar{c}_p, \ldots, \bar{c}_p)$*, where* $\bar{c}_j$ *is repeated* k_j *times, with* $\bar{c}_1 < \cdots < \bar{c}_p$*, and* $k_1 + \cdots + k_p = n-1$*. Let* $\vec{k} = (k_1, \ldots, k_p)$*. Consider the function*

$$\mathfrak{S}_{\vec{k}}(x) = |\vec{h}(c_1)\ldots\vec{h}^{(k_1-1)}(c_1)\ \ldots\ \vec{h}(c_p)\ldots\vec{h}^{(k_p-1)}(c_p)\vec{h}(x)|\,, \tag{35}$$

where $\vec{h}^{(m)}(x)$ *is the vector* $\left(h_1^{(m)}(x), \ldots, h_n^{(m)}(x)\right)$*, and where the superscript* (m) *denotes the* m*-th derivative. Then, the function* $\mathfrak{S}_{\vec{k}}$ *is not identically zero, and vanishes at exactly order* k_j *at* $\bar{c}_j$*. Furthermore, the vectors* $\vec{h}(c_1), \ldots, \vec{h}^{(k_1-1)}(c_1), \ldots, \vec{h}(c_p), \ldots, \vec{h}^{(k_p-1)}(c_p)$*, are linearly independent.*

Proof. It suffices to consider the case of $\bar{c}_1$. Clearly, $\mathfrak{S}_{\vec{k}}$ vanishes at least at order k_1 at $\bar{c}_1$. We consider the k_1-th derivative of this function. We have

$$\mathfrak{S}_{\vec{k}}^{(k_1)}(x) = \pm\,|\vec{h}(c_1)\ldots\vec{h}^{(k_1-1)}(c_1)\vec{h}(c_2)\ldots\vec{h}^{(k_2-1)}(c_2)\ \ldots\ \vec{h}(c_p)\ldots\vec{h}^{(k_p-1)}(c_p)\vec{h}^{(k_1)}(x)|\,.$$

Claim: the value of this determinant at $x = \bar{c}_1$ *is different from zero.* Indeed, consider the vector $\vec{c} = (\bar{c}_1, \ldots, \bar{c}_1, \ldots, \bar{c}_p, \ldots, \bar{c}_p)$, where $\bar{c}_1$ is repeated k_1+1 times, and for $2 \le j \le p$, $\bar{c}_j$ is repeated k_j times. Then $\mathfrak{S}_{\vec{k}}^{(k_1)}(\bar{c}_1)$ is a higher order derivative of $\mathfrak{S}_n$ at $\vec{c}$. More precisely, using the relabeling of variables associated with $\vec{c}$, as given in (7)–(9), $\mathfrak{S}_{\vec{k}}^{(k_1)}(\bar{c}_1)$ is, up to sign, the derivative

$$\partial^{k_1}_{\xi_{1,k_1+1}}\ldots\partial_{\xi_{1,2}}\,\partial^{k_2-1}_{\xi_{2,k_2}}\ldots\partial_{\xi_{2,2}}\ \ldots\ \partial^{k_p-1}_{\xi_{p,k_p}}\ldots\partial_{\xi_{p,2}}\,\mathfrak{S}_n(\vec{c}+\vec{\xi}\,)\Big|_{\vec{\xi}=0}\,,$$

or, using the notation (14),

$$D_{k_1}(\partial_{\xi^{(1)}})\ldots D_{k_p}(\partial_{\xi^{(p)}})\,\mathfrak{S}_n(\vec{c}+\vec{\xi}\,)\Big|_{\vec{\xi}=0}\,.$$

The claim then follows from Lemma 4.1, Equation (13) and Lemma 4.6, Equation (19). The second assertion follows immediately. ☐

As a by product of the preceding proof, we have,

[11] Wolfgang Ernst Pauli (1900–1958).

Corollary 5.4. *Given, p, $1 \le p \le n$, let $k_1, \dots, k_p$ be p positive integers such that $k_1 + \cdots + k_p = n$. Let $\bar{c}_1 < \cdots < \bar{c}_p$ be real numbers. Then, the determinant*

$$|\vec{h}(c_1) \dots \vec{h}^{(k_1-1)}(c_1)\vec{h}(c_2) \dots \vec{h}^{(k_2-1)}(c_2) \ \dots \ \vec{h}(c_p) \dots \vec{h}^{(k_p-1)}(c_p)| \tag{36}$$

is nonzero, so that the corresponding vectors are linearly independent.

Proposition 5.5. *For any $n \ge 1$, a nontrivial linear combination $S_{\vec{b}}$ of the eigenfunctions $h_1, \dots, h_n$ of the harmonic operator $\mathfrak{H}^{(1)}$ has at most $(n-1)$ zeros on the real line, counted with multiplicities. Assume that $S_{\vec{b}}$ has p zeros, $c_1 < \cdots < c_p$ on the real line, with multiplicities k_j's, such that $k_1 + \cdots + k_p = n - 1$. Then, there exists a nonzero constant C such that*

$$S_{\vec{b}}(x) = C\,|\vec{h}(c_1) \dots \vec{h}^{(k_1-1)}(c_1) \ \dots \ \vec{h}(c_p) \dots \vec{h}^{(k_p-1)}(c_p)\vec{h}(x)|\,.$$

Proof. The first assertion is a particular case of Sturm's upper bound, Theorem 1.1. Here is a proof, à la Gelfand, of this elementary property. Assume that a linear combination $S_{\vec{b}}$ has a least n zeros on the real line, counted with multiplicities. From these zeros, one can determine some positive integer p, and sequences $\bar{c}_1 < \cdots < \bar{c}_p$, $k_1, \dots, k_p$ satisfying the assumptions of Corollary 5.4, and such that $S_{\vec{b}}$ vanishes at order (at least) k_j at $\bar{c}_j$, $1 \le p$. This last condition implies that the n entries of the vector $\vec{b}$ satisfy a system of n equations, whose determinant is precisely

$$|\vec{h}(c_1) \dots \vec{h}^{(k_1-1)}(c_1)\vec{h}(c_2) \dots \vec{h}^{(k_2-1)}(c_2) \ \dots \ \vec{h}(c_p) \dots \vec{h}^{(k_p-1)}(c_p)|.$$

Corollary 5.4 then implies that $\vec{b} = 0$, so that a nontrivial linear combination $S_{\vec{b}}$ can have at most $(n-1)$ zeros on the real line, counted with multiplicities.

Here is a trivial proof. The function $S_{\vec{b}}$ is a linear combination of Hermite polynomials $H_0, \dots, H_{n-1}$, times the positive function $\exp(-|\vec{x}\,|^2/2)$. This immediately implies that the number of zeros of $S_{\vec{b}}$ on the real line, counted with multiplicities, is at most $(n-1)$.

The second assertion is a consequence of (the proof of) Lemma 5.3. □

6 The Dirichlet Sturm-Liouville operator

In this section, we show how Gelfand's strategy, Section 5, can be applied to the general Dirichlet Sturm-Liouville problem (1).

6.1 Notation

Let q be a C^∞ real function defined in a neighborhood of the interval $\mathrm{I} :=]0,1[$. We consider the 1-particle operator

$$\mathfrak{h}^{(1)} := -\frac{d^2}{dx^2} + q(x)\,, \tag{37}$$

and, more precisely, its Dirichlet realization in I, i.e. the Dirichlet boundary value problem

$$\begin{cases} -\frac{d^2y}{dx^2} + q\,y = \lambda\,y\,, \\ y(0) = y(1) = 0\,. \end{cases} \tag{38}$$

Let $\{(\lambda_j, h_j), j \geq 1\}$ be the eigenpairs of $\mathfrak{h}^{(1)}$, with

$$\lambda_1 < \lambda_2 < \lambda_2 < \cdots , \tag{39}$$

and $\{h_j, j \geq 1\}$ an associated orthonormal basis of eigenfunctions.

We also consider the Dirichlet realization $\mathfrak{h}^{(n)}$ of the n-particle operator in I^n,

$$\mathfrak{h}^{(n)} := -\sum_{j=1}^{n} \big(\frac{\partial^2}{\partial x_j^2} + q(x_j)\big) = -\Delta + Q\,, \tag{40}$$

where $Q(x_1, \ldots, x_n) = q(x_1) + \cdots + q(x_n)$.

Denote by $\vec{k} = (k_1, \cdots, k_n)$ a vector with positive integer entries, and by $\vec{x} = (x_1, \cdots, x_n)$ a vector in I^n. The eigenpairs of $\mathfrak{h}^{(n)}$ are the $(\Lambda_{\vec{k}}, H_{\vec{k}})$, with

$$\begin{cases} \Lambda_{\vec{k}} = \lambda_{k_1} + \cdots + \lambda_{k_n}\,, \text{ and} \\ H_{\vec{k}}(\vec{x}) = h_{k_1}(x_1) \cdots h_{k_n}(x_n)\,, \end{cases} \tag{41}$$

where $H_{\vec{k}}$ is seen as a function in $L^2(\mathrm{I}^n, dx)$ identified with $\widehat{\bigotimes} L^2(\mathrm{I}, dx_j)$.

The symmetric group $\mathfrak{s}_n$ acts on I^n by $\sigma(\vec{x}) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, if $\vec{x} = (x_1, \cdots, x_n)$. It consequently acts on $L^2(\mathrm{I}^n)$, and on the functions $H_{\vec{k}}$ as well. A fundamental domain of the action of $\mathfrak{s}_n$ on I^n is the n-simplex

$$\Omega_n^{\mathrm{I}} := \{0 < x_1 < x_2 < \cdots < x_n < 1\}\,. \tag{42}$$

In analogy with (33), we introduce the Slater determinant $\mathfrak{S}_n$ defined by,

$$\mathfrak{S}_n(x_1, \ldots, x_n) = \begin{vmatrix} h_1(x_1) & h_1(x_2) & \ldots & h_1(x_n) \\ h_2(x_1) & h_2(x_2) & \ldots & h_2(x_n) \\ \vdots & \vdots & & \vdots \\ h_n(x_1) & h_n(x_2) & \ldots & h_n(x_n) \end{vmatrix}. \tag{43}$$

Let $\overrightarrow{c} = (c_1, \ldots, c_{n-1}) \in \mathrm{I}^{n-1}$. We consider the function $x \mapsto \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)$. Developing the determinant with respect to the last column, we see that this function is a linear combination of the functions $h_1, \ldots, h_n$, which we write as

$$S_{s(\overrightarrow{c})}(x) = \sum_{j=1}^{n} s_j(\overrightarrow{c})\, h_j(x) \tag{44}$$

where $s(\overrightarrow{c}) = \Big(s_1(\overrightarrow{c}), \ldots, s_n(\overrightarrow{c})\Big)$, and

$$s_j(\overrightarrow{c}) = s_j(c_1, \ldots, c_{n-1}) = \begin{vmatrix} h_1(c_1) & \ldots & h_1(c_{n-1}) \\ \vdots & & \vdots \\ h_{j-1}(c_1) & \ldots & h_{j-1}(c_{n-1}) \\ h_{j+1}(c_1) & \ldots & h_{j+1}(c_{n-1}) \\ \vdots & & \vdots \\ h_n(c_1) & \ldots & h_n(c_{n-1}) \end{vmatrix} \tag{45}$$

so that $s(\overrightarrow{c})$ is computed in terms of Slater determinants of size $(n-1) \times (n-1)$.

6.2 Weak upper bound

We now prove Statement 1.6 using Gelfand's strategy, as explained in Section 5.

Lemma 6.1. *The function $\mathfrak{S}_n$ is not identically zero.*

Proof. The proof relies on the fact that the functions h_j, $1 \leq j \leq n$ are linearly independent. Clearly, $\mathfrak{S}_1(x_1) = h_1(x_1) \not\equiv 0$. We now use induction on n. Assume that $\mathfrak{S}_{n-1}(x_1, \ldots, x_{n-1}) \not\equiv 0$. Develop the determinant $\mathfrak{S}_n(x_1, \ldots, x_n)$ with respect to the last column,

$$\mathfrak{S}_n(x_1, \ldots, x_n) = \mathfrak{S}_{n-1}(x_1, \ldots, x_{n-1})\, h_n(x) + \cdots .$$

By the induction hypothesis, there exists $(x_1^0, \ldots, x_{n-1}^0) \in \mathrm{I}^{n-1}$, such that $\mathfrak{S}_{n-1}(x_1^0, \ldots, x_{n-1}^0) \neq 0$. Then, $\mathfrak{S}_n(x_1^0, \ldots, x_{n-1}^0, x_n) \not\equiv 0$ because the h_j's are linearly independent, and the lemma follows. ☐

Lemma 6.2. *The function $\mathfrak{S}_n$ is the first Dirichlet eigenfunction of $\mathfrak{h}^{(n)}$ in Ω_n^{I}, with corresponding eigenvalue $\Lambda^{(n)} := \lambda_1 + \cdots + \lambda_n$. In particular, the function $\mathfrak{S}_n$ does not vanish in Ω_n^{I}. More precisely, one can choose the signs of the functions h_j, $1 \leq j \leq n$, such that $\mathfrak{S}_k$ is positive in Ω_k^{I} for $1 \leq k \leq n$. As a consequence, for any $c_1 < \cdots < c_{n-1}$ in I, the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_n)$, are linearly independent.*

Proof. An eigenfunction Ψ of $\mathfrak{h}_F^{(n)}$ is given by a (finite) linear combination $\Psi = \sum \alpha_{\vec{k}} H_{\vec{k}}$ of eigenfunctions of $\mathfrak{h}^{(n)}$, such that the corresponding $\Lambda_{\vec{k}}$ are equal, and such that Ψ is antisymmetric. If $\vec{k} = (k_1, \cdots, k_n)$ is such that $k_i = k_j$ for some pair $i \neq j$, using the permutation which exchanges i and j, we see that the corresponding $\alpha_{\vec{k}}$ vanishes. It follows that the eigenvalues of $\mathfrak{h}_F^{(n)}$ are the $\Lambda_{\vec{k}}$ such that the entries of $\vec{k}$ are all different. It then follows that the ground state energy of $\mathfrak{h}_F^{(n)}$ is $\Lambda^{(n)}$.

It is clear that $\mathfrak{S}_n$ vanishes on $\partial\Omega_n^{\mathrm{I}}$. Its restriction $\mathfrak{S}_{\Omega_n^{\mathrm{I}}}$ to Ω_n^{I} satisfies the Dirichlet condition on $\partial\Omega_n^{\mathrm{I}}$, and is an eigenfunction of $\mathfrak{h}_F^{(n)}$ corresponding to $\Lambda^{(n)}$. Suppose that $\mathfrak{S}_{\Omega_n^{\mathrm{I}}}$ is not the ground state. Then, it has a nodal domain ω strictly included in Ω_n. Define the function U which is equal to $\mathfrak{S}_{\Omega_n^{\mathrm{I}}}$ in ω, and to 0 elsewhere in I^n. It is clearly in $H_0^1(\Omega_n^{\mathrm{I}})$. Using $\mathfrak{s}_n$, extend the function U to a Fermi state U_F on I^n. Its energy is $\Lambda^{(n)}$ which is the bottom of the spectrum of $\mathfrak{h}_F^{(n)}$. It follows that U_F is an eigenfunction of $\mathfrak{h}_F^{(n)}$, and a fortiori of $\mathfrak{h}^{(n)}$. This would imply that $\mathfrak{S}_n$ is identically zero, a contradiction with Lemma 6.1.

The fact that one can choose the $\mathfrak{S}_n$ to be positive in Ω_n^{I} follows immediately.

If the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_n)$ were linearly dependent, the function given by (44) would be identically zero, contradicting the fact that the coefficient of h_n in this linear combination is $\mathfrak{S}_{n-1}(c_1, \ldots, c_{n-1}) > 0$. ☐

The following proposition provides a *weak* form of Sturm's upper bound on the number of zeros of a linear combination of eigenfunctions of (38) ("weak" in the sense that we here do not count zeros with their multiplicities).

Proposition 6.3. *Let $\vec{b} \in \mathbb{R}^n$, with $\vec{b} \neq \vec{0}$. Then, the linear combination $S_{\vec{b}}$ has a most $(n-1)$ distinct zeros in $\mathrm{I} =]0,1[$. If $S_{\vec{b}}$ has exactly $(n-1)$ zeros in I, $c_1 < \cdots < c_{n-1}$, then there exists a nonzero constant C such that*

$$S_{\vec{b}}(x) = C\,\mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)\,.$$

Furthermore, each zero c_j has order 1.

Proof. Given $\vec{b}$, assume that $S_{\vec{b}}$ has at least n distinct zeros $c_1 < \ldots c_n$ in I. This means that the n components $b_j, 1 \leq j \leq n$, satisfy the system of n equations,

$$\begin{cases} b_1 h_1(c_1) + \cdots + b_n h_n(c_1) = 0, \\ \cdots \\ b_1 h_1(c_n) + \cdots + b_n h_n(c_n) = 0. \end{cases}$$

By Lemma 6.2, the determinant of this system is positive, and hence the unique possible solution is $\vec{0}$. This proves the first assertion.

Assume that $S_{\vec{b}}$ has precisely $(n-1)$ distinct zeros, $c_1 < \ldots c_{n-1}$, in I. By Lemma 6.2, the vectors $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$, are linearly independent. The function $x \mapsto \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)$ can be written as the linear combination $S_{\vec{s}(\vec{c})}$, where the vector $\vec{s}(\vec{c})$ is given by (45). It follows that the vectors $\vec{b}$ and $\vec{s}(\vec{c})$ are both orthogonal to the family $\vec{h}(c_1), \ldots, \vec{h}(c_{n-1})$, and must therefore be proportional. This proves the second assertion.

Assume that $x \mapsto \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)$ vanishes at order at least 2 at c_1. Then

$$\frac{d}{dx}\Big|_{x=c_1} \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x) = 0.$$

This implies that $\frac{\partial \mathfrak{S}_n}{\partial x_2}(c_1, c_1, c_2, \ldots, c_{n-1}) = 0$, and hence that $\frac{\partial \mathfrak{S}_n}{\partial \nu}(c_1, c_1, c_2, \ldots, c_{n-1})$, where ν is the unit normal to the boundary $\partial\Omega_n^{\mathrm{I}}$, which contradicts Hopf's lemma. This proves the last assertion, as well as the corollary. $\square$

For completeness, we state the following immediate corollaries.

Corollary 6.4. *Given $c_1 < \cdots < c_{n-1}$ in I, the function*

$$x \mapsto \mathfrak{S}_n(c_1, \ldots, c_{n-1}, x)\,,$$

vanishes exactly at order 1, changes sign at each c_j, and does not vanish elsewhere in I.

Corollary 6.5. *Let $\vec{b} \in \mathbb{R}^n \backslash \{0\}$. If the linear combination $S_{\vec{b}}$ has k distinct zeros, and if one of the zeros has order at least 2, then $k \leq n-2$.*

Remark 6.6. Note that for $x \in]c_j, c_{j+1}[$, $1 \leq j \leq n-1$,

$$\mathfrak{S}_n(c_1, \ldots, c_{n-1}, x) = (-1)^{n-1-j}\,\mathfrak{S}_n(c_1, \ldots, c_j, x, c_{j+1}, \ldots, c_{n-1})\,,$$

so that, according to Lemma 6.2, it has the sign of $(-1)^{n-1-j}$. This also shows that this function of x changes sign when x passes one of the c_j's.

Definition 6.7. Let S be a continuous function in I. Let $c \in \mathrm{I}$ be a zero of S. Following [11, Chap. III.5], call c a *node* of S, if for any $\varepsilon > 0$ small enough, there exists some $x_\varepsilon^\pm$ such that $c - \varepsilon < x_\varepsilon^- < c < x_\varepsilon^+ < c + \varepsilon$, with $S(x_\varepsilon^-)\, S(x_\varepsilon^+) < 0$; call c an *antinode* of S, if for any $\varepsilon > 0$, small enough, S does not change sign in $]c-\varepsilon, c+\varepsilon[$, and does not vanish identically in $]c-\varepsilon, c[$ and in $]c, c+\varepsilon[$.

This definition applies to any continuous function S. An isolated zero of S is either a node or and antinode. In our case, according to Lemma 2.2, any zero of a nontrivial $S_{\vec{b}}$ is isolated, and one can determine whether this is a node or an antinode by looking at the first nonzero coefficient in its Taylor series at c. The following result appears in [11, Chap. III.5] in the more general framework of Chebyshev systems of continuous functions. We sketch the proof for completeness.

Proposition 6.8. *Let $\vec{b} \in \mathbb{R}^n \backslash \{0\}$. Let $N_{\vec{b}}$ be the number of nodes of $S_{\vec{b}}$, resp. $A_{\vec{b}}$ the number of antinodes. Then,*

$$N_{\vec{b}} + 2\, A_{\vec{b}} \leq n - 1\,.$$

Proof. In this proof, we use the notation S for $S_{\vec{b}}$. We already know, Proposition 6.3, that S has at most $(n-1)$ distinct zeros.

We say that a set $z_1 < z_2 < \cdots < z_s$ has the property $\mathcal{A}$ with respect to S (alternating property) if there exists $\kappa \in \{0, 1\}$ such that for any $k \in \{1, \ldots, s\}$, $(-1)^{k+\kappa}\, S(z_k) \geq 0$.

Lemma 6.9. *Assume that $\{z_1 < z_2 < \cdots < z_s\}$ has the property $\mathcal{A}$ with respect to S. If $\xi \notin \{z_1 < z_2 < \cdots < z_s\}$ is an antinode of S, then, for ε small enough, one of the sets $\{z_1 < z_2 < \cdots < z_s\} \cup \{\xi - \varepsilon, \xi\}$ or $\{z_1 < z_2 < \cdots < z_s\} \cup \{\xi, \xi + \varepsilon\}$, properly reordered, has the property $\mathcal{A}$, with $S(\xi - \varepsilon)\, S(\xi + \varepsilon) > 0$.*

Proof of the lemma. We examine the case in which there exists $1 < j < s - 1$, such that $z_j < \xi < z_{j+1}$. We choose ε such that

$$z_j < \xi - \varepsilon < \xi < \xi + \varepsilon < z_{j+1}\,,$$

with $S(\xi \pm \varepsilon) \neq 0$.

We know that $(-1)^{\kappa+j} S(z_j) \geq 0$. We have two cases,

- if $(-1)^{\kappa+j+1} S(\xi - \varepsilon) > 0$, then $(-1)^{\kappa+j+2} S(\xi) \geq 0$, and $(-1)^{\kappa+j+3} S(z_{j+1}) \geq 0$,
- if $(-1)^{\kappa+j+1} S(\xi - \varepsilon) < 0$, then $(-1)^{\kappa+j+1} S(\xi) \geq 0$, and $(-1)^{\kappa+j+2} S(\xi + \varepsilon) \geq 0$,

From the set $\{z_1, \ldots, z_s, \xi - \varepsilon, \xi, \xi + \varepsilon\}$, we construct an ordered list $\{z'_1 < \cdots < z'_{s+2}\}$ as follows

- for $k \leq j$, $z'_k = z_k$, and for $k \geq j+3$, $z'_k = z_{k+2}$, and we choose $z'_{j+1} < z'_{j+2}$ in the interval $]z_j, z_{j+1}[$, as follows:
- if $(-1)^{\kappa+j+1} S(\xi - \varepsilon) > 0$, we choose

$$\begin{cases} z'_{j+1} = \xi - \varepsilon\,, \text{ so that } (-1)^{\kappa+j+1} S(z'_{j+1}) > 0, \\ z'_{j+2} = \xi\,, \text{ so that } (-1)^{\kappa+j+2} S(z'_{j+2}) = 0, \end{cases}$$

- if $(-1)^{\kappa+j+1}S(\xi-\varepsilon)<0$, then $(-1)^{\kappa+j+2}S(\xi+\varepsilon)>0$, and we choose

$$\begin{cases} z'_{j+1}=\xi\,, \text{ so that } (-1)^{\kappa+j+1}S(z'_{j+1})=0\,,\\ z'_{j+2}=\xi+\varepsilon\,, \text{ so that } (-1)^{\kappa+j+2}S(z'_{j+2})>0\,. \end{cases}$$

The case $\xi<z_1$ or $\xi>z_s$ are dealt with similarly. This proves the lemma. ☐

Proof of the proposition continued. Call $z_1<\cdots<z_p$, $p=N_{\vec{b}}$, the nodes of S. Then, one can choose numbers α_j such that,

$$\alpha_1<z_1<\alpha_2<z_2<\cdots<\alpha_p<z_p<\alpha_{p+1}\,,$$

and some $\kappa\in\{0,1\}$ such that $(-1)^{\kappa+j}S(\alpha_j)>0\,.$

By applying Lemma 6.9 recursively for the $q=A_{\vec{b}}$ antinodes, we obtain a set

$$\beta_1<\beta_2<\cdots<\beta_{p+2q+1}$$

such that $(-1)^{\kappa+j}S(\beta_j)\geq 0$ (Property $\mathcal{A}$).

Assume that $p+2q>n-1$, i.e. $p+2q+1\geq n+1$. Consider the vector $\vec{H}=(h_1,\ldots,h_n,S)$. Then, the determinant

$$|\vec{H}(\beta_1)\ldots\vec{H}(\beta_{n+1})| \text{ is identically } 0,$$

because S is a linear combination of $h_1,\ldots,h_n$. Developing this determinant with respect to the last row, we find that

$$0\equiv\sum_{k=1}^{n}(-1)^{n+1+k}S(\beta_k)\,|\vec{h}(\beta_1)\ldots\vec{h}(\beta_k-1)\vec{h}(\beta_k+1)\ldots\vec{h}(\beta_n)|\,.$$

For each k, we have $(-1)^{n+1+k}S(\beta_k)\geq 0$ by construction of the set $\{\beta_j, 1\leq j\leq n+1\}$, and

$$|\vec{h}(\beta_1)\ldots\vec{h}(\beta_{k-1})\vec{h}(\beta_{k+1})\ldots\vec{h}(\beta_n)|=\mathfrak{S}_n(\beta_1\ldots\beta_{k-1},\beta_{k+1},\ldots,\beta_{n+1})>0$$

according to Lemma 6.2 . This implies that S has a least n distinct zeros, a contradiction with Proposition 6.3. ☐

6.3 Local behaviour of $\mathfrak{S}_n$ near a zero

We begin by treating two particular examples which are similar to Examples 4.3 and 4.4. We then deal with the general case.

Consider $\mathfrak{S}_5$. Let $\vec{c}\in\partial\Omega_5^{\mathrm{I}}$ be a boundary point. Write $\vec{x}=\vec{c}+\vec{\xi}$, with $\vec{\xi}$ close to 0. The function $\mathfrak{S}_5$ is an eigenfunction of the operator $-\Delta+Q$, and vanishes at the point $\vec{c}\in\mathrm{I}^n$. By Bers' theorem [8], there exists a *harmonic* homogeneous polynomial $\widehat{P}_k$, of degree k, such that

$$\mathfrak{S}_5(\vec{c}+\vec{\xi})=\widehat{P}_k(\vec{\xi})+\omega_{k+1}(\vec{\xi})\,, \tag{46}$$

where $\omega_{k+1}(t\vec{\xi})=O(t^{k+1})$. Note that, for the time being, we have no a priori information on k.

6.3.1 Example 1

In this example, we take $\vec{c} = (\bar{c}_1, \bar{c}_1, \bar{c}_2, \bar{c}_2, c_5)$, with $\bar{c}_1 < \bar{c}_2 < c_5$. Call $\widehat{P}_k$ the polynomial given by (46) for this particular case.

Lemma 6.10. *The polynomial $\widehat{P}_k$ satisfies*

$$\widehat{P}_k(\vec{\xi}\,) = \rho\,(\xi_1 - \xi_2)(\xi_3 - \xi_4)\,, \tag{47}$$

where ρ is a nonzero constant, and

$$\mathfrak{S}_5(\vec{c} + \vec{\xi}\,) = \rho\, P_2(\xi_1, \xi_2)\, P_2(\xi_3, \xi_4)\, \big(1 + \omega(\vec{\xi}\,)\big)\,. \tag{48}$$

Proof. According to (46), we have

$$\mathfrak{S}_5(\bar{c}_1 + \xi_1, \bar{c}_1 + \xi_2, \bar{c}_2 + \xi_3, \bar{c}_2 + \xi_4, c_5 + \xi_5) = \widehat{P}_k(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) + \omega_{k+1}(\vec{\xi}\,)\,.$$

Using the anti-symmetry of $\mathfrak{S}_5$, taking $\vec{\xi} = t\,\vec{\eta}$, using the fact that $\omega_{k+1}(t\,\vec{\eta}\,)$ is of order $k+1$, and letting t tend to zero, we see that $\widehat{P}_k$ is anti-symmetric with respect to the pair (ξ_1, ξ_2). A similar argument applies to the pair (ξ_3, ξ_4). This proves that

$$\widehat{P}_k(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = -\widehat{P}_k(\xi_2, \xi_1, \xi_3, \xi_4, \xi_5) = -\widehat{P}_k(\xi_1, \xi_2, \xi_4, \xi_3, \xi_5)\,, \tag{49}$$

and hence, that $\widehat{P}_k(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = 0$ when $(\xi_1 - \xi_2)(\xi_3 - \xi_4) = 0$.

We claim that the converse statement is true in a neighborhood of 0. Indeed, assume that $\widehat{P}_k(\vec{\eta}\,) = 0$, where $\eta_1 \neq \eta_2$ and $\eta_3 \neq \eta_4$. Using (49), we can assume that $\eta_1 < \eta_2$ and $\eta_3 < \eta_4$. Because $\widehat{P}_k$ is a nonzero harmonic polynomial which vanishes at $\vec{\eta}$, in any neighborhood of $\vec{\eta}$, there exist points $\vec{\eta}^{\,\pm}$ such that $\widehat{P}_k(\vec{\eta}^{\,+}\,)\widehat{P}_k(\vec{\eta}^{\,-}\,) < 0$. For t positive small enough, the function $\mathfrak{S}_n(\vec{c} + t\,\vec{\eta}^{\,\pm}\,)$ has the sign of $\widehat{P}_k(\vec{c} + t\,\vec{\eta}^{\,\pm}\,)$, and this contradicts the fact that the function $\mathfrak{S}_5$ is positive in Ω_n^{I}.

We have just proved that, in a neighborhood of zero, $\widehat{P}_k$ vanishes if and only if $(\xi_1 - \xi_2)(\xi_3 - \xi_4)$ vanishes. The polynomials $\widehat{P}_k$ and $(\xi_1 - \xi_2)(\xi_3 - \xi_4)$ are both harmonic and homogeneous, and they have the same zero set in some neighborhood of zero. According to [17, Lemma 2.1], they divide each other, so that there exists a nonzero constant ρ such that $\widehat{P}_k = \rho\,(\xi_1 - \xi_2)(\xi_3 - \xi_4)$. □

6.3.2 Example 2

In this example, we choose $\vec{c} = (\bar{c}_1, \bar{c}_1, \bar{c}_1, c_4, c_5)$, with $\bar{c}_1 < c_4 < c_5$. Call $\widehat{P}_k$ the polynomial given by (46).

Lemma 6.11. *The polynomial $\widehat{P}_k$ has the following properties. For any permutation $\sigma \in \mathfrak{s}_3(\xi_1, \xi_2, \xi_3)$, of the first three variables,*

$$\begin{cases} \widehat{P}_k(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \varepsilon(\sigma)\widehat{P}_k(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}, \xi_4, \xi_5)\,, \\ \widehat{P}_k = 0 \Leftrightarrow \xi_1 = \xi_2 \text{ or } \xi_1 = \xi_3 \text{ or } \xi_2 = \xi_3\,, \\ \widehat{P}_k(\vec{\xi}\,) = \rho\, P_3(\xi_1, \xi_2, \xi_3)\,, \end{cases} \tag{50}$$

where ρ is a nonzero constant. This means that $\widehat{P}_k$ has degree 3, and that

$$\mathfrak{S}_5(\vec{c} + \vec{\xi}\,) = \rho\, P_3(\xi_1, \xi_2, \xi_3)\big(1 + \omega(\vec{\xi}\,)\big)\,. \tag{51}$$

Proof. Similar to the previous one. □

6.3.3 General case

Let $\vec{c} \in \partial\Omega_n^{\mathrm{I}}$ be a boundary point, i.e. a point of the form $\vec{c} = (\bar{c}_1, \ldots, \bar{c}_1, \ldots, \bar{c}_p, \ldots, \bar{c}_p)$, where p is a positive integer, where $\bar{c}_1 < \bar{c}_2 < \cdots < \bar{c}_p$, are points in I, and where $\vec{c}$ is such that $\bar{c}_j$ is repeated k_j times, with $k_1 + \cdots + k_n = n$.

We write $\vec{x} = \vec{c} + \vec{\xi}$, with $\vec{\xi}$ close to 0. The function $\mathfrak{S}_n$ is an eigenfunction of the operator $-\Delta + Q$, and vanishes at the point $\vec{c} \in \mathrm{I}^n$. By Bers's theorem [8], there exists a *harmonic* homogeneous polynomial $\widehat{P}_k$, of degree k, such that

$$\mathfrak{S}_n(\vec{c} + \vec{\xi}\,) = \widehat{P}_k(\vec{\xi}\,) + \omega_{k+1}(\vec{\xi}\,)\,, \tag{52}$$

where $\omega_{k+1}(t\vec{\xi}) = O(t^{k+1})$. Note that, for the time being, we have no a priori information on k.

We relabel the coordinates $\vec{\xi}$, according to (7) – (9), and we write this vector as

$$\vec{\xi} = \big(\xi^{(1)}, \ldots, \xi^{(p)}\big)\,, \tag{53}$$

where $\xi^{(j)} = (\xi_{j,1}, \ldots, \xi_{j,k_j})$.

The permutation group $\mathfrak{s}_{k_j}$ acts by permuting the entries of $\xi^{(j)}$. Given $\sigma_j \in \mathfrak{s}_{k_j}, 1 \le j \le p$, we denote by $\sigma = (\sigma_1, \ldots, \sigma_p) \in \mathfrak{s}_{k_1} \times \cdots \times \mathfrak{s}_{k_p}$ the permutation in $\mathfrak{s}_n$ which permutes the entries of $\xi^{(j)}$ by σ_j.

For the same vector $\vec{c}$, we look at the local behavior of P_n, and we rewrite (19) as

$$P_n(\vec{c} + \vec{\xi}\,) = \rho_1(\vec{c}\,)\, P_{k_1}\big(\xi^{(1)}\big) \cdots P_{k_p}\big(\xi^{(p)}\big) \Big(1 + \omega(\vec{c}, \vec{\xi}\,)\Big)\,. \tag{54}$$

Lemma 6.12. *The polynomial $\widehat{P}_k$ given by (52) has the following properties.*

1. *For any permutation $\sigma = (\sigma_1, \ldots, \sigma_p) \in \mathfrak{s}_{k_1} \times \cdots \times \mathfrak{s}_{k_p} \subset \mathfrak{s}_n$,*

$$\widehat{P}_k(\sigma \cdot \vec{\xi}\,) = \varepsilon(\sigma)\, \widehat{P}_k(\vec{\xi}\,)\,. \tag{55}$$

2. *The zero set of $\widehat{P}_k$ is characterized by*

$$\widehat{P}_k(\vec{\xi}\,) = 0 \Leftrightarrow \prod_{j=1}^{p} P_{k_j}\big(\xi^{(j)}\big) = 0\,. \tag{56}$$

3. *There exists a nonzero constant $\rho(\vec{c}\,)$ such that*

$$\widehat{P}_k(\vec{\xi}\,) = \rho(\vec{c})\, P_{k_1}(\xi^{(1)}) \ldots P_{k_p}(\xi^{(p)})\,. \tag{57}$$

This means that $\widehat{P}_k$ has degree $k = \sum_j \frac{k_j(k_j-1)}{2}$, and that

$$\mathfrak{S}_n(\vec{c} + \vec{\xi}\,) = \rho(\vec{c}\,)\, P_{k_1}(\xi^{(1)}) \ldots P_{k_p}(\xi^{(p)})\big(1 + \omega(\vec{\xi}\,)\big)\,. \tag{58}$$

Proof. Assertion 1. From the form of $\vec{c}$, and the definition of $\sigma = (\sigma_1, \ldots, \sigma_p)$, we have the relations,

$$\varepsilon(\sigma)\, \mathfrak{S}_n(\vec{c} + t\vec{\xi}\,) = \mathfrak{S}_n(\sigma\cdot(\vec{c} + t\vec{\xi}\,)) = \mathfrak{S}_n(\vec{c} + t\sigma\cdot\vec{\xi}\,)\,.$$

It follows that

$$\widehat{P}_k(t\sigma\cdot\vec{\xi}\,) + \omega_{k+1}(t\sigma\cdot\vec{\xi}\,) = \varepsilon(\sigma)\left(\widehat{P}_k(t\vec{\xi}\,) + \omega_{k+1}(t\vec{\xi}\,)\right).$$

The assertion follows by dividing by t and letting t tend to zero.

Assertion 2. The first assertion implies that the polynomial $\widehat{P}_k$ vanishes whenever the polynomial $\prod_{j=1}^{p} P_{k_j}\big(\xi^{(j)}\big)$ vanishes. Part $(\Leftarrow)$ of the second assertion follows.

Assume that there exists some $\vec{\eta} = (\eta^{(1)}, \ldots, \eta^{(p)})$ such that

$$\widehat{P}_k(\vec{\eta}\,) = 0 \text{ and } \prod_{j=1}^{p} P_{k_j}\big(\eta^{(j)}\big) \neq 0.$$

Since $\widehat{P}_k$ is harmonic, nonconstant, and vanishes at $\vec{\eta}$, it must change sign, and there exist $\vec{\eta}^{\,\pm}$ such that $\widehat{P}_k(\vec{\eta}^{\,+})\widehat{P}_k(\vec{\eta}^{\,-}) < 0$. Using the first assertion and the properties of the Vandermonde polynomials, we see that one can choose $\vec{\eta}^{\,\pm} \in \Omega_n$, with Ω_n as in (32). It follows that for t small enough, the vectors $\vec{c} + t\vec{\eta}^{\,\pm}$ are in Ω_n^{I}, defined in (42). For these vectors, one has

$$\mathfrak{S}_n(\vec{c} + t\vec{\eta}^{\,\pm}\,) = \widehat{P}_k(\vec{c} + t\vec{\eta}^{\,\pm}) + \omega(\vec{c} + t\vec{\eta}^{\,\pm})\,.$$

This equality contradicts the fact that $\mathfrak{S}_n$ is positive in Ω_n^{I}.

Assertion 3. Notice that the polynomials $\widehat{P}_k(\xi)$ and $\prod_{j=1}^{p} P_{k_j}\big(\xi^{(j)}\big)$ are both harmonic and homogeneous, with the same zero set in a neighborhood of 0. We can then apply [17, Lemma 2.1], which implies that they divide each other, so that these polynomials must be proportional. The lemma is proved. $\square$

As a consequence of the preceding lemma, we have,

Corollary 6.13. *Let $\vec{c} \in \partial\Omega_n^{\mathrm{I}}$ be as above. with the notation (14), we have the relations,*

$$D_{k_1}(\partial_{x^{(1)}}) \cdots D_{k_p}(\partial_{x^{(p)}})\, \mathfrak{S}_n(\vec{x}\,)\Big|_{\vec{x}=\vec{c}} = D_{k_1}(\partial_{\xi^{(1)}}) \cdots D_{k_p}(\partial_{\xi^{(p)}})\, \mathfrak{S}_n(\vec{c} + \vec{\xi}\,)\Big|_{\vec{\xi}=0} \neq 0\,. \tag{59}$$

6.4 Strong upper bound

We can now prove Assertion 3a in Theorem 1.1, using Gelfand's strategy, as explained in Section 5.

Proposition 6.14. *Let $\vec{b} \in \mathbb{R}^n \backslash \{0\}$. Call $\bar{c}_1 < \cdots < \bar{c}_p$ the zeros of the linear combination $S_{\vec{b}}$ of the first n eigenfunctions of problem (38). Call k_j the order of vanishing of $S_{\vec{b}}$ at $\bar{c}_j$. Call $\vec{c}$ the vector $(\bar{c}_1, \ldots, \bar{c}_1, \ \ldots\ , \bar{c}_p, \ldots, \bar{c}_p)$, where $c_j, 1 \leq j \leq p$ is repeated k_j times. Then,*

1. *$k_1 + \cdots + k_p \leq (n-1)$,*

2. *If $k_1 + \cdots + k_p = (n-1)$, then there exists a nonzero constant C such that*

$$S_{\vec{b}} = C\, S_{\vec{s}(\vec{c})}\,,$$

where the linear combination $S_{\vec{s}(\vec{c})}$ is given by developing the determinant

$$\left|\vec{h}(c_1)\ldots\vec{h}^{(k_1-1)}(c_1)\ \ldots\ \vec{h}(c_p)\ldots\vec{h}^{(k_p-1)}(c_1)\vec{h}(x)\right|\,, \tag{60}$$

and where $\vec{h}^{(m)}(a)$ is the vector $\left(h_1^{(m)}(a),\ldots,h_n^{(m)}(a)\right)$ of the mth derivatives of the h_j's evaluated at the point a.

Proof. Assertion 1. Assume that $k_1 + \cdots + k_p \geq n$. Without loss of generality, one can assume that $k_1 + \cdots + k_p = n$. The coefficients $b_1, \ldots, b_n$, satisfy the system of n equations,

$$(b_1,\ldots,b_n)\left(\vec{h}(c_1)\ldots\vec{h}^{(k_1-1)}(c_1)\ \ldots\ \vec{h}(c_p)\ldots\vec{h}^{(k_p-1)}(c_p)\right) = 0$$

where the left hand side is the product of the row matrix $(b_1,\ldots,b_n)$ by the $n \times n$ matrix

$$\left(\vec{h}(c_1)\ldots\vec{h}^{(k_1-1)}(c_1)\ \ldots\ \vec{h}(c_p)\ldots\vec{h}^{(k_p-1)}(c_p)\right)\,.$$

Using (59), we see that the determinant of the latter matrix is nonzero. This implies that $\vec{b} = 0$, a contradiction.

Assertion 2. Using (59) again (with $n-1$ instead of n), we see that the coefficient of $h_n(x)$ in the linear combination $S_{\vec{s}(\vec{c})}$ is nonzero, so that $S_{\vec{s}(\vec{c})}$ is not identically zero. It follows that the family of $(n-1)$ vectors $\mathcal{F} := \left\{\vec{h}(c_1),\ldots,\vec{h}^{(k_1-1)}(c_1),\ \ldots,\vec{h}(c_p),\ldots,\vec{h}^{(k_p-1)}(c_p)\right\}$ is free. Both functions $S_{\vec{b}}$ and $S_{\vec{s}(\vec{c})}$ vanish at order k_j at $\bar{c}_j$, for $1 \leq j \leq p$. This means that the vectors $\vec{b}$ and $\vec{s}(\vec{c})$ are both orthogonal to $\mathcal{F}$, which implies that they are proportional. The proposition is proved. □

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