An Integrated Design for Robust Actuator Fault Accommodation based on $\mathscr{H}_\infty$ Proportional-Integral Observer

Manh-Hung Do, Damien Koenig, and Didier Theilliol

***Abstract*— The main contribution of this paper is an integrated design of robust fault estimation (FE) and fault accommodation (FA), applied to uncertain linear time invariant (LTI) systems. In this design, the robust $\mathscr{H}_\infty$ proportional-integral (PI) observer allows a precise estimation of actuator fault by dealing with system disturbances and uncertainties, while the feed-back controller compensates the actuator fault, and therefore assuring the closed-loop stability. Thanks to the application of majoration and Young inequalities, the observer-controller decoupling problem is solved and both above objectives are combined into only one linear matrix inequalities (LMI) allowing a simultaneous solution for both observer and controller. Finally, an example of vehicle suspension system is illustrated to highlight the performance of the proposed method.**

***Index Terms*— Uncertain linear system, PI Observer, Fault diagnosis, Fault tolerant control, Vehicles.**

I. INTRODUCTION

Nowadays, fault tolerant control (FTC) plays an important role in assuring the normal system behavior against fault occurences. One of many interesting approaches to meet FTC objective is the observer-based controller, in which the observer is used for fault diagnosis or state estimation, while the controller applies the reconfiguration mechanism based on information obtained from the observer (see [1]). Authors in [2] has reformulated the fault and system states into a descriptor system and then used Luenberger observer and state-feedback controller for fault estimation and fault compensation, respectively. In [3], a joint design for FTC has also been developed, where a proportional-integral (PI) observer is used to estimate the actuator fault; the tracking error is assured by using $H_2/\mathscr{H}_\infty$ norm, and then a state-feedback controller based on estimated states, faults and tracking error is applied for FTC purpose. The sliding-mode observer has also been developed for actuator fault estimation and a fault compensation has been applied to deal with the existing fault in [4]. Authors in [5] has developed virtual observers with immeasurable input for fault estimation and reformulated them into the normal observer form, so that the actuator and sensor fault can be estimated simultaneously.

However, the above solutions have only taken into account the observer and controller in two separate designs, which may cause the system instability in fault compensation where the time for estimation and reconfiguration is critical (see [6]). To solve this problem, an integrated design has been developed in [7], where the observer and controller are designed together. With this approach, the unknown input (UI) observer is applied to eliminate disturbance impact on fault estimation while $\mathscr{H}_\infty$ sliding mode control assures the quality of controlled output against system disturbance and uncertainties. In [8], for non-UI descriptor system, a PI observer-based controller has been developed to solve uncertainties existing in the observer gain and handle FTC problem. Although the above approach has justified its performance, there are still some obstacles to overcome. Firstly, the controller design demands restrictive conditions for the positive definite matrix P of Lyapunov function, which allows the decoupling of observer and controller design, but may also lead to an unfeasible solution for LMI optimization. More discussions on this problem can be found in [9]. Secondly, the decoupling condition between UI and estimation errors may be not met due to UI observer exigences. Therefore, this paper will examine these difficulties with the following contributions:

- The integrated design of observer and controller, where $\mathscr{H}_\infty$ PI observer attenuates the disturbance impact on fault estimation and the observer-controller coupling problem is solved by the Young relation, is studied;
- Fault estimation is robust against the uncertainties in system states (matrices A and C), which is counted in the integrated design by applying the majoration lemma;
- Both above objectives and the robust observer-controller interaction are resumed in only one LMI problem.

An application of vehicle suspension model is also presented in this study. In which, the PI observer promotes the estimation of actuator fault while the combination of the state-feedback controller and fault compensation assures the normal behavior of suspension.

The paper is organized as follows. Firstly, the problem formulation is presented in Section 2. In Section 3, under fault occurrence, the integrated design of the observer and controller against system uncertainties is constructed by using LMI optimization. The modeling for suspension system is considered under actuator fault occurrence in Section 4. Finally, the conclusion with remarks and future work are presented in Section 5.

Notations: $\mathbb{R}^n$ represents the n-dimensional Euclidean space; X^T is the transpose of the matrix X; 0 and I denote respectively zeros and identity matrix with appropriate

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dimensions; the symbol $(*)$ denotes the transposed block in the symmetric position; and $X^\dagger$ is the pseudo-inverse matrix of X which satisfies that $XX^\dagger X = X$.

II. Problem Formulation

Consider the following uncertain faulty LTI system:

$$\begin{cases} \dot{x} &= (A+\Delta A)x + Bu + E_d d + E_f f \\ y &= (C+\Delta C)x + Du + F_d d + F_f f \end{cases} \tag{1}$$

In which,

- $x \in \mathbb{R}^{n_x}$ is the state vector; $y \in \mathbb{R}^{n_y}$ is the measurement output vector; $u \in \mathbb{R}^{n_u}$ is the input vector; $d \in \mathbb{R}^{n_d}$ is the disturbance vector; $f \in \mathbb{R}^{n_f}$ is the actuator fault vector to be detected and estimated.
- Matrices A, B, C, D, E_d, E_f, F_d, and F_f are constant matrices corresponding to the nominal system with appropriate dimensions. In which, $E_f = B$ and $F_f = D$ in case of actuator faults.
- The terms ΔA and ΔC are time-varying parameter matrices corresponding to the uncertainty of nominal system. They can be represented as:

$$\begin{cases} \Delta A = M_a \Delta_a N_a \\ \Delta C = M_c \Delta_c N_c \end{cases} \tag{2}$$

where M_a, N_a, M_c, and N_c are known real constant matrices with appropriate dimension. Δ_a and Δ_c are unknown real-valued matrices satisfying that $\Delta_a^T \Delta_a \leq I$ and $\Delta_c^T \Delta_c \leq I$.

Remark 1: In this study, only the terms ΔA and ΔC are considered for the simplicity of design model (2). In case of uncertainties existing in input matrices B and D, the solution can also be obtained with the same principle with the use of majoration lemma.

The fault accommodation strategy for system (1) is presented in the next section.

III. Fault Tolerant Control design

Assumption 1. As discussed in [10], the faults are considered to be bounded and supposed to be in low frequency domain, i.e., $\dot{f} \simeq 0$. In fact, most system faults, such as an actuator jam or a stuck, exist in this zone (see [11], and [12]).

The integrated design combines 2 components:

- The PI Observer for actuator fault estimation:

$$\begin{cases} \dot{\hat{x}} &= A\hat{x} + Bu + L_P(y-\hat{y}) + E_f\hat{f} \\ \dot{\hat{f}} &= L_I(y-\hat{y}) \\ \hat{y} &= C\hat{x} + Du + F_f\hat{f} \end{cases} \tag{3}$$

  where L_P and L_I are the proportional and integral gains of observer, respectively.
- The state-feedback controller $u = -K\hat{x}$ with the gain K assuring the closed-loop stability and an input $u_{add} = -B^\dagger B\hat{f}$ which compensates the actuator fault. This can be expressed as:

$$u = -K\hat{x} - B^\dagger B\hat{f} \tag{4}$$

Defining the state and fault errors as:

$$\begin{cases} e_x = x - \hat{x} \\ e_f = f - \hat{f} \end{cases} \tag{5}$$

Substituting (4) into (1), the dynamic of system states can be presented as:

$$\dot{x} = (A+\Delta A)x - BK\hat{x} + E_d d - B\hat{f} + E_f f \tag{6}$$
$$= (A+\Delta A)x - BK(x-e_x) + E_d d - B\hat{f} + E_f f \tag{7}$$
$$= (A+\Delta A - BK)x + (BKe_x + Be_f) + E_d d \tag{8}$$

From (1) and (3), the error dynamics are expressed as:

$$\dot{e}_x = \dot{x} - \dot{\hat{x}} = (A - L_P C)e_x + (E_f - L_P F_f)e_f + (E_d - L_P F_d)d + (\Delta A x - L_P \Delta C)x \tag{9}$$

$$\dot{e}_f = \dot{f} - \dot{\hat{f}} = -L_I C e_x - L_I F_f e_f - L_I F_d d - L_I \Delta C x \tag{10}$$

In other words:

$$\dot{e} = (A_a - L_a C_a)e + (E_{da} - L_a F_d)d + (\Delta A - L_a \Delta C)x \tag{11}$$

where estimation error: $e = \begin{bmatrix} e_x \\ e_f \end{bmatrix}$ $M_{ap} = \begin{bmatrix} M_a \\ 0_{n_f \times n_{1a}} \end{bmatrix}$, $A_a = \begin{bmatrix} A & E_f \\ 0 & 0 \end{bmatrix}$, $L_a = \begin{bmatrix} L_P \\ L_I \end{bmatrix}$, $E_{da} = \begin{bmatrix} E_d \\ 0_{n_f \times n_d} \end{bmatrix}$, and $C_a = \begin{bmatrix} C & F_f \end{bmatrix}$.

Assumption 2. For the existence of the PI observer in (4), the pair (A_a, C_a) in (11) is assumed to be detectable, which can be expressed as the following condition (see [10]):

$$rank \begin{bmatrix} sI - A_a \\ C_a \end{bmatrix} = rank \begin{bmatrix} sI - A & -E_f \\ 0 & sI_{n_f} \\ C & F_f \end{bmatrix} = n_x + n_f \quad \forall Re(s) \geq 0 \tag{12}$$

The closed-loop system can be expressed as:

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{e}_x \\ \dot{e}_f \end{bmatrix} &= \left[\begin{array}{c|cc} A+\Delta A - BK & BK & B \\ \hline \Delta A - L_P \Delta C & A - L_P C & E_f - L_P F_f \\ -L_I \Delta C & -L_I C & -L_I F_f \end{array}\right] \begin{bmatrix} x \\ e_x \\ e_f \end{bmatrix} \\ &\quad + \begin{bmatrix} E_d \\ E_d - L_P F_d \\ -L_I F_d \end{bmatrix} d \\ e_f &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ e_x \\ e_f \end{bmatrix} \end{cases} \tag{13}$$

Remark 2: Due to the certainty existence, the terms $(\Delta A - L_P \Delta C)$ and $(-L_I \Delta C)$ in (13) are not null, which presents a challenge for the separated design to assure the robustness of observer-controller interaction. Therefore, the existence of an integrated solution is necessary. More details on performance of 2 approaches are found in simulation results of Section 4.

As usual (see [10]), the main objective of robust PI $\mathscr{H}_\infty$ observer is to minimize the effect of disturbance d on the fault estimation error e_f in (13), which can be rewritten as:

$$\min_{\gamma > 0} \|S_{e_f d}\|_\infty = \min_{\gamma > 0} \frac{\|e_f\|_2}{\|d\|_2} \leq \gamma \tag{14}$$

The robust stability of the closed-loop system (13) and the disturbance attenuation objective (14) can be achieved if the following condition (15) is satisfied:

$$\dot{V} + e_f^T e_f - \gamma^2 d^T d < 0 \tag{15}$$

where the Lyapunov candidate function is chosen as:

$$V = \begin{bmatrix} x^T & e_x^T & e_f^T \end{bmatrix} P \begin{bmatrix} x \\ e_x \\ e_f \end{bmatrix} \quad , P > 0 \tag{16}$$

Some remarks on LMI and inequality problems are presented by the following lemmas:

Lemma 1. (Majoration lemma) (see [13])

If there exists $F^T F \leq I$, for given matrices X and Y with appropriate dimensions, the following statement is always true with an arbitrary scalar $\sigma > 0$:

$$XFY + Y^T F^T X^T \leq \sigma X X^T + \sigma^{-1} Y^T Y \tag{17}$$

Lemma 2. (Schur complement) (see [14])

Let $Q < 0$, S and R be given matrices, the following statements are equivalents:

$$\begin{aligned} \begin{bmatrix} Q & R \\ R^T & S \end{bmatrix} < 0 \\ Q - RS^{-1}R^T < 0 \end{aligned} \tag{18}$$

Lemma 3. (Young relation) (see [14])

For given matrices X and Y with appropriate dimensions, we have for any invertible matrix F and scalar $\varepsilon > 0$ that:

$$X^T Y + Y^T X \leq \varepsilon X^T F X + \varepsilon^{-1} Y^T F^{-1} Y \tag{19}$$

Theorem 1. If there exist such symmetric positive definite matrices P_1, P_2 and P_3; matrices Q_1, Q_2, and Q_3; and given positive scalars $\varepsilon, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ that the following LMI is satisfied:

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \begin{bmatrix} 0 \\ 0 \\ Q_3 M_c \\ 0 \end{bmatrix} \\ (*) & \Omega_{22} & 0 & 0 & 0 \\ (*) & 0 & \Omega_{33} & 0 & 0 \\ (*) & 0 & 0 & \Omega_{44} & 0 \\ (*) & 0 & 0 & 0 & -\sigma_4^{-1} \end{bmatrix} < 0 \tag{20}$$

where

$$\Omega_{11} = \begin{bmatrix} \Gamma''_{11} & 0 & B & E_d \\ 0 & \Gamma'_{22} & \Gamma_{23} & \Gamma_{24} \\ (*) & (*) & \Gamma'_{33} & \Gamma_{34} \\ (*) & (*) & (*) & \Gamma_{44} \end{bmatrix} \quad \Omega_{12} = \begin{bmatrix} BQ_1 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

$$\Omega_{13} = \begin{bmatrix} P_1 N_a^T & P_1 N_c^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Omega_{14} = \begin{bmatrix} 0 & 0 \\ P_2 M_a & Q_2 M_c \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

$$\Omega_{22} = \begin{bmatrix} -\varepsilon^{-1} P_1 & 0 \\ 0 & -\varepsilon P_1 \end{bmatrix} \tag{23}$$

$$\Omega_{33} = \begin{bmatrix} -\frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} & 0 \\ 0 & -\frac{\sigma_3 \sigma_4}{\sigma_3 + \sigma_4} \end{bmatrix} \tag{24}$$

$$\Omega_{44} = \begin{bmatrix} -\sigma_2^{-1} & 0 \\ 0 & -\sigma_3^{-1} \end{bmatrix} \tag{25}$$

$$\Gamma''_{11} = AP_1 + P_1 A^T - BQ_1 - Q_1^T B^T + \sigma_1 M_a M_a^T \tag{26}$$

$$\Gamma'_{22} = P_2 A + A^T P_2 + Q_2 C + C^T Q_2^T \tag{27}$$

$$\Gamma'_{33} = Q_3 F_f + F_f^T Q_3^T + I \tag{28}$$

$$\Gamma_{23} = P_2 E_f + Q_2 F_f + C^T Q_3^T \tag{29}$$

$$\Gamma_{24} = P_2 E_d + Q_2 F_d \tag{30}$$

$$\Gamma_{34} = Q_3 F_d \tag{31}$$

$$\Gamma_{44} = -\gamma^2 I \tag{32}$$

then the gains $K = Q_1 P_1^{-1}$, $L_P = -P_2^{-1} Q_2$, and $L_I = -P_3^{-1} Q_3$ can assure the asymptotically robust stability of system (13).

Proof: Assuming that P in (16) has the form:

$$P = diag(P_1^{-1}, P_2, P_3) \tag{33}$$

where P_1^{-1}, P_2, and P_3 are symmetric positive definite matrices. As a result, (16) can be expressed as:

$$V = V_1 + V_2 + V_3 = x^T P_1^{-1} x + e_x^T P_2 e_x + e_f^T P_3 e_f \tag{34}$$

The condition (15) is rewritten as:

$$\dot{V}_1 + \dot{V}_2 + \dot{V}_3 + e_f^T e_f - \gamma^2 d^T d < 0 \tag{35}$$

In which,

$$\begin{aligned} \dot{V}_1 &= \dot{x}^T P_1^{-1} x + x^T P_1^{-1} \dot{x} \\ &= x^T (P_1^{-1} A + A^T P_1^{-1} - P_1^{-1} B K - K^T B^T P_1^{-1}) x \\ &+ x^T (P_1^{-1} M_a \Delta_a N_a + N_a^T \Delta_a^T M_a^T P_1^{-1}) x \\ &+ x^T (P_1^{-1} B K) e_x + e_x^T (K^T B^T P_1^{-1}) x \\ &+ x^T (P_1^{-1} B) e_f + e_f^T (B^T P_1^{-1}) x \\ &+ x^T P_1^{-1} E_d d + d^T E_d^T P_1^{-1} x \end{aligned} \tag{36}$$

$$\begin{aligned} \dot{V}_2 &= \dot{e}_x^T P_2 e_x + e_x^T P_2 \dot{e}_x \\ &= e_x^T (P_2 A + A^T P_2 + Q_2 C + C^T Q_2^T) e_x \\ &+ e_x^T (P_2 E_f + Q_2 F_f) e_f + e_f^T (E_f^T P_2 + F_f^T Q_2^T) e_x \\ &+ e_x^T (P_2 E_d + Q_2 F_d) d + d^T (E_d^T P_2 + F_d^T Q_2^T) e \\ &+ (e_x^T P_2 M_a) \Delta_a (N_a x) + (x^T N_a^T) \Delta_a^T (M_a^T P_2 e_x) \end{aligned} \tag{37}$$

$$+ (e_x^T Q_2 M_c) \Delta_c (N_c x) + (x^T N_c^T) \Delta_c^T (M_c^T Q_2^T e_x) \tag{39}$$

$$\begin{aligned} \dot{V}_3 &= \dot{e}_f^T P_3 e_f + e_f^T P_3 \dot{e}_f \\ &= e_f^T (Q_3 F_f + F_f^T Q_3^T) e_f + e_f^T Q_3 C e_x + e_x^T C^T Q_3^T e_f \\ &+ e_f^T (Q_3 F_d) d + d^T (F_d^T Q_3^T) e_f \end{aligned} \tag{40}$$

$$+ (e_f^T Q_3 M_c) \Delta_c (N_c x) + (x^T N_c^T) \Delta_c^T (M_c^T Q_3^T e_f) \tag{41}$$

where $Q_2 = P_2 L_P$ and $Q_3 = P_3 L_I$

Applying lemma 1, $\forall \sigma_1, \sigma_2, \sigma_3, \sigma_4 > 0$:

$$\begin{aligned} \dot{V}_1 \leq\ & x^T (P_1^{-1} A + A^T P_1^{-1} - P_1^{-1} B K - K^T B^T P_1^{-1} \\ & + \sigma_1 P_1^{-1} M_a M_a^T P_1^{-1} + \sigma_1^{-1} N_a^T N a) x \\ & + x^T (P_1^{-1} B K) e_x + e_x^T (K^T B^T P_1^{-1}) x \\ & + x^T (P_1^{-1} B) e_f + e_f^T (B^T P_1^{-1}) x \\ & + x^T P_1^{-1} E_d d + d^T E_d^T P_1^{-1} x \end{aligned} \tag{42}$$

$$\dot{V}_2 \le e_x^T(P_2A + A^TP_2 + Q_2C + C^TQ_2^T + \sigma_2 P_2 M_a M_a^T P_2 + \sigma_3 Q_2 M_c M_c^T Q_2^T)e_x + x^T(\sigma_2^{-1}N_a^T N_a + \sigma_3^{-1}N_c^T N_c)x + e_x^T(P_2E_f + Q_2F_f)e_f + e_f^T(E_f^T P_2 + F_f^T Q_2^T)e_x + e_x^T(P_2E_d + Q_2F_d)d + d^T(E_d^T P_2 + F_d^T Q_2^T)e_x \quad (43)$$

$$\dot{V}_3 \le e_f^T(Q_3F_f + F_f^T Q_3^T + \sigma_4 Q_3 M_c M_c^T Q_3^T)e_f + e_f^T Q_3 C e_x + e_x^T C^T Q_3^T e_f + x^T(\sigma_4^{-1}N_c^T N_c)x + e_f^T(Q_3F_d)d + d^T(F_d^T Q_3^T)e_f \quad (44)$$

The left side of (35) can be expressed as:

$$\dot{V}_1 + \dot{V}_2 + \dot{V}_3 + e_f^T e_f - \gamma^2 d^T d \le \begin{bmatrix} x^T & e_x^T & e_f^T & d^T \end{bmatrix} \Gamma \begin{bmatrix} x \\ e_x \\ e_f \\ d \end{bmatrix} \quad (45)$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ (*) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ (*) & (*) & \Gamma_{33} & \Gamma_{34} \\ (*) & (*) & (*) & \Gamma_{44} \end{bmatrix} \quad (46)$$

$$\Gamma_{11} = P_1^{-1}A + A^TP_1^{-1} - P_1^{-1}BK - K^TB^TP_1^{-1} + \sigma_1 P_1^{-1} M_a M_a^T P_1^{-1} + (\sigma_1^{-1} + \sigma_2^{-1})N_a^T N_a + (\sigma_3^{-1} + \sigma_4^{-1})N_c^T N_c \quad (47)$$

$$\Gamma_{12} = P_1^{-1}BK \quad (48)$$

$$\Gamma_{13} = P_1^{-1}B \quad (49)$$

$$\Gamma_{14} = P_1^{-1}E_d \quad (50)$$

$$\Gamma_{22} = P_2A + A^TP_2 + Q_2C + C^TQ_2^T + \sigma_2 P_2 M_a M_a^T P_2 + \sigma_3 Q M_c M_c^T Q_2^T \quad (51)$$

$$\Gamma_{33} = Q_3F_f + F_f^T Q_3^T + \sigma_4 Q_3 M_c M_c^T Q_3^T + I \quad (52)$$

The sufficient condition for the robust stability of (13) is that:

$$\Gamma < 0 \quad (53)$$

Pre- and post-multiplying this inequality by $diag\{P_1, I, I\}$:

$$\Gamma_1 = \begin{bmatrix} \Gamma'_{11} & BK & B & E_d \\ (*) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ (*) & (*) & \Gamma_{33} & \Gamma_{34} \\ (*) & (*) & (*) & \Gamma_{44} \end{bmatrix} < 0 \quad (54)$$

where

$$\Gamma'_{11} = AP_1 + P_1A^T - BQ_1 - Q_1^TB^T + \sigma_1 M_a M_a^T + (\sigma_1^{-1} + \sigma_2^{-1})P_1 N_a^T N_a P_1 + (\sigma_3^{-1} + \sigma_4^{-1})P_1 N_c^T N_c P_1 \quad (55)$$

$$Q_1 = KP_1 \quad (56)$$

The term Γ_1 can be expressed as:

$$\Gamma_1 = \Omega_1 + \begin{bmatrix} BK \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (BK)^T & 0 & 0 & 0 \end{bmatrix} \quad (57)$$

where $\Omega_1 = \begin{bmatrix} \Gamma'_{11} & 0 & B & E_d \\ 0 & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ (*) & (*) & \Gamma_{33} & \Gamma_{34} \\ (*) & (*) & (*) & \Gamma_{44} \end{bmatrix}$

Applying lemma 3 with $F = P_1$, $X = \begin{bmatrix} (BK)^T & 0 & 0 & 0 \end{bmatrix}$, and $Y = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$:

$$\Gamma_1 \le \Gamma_2 \quad (58)$$

$$\Gamma_2 = \Omega_1 + \varepsilon \begin{bmatrix} BK \\ 0 \\ 0 \\ 0 \end{bmatrix} P_1 \begin{bmatrix} (BK)^T & 0 & 0 & 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} P_1^{-1} \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \quad (59)$$

$$= \Omega_1 + \begin{bmatrix} BQ_1 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon P_1^{-1} & 0 \\ 0 & \varepsilon^{-1}P_1^{-1} \end{bmatrix} \begin{bmatrix} (BQ_1)^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \quad (60)$$

The condition (53) holds if:

$$\Gamma_2 < 0 \quad (61)$$

Applying Schur complement on Γ_2, we obtain:

$$\begin{bmatrix} \Omega_1 & \begin{bmatrix} BQ_1 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ (*) & \begin{bmatrix} -\varepsilon^{-1}P_1 & 0 \\ 0 & -\varepsilon P_1 \end{bmatrix} \end{bmatrix} < 0 \quad (62)$$

Applying Schur complement many times on Ω_1 in order to rewrite the above inequality as LMI form in (20). That completes the proof for robust stability of (13).

An application of the proposed design to vehicle suspension system is presented in the next section to prove its performance against the actuator fault.

IV. Application to vertical car system

A. Suspension modeling

The quarter-car, or semi-active suspension in our example, can be modeled by a mass-spring-damper system (see Fig. 1), which contains: the sprung mass m_s represents a quarter of the chassis body and z_s is the vertical displacement around the equilibrium point of m_s; the sprung mass m_{us} represents the wheel / the tire of the vehicle and z_{us} is the vertical displacement around the equilibrium point of m_{us}; the semi-active suspension is composed of a spring with the stiffness coefficient k_s and a controllable damper with the damping coefficient c, in which $c_{min} \le c \le c_{max}$; the tire is modeled by a spring with the stiffness coefficient k_t; and the road profile z_r is considered as unknown input d for the suspension.

The suspension dynamics are described by the following equations ([15]):

$$\begin{cases} m_s\ddot{z}_s & = -k_s(z_s - z_{us}) - F_c \\ m_{us}\ddot{z}_{us} & = k_s(z_s - z_{us}) + F_c - k_t(z_{us} - z_r) \end{cases} \quad (63)$$

Where $F_c = c\dot{z}_{def}$ is the damper force; $z_{def} = z_s - z_{us}$ is the displacement (deflection) between the chassis and the tire position; and $\dot{z}_{def}$ is the deflection speed.

In order to obtain LTI model of suspension system, the damper force F_c can be decomposed into 2 components:

$$F_c = c\dot{z}_{def} = c_0\dot{z}_{def} + u \quad (64)$$

In which, u is the model input corresponding to the varying part of semi-active damper force F_c; c_0 is the nominal value of damper, which corresponds to a passive damper when there is no control input u.

According to [16], in case of semi-active suspension, the authors chose $c_0 = (c_{min} + c_{max})/2$ as the nominal damping value, so the control input u in (64) is supposed to be limited in the symmetric region $[-u^*_{max}, u^*_{max}]$, where $u^*_{max} = (c_{max} - c_{min})\dot{z}_{def}/2$. However, in this example, we only focus on the performance of FTC system, not the saturation problem.

The faulty LTI model has been considered for the study as:

$$\begin{cases} \dot{x} & = Ax + Bu + E_d d + E_f f_a \\ y & = Cx + Du + F_d d + F_f f_a \end{cases} \quad (65)$$

In which, $x = \begin{bmatrix} z_s & \dot{z}_s & z_{us} & \dot{z}_{us} \end{bmatrix}^T$ is the state vector; $y = \begin{bmatrix} z_{def} & \ddot{z}_{us} \end{bmatrix}^T$ is the output vector; d is the road profile z_r considered as unknown input; u is the control input; f_a is the actuator fault;

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{c_0}{m_s} & \frac{k_s}{m_s} & \frac{c_0}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_{us}} & \frac{c_0}{m_{us}} & -\frac{kt+k_s}{m_{us}} & -\frac{c_0}{m_{us}} \end{bmatrix}, F_d = \begin{bmatrix} 0 & \frac{k_t}{m_{us}} \end{bmatrix}^T,$$

$$C = \begin{bmatrix} 1 & 0 & -1 & 0 \\ \frac{k_s}{m_{us}} & \frac{c_0}{m_{us}} & -\frac{kt+k_s}{m_{us}} & -\frac{c_0}{m_{us}} \end{bmatrix}, E_d = \begin{bmatrix} 0 & 0 & 0 & \frac{k_t}{m_{us}} \end{bmatrix}^T,$$

$$F_f = D = \begin{bmatrix} 0 & \frac{1}{m_{us}} \end{bmatrix}^T, \text{ and } E_f = B = \begin{bmatrix} 0 & -\frac{1}{m_s} & 0 & \frac{1}{m_{us}} \end{bmatrix}^T.$$

The parameters of suspension, which satisfy the *Assumption 2*, are presented in the table I.

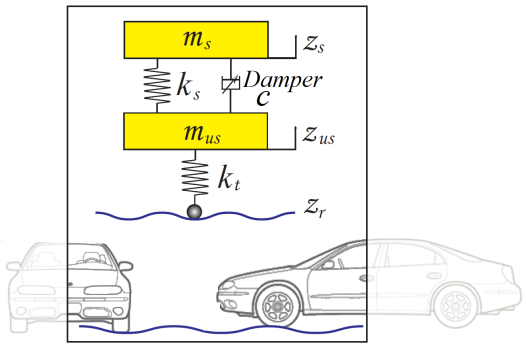


Fig. 1. The quarter-car scheme.

TABLE I

SUSPENSION PARAMETERS

Parameters	Unit	Value	Description
m_s	kg	2.578	A quarter-car chassis mass
m_{us}	kg	0.485	Rear tire mass
k_s	N/m	349	Suspension stiffness
c_{min}	$N.s/m$	18	Minimum damping coefficient
c_{max}	$N.s/m$	85	Maximum damping coefficient
k_t	N/m	3067.5	Tire stiffness

B. System uncertainty

The damping coefficient c is varying from c_{min} to c_{max}, so it has the uncertainty of $\Delta c_0 = (c_{max} - c_{min})/2$ comparing to the nominal value c_0.

The uncertainty effect on system dynamics are presented by the eigenvalues of $(A \pm \Delta A)$ with $\Delta A = 0$ and with both c_{max} and c_{min} cases, as presented as following:

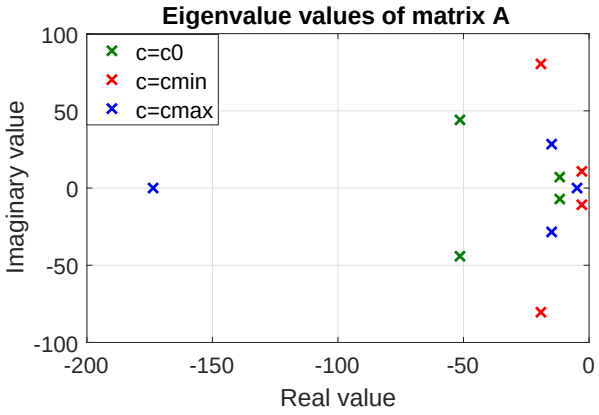


Fig. 2. Dynamics of uncertain system.

Fig. 2 proves an important influence of the damping coefficient's variation on the system dynamics. Therefore, the application of the robust observer and controller is necessary to deal with this uncertainty.

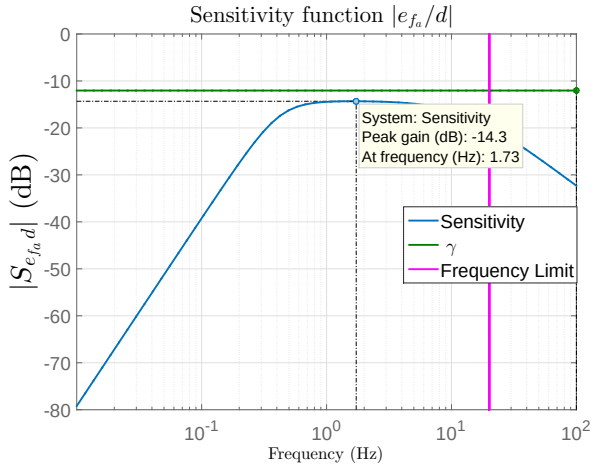


Fig. 3. Sensitivity function $|S_{e_{f_a}d}| = |e_{f_a}/d|$.

C. PI Observer Performance Analysis

Following from the assumption 1 and authors in [15], the actuator fault f_a is supposed to exist in low frequency, while the working frequency of road profile d is from 0 to 20 Hz. By solving LMI optimization problem, optimal $\mathscr{H}_\infty$ performance of the PI observer is calculated: $\gamma = 0.25$.

The frequency analysis is illustrated in Fig. 3 and Fig. 4 to evaluate the sensitivity of the fault estimation error e_{f_a} to the disturbance and the estimation performance of PI observer.

Fig. 3 shows an efficient attenuation of the road profile d (cm) impact on the fault estimation error e_{f_a}. It has a maximum at 1.73 Hz, which emphasizes the worst case of -14.3 dB in its frequency range.

In Fig. 4, $|T_{\hat{f}_a f_a}| = 0$ from 0 to 0.1 Hz, so the actuator fault f_a can be well estimated if its bandwidth is less than 0.1 Hz.

D. Simulation condition in closed loop

The closed-loop system is simulated during 12 seconds.

Road profile d: is modeled as a sinus signal. The frequency f_c is chosen as 1.73 Hz, which represents one of the worst case that the suspension has to handle (see Fig. 3)

$$d = z_r = 0.2 sin(2\pi f_c) \quad (cm) \tag{66}$$

Fault Scenario: Abrupt fault (stepwise) has been considered for the test. A step up is chosen for the sake of modeling f_a existing in damper actuator.

$$f_a = \begin{cases} 6 & \text{if } (2s \leq t \leq 9s) \\ 0 & \text{otherwise} \end{cases} \quad (N) \tag{67}$$

System uncertainty: is modified by Δ_a and Δ_c, which represent the uncertainty in the damping coefficient. Those terms are chosen as sinus signals to satisfy the condition that $\Delta_a^T \Delta_a \leq I$ and $\Delta_c^T \Delta_c \leq I$

$$\Delta_a = \Delta_c = sin(2\pi 30) \tag{68}$$

The frequency of uncertainty is chosen as 30 Hz, which depends on the characteristics of suspension.

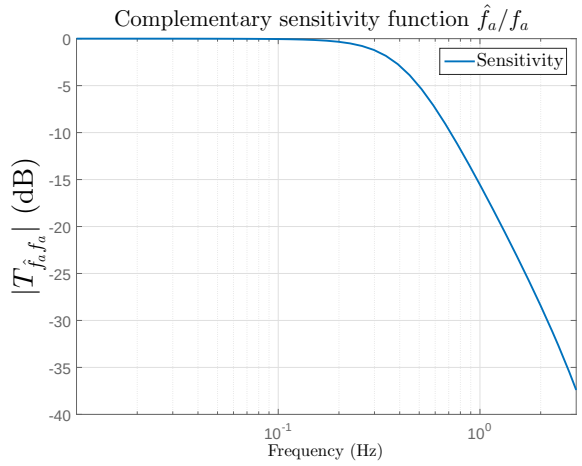


Fig. 4. Sensitivity function $|T_{\hat{f}_a f_a}| = |\hat{f}_a/f_a|$.

The distribution matrices for system uncertainty are presented below:

$$M_a = \begin{bmatrix} 0 & -\frac{\Delta c_0}{m_s} & 0 & \frac{\Delta c_0}{m_{us}} \end{bmatrix}^T \quad M_c = \begin{bmatrix} 0 & \frac{\Delta c_0}{m_{us}} \end{bmatrix}^T \tag{69}$$

$$N_a = N_c = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} \tag{70}$$

Initial conditions: initial system states are chosen as:

$$x_{(0)} = \begin{bmatrix} z_{s(0)} & \dot{z}_{s(0)} & z_{us(0)} & \dot{z}_{us(0)} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{71}$$

E. Simulation result in closed loop

In order to evaluate FTC performance, 2 designs of controller are considered:

- **Integrated design:** is the proposed method where observer and controller gains are designed simultaneously.
- **Separated design:** the robustness interaction between the observer and controlled system is neglected. As mentioned in [2], the controller is assumed as : $u' = -K'x - B^\dagger B f_a$, so (1) becomes:

$$\begin{cases} \dot{x} &= (A + \Delta A - BK')x + E_d d \\ y &= (C + \Delta C - BK')x + F_d d \end{cases} \tag{72}$$

The stability of the closed-loop system (72) is assured if the gain K' stabilizes the term $(A + \Delta A - BK)$. On the other hand, the state x and the fault f_a can only be estimated from PI observer (obtained from *Theorem 1*), so the control input for separated design is rewritten as: $u' = -K'\hat{x} - B^\dagger B \hat{f}_a$.

As observed in Fig. 5, the displacement position z_{def} is stabilized under actuator fault occurrence in both cases, which means that FTC objective is achieved. On the other hand, the separated design takes more time to stabilize the controlled signal and its peak in displacement output is also higher when the actuator fault occurs. Therefore, the performance of the integrated design is proved to be better than that of the separated design.

The actuator fault estimation is presented in Fig. 6. In both design methods, under the influence of road profile disturbance, the actuator fault f_a is well estimated with the rising time about 2 seconds. In other words, the $\mathscr{H}_\infty$ PI observer has justified its effectiveness in fault estimation.

The control input is also reconfigured by FTC mechanism. In Fig. 7, the control input has to decrease to compensate for the rise of actuator fault from 2 to 9 seconds, and then it comes back to normal behavior when there is no fault. In spite of that, the control designed by the separated method has slower reaction comparing to integrated design.

V. Conclusion and Future Works

Through the simulation, the integrated observer-controller design has proven its performance in fault tolerant control. In which, by applying $\mathscr{H}_\infty$ norm, the PI observer attenuates the disturbance impact on actuator fault estimation, while the combination of state-feedback control and fault compensation has assured the normal system behavior. Moreover, the system uncertainty and the decoupling condition for

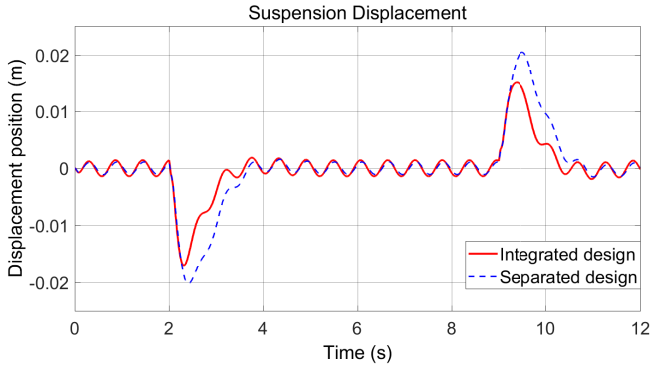


Fig. 5. Fault tolerant control result from 0 to 12 seconds.

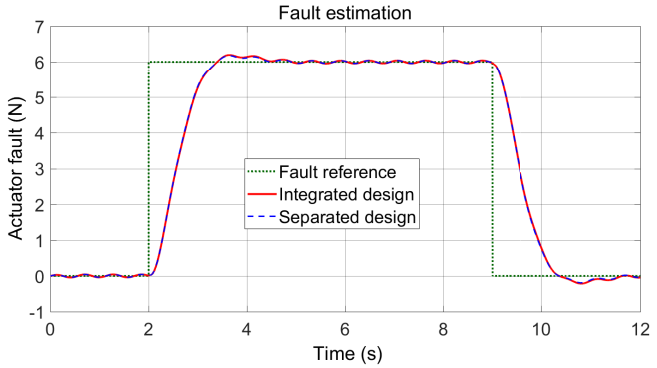


Fig. 6. Actuator fault estimation.

observer-controller design have been taken into account for the proposed design and LMI optimization, which allows a robust stability of the closed-loop system. For future work, the application of the proposed design to linear parameter-varying system is also an interesting research topic.

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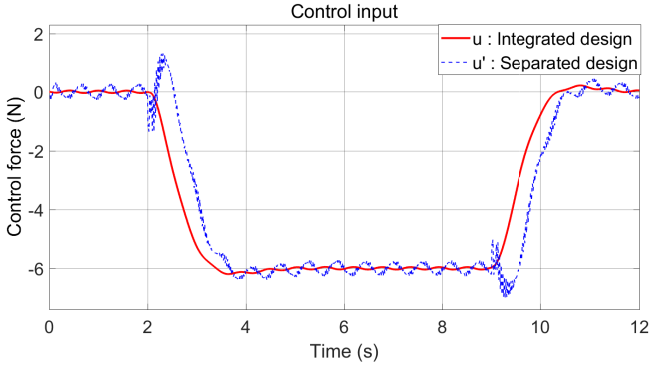


Fig. 7. Control input.

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