On the determination of nonlinear terms appearing in semilinear hyperbolic equations

Yavar Kian

ON THE DETERMINATION OF NONLINEAR TERMS APPEARING IN SEMILINEAR HYPERBOLIC EQUATIONS

YAVAR KIAN

Abstract. We consider the inverse problem of determining a general nonlinear term appearing in a semilinear hyperbolic equation on a Riemannian manifold with boundary (M, g) of dimension $n = 2, 3$. We prove results of unique recovery of the nonlinear term $F(t, x, u)$, appearing in the equation $\partial_t^2 u - \Delta_g u + F(t, x, u) = 0$ on $(0, T) \times M$ with $T > 0$, from some partial knowledge of the solutions u on the boundary of the time-space cylindrical manifold $(0, T) \times M$ or on the lateral boundary $(0, T) \times \partial M$. We determine the expression $F(t, x, u)$ both on the boundary $x \in \partial M$ and inside the manifold $x \in M$.
Keywords: Inverse problems, nonlinear wave equation, semilinear equation, equations on manifold.

Mathematics subject classification 2010 : 35R30, 35L71, 35L20.

1. Introduction

1.1. **Statement of the problem.** Let (M, g) be a smooth compact Riemannian manifold with boundary and let $T > 0$. We introduce the Laplace and wave operators

$$\Delta_g u = |g|^{-1/2}\partial_{x_j}\left(g^{jk}|g|^{1/2}\partial_{x_k} u\right), \quad \Box_g = \partial_t^2 - \Delta_g, \tag{1.1}$$

where $|g|$ and g^{jk} denote the absolute of value of the determinant and the inverse of g in local coordinates, and consider, for $T > 0$, the semilinear wave equation

$$\Box_g u + F(t, x, u) = 0, \quad (t, x) \in (0, T) \times M, \tag{1.2}$$

with a nonlinear term F suitably chosen. In this paper, we consider the inverse problem of determining F from observations of solutions of (1.2) on the boundary of the manifold $(0, T) \times M$.

1.2. **Motivations.** Let us first observe that nonlinear wave equations of the form (1.2) can be associated with different models where the transmission of waves is perturbed by a semilinear expression. Such phenomenon can occur in many mechanical and electromagnetic models. For instance, we can mention the study of vibrating systems where the expression $F(t, x, u)$ can be seen as a nonlinear perturbation of the system. The semilinear term $F(t, x, u)$ can also be associated with other perturbations arising in electronics like in the telegraph equation of for semi-conductors (see for instance [4]). In this context, the goal of our inverse problem is to recover the nonlinear expression $F(t, x, u)$ which describes the underlying physical law of the perturbed system.

Beside these physical motivations, we mention that there is a natural mathematical motivation for the study of such inverse problems which are highly nonlinear and ill-posed.

1.3. **Known results.** Let us first mention that, to our best knowledge, there is only a small number of papers dealing with inverse problems for nonlinear partial differential equations. Among them we can mention the work [12, 13, 14] of Isakov dedicated to the recovery of nonlinear terms appearing in elliptic or parabolic equations. The method developed by Isakov is based on a linearization of the inverse problem for nonlinear equations and results based on recovery of coefficients for linear equations. This approach has been applied in different other context. For instance, we can mention the work of [15, 33], dealing with the unique recovery of nonlinear terms appearing in nonlinear elliptic equations and the work of [7] dealing with the stable recovery of a semilinear term appearing in a parabolic equation. For more specific nonlinear terms, we can mention the work of [6, 8, 22], who have considered similar problems with single measurements.

For hyperbolic equations we refer to the work of [29, 30] dealing with the recovery of a conductivity and quadratic coefficients appearing in a non-linear wave equation of divergence form. We mention also the recent works of [9, 23, 24], who have considered inverse problems for semilinear hyperbolic equations on a general Lorentzian manifold. To our best knowledge, beside the present paper, the recovery of a general nonlinear term appearing in hyperbolic equations has not been addressed so far.

1.4. **Preliminary results.** Before the statement of our main result let us first state some properties of solutions of (1.2), that will be required for the statement of our main results. Let us first fix the class of nonlinear terms under consideration. Let $b>0$ be such that, for $n=2$, $b>1$ and, for $n=3$, $b\in\left(1,\frac{13}{3}\right]$. For $a_1>0$ a fixed constant, we consider $\mathcal A$ the set of functions $F\in\mathcal C^3(\mathbb R_+\times M\times\mathbb R)$ satisfying

$$|\partial_t^k\partial_x^\alpha\partial_u^jF(t,x,u)|\leqslant a_1(1+|u|^{b-j}),\quad (t,x,u)\in\mathbb R_+\times M\times\mathbb R,\ k+|\alpha|+j\leqslant 3. \tag{1.3}$$

We fix also $\mathcal H$ the space of elements

$$G=(f,u_0,u_1)\in H^{\frac{11}{2}}((0,T)\times\partial M)\times H^{\frac{11}{2}}(M)\times H^{\frac{9}{2}}(M)$$

satisfying the compatibility conditions

$$f_{|t=0}=u_{0|\partial M},\quad \partial_tf_{|t=0}=u_{1|\partial M},\quad \partial_t^2f_{|t=0}=\Delta_gu_{0|\partial M},\quad \partial_t^3f_{|t=0}=\Delta_gu_{1|\partial M},\quad \partial_t^4f_{|t=0}=\Delta_g^2u_{0|\partial M}. \tag{1.4}$$

Then, for $F\in\mathcal A$ and $(f,u_0,u_1)\in\mathcal H$, we consider the following problem

$$\begin{cases}\partial_t^2u-\Delta_gu+F(t,x,u)=0, & \text{in } (0,T)\times M,\\ u=f, & \text{on } (0,T)\times\partial M,\\ u(0,\cdot)=u_0,\quad \partial_tu(0,\cdot)=u_1 & \text{in } M.\end{cases} \tag{1.5}$$

We prove in Section 2 (see Lemma 2.2), that for

$$\|f\|_{H^{\frac52}((0,T)\times\partial M)}+\|u_0\|_{H^{\frac52}(M)}+\|u_1\|_{H^{\frac32}(M)}\leqslant L$$

and for $p>b$, when $n=2$ and $p=5$ when $n=3$, there exists $T_*(2L)\in(0,+\infty]$ such that, for all $T<T_*(2L)$ and all $F\in\mathcal A$, the problem (1.5) admits a unique solution $u\in W^{1,\frac{p}{b-1}}(0,T;H^2(M))\cap W^{3,\frac{p}{b-1}}(0,T;L^2(M))$.

We fix also $\mathcal H_*$ the space of elements $f\in H^{\frac{11}{2}}((0,T)\times\partial M)$ satisfying the compatibility conditions

$$f_{|t=0}=\partial_tf_{|t=0}=\partial_t^2f_{|t=0}=\partial_t^3f_{|t=0}=\partial_t^4f_{|t=0}=0. \tag{1.6}$$

Then, for $G\in\mathcal H$ we denote by $u_{F,G}\in W^{1,\frac{p}{b-1}}(0,T;H^2(M))\cap W^{3,\frac{p}{b-1}}(0,T;L^2(M))$ the solution of (1.5). In the same way, we denote by $u_{F,f}\in W^{1,\frac{p}{b-1}}(0,T;H^2(M))\cap W^{3,\frac{p}{b-1}}(0,T;L^2(M))$ the solution of (1.5) with $u_0=u_1=0$. Then, for some $L>0$, $\varepsilon\in(0,1)$, fixing $T<T_*(2(L+3\varepsilon))$ and the set

$$\mathcal K_L:=\{G\in\mathcal H:\ \|G\|_{H^{\frac52}((0,T)\times\partial M)\times H^{\frac52}(M)\times H^{\frac32}(M)}\leqslant L+3\varepsilon\},$$

we define the boundary maps

$$\mathcal B_{F,\gamma_1}:\mathcal K_L\ni G\longmapsto(\partial_\nu u_{F,G|(0,T)\times\gamma_1},u_{F,G}(T,\cdot)_{|M})\in L^2((0,T)\times\gamma_1)\times H^1(M),$$

$$\mathcal N_{F,\gamma_1}:\{h\in\mathcal H_*:\ \|h\|_{H^{\frac52}((0,T)\times\gamma_1)}\leqslant L+\varepsilon\}\ni f\longmapsto\partial_\nu u_{F,f|(0,T)\times\gamma_1}\in L^2((0,T)\times\gamma_1),$$

with γ_1 an open subset of ∂M and ν the outward unit normal vector to ∂M. We prove in Theorem 2.1 that the maps $\mathcal B_{F,\gamma_1}$ and $\mathcal N_{F,\gamma_1}$ admit a continuous Fréchet derivative denoted by $\mathcal B'_{F,\gamma_1}$ and $\mathcal N'_{F,\gamma_1}$. The observation of our inverse problem will be given by some partial information of the Fréchet derivative of the map $\mathcal B_{F,\gamma_1}$ and $\mathcal N_{F,\gamma_1}$.

1.5. **Main results.** In our first result we consider the recovery of the nonlinear term $F(t,x,u)$ restricted to a portion of the lateral boundary $(0,T)\times\partial M$. More precisely, we fix γ an arbitrary open subset of ∂M, $\delta>0$, $\chi\in\mathcal C^\infty_0((0,+\infty)\times\partial M)$ satisfying $\chi=1$ on $[\delta,+\infty)\times\gamma$ and

$$\mathcal H_{*,\gamma}:=\{f\in\mathcal H_*:\ \mathrm{supp}(f)\subset[0,T]\times\gamma\}.$$

Then, we consider the recovery of F restricted to $[\delta,T)\times\gamma\times[-L,L]$ from the data

$$N'_{F,\gamma}(\lambda\chi)h,\quad h\in\mathcal H_{*,\gamma},\quad \lambda\in I,$$

with I an interval of $\mathbb R$. This result can be stated as follows.

Theorem 1.1. *Let $n=2,3$, $F_1,F_2\in\mathcal A$ and fix $T<T_*(2(L+3\varepsilon))$. Consider also $\delta>0$ and $\chi\in\mathcal C^\infty_0((0,+\infty)\times\partial M)$ satisfying $\chi=1$ on $[\delta,+\infty)\times\gamma$, $L_1:=\frac{L}{\|\chi\|_{H^{\frac52}((0,T)\times\partial M)}}$ and $T<T_*(2(L+\varepsilon))$. Then the conditions*

$$F_1(t,x,0)=F_2(t,x,0),\quad (t,x)\in[\delta,T]\times\gamma, \tag{1.7}$$

$$N'_{F_1,\gamma}(\lambda\chi)h=N'_{F_2,\gamma}(\lambda\chi)h,\quad \lambda\in[-L_1,L_1],\ h\in\mathcal H_{*,\gamma}, \tag{1.8}$$

imply

$$F_1(t,x,\lambda)=F_2(t,x,\lambda),\quad (t,x,\lambda)\in[\delta,T]\times\gamma\times[-L_1,L_1]. \tag{1.9}$$

This first result corresponds to the recovery of the nonlinear term F restricted to a portion γ of the boundary of M. In order to recover F inside M we will first need additional information about M. Let us first recall the definition of simple manifold.

Definition 1.1. *A compact smooth Riemannian manifold with boundary (M,g) is simple if it is simply connected, the boundary ∂M is strictly convex in the sense of the second fundamental form, and M has no conjugate points.*

With this additional assumption, we can extend Theorem 1.1 in the following way.

Theorem 1.2. *Let $n=2,3$, M be a simple manifold, $F_1,F_2\in\mathcal A$ and fix $T<T_*(2(L+3\varepsilon))$. Then the conditions*

$$F_1(t,x,0)=F_2(t,x,0),\quad (t,x)\in(\{0\}\times M)\cup((0,T)\times\partial M), \tag{1.10}$$

$$B'_{F_1,\partial M}(\lambda,\lambda,0)H=B'_{F_2,\partial M}(\lambda,\lambda,0)H,\quad \lambda\in\left[-\frac{L}{(2T|M|)^{\frac12}},\frac{L}{(2T|M|)^{\frac12}}\right],\ H\in\mathcal H \tag{1.11}$$

imply

$$F_1(t,x,\lambda)=F_2(t,x,\lambda),\quad (t,x,\lambda)\in[0,T]\times\partial M\times\left[-\frac{L}{(2T|M|)^{\frac12}},\frac{L}{(2T|M|)^{\frac12}}\right], \tag{1.12}$$

$$F_1(0,x,\lambda)=F_2(0,x,\lambda),\quad (x,\lambda)\in M\times\left[-\frac{L}{(2T|M|)^{\frac12}},\frac{L}{(2T|M|)^{\frac12}}\right]. \tag{1.13}$$

Here $(\lambda,\lambda,0)$ denotes the element of $\mathcal H$ corresponding to the different traces of the constant map $(t,x)\mapsto\lambda$.

In the specific case of a bounded domain of $\mathbb R^n$, $n=2,3$, with Euclidean metric, we can give a more precise result with restriction of the data to some portion of the boundary and solutions with constant values at $t=0$. To state this result which will be our last main result, we consider first the following tools. For any $\omega\in\mathbb S^{n-1}=\{y\in\mathbb R^n:\ |y|=1\}$ we consider the ω-shadowed and ω-illuminated faces of $\partial\Omega$

$$\partial\Omega_{+,\omega}=\{x\in\partial\Omega:\ \nu(x)\cdot\omega\geqslant0\},\quad \partial\Omega_{-,\omega}=\{x\in\partial\Omega:\ \nu(x)\cdot\omega\leqslant0\}.$$

Here, for all $k\in\mathbb N^*$, $\cdot$ denotes the scalar product in $\mathbb R^k$ defined by

$$x\cdot y=x_1y_1+\ldots+x_ky_k,\quad x=(x_1,\ldots,x_k)\in\mathbb R^k,\ y=(y_1,\ldots,y_k)\in\mathbb R^k.$$

We fix $\omega_0\in\mathbb S^{n-1}$ and we consider $U=[0,T]\times U'$ (resp $V=(0,T)\times V'$) with U' (resp V') an open neighborhood of $\partial\Omega_{+,\omega_0}$ (resp $\partial\Omega_{-,\omega_0}$) in $\partial\Omega$. Let us also consider the following restriction of the space $\mathcal H$ given by

$$\mathcal H_U:=\{H=(h,h_0,h_1)\in\mathcal H:\ h_0=0,\ \mathrm{supp}(h)\subset U\}.$$

Theorem 1.3. *Let $n = 2, 3$, $M = \overline{\Omega}$ with Ω an open connected and smooth domain of $\mathbb{R}^n$ with the Euclidean metric, let $F_1, F_2 \in \mathcal{A}$ and fix $T < T_*(2(L + 3\varepsilon))$. Then the conditions* (1.10) *and*

$$B'_{F_1,V}(\lambda, \lambda, 0)H = B'_{F_2,V}(\lambda, \lambda, 0)H, \quad \lambda \in \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right], \ H \in \mathcal{H}_U \tag{1.14}$$

imply

$$F_1(t, x, \lambda) = F_2(t, x, \lambda), \quad (t, x, \lambda) \in [0, T] \times \partial M \times \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right], \tag{1.15}$$

$$F_1(0, x, \lambda) = F_2(0, x, \lambda), \quad (x, \lambda) \in M \times \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right]. \tag{1.16}$$

1.6. **Comments about the main results.** To our best knowledge Theorem 1.1, 1.2 and 1.3 are the first results of recovery of a general semilinear term appearing in a hyperbolic nonlinear equation. Indeed, to our best knowledge one can only find results dealing with recovery of coefficients, appearing in a nonlinear hyperbolic equation, in the mathematical literature (see e.g. [29, 30]). It seems that such results have only been considered for parabolic or elliptic equations (e.g. [7, 12, 13, 14, 15, 33]). It seems also that Theorem 1.1, 1.2 are the first results dealing with the recovery of nonlinear terms appearing in a PDE of order two with variable second order coefficients that are not analytic in the case $n = 3$ ([33] considered this situation for quasilinear elliptic equations in dimension $n = 2$ but they make the assumption of analyticity for the dimension $n \geqslant 3$). Note also that like [7, 12, 13], we manage to recover the nonlinear term at the lateral boundary $(0, T) \times \partial M$, with data restricted to the lateral boundary, but also inside the domain.

The proof of Theorem 1.1, 1.2 and 1.3, are based on a linearization procedure inspired by [7, 12, 13, 14]. The idea consists in transforming the recovery of the nonlinear term $F(t, x, u)$ into the recovery of time-dependent coefficients $q(t, x) = \partial_u F(t, x, u(t, x))$, where u solves (1.5) with suitable choice of the data (f, u_0, u_1), appearing in a linear hyperbolic equation. So far this approach has been considered only with Hölder continuous solutions of some nonlinear parabolic or elliptic equations. For hyperbolic equations, the existence of such smooth solutions seems to require at least strong assumptions on the semilinear term under consideration. For this reason, in this paper, we provide, for what seems to be the first time, the extension of the linearization procedure considered for the first time by [12], to solutions lying in Sobolev space instead of Hölder continuous space. This extension of the analysis of [12] allows us to consider the case of nonlinear hyperbolic equations.

As mentioned above, our approach consists in transforming our inverse problem into the recovery of a time-dependent potential of the form $q(t, x) = \partial_u F(t, x, u(t, x))$, where u solves (1.5). This means that the regularity of the coefficient q will depend explicitly on the solution of the nonlinear problem (1.5). For this reason, we can not apply results dealing with recovery of smooth time-dependent coefficients. In Theorem 1.2 and 1.3, we use the results of [10, 18, 19, 20] dealing with the global recovery of such coefficients with low regularity assumptions. For Theorem 1.1, we need to use results of recovery of time-dependent coefficients on the portion $(0, T) \times \gamma$ of the lateral boundary $(0, T) \times \partial M$ from measurement restricted also to $(0, T) \times \gamma$. Moreover, we need to consider such results on some general Riemannian manifold. To our best knowledge [32] is the only work dealing with results close to the one needed for Theorem 1.1 (see also [31] for time-independent coefficients). However, the approach of [32], based on local properties of general geometric optics solutions, requires strong smoothness assumptions and it can not be applied in the context of Theorem 1.1. For this reason we introduce a new approach for the recovery of less-regular coefficients in the proof of Theorem 3.1 (see Section 3). The result of Theorem 3.1 is based on a global construction of particular solutions of the linear problem (3.1), with a control of their behavior close to the boundary. In contrast to other related results (e.g. [31, 32]) we do not restrict our analysis on some local properties of general geometric optics solutions associated with (3.1), but some global construction in boundary normal coordinates suitable designed for any point $(t, x) \in (0, T) \times \gamma$.

In contrast to other related results for parabolic or elliptic equations (e.g. [7, 12, 13, 14]), we make only small restrictions on the class of nonlinear terms under consideration. Indeed, we even consider semilinear equations with solutions that may blow-up at finite time. For this purpose, we state our result on, what can correspond to, the infimum of the final time of existence, denoted by T_*, of maximal solutions associated with all possible semi-linear terms lying in $\mathcal{A}$. Here T_* is a function of the size of the data (f, u_0, u_1) and it is well defined thanks to the lower bound (2.3) that we derive for this expression in Lemma 2.1. We believe that with additional assumptions on the class of admissible nonlinear terms $\mathcal{A}$ (see [3, 16, 11]) our result would be equivalent to the one stated by [7, 12, 13, 14] for global solutions of some nonlinear parabolic equations. However, in order to preserve the generality of our results, we prefer to keep this statement.

Let us observe, that, to our best knowledge, contrary to all other works dealing with recovery of nonlinear terms (e.g. [7, 12, 13, 14, 15, 33]), we do not state our results with boundary map $\mathcal{B}_{F,\gamma_1}$ or $\mathcal{N}_{F,\gamma_1}$ associated with the nonlinear problem (1.5), but with some partial knowledge of their Fréchet derivative. By taking into account the important amount of data contained into $\mathcal{B}_{F,\gamma_1}$ or $\mathcal{N}_{F,\gamma_1}$, this statement of the main results makes an important difference in terms of restriction of the data used for solving the inverse problem.

Our analysis is restricted to dimension of space $n = 2, 3$, but we believe that with suitable assumptions it could be extended to higher dimension. This restriction is due the application of the Sobolev embedding theorem in the linearization procedure.

1.7. **Outline.** This paper is organized as follows. In Section 2, we prove existence of sufficiently smooth solutions of (1.5). Then, we define the maps $\mathcal{B}_{F,\gamma_1}$ and $\mathcal{N}_{F,\gamma_1}$ and we prove that they admit a Fréchet derivative associated with solutions of linear wave equations with time-dependent coefficients. In Section 3, we establish the recovery on the portion $(0,T) \times \gamma$ of a time-dependent potential from measurements of solutions of the linear problem restricted to $(0,T) \times \gamma$. We prove this result, which is stated in Theorem 3.1, for coefficients $q \in H^2((0,T) \times M) \cap \mathcal{C}([0,T] \times M)$. In Section 4, we recall some results about recovery of time-dependent coefficients appearing in hyperbolic equations borrowed from [19, 20]. Finally, in Section 5, we combine all the arguments introduced in the preceding sections of the paper in order to complete the proof of Theorem 1.1, 1.2 and 1.3.

2. Forward problem and linearization of the inverse problem

In this section we will consider results related to existence and uniqueness of sufficiently smooth solutions of (1.5). Then we will use these results in order to prove that the maps $\mathcal{B}_{F,\gamma_1}$ and $\mathcal{N}_{F,\gamma_1}$ are well defined and admit a continuous Fréchet derivative.

We start with a result of local well-posedness for the problem (1.5) that can be proved by mean of Strichartz estimates stated in this context.

Lemma 2.1. *Assume that $n = 2$ or $n = 3$. Let $F \in \mathcal{A}$ and let $f \in H^{\frac{5}{2}}((0,+\infty) \times \partial M)$, $u_0 \in H^{\frac{5}{2}}(M)$ and $u_1 \in H^{\frac{3}{2}}(M)$ satisfy $f_{|t=0} = u_{0|\partial M}$, $\partial_t f_{|t=0} = u_{1|\partial M}$. Assume also that there exists $L > 0$ such that*

$$\|f\|_{H^{\frac{5}{2}}((0,+\infty)\times\partial M)} + \|u_0\|_{H^{\frac{5}{2}}(M)} + \|u_1\|_{H^{\frac{3}{2}}(M)} \leqslant L. \tag{2.1}$$

We consider the estimate

$$\|u\|_{\mathcal{C}([0,T];H^1(M))} + \|u\|_{L^p(0,T_1;L^{2p}(M))} \leqslant C_1 L, \tag{2.2}$$

with C_1 depending only on b, M, a_1, and we define the sets

$$\mathcal{T}_{F,L} := \{T > 0 : \text{ for all data } (f, u_0, u_1) \text{ satisfying (2.1), (1.5) admits a unique solution}$$
$$u \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H^1(M)) \cap L^p(0,T_1; L^{2p}(M)) \text{ satisfying (2.2)}\},$$

$$\mathcal{T}_L := \bigcap_{F\in\mathcal{A}} \mathcal{T}_{F,L},$$

with $p > b$, when $n = 2$ and $p = 5$ when $n = 3$. Then the set $\mathcal{T}$ is not empty and $\sup \mathcal{T}_L = T_*(L) \in (0,+\infty]$ depends on L, b, M and a_1. Moreover, $L \mapsto T_*(L)$ is non-increasing and, for any $\varepsilon_1 \in (0,1)$, we have the following lower bound for $T_*(L)$ given by

$$T_*(L) \geqslant C \min\left(L^{\frac{p(1-b)}{(p-b)}}, L^{\frac{2(1-b)}{(2+b)}}, L^{1+\varepsilon_1}, 1\right) \tag{2.3}$$

with C depending only on b, M, ε_1 and a_1. In addition, (1.5)*, with $T = T_*(L)$, admits a unique solution lying in $\mathcal{C}^1([0,T_*(L)); L^2(M)) \cap \mathcal{C}([0,T_*(L)); H^1(M)) \cap L^p_{loc}(0,T_*(L); L^{2p}(M))$.*

Proof. We prove this result by applying some arguments of [16, 18] that we adapt to problems stated with non-homogeneous Dirichlet boundary conditions. According to [28, Theorem 2.3, Chapter 4], there exists $G \in H^3((0,+\infty)\times M)$ satisfying

$$G_{|(0,+\infty)\times\partial M} = f, \quad G_{|t=0} = u_0, \quad \partial_t G_{|t=0} = u_1,$$
$$\|G\|_{H^3((0,+\infty)\times M)} \leqslant C(\|f\|_{H^{\frac{5}{2}}((0,+\infty)\times\partial M)} + \|u_0\|_{H^{\frac{5}{2}}(M)} + \|u_1\|_{H^{\frac{3}{2}}(M)}) \leqslant CL \tag{2.4}$$

where C depends only on M. From now on and in all the remaining part of this proof, we denote by C a constant depending on M and b. Moreover, by the Sobolev embedding theorem, we have $G \in \mathcal{C}([0,T]\times M)$ and fixing

$$G_1 := -(\partial_t^2 G - \Delta_g G),$$

one can check that $G_1 \in H^1(0,T; L^2(M))$. Then, we can split the solutions of (1.5) into two terms $u = G + v$ with v solving

$$\begin{cases} \partial_t^2 v - \Delta_g v + F(t,x,v+G) = G_1(t,x), & \text{in } (0,T_1)\times M,\\ v = 0, & \text{on } (0,T_1)\times\partial M,\\ v(0,\cdot) = 0, \quad \partial_t v(0,\cdot) = 0 & \text{in } M.\end{cases} \tag{2.5}$$

We will prove existence of a solution of (2.5) by mean of a fixed point argument. We denote by A the operator $-\Delta_g$ in M with Dirichlet boundary condition. Now consider, for some $T_1 > 0$ to be determined, the map $\mathcal{G}$ defined on $\mathcal{C}([0,T_1]; H^1_0(M)) \cap L^p(0,T_1; L^{2p}(M))$ by

$$\mathcal{G}[v)](t) := -\int_0^t \sin((t-s)A^{\frac12})A^{-\frac12}F(s,\cdot,v(s,\cdot)+G(s,\cdot))ds + \int_0^t \sin((t-s)A^{\frac12})A^{-\frac12}G_1(s,\cdot)ds$$

Combining the Christ-Kieslev lemma (see for instance [18, Lemma 1] and also [7] for the original result) with the Strichartz estimates on manifolds stated in [2, Theorem 1] and following [18, Lemma 2], we deduce that

$$\begin{aligned}
&\|G(v)\|_{\mathcal{C}([0,T_1];H^1(M))} + \|G(v)\|_{L^p(0,T_1;L^{2p}(M))}\\
&\leqslant C\|H\|_{L^1(0,T;L^2(M))} + 2\|G_1\|_{L^1(0,T_1;L^2(M))}\\
&\leqslant Ca_1\left\|(1+|v+G|^b)\right\|_{L^1(0,T;L^2(M))} + 2T_1\|G_1\|_{L^\infty(0,T_1;L^2(M))}\\
&\leqslant Ca_1 2^{b-1}\left\||v|^b\right\|_{L^1(0,T;L^2(M))} + Ca_1 2^{b-1}\left\||G|^b\right\|_{L^1(0,T;L^2(M))} + 2T_1^{\frac32}\|G_1\|_{H^1(0,T_1;L^2(M))} + Ca_1T_1\\
&\leqslant Ca_1 2^{b-1}\|v\|^b_{L^b(0,T;L^{2b}(M))} + Ca_1 2^{b-1}\|G\|^b_{L^b(0,T;L^{2b}(M))} + 4T_1^{\frac32}L + Ca_1T_1
\end{aligned}$$

with $H(t,x) := F(t,x,v(t,x)+G(t,x))$. On the other hand, by the Sobolev embedding theorem, we have

$$\|G\|_{L^b(0,T;L^{2b}(M))} \leqslant C\|G\|_{L^b(0,T;L^\infty(M))} \leqslant C\|G\|_{L^b(0,T;H^2(M))} \leqslant CT_1^{\frac1b}\|G\|_{L^\infty(0,T;H^2(M))} \leqslant CT_1^{\frac{2+b}{2b}}L$$

and the Hölder inequality implies

$$\|v\|_{L^b(0,T;L^{2b}(M))} \leqslant CT_1^{\frac{p-b}{pb}}\|v\|_{L^p(0,T;L^{2p}(M))}.$$

Thus, we have

$$\begin{aligned}
&\|G(v)\|_{\mathcal{C}([0,T_1];H^1(M))} + \|G(v)\|_{L^p(0,T_1;L^{2p}(M))}\\
&\leqslant Ca_1T_1^{\frac{p-b}{p}}\|v\|^b_{L^p(0,T;L^{2p}(M))} + Ca_1T_1^{\frac{2+b}{2}}L^b + 4T_1^{\frac32}L + Ca_1T_1.
\end{aligned} \tag{2.6}$$

In the same way, fixing $v_1, v_2 \in \mathcal{C}([0,T_1]; H^1_0(M)) \cap L^p(0,T_1; L^{2p}(M))$, we find

$$\begin{aligned}
&\|G(v_1)-G(v_2)\|_{\mathcal{C}([0,T_1];H^1(M))} + \|G(v_1)-G(v_2)\|_{L^p(0,T_1;L^{2p}(M))}\\
&\leqslant Ca_1T_1^{\frac{p-b}{p}}\|v_1-v_2\|_{L^p(0,T;L^{2p}(M))}(\|v_1\|^{b-1}_{L^p(0,T;L^{2p}(M))} + \|v_2\|^{b-1}_{L^p(0,T;L^{2p}(M))} + 1)
\end{aligned} \tag{2.7}$$

Combining (2.6)-(2.7) with the Poincaré fixed point theorem, we deduce that for

$$T_1 := C \min\left(L^{\frac{p(1-b)}{(p-b)}}, L^{\frac{2(1-b)}{(2+b)}}, L^{1+\varepsilon_1}, 1\right),$$

with C some suitable constant depending only on b, M, ε_1 and a_1, the map $\mathcal{G}$ admits a unique fixed point v in the set

$$\{w \in \mathcal{C}([0,T_1]; H_0^1(M)) \cap L^p(0,T_1; L^{2p}(M)) : \ \|w\|_{\mathcal{C}([0,T_1];H^1(M))} + \|w\|_{L^p(0,T_1;L^{2p}(M))} \leqslant C_1 L\},$$

where C_1 is also a constant depending only on b, M and a_1. One can easily deduce that this fixed point v is also lying in $\mathcal{C}^1([0,T_1]; L^2(M))$, it satisfies (2.2) and it solves (2.5). This proves the existence of local solutions for (1.5). The uniqueness can be deduced from arguments inspired by [16, Theorem 2.1] (see also [21, page 134] for similar arguments). This result gives also the lower bound for $T_*(L)$ and it completes the proof of the lemma. □

This result gives us the existence and uniqueness of variational solutions of (1.5) on $(0,T)$, provided $T < T_*(L)$. We believe that, under some suitable restriction imposed to the set $\mathcal{A}$ (see for instance [3, 4, 16]), this result can be extended to a global existence result corresponding to the condition $T_*(L) = +\infty$, $L > 0$. However, in the general setting, there is counterexamples to the global existence of solutions due to the blow up at finite time of some of them (e.g. [4, Proposition 6.4.1]). In order to preserve the generality of our results, we do not consider possible restriction of the class $\mathcal{A}$ of nonlinear terms which would allow the extension of our local well-posedness result to existence of global solutions by proving that $T_*(L) = +\infty$, $L > 0$.

By mean of suitable conditions, we can increase the regularity of the solution u of (1.5) in the following way.

Lemma 2.2. *Assume that $n = 2$ or $n = 3$ and for $L > 0$ fix $T < T_*(2L)$. Let $F \in \mathcal{A}$ and let $(f, u_0, u_1) \in \mathcal{H}$. Assume also that*

$$\|f\|_{H^{\frac{5}{2}}((0,T)\times\partial M)} + \|u_0\|_{H^{\frac{5}{2}}(M)} + \|u_1\|_{H^{\frac{3}{2}}(M)} \leqslant L.$$

Then, (1.5) admits a unique solution lying in $W^{1,\frac{p}{b-1}}(0,T; H^2(M)) \cap W^{2,\frac{p}{b-1}}(0,T; H^1(M)) \cap L^p(0,T; L^{2p}(M))$ satisfying

$$\begin{aligned} &\|u\|_{W^{1,\frac{p}{b-1}}(0,T;H^2(M))} + \|u\|_{W^{2,\frac{p}{b-1}}(0,T;H^1(M))} + \|u\|_{L^p(0,T;L^{2p}(M))} \\ &\leqslant C\left(\|f\|_{H^{\frac{11}{2}}((0,T)\times\partial M)} + \|u_0\|_{H^{\frac{11}{2}}(M)} + \|u_1\|_{H^{\frac{9}{2}}(M)}\right), \end{aligned} \tag{2.8}$$

with C depending on L, M, b, n and a_1.

Proof. We start by extending f to an element of $H^{\frac{5}{2}}((0,+\infty)\times\partial M)$ satisfying

$$\|f\|_{H^{\frac{5}{2}}((0,+\infty)\times\partial M)} \leqslant 2\|f\|_{H^{\frac{5}{2}}((0,T)\times\partial M)}.$$

According to [28, Theorem 2.3, Chapter 4], in view of the compatibility condition (1.4), there exists $G \in H^6((0,+\infty)\times M)$ satisfying

$$\begin{aligned} &G_{|(0,+\infty)\times\partial M} = f, \quad G_{|t=0} = u_0, \quad \partial_t G_{|t=0} = u_1, \ \partial_t^2 G_{|t=0} = \Delta_g u_0, \\ &\partial_t^3 G_{|t=0} = \Delta_g u_1, \quad \partial_t^4 G_{|t=0} = \Delta_g^2 u_0, \end{aligned} \tag{2.9}$$

$$\|G\|_{H^6((0,+\infty)\times M)} \leqslant C\left(\|f\|_{H^{\frac{11}{2}}((0,+\infty)\times\partial M)} + \|u_0\|_{H^{\frac{11}{2}}(M)} + \|u_1\|_{H^{\frac{9}{2}}(M)}\right). \tag{2.10}$$

Then, following Lemma 2.1, the solution $u \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H^1(M)) \cap L^p(0,T; L^{2p}(M))$ of (1.5) takes the form $u = v + G$ with $v \in \mathcal{C}^1([0,T]; L^2(M)) \cap \mathcal{C}([0,T]; H_0^1(M)) \cap L^p(0,T; L^{2p}(M))$ solving (2.5). Thus, the proof will be completed if we prove that $v \in W^{1,\frac{p}{b-1}}(0,T; H^2(M)) \cap W^{3,\frac{p}{b-1}}(0,T; L^2(M)) \cap L^p(0,T; L^{2p}(M))$ satisfies

$$\|v\|_{W^{1,\frac{p}{b-1}}(0,T;H^2(M))} + \|v\|_{W^{2,\frac{p}{b-1}}(0,T;H^1(M))} + \|v\|_{L^p(0,T;L^{2p}(M))} \leqslant C\|G\|_{H^6((0,+\infty)\times M)}. \tag{2.11}$$

For this purpose, we remark first that since $v\in \mathcal C^1([0,T];L^2(M))\cap L^p(0,T;L^{2p}(M))$, for

$$q(t,x):=\partial_uF(t,x,u(t,x)),\quad (t,x)\in[0,T]\times M,$$

we have

$$\|q\|_{L^{\frac{p}{b-1}}(0,T;L^3(M))}\leqslant a_1\left\|1+|u|^{b-1}\right\|_{L^{\frac{p}{b-1}}(0,T;L^3(M))}\leqslant C(\|u\|_{L^p(0,T;L^{3(b-1)}(M))}^{b-1}+1)$$

and using the fact that, for $n=3$, $2p=10\geqslant 3(\frac{13}{3}-1)\geqslant 3(b-1)$, and the fact that $p>b$ is arbitrary for $n=2$, we have $q\in L^{\frac{p}{b-1}}(0,T;L^3(M))$. Thus, by the Sobolev embedding theorem, we deduce that $q\partial_tv\in L^{\frac{p}{b-1}}(0,T;H^{-1}(M))$. Moreover, using the fact that by density, for a.e $(t,x)\in(0,T)\times M$, we have

$$\partial_t[F(t,x,u(t,x))]=\partial_tF(t,x,u(t,x))+\partial_uF(t,x,u(t,x))\partial_tv(t,x)+\partial_uF(t,x,u(t,x))\partial_tG(t,x)$$

and the fact that for

$$G_2(t,x):=-\partial_uF(t,x,u(t,x))\partial_tG(t,x)-\partial_tF(t,x,u(t,x))-\partial_t^3G(t,x)+\Delta_g\partial_tG(t,x),\quad (t,x)\in[0,T]\times M,$$

we have

$$\|G_2\|_{L^{\frac{p}{b}}(0,T;L^2(M))}\leqslant C(\|u\|_{L^p(0,T;L^{2p}(M))}+1),$$

we deduce that $E:(t,x)\mapsto F(t,x,u(t,x))\in W^{1,\frac{p}{b}}(0,T;H^{-1}(M))\subset\mathcal C([0,T];H^{-1}(M))$ and $v_1:=\partial_tv\in\mathcal C([0,T];L^2(M))\cap\mathcal C^1([0,T];H^{-1}(M))$. Moreover, in view of (2.9), we have

$$G_1(0,x)=-\partial_t^2G(0,x)+\Delta_gG(0,x)=0,\quad x\in M.$$

Therefore, fixing $w\in\mathcal C^\infty([0,T]\times M)\cap\mathcal C([0,T];H^1_0(M))$ satisfying $w(T,\cdot)=\partial_tw(T,\cdot)=0$, we find

$$\begin{aligned}
&\left\langle v_1,\partial_t^2w-\Delta_gw\right\rangle_{L^2(0,T;L^2(M))}\\
&=\left\langle \partial_tv,\partial_t^2w-\Delta_gw\right\rangle_{L^2(0,T;L^2(M))}\\
&=-\left\langle \partial_t^2v-\Delta_gv,\partial_tw\right\rangle_{L^{\frac{p}{b}}(0,T;L^2(M)),L^{\frac{p}{p-b}}(0,T;L^2(M))}\\
&=\left\langle E-G_1,\partial_tw\right\rangle_{L^{\frac{p}{b}}(0,T;L^2(M)),L^{\frac{p}{p-b}}(0,T;L^2(M))}\\
&=-\int_MF(0,x,u_0(x))w(0,x)dV_g(x)-\left\langle \partial_tE-\partial_tG_1,w\right\rangle_{L^{\frac{p}{b}}(0,T;H^{-1}(M)),L^{\frac{p}{p-b}}(0,T;H^1_0(M))}\\
&=-\int_MF(0,x,u_0(x))w(0,x)dV_g(x)-\left\langle qv_1,w\right\rangle_{L^{\frac{p}{b}}(0,T;H^{-1}(M)),L^{\frac{p}{p-b}}(0,T;H^1_0(M))}+\left\langle G_2,w\right\rangle_{L^{\frac{p}{b}}(0,T;L^2(M)),L^{\frac{p}{p-b}}(0,T;L^2(M))}\\
&=-\int_MF(0,x,u_0(x))w(0,x)dV_g(x)-\left\langle v_1,qw\right\rangle_{L^{\frac{p}{p-b}}(0,T;L^2(M)),L^{\frac{p}{b}}(0,T;L^2(M))}+\left\langle G_2,w\right\rangle_{L^{\frac{p}{b}}(0,T;L^2(M)),L^{\frac{p}{p-b}}(0,T;L^2(M))}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left\langle v_1,\partial_t^2w-\Delta_gw+qw\right\rangle_{L^{\frac{p}{p-b}}(0,T;L^2(M)),L^{\frac{p}{b}}(0,T;L^2(M))}\\
&=-\int_MF(0,x,u_0(x))w(0,x)dV_g(x)+\left\langle G_2,w\right\rangle_{L^{\frac{p}{b}}(0,T;L^2(M)),L^{\frac{p}{p-b}}(0,T;L^2(M))}.
\end{aligned}\tag{2.12}$$

By the Sobolev embedding theorem, for any $w\in\mathcal C([0,T];H^1(M))$, we find $qw\in L^{\frac{p}{b-1}}(0,T;L^2(M))$ and, by density, the identity (2.12) holds true for any $w\in X$, where X denotes the space of all elements $w\in\mathcal C^1([0,T];L^2(M))\cap\mathcal C([0,T];H^1_0(M))\cap W^{2,\frac{p}{b}}(0,T;H^{-1}(M))$ satisfying

$$\partial_t^2w-\Delta_gw+qw\in L^{\frac{p}{b}}(0,T;L^2(M)),\quad w(T,\cdot):=\partial_tw(T,\cdot)=0.$$

Moreover, according to [27, Theorem 9.1, Chapter 3] combined with [10, Proposition 1][1], v_1 is the unique element of $L^{\frac{p}{p-b}}(0,T;L^2(M))$ satisfying (2.12) for any $w\in X$. On the other hand, by the Sobolev embedding theorem we have $u_0\in\mathcal C^1(M)$ and it follows that $x\mapsto F(0,x,u_0(x))\in\mathcal C^1(M)$. Therefore, using the fact

[1]The result [10, Proposition 1] is stated for a bounded subdomain of $\mathbb R^n$ but it can be extended without any difficulty to a compact Riemannian manifold of dimension n.

that $n \leqslant 3$ and applying [10, Proposition 1], we deduce that there exists a unique $z \in \mathcal{C}^1([0,T];L^2(M)) \cap \mathcal{C}([0,T];H^1_0(M))$ solving the linear problem

$$\begin{cases} \partial_t^2 z - \Delta_g z + q(t,x)z = G_2(t,x), & \text{in } (0,T)\times M,\\ z=0, & \text{on } (0,T)\times\partial M,\\ z(0,\cdot)=0, \quad \partial_t z(0,\cdot) = -F(0,x,u_0(x)) & \text{in } M.\end{cases}$$

Moreover, integrating by parts, for any $w \in X$, we find

$$\begin{aligned}&\left\langle z, \partial_t^2 w - \Delta_g w + qw\right\rangle_{L^{\frac{p}{p-b}}(0,T;L^2(M)),L^{\frac{p}{b}}(0,T;L^2(M))}\\ &= -\int_M F(0,x,u_0(x))w(0,x)dV_g(x) + \langle G_2, w\rangle_{L^{\frac{p}{b}}(0,T;L^2(M)),L^{\frac{p}{p-b}}(0,T;L^2(M))}\end{aligned}$$

and by the uniqueness of the elements of $L^{\frac{p}{p-b}}(0,T;L^2(M))$ satisfying (2.10), we deduce that $v_1 = z$. It follows that $v_1 \in \mathcal{C}^1([0,T];L^2(M)) \cap \mathcal{C}([0,T];H^1_0(M))$ and v_1 solves

$$\begin{cases} \partial_t^2 v_1 - \Delta_g v_1 + q(t,x)v_1 = G_2(t,x), & \text{in } (0,T)\times M,\\ v_1=0, & \text{on } (0,T)\times\partial M,\\ v_1(0,\cdot)=0, \quad \partial_t v_1(0,\cdot) = -F(0,x,u_0(x)) & \text{in } M.\end{cases} \tag{2.13}$$

In the same way, we can prove that $v_2 = \partial_t v_1 = \partial_t^2 v$ is lying in $\mathcal{C}^1([0,T];L^2(M)) \cap \mathcal{C}([0,T];H^1_0(M))$ and it solves the linear problem

$$\begin{cases} \partial_t^2 v_2 - \Delta_g v_2 + q(t,x)v_2 = G_3(t,x), & \text{in } (0,T)\times M,\\ v_2=0, & \text{on } (0,T)\times\partial M,\\ v_2(0,\cdot)=-F(0,x,u_0(x)), \quad \partial_t v_2(0,\cdot) = -\partial_u F(0,x,u_0(x))u_1(x) - \partial_t F(0,x,u_0(x)) & \text{in } M,\end{cases} \tag{2.14}$$

with

$$\begin{aligned}G_3(t,x) :=& -\partial_u F(t,x,u(t,x))\partial_t^2 G(t,x) - 2\partial_u\partial_t F(t,x,u(t,x))[\partial_t G(t,x) + v_1(t,x)] - \partial_t^2 F(t,x,u(t,x))\\ &- \partial_t^4 G(t,x) + \Delta_g \partial_t^2 G(t,x) - \partial_u^2 F(t,x,u(t,x))[v_1(t,x) + \partial_t G(t,x)]^2, \quad (t,x)\in[0,T]\times M.\end{aligned}$$

Here we use the fact that, by the Sobolev embedding theorem, $G \in \mathcal{C}^3([0,T];H^2(M)) \subset \mathcal{C}^3([0,T];L^\infty(M))$ and $G_3 \in W^{1,\frac{p}{b}}(0,T;L^2(M))$. Finally, using similar arguments, we can prove that $v_3 = \partial_t v_2 = \partial_t^2 v_1 \in \mathcal{C}^1([0,T];L^2(M)) \cap \mathcal{C}([0,T];H^1_0(M))$ solves the linear problem

$$\begin{cases} \partial_t^2 v_3 - \Delta_g v_3 + q(t,x)v_3 = G_4(t,x), & \text{in } (0,T)\times M,\\ v_3=0, & \text{on } (0,T)\times\partial M,\\ v_3(0,x)=\partial_t v_2(0,x), \quad \partial_t v_3(0,x) = -\Delta_g[F(0,x,u_0(x)] + q(0,x)F(0,x,u_0(x) + G_3(0,x), & x\in M,\end{cases} \tag{2.15}$$

with

$$G_4(t,x) := \partial_t G_3(t,x) - \partial_u^2 F(t,x,u(t,x))(v_1(t,x) + \partial_t G(t,x))v_2(t,x).$$

This proves that, for a.e. $t \in (0,T)$, $v_1(t,\cdot)$ solves the boundary value problem

$$\begin{cases} -\Delta_g v_1(t,\cdot) = -v_3(t,\cdot) - q(t,\cdot)v_1(t,\cdot) + G_2(t,\cdot), & \text{in } M,\\ v_1 = 0, & \text{on } (0,T)\times\partial M\end{cases}$$

and using the fact that $-v_3 + qv_1 + G_2 \in L^{\frac{p}{b-1}}(0,T;L^2(M))$, we deduce that $v_1 \in L^{\frac{p}{b-1}}(0,T;H^2(M))$. It follows that $v \in W^{1,\frac{p}{b-1}}(0,T;H^2(M)) \cap W^{2,\frac{p}{b-1}}(0,T;H^1(M)) \cap L^p(0,T;L^{2p}(M))$ and we deduce the required regularity result as well as (2.11). $\square$

Now let us consider the following linear initial boundary value problem

$$\begin{cases} \partial_t^2 w - \Delta_g w + qw = 0, & \text{in } (0,T)\times M,\\ w=h, & \text{on } (0,T)\times\partial M,\\ w(0,\cdot)=h_0, \quad \partial_t w(0,\cdot) = h_1 & \text{in } M,\end{cases} \tag{2.16}$$

to which we associate the linear operator

$$\mathcal{D}_{q,\gamma_1}:\mathcal{H}\ni H=(h,h_0,h_1)\longmapsto(\partial_\nu w_{|(0,T)\times\gamma_1},w(T,\cdot)_{|M})\in L^2((0,T)\times\gamma_1)\times H^1(M),$$

and for w solutions of (2.16), with $h_0=h_1=0$, the linear operator

$$\Lambda_{q,\gamma_1}:\mathcal{H}_*\ni h\longmapsto\partial_\nu w_{|(0,T)\times\gamma_1}\in L^2((0,T)\times\gamma_1).$$

From now on, for any $H=(h,h_0,h_1)\in\mathcal{H}$, we denote by $\|H\|_{\mathcal{H}}$ the norm defined by

$$\|H\|_{\mathcal{H}}^2:=\|h\|^2_{H^{\frac{11}{2}}((0,T)\times\partial M)}+\|u_0\|^2_{H^{\frac{11}{2}}(M)}+\|u_1\|^2_{H^{\frac{9}{2}}(M)}$$

We proceed now to the following linearization of the maps $\mathcal{B}_{F,\gamma_1}$ and $\mathcal{N}_{F,\gamma_1}$ introduced in Section 1.1.

Theorem 2.1. *Assume that $n=2$ or $n=3$ and let $F\in\mathcal{A}$. Then, the maps $\mathcal{B}_{F,\gamma_1}$ and $\mathcal{N}_{F,\gamma_1}$ admit a continuous Fréchet derivative $\mathcal{B}'_{F,\gamma_1}$ and $\mathcal{N}'_{F,\gamma_1}$ on*

$$\{G\in\mathcal{H}:\ \|G\|_{H^{\frac52}((0,T)\times\partial M)\times H^{\frac52}(M)\times H^{\frac32}(M)}\leqslant L\},$$

$$\{h\in\mathcal{H}_*:\ \|h\|_{H^{\frac52}((0,T)\times\partial M)}\leqslant L\}.$$

Moreover, fixing

$$G\in\{K\in\mathcal{H}:\ \|K\|_{H^{\frac52}((0,T)\times\partial M)\times H^{\frac52}(M)\times H^{\frac32}(M)}\leqslant L\},$$

$$f\in\{h\in\mathcal{H}_*:\ \|h\|_{H^{\frac52}((0,T)\times\partial M)}\leqslant L\},$$

$q_{F,G}(t,x):=\partial_uF(t,x,u_{F,G}(t,x))$ and $q_{F,f}(t,x):=\partial_uF(t,x,u_{F,f}(t,x))$, we find

$$\mathcal{B}'_{F,\gamma_1}(G)H=\mathcal{D}_{q_{F,G},\gamma_1}H,\quad \mathcal{N}'_{F,\gamma_1}(f)h=\Lambda_{q_{F,f},\gamma_1}h,\quad H\in\mathcal{H},\ h\in\mathcal{H}_*.\tag{2.17}$$

Proof. Since the proof for $\mathcal{B}_{F,\gamma_1}$ and $\mathcal{N}_{F,\gamma_1}$ are similar, we will only prove this result for B_{F,γ_1}. Moreover, without lost of generality, we assume that $\gamma_1=\partial M$. For this purpose, we fix $H:=(h,h_0,h_1)\in\mathcal{H}$ satisfying $\|H\|_{H^{\frac52}((0,T)\times\partial M)\times H^{\frac52}(M)\times H^{\frac32}(M)}+\|H\|_{\mathcal{H}}\leqslant\varepsilon$ and we consider $v=u_{F,G+H}-u_{F,G}-w_{F,G,H}$, with w solving (2.16) with $q=q_{F,G}$. By taylor expansion in u of F, we find

$$\begin{aligned}&F(t,x,u_{F,G+H}(t,x))\\&=F(t,x,u_{F,G}(t,x))+\partial_uF(t,x,u_{F,G}(t,x))((u_{F,G+H}(t,x)-u_{F,G}(t,x))\\&\quad+\left(\int_0^1(1-s)\partial_u^2F(t,x,u_{F,G}(t,x)+s(u_{F,G+H}(t,x)-u_{F,G}(t,x)))ds\right)((u_{F,G+H}(t,x)-u_{F,G}(t,x))^2.\end{aligned}$$

Then, v solves the linear problem

$$\left\{\begin{array}{ll}\partial_t^2v-\Delta_gv+q_{F,G}v=R_{F,G,H}, & \text{in } (0,T)\times M,\\ v=0, & \text{on } (0,T)\times\partial M,\\ v(0,\cdot)=0,\quad \partial_tv(0,\cdot)=0 & \text{in } M,\end{array}\right.\tag{2.18}$$

with

$$\begin{aligned}&R_{F,G,H}(t,x)\\&:=\left(\int_0^1(1-s)\partial_u^2F(t,x,u_{F,G}(t,x)+s(u_{F,G+H}(t,x)-u_{F,G}(t,x)))ds\right)((u_{F,G+H}(t,x)-u_{F,G}(t,x))^2.\end{aligned}$$

By the Sobolev embedding theorem, the space $W^{1,\frac{p}{b-1}}(0,T;H^2(M))$ embedded continuously into $\mathcal{C}([0,T]\times M)$ and we deduce that

$$\|R_{F,G,H}\|_{L^2(0,T;L^2(M))}\leqslant C\,\|R_{F,G,H}\|_{L^\infty((0,T)\times M)}\leqslant C\,\|u_{F,G+H}-u_{F,G}\|^2_{L^\infty((0,T)\times M)}.$$

Combining this with [1, Theorem A.2], [10, Proposition 1], (2.2) and applying the Sobolev embedding theorem, we obtain

$$\begin{aligned}\|\partial_\nu v\|_{L^2((0,T)\times\partial M)}+\|v\|_{\mathcal{C}([0,T];H^1(M))} &\leqslant C\left(\|R_{F,G,H}\|_{L^1(0,T;L^2(M))}+\|q_{F,G}v\|_{L^1(0,T;L^2(M))}\right)\\ &\leqslant C\left(\|R_{F,G,H}\|_{L^2(0,T;L^2(M))}+\|q_{F,G}\|_{L^1(0,T;L^3(M))}\|v\|_{L^\infty(0,T;H^1(M))}\right)\\ &\leqslant C\|R_{F,G,H}\|_{L^2(0,T;L^2(M))}\\ &\leqslant C\|u_{F,G+H}-u_{F,G}\|^2_{L^\infty((0,T)\times M)}.\end{aligned} \tag{2.19}$$

On the other hand, $y := u_{F,G+H} - u_{F,G}$ solves the problem

$$\begin{cases} \partial_t^2 y-\Delta_g y+V_{F,G,H}y=0, & \text{in } (0,T)\times M,\\ y=h, & \text{on } (0,T)\times\partial M,\\ y(0,\cdot)=h_0,\quad \partial_t y(0,\cdot)=h_1 & \text{in } M,\end{cases} \tag{2.20}$$

with

$$V_{F,G,H}(t,x):=\int_0^1 \partial_u F(t,x,u_{F,G}(t,x)+s(u_{F,G+H}(t,x)-u_{F,G}(t,x)))ds.$$

Using the fact that $u\in W^{1,\frac{b}{b-1}}(0,T;H^2(M))\subset W^{1,\frac{b}{b-1}}(0,T;L^\infty(M))$, we deduce that $V_{F,G,H}\in W^{1,\frac{b}{b-1}}(0,T;L^\infty(M))$. Thus, $y_1=\partial_t y$ solves

$$\begin{cases} \partial_t^2 y_1-\Delta_g y_1+V_{F,G,H}y_1=\partial_t V_{F,G,H}y, & \text{in } (0,T)\times M,\\ y_1=\partial_t h, & \text{on } (0,T)\times\partial M,\\ y_1(0,\cdot)=h_1,\quad \partial_t y_1(0,\cdot)=\Delta_g h_0-V_{F,G,H}(0,\cdot)h_0 & \text{in } M,\end{cases}$$

where one can check that

$$V_{F,G,H}(0,x)=\int_0^1 \partial_u F(0,x,u_0(x)+sh_0(x))ds,\quad x\in\Omega.$$

Combining this with the fact that $G,H=(h,h_0,h_1)\in\mathcal{H}$, we deduce from [10, Proposition 1] that this problem admits a unique solution $y_1\in\mathcal{C}([0,T];H^1(M))\cap\mathcal{C}^1([0,T];L^2(M))$, satisfying

$$\begin{aligned}\|y_1\|_{\mathcal{C}^1([0,T];L^2(M))} &\leqslant C(\|H\|_{\mathcal{H}}+\|V_{F,G,H}\|_{L^\infty((0,T)\times M)}\|y\|_{\mathcal{C}([0,T];L^2(M))})\\ &\leqslant C\|H\|_{\mathcal{H}}.\end{aligned}$$

Note that here we use the fact that for $\|H\|_{\mathcal{H}}\leqslant\varepsilon$, $\|V_{F,G,H}\|_{L^\infty((0,T)\times M)}$ is upper bounded by a constant depending only on ε, G, T and M. Thus, we have $y\in\mathcal{C}^2([0,T];L^2(M))$ and

$$\|\Delta y\|_{\mathcal{C}([0,T];L^2(M))}\leqslant\left\|\partial_t^2 y\right\|_{\mathcal{C}([0,T];L^2(M))}+\|V_{F,G,H}\|_{L^\infty((0,T)\times M)}\|y\|_{\mathcal{C}([0,T];L^2(M))}\leqslant C\|H\|_{\mathcal{H}}.$$

Combining this with the fact that for all $t\in[0,T]$, $y(t,\cdot)$ solves the boundary value problem

$$\begin{cases} -\Delta_g y(t,\cdot)=\partial_t^2 y(t,\cdot)-V_{F,G,H}y(t,\cdot), & \text{in } M,\\ y(t,\cdot)=h(t,\cdot), & \text{on } \partial M,\end{cases}$$

we deduce that $y\in\mathcal{C}([0,T];H^2(M))$ satisfies the estimate

$$\|y\|_{\mathcal{C}([0,T];H^2(M))}\leqslant C\|H\|_{\mathcal{H}}.$$

Then, by the Sobolev embedding theorem, we get

$$\|u_{F,G+H}-u_{F,G}\|_{L^\infty((0,T)\times M)}=\|y\|_{L^\infty((0,T)\times M)}\leqslant C\|H\|_{\mathcal{H}}$$

and, from (2.19), we get

$$\begin{aligned}&\|\partial_\nu u_{F,G+H}-\partial_\nu u_{F,G}-\partial_\nu w_{F,G,H}\|_{L^2((0,T)\times\partial M)}+\|u_{F,G+H}-u_{F,G}-w_{F,G,H}\|_{\mathcal{C}([0,T];H^1(M))}\\ &\leqslant C\|H\|^2_{\mathcal{H}}.\end{aligned}$$

This proves that $\mathcal{B}_{F,\gamma_1}$ is Fréchet differentiable at G and

$$\mathcal{B}'_{F,\gamma_1}(G)H = (\partial_\nu w_{F,G,H}{}_{|(0,T)\times\gamma}, w_{F,G,H}(T,\cdot)_{|M}) = \mathcal{D}_{q_{F,G},\gamma_1}H.$$

Now let us prove the continuity of the map $G \mapsto \mathcal{B}'_{F,\gamma_1}(G) = \mathcal{D}_{q_{F,G},\gamma_1}$. For this purpose, we fix $z := w_{F,G+K,H} - w_{F,G,H}$, with $K = (k, k_0, k_1) \in \mathcal{H}$,

$$\|H\|_{H^{\frac{5}{2}}((0,T)\times\partial M)\times H^{\frac{5}{2}}(M)\times H^{\frac{3}{2}}(M)} + \|K\|_{H^{\frac{5}{2}}((0,T)\times\partial M)\times H^{\frac{5}{2}}(M)\times H^{\frac{3}{2}}(M)} + \|H\|_{\mathcal{H}} + \|K\|_{\mathcal{H}} \leqslant \varepsilon,$$

and we remark that z solves the problem

$$\begin{cases} \partial_t^2 z - \Delta_g z + q_{F,G} z = S_{F,G,H}, & \text{in } (0,T)\times M,\\ z = k, & \text{on } (0,T)\times\partial M,\\ z(0,\cdot) = k_0, \quad \partial_t z(0,\cdot) = k_1 & \text{in } M.\end{cases} \tag{2.21}$$

with

$$S_{F,G,H} = -(q_{F,G+K} - q_{F,G}) w_{F,G+K,H}.$$

On the other hand, we can prove that $\|w_{F,G+K,H}\|_{\mathcal{C}([0,T];H^2(M))} \leqslant C$ with C depending on G, ε, M, T. Therefore, we find

$$\|S_{F,G,H}\|_{L^2((0,T)\times M)} \leqslant C \|q_{F,G+K} - q_{F,G}\|_{L^\infty((0,T)\times M)}. \tag{2.22}$$

Using the Taylor expansion of $\partial_u F$ in u, we find

$$q_{F,G+K}(t,x) - q_{F,G}(t,x) = \left(\int_0^1 \partial_u^2 F(t,x,u_{F,G} + s(u_{F,G+K} - u_{F,G}))ds\right)(u_{F,G+K} - u_{F,G})$$

and repeating the above arguments, we obtain

$$\|q_{F,G+K} - q_{F,G}\|_{L^\infty((0,T)\times M)} \leqslant C\|K\|_{\mathcal{H}}.$$

Combining this with (2.22) and the estimate

$$\|\partial_\nu z\|_{L^2((0,T)\times\partial M)} + \|z\|_{\mathcal{C}([0,T];H^1(M))} \leqslant C\|S_{F,G,H}\|_{L^2((0,T)\times M)},$$

we deduce the continuity of $G \mapsto \mathcal{B}'_{F,\gamma_1}(G) = \mathcal{D}_{q_{F,G},\gamma_1}$. This completes the proof of the theorem. ☐

3. Recovery of a time-dependent coefficient on parts of the boundary

For $T > 0$ and $q \in L^\infty((0,T)\times M)$ we consider the initial boundary value problem

$$\begin{cases} \partial_t^2 u - \Delta_g u + qu = 0, & \text{in } (0,T)\times M,\\ u = f, & \text{on } (0,T)\times\partial M,\\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = 0 & \text{in } M,\end{cases} \tag{3.1}$$

with non-homogeneous Dirichlet data f. According to [26], for $f \in H^1((0,T)\times\partial M)$ satisfying $f_{|t=0} = 0$ this problem admits a unique solution $u \in \mathcal{C}([0,T];H^1(M)) \cap \mathcal{C}^1([0,T];L^2(M))$ satisfying $\partial_\nu u \in L^2((0,T)\times\partial M)$. Thus, fixing γ an open portion of ∂M, we can define the partial hyperbolic Dirichlet-to-Neumann map in the following way

$$\Lambda_{q,\gamma,*} : \mathcal{H}_{*,\gamma} \ni f \mapsto \partial_\nu u_{|(0,T)\times\gamma}, \quad \text{supp}(f) \subset (0,T]\times\gamma,$$

with $\mathcal{H}_{*,\gamma} := \{f \in \mathcal{H}_* : \text{ supp}(f) \subset (0,T]\times\gamma\}$ and with u solving problem (3.1). In this section, we consider the problem of recovering q restricted to $(0,T)\times\gamma$ from the knowledge of $\Lambda_{q,\gamma,*}$.

Theorem 3.1. *Let (M,g) be a smooth connected and compact Riemannian manifold of dimension $n \geqslant 2$ and let q_1, $q_2 \in \mathcal{C}([0,T]\times M) \cap H^2((0,T)\times M)$. Then $\Lambda_{q_1,\gamma,*} = \Lambda_{q_2,\gamma,*}$ implies that $q_1 = q_2$ on $(0,T)\times\gamma$.*

We mention that [31] established results similar to Theorem 3.1 for time-independent coefficients and [32] treated the case of time-dependent coefficients from some measurements associated with some general hyperbolic equation on a Lorentzian manifold. Both of these results require strong smoothness assumptions on the coefficients under consideration. In Theorem 3.1, we extend such results to time-dependent potentials q lying in $\mathcal{C}([0,T]\times M)\cap H^2((0,T)\times M)$. To prove this result, like in [31, 32], we consider specific solutions of the problem (3.1) also called geometric optics. However, since we restrict the regularity of the coefficients under consideration, in contrast to [31, 32], we will use a new global construction involving some approximation of the potential q. We mention that the recovery of coefficients lying in $\mathcal{C}([0,T]\times M)\cap H^2((0,T)\times M)$ will be a crucial point in the proof of Theorem 1.1.

3.1. **Geometric optics solutions.** Let $t_0\in(0,T)$ and consider $\delta>0$ a constant that will be fixed later. The goal of this subsection is to construct energy class solutions u_j of the equation

$$\begin{cases} \partial_t^2 u_j-\Delta_g u_j+q_j u_j=0, & \text{in } (0,T)\times M,\\ u_j(0,\cdot)=0,\quad \partial_t u_j(0,\cdot)=0 & \text{in } M,\end{cases} \tag{3.2}$$

whose restriction to $[0,t_0+\delta]\times M$ takes the form

$$u_j(t,x)=e^{i\rho(t-\psi(x))}\left(a_0(t,x)+\frac{a_{j,1}(t,x)}{\rho}+\frac{a_{j,2,\rho}(t,x)}{\rho^2}\right)+R_{j,\rho}(t,x),\quad \rho>1, \tag{3.3}$$

with the remainder term $R_{j,\rho}\in\mathcal{C}([0,t_0+\delta];H^1(M))\cap\mathcal{C}^1([0,t_0+\delta];L^2(M))$ satisfying

$$\partial_t^2R_{j,\rho}-\Delta_gR_{j,\rho}\in L^2((0,T)\times M),$$

$$R_{j,\rho}=0 \text{ on } (0,t_0+\delta)\times\partial M,\quad R_{j,\rho}(0,\cdot)=\partial_tR_{j,\rho}(0,\cdot)=0 \text{ on } M, \tag{3.4}$$

$$\lim_{\rho\to+\infty}\rho\left\|\partial_\nu R_{j,\rho}\right\|_{L^2((0,t_0+\delta)\times\partial M)}=0. \tag{3.5}$$

More precisely, we fix $x_0\in\gamma$ and we want to construct solutions of the form (3.3) on $[0,t_0+\delta]\times M$ that will allow us to recover $q(t_0,x_0)=q_1(t_0,x_0)-q_2(t_0,x_0)$.

In order to get the decay (3.5), we choose ψ, a_0, $a_{j,1}$ and $a_{j,2,\rho}$, $j=1,2$, so that they satisfy the following eikonal and transport equations

$$\sum_{i,j=1}^d g^{ij}(x)\partial_{x_i}\psi\partial_{x_j}\psi=|\nabla_g\psi|_g^2=1, \tag{3.6}$$

$$2i\partial_ta_0+2i\sum_{i,j=1}^d g^{ij}(x)\partial_{x_i}\psi\partial_{x_j}a_0+i(\Delta_g\psi)a_0=0, \tag{3.7}$$

$$2i\partial_ta_{k,1}+2i\sum_{i,j=1}^d g^{ij}(x)\partial_{x_i}\psi\partial_{x_j}a_{k,1}+i(\Delta_g\psi)a_{k,1}=-(\partial_t^2-\Delta_g+q_k)a_0,\quad k=1,2, \tag{3.8}$$

$$2i\partial_ta_{k,2,\rho}+2i\sum_{i,j=1}^d g^{ij}(x)\partial_{x_i}\psi\partial_{x_j}a_{k,2,\rho}+i(\Delta_g\psi)a_{k,2,\rho}=-(\partial_t^2-\Delta_g+q_k)a_{k,1,\rho},\quad k=1,2, \tag{3.9}$$

on some neighborhood of $[0,t_0+\delta]\times\partial M$. Here $a_{k,1,\rho}$ is a smooth approximation of $a_{k,1}$ that we will precise later.

Using boundary coordinates we will solve the equations (3.6)-(3.9). For any $y\in M$ and $\theta\in S_yM$, we denote by $\gamma_{y,\theta}$ the maximal geodesic starting at y in the direction θ. Then, for some $\varepsilon>0$ small enough, we define the map $\exp_{\partial M}:\partial M\times[0,\varepsilon)\longrightarrow M$ given by

$$\exp_{\partial M}(x',x_n):=\gamma_{x',-\nu(x')}(x_n),\quad (x',x_n)\in\partial M\times[0,\varepsilon).$$

For any $s>0$, we define the submanifold $M_s:=\{x\in M:\ \mathrm{dist}(x,\partial M)<s\}$. It is well known (e.g. [17, Section 2.1.16]) that, for ε sufficiently small, $\exp_{\partial M}$ is a diffeomorphism from $\partial M\times[0,\varepsilon)$ to M_ε with

$$\exp_{\partial M}^{-1}(x):=(x',x_n),\quad x_n=\mathrm{dist}(x,\partial M),\quad x\in M_\varepsilon.$$

Here dist denotes the Riemanian distance function on (M,g). Thus, we can consider the boundary normal coordinates (x',x_n) on M_ε given by $x=\exp_{\partial M}(x',x_n)$ where $x_n\geqslant 0$ and $x'\in\partial M$. It is well known (see e.g. [17, Section 2.1.18]) that in these coordinates the metric takes the form $g(x',x_n)=g_0(x',x_n)+dx_n^2$ with $g_0(x',x_n)$ a metric on ∂M that depends smoothly on x_n. We choose

$$\psi(x)=\mathrm{dist}(x,\partial M),\quad x\in M_\varepsilon. \tag{3.10}$$

As ψ is given by x_n in the boundary normal coordinates, one can easily check that ψ solves (3.6) in M_ε.

Let us now turn to the transport equation. We write $a(t,x',x_n)=a(t,\exp_{\partial M}(x',x_n))$ and use this notation to indicate the representation in the boundary normal coordinates also for other functions. Moreover, we define $\beta(x',x_n)=\mathrm{det}g_0(x',x_n)$, and transform (3.7) into

$$\partial_t a_0+\partial_{x_n}a_0+\left(\frac{\partial_{x_n}\beta}{4\beta}\right)a_0=0.$$

From now on we fix $\delta\in\left(0,\frac{\min(\varepsilon,t_0,T-t_0)}{16}\right)$. Then, we consider $\chi\in\mathcal{C}_0^\infty((-2\delta,2\delta))$ such that $\chi=1$ on $[-\delta,\delta]$, $\chi_1\in\mathcal{C}_0^\infty((-3\delta,3\delta))$ such that $\chi_1=1$ on $[-2\delta,2\delta]$, $\varphi\in\mathcal{C}_0^\infty(\gamma)$ such that $\varphi=1$ on a neighborhood of x_0 and $\varphi_1\in\mathcal{C}_0^\infty(\gamma)$ such that $\varphi_1=1$ on a neighborhood of $\mathrm{supp}(\varphi)$. We choose

$$a_0(t,x',x_n):=\chi((t-t_0)-x_n)\varphi(x')\beta(x',x_n)^{-1/4}. \tag{3.11}$$

Using the fact that

$$a_0(t,x',x_n)=0,\quad t\in[0,t_0+\delta),\quad x_n\in[3\delta,+\infty),$$

we can extend a_0 by zero to a function defined on $[0,t_0+\delta]\times M$ solving (3.7) on $(0,t_0+\delta)\times M$. With this choice of a_0, (3.8) is transformed into

$$\partial_t a_{j,1}+\partial_{x_n}a_{j,1}+\left(\frac{\partial_{x_n}\beta}{4\beta}\right)a_{j,1}=\frac{i}{2}[(\partial_t^2-\Delta_g)a_0(t,x',x_n)+q_ja_0(t,x',x_n)].$$

From now on, we use the notation

$$\widetilde{f}(s_1,s_2,x'):f\left(\frac{s_1+s_2}{2}+t_0,x',\frac{s_1-s_2}{2}\right),\quad s_1,s_2\in[-3\delta,3\delta],\ x'\in\partial M.$$

We choose

$$a_{j,1}(t,x',x_n):=\frac{i}{2}\chi_1((t-t_0)-x_n)\varphi_1(x')\beta(x',x_n)^{-\frac14}\left(a_{1*}(t,x',x_n)+a_{j,1,*}(t,x',x_n)\right), \tag{3.12}$$

on $(0,t_0+\delta)\times M_{2\delta}$, where

$$\widetilde{a_{j,1,*}}(s_1,s_2,x'):=\chi(s_2)\varphi(x')\frac12\left(\int_{s_2}^{s_1}\beta\left(x',\frac{\tau-s_2}{2}\right)^{\frac14}\widetilde{q_j}(\tau,s_2,x')d\tau\right),\quad s_1,s_2\in[-2\delta,2\delta],\ x'\in\partial M,$$

and a_{1*} given by

$$\widetilde{a_{1*}}(s_1,s_2,x')=\frac12\left(\int_{s_2}^{s_1}\beta\left(x',\frac{\tau-s_2}{2}\right)^{\frac14}\widetilde{d_1}(\tau,s_2,x')d\tau\right),\quad s_1,s_2\in[-3\delta,3\delta],\ x'\in\partial M$$

with $d_1=(\partial_t^2-\Delta_g)a_0$. It is clear that

$$a_{j,1,*}(t,x',0)=\widetilde{a_{j,1,*}}(t-t_0,t-t_0,x')=0.$$

Thus, one can check that

$$a_{1,1}(t,x)=a_{2,1}(t,x),\quad (t,x)\in(0,T)\times\partial M. \tag{3.13}$$

For the construction of $a_{j,2,\rho}$, we need first to define the expression $a_{j,1,\rho}$ which is an approximation of $a_{j,1}$. For this purpose, we consider an approximation of q_j given by the following lemma.

Lemma 3.1. *There exists $q_{j,\rho} \in \mathcal{C}^\infty([0,T]\times M)$ such that*

$$\lim_{\rho\to+\infty} \|q_{j,\rho} - q_j\|_{H^2(M)} = 0, \tag{3.14}$$

$$\|q_{j,\rho}\|_{H^\ell(\mathbb{R}\times M_1)} \leqslant C_\ell \rho^{\frac{\ell-2}{4}},\ \ \ell \geqslant 2. \tag{3.15}$$

with C_ℓ independent of ρ.

Proof. We consider first (M_j, g), $j = 1, 2$, two compact an smooth connected manifolds such that M is contained into $\mathrm{Int}(M_1)$, M_1 is contained into $\mathrm{Int}(M_2)$. Then, we fix $q_{j*} \in H^2(\mathbb{R}\times M_1)$ supported on $(-1, T+1)\times Int(M_1)$, which coincides with q_j on $(0,T)\times M$ such that

$$\|q_{j*}\|_{H^2(\mathbb{R}\times M_1)} \leqslant C\, \|q_j\|_{H^2((0,T)\times M)}\,,$$

with $C>0$ depending only on M_1, T. We fix the following local coordinates in M_2:

$$(\varphi_1, U_1), \dots, (\varphi_m, U_m)$$

such that

$$M_1 \subset \bigcup_{k=1}^n U_k \subset \mathrm{Int}(M_2).$$

We fix also $\psi_k \in \mathcal{C}_0^\infty(U_k)$, $k = 1,\dots,m$, such that

$$\sum_{k=1}^n \psi_k(x) = 1, \quad x \in M_1$$

and $\psi_{k,\sharp} \in \mathcal{C}_0^\infty(U_k)$, $k = 1,\dots,m$, satisfying $\psi_{k,\sharp} = 1$ on $\mathrm{supp}(\psi_k)$. Then, we set $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^{1+n})$ such that $\mathrm{supp}(\zeta) \subset \{(t,x):\ |(t,x)| \leqslant 1\}$, $\zeta \geqslant 0$ and

$$\int_{\mathbb{R}^{1+n}} \zeta(t,x)dxdt = 1.$$

We consider also $\zeta_\rho(t,x) = \rho^{\frac{n+1}{4}}\zeta(\rho^{\frac14}t, \rho^{\frac14}x)$ and, for $j = 1,2$ and $k = 1,\dots,m$, we define

$$\begin{aligned} q_{j,k,\rho}(t,y) &= \zeta_\rho * ((\varphi_k^{-1})^*\psi_{k,\sharp}q_{j*})(t,y) \\ &= \int_{\mathbb{R}^{1+n}} \zeta_\rho(t-s, y-z)\psi_{k,\sharp}(\varphi_k^{-1}(z))\tilde{q}_j(s,(\varphi_k^{-1}(z))dsdz, \quad j=1,2,\ (t,y)\in\mathbb{R}^{1+n}, \end{aligned}$$

and we consider

$$q_{j,\rho}(t,x) = \sum_{k=1}^m q_{j,k,\rho}(t,\varphi_k(x))\psi_k(x), \quad (t,x)\in\mathbb{R}\times M_1,\ j=1,2.$$

Note that

$$\begin{aligned} \|q_{j,\rho} - q_j\|_{L^2((0,T)\times M)} &= \left\| \sum_{k=1}^m (\varphi_k^* q_{j,k,\rho} - q_j\psi_{k,\sharp})\psi_k \right\|_{L^2((0,T)\times M)} \\ &\leqslant \sum_{k=1}^m \|\varphi_k^* q_{j,k,\rho} - q_j\psi_{k,\sharp}\|_{L^2((0,T)\times U_k)} \\ &\leqslant \sum_{k=1}^m \left\|q_{j,k,\rho} - (\varphi_k^{-1})^* q_j\psi_{k,\sharp}\right\|_{L^2((0,T)\times\varphi_k(U_k))} \\ &\leqslant \sum_{k=1}^m \left\|q_{j,k,\rho} - (\varphi_k^{-1})^* q_{j*}\psi_{k,\sharp}\right\|_{L^2(\mathbb{R}^{1+n})}. \end{aligned}$$

Combining this with the fact that

$$\limsup_{\rho\to+\infty}\left\|q_{j,k,\rho}-(\varphi_k^{-1})^*q_{j*}\psi_{k,\sharp}\right\|_{L^2(\mathbb R^{1+n})}=\limsup_{\rho\to+\infty}\left\|\zeta_\rho*((\varphi_k^{-1})^*\psi_{k,\sharp}q_{j*})-(\varphi_k^{-1})^*q_{j*}\psi_{k,\sharp}\right\|_{L^2(\mathbb R^{1+n})}=0$$

we deduce that

$$\lim_{\rho\to+\infty}\|q_{j,\rho}-q_j\|_{L^2((0,T)\times M)}=0.$$

In the same way, using the fact that $q_{j*}\in H^2(\mathbb R\times M_1)$, we deduce (3.14)-(3.15). □

Using this result we define $a_{j,1,\rho}$ as follows

$$a_{j,1,\rho}(t,x',x_n):=\frac{i}{2}\chi_1((t-t_0)-x_n)\varphi_1(x')\beta(x',x_n)^{-\frac14}\left(a_{1*}(t,x',x_n)+a_{j,1,*,\rho}(t,x',x_n)\right),$$

on $(0,t_0+\delta)\times M_{2\delta}$, where

$$\widetilde{a_{j,1,*,\rho}}(s_1,s_2,x'):=\chi(s_2)\varphi(x')\frac12\left(\int_{s_2}^{s_1}\beta\left(x',\frac{\tau-s_2}{2}\right)^{\frac14}\widetilde{q_{j,\rho}}(\tau,s_2,x')d\tau\right),\quad s_1,s_2\in[-2\delta,2\delta],\ x'\in\partial M.$$

Then according to (3.14)-(3.15) and the expression (3.12) of $a_{j,1}$, we have

$$\lim_{\rho\to+\infty}\|a_{j,1,\rho}-a_{j,1}\|_{H^2((0,t_0+\delta)\times M)}=0,\tag{3.16}$$

$$\|a_{j,1,\rho}\|_{H^\ell((0,t_0+\delta)\times M_1)}\leqslant C_\ell\rho^{\frac{\ell-2}{4}},\ \ell\geqslant 2.\tag{3.17}$$

Finally, for all $s_1,s_2\in[-3\delta,3\delta],\ x'\in\partial M$, we fix

$$\widetilde{a_{j,2,\rho}}(s_1,s_2,x'):=\chi_1(s_2)\varphi_1(x')\beta\left(x',\frac{s_1-s_2}{2}\right)^{-\frac14}\frac{1}{4i}\left(\int_{s_2}^{s_1}\beta\left(x',\frac{\tau-s_2}{2}\right)^{\frac14}\widetilde{b_{j,1,\rho}}(\tau,s_2,x')d\tau\right),\tag{3.18}$$

where

$$b_{j,1,\rho}(t,x',x_n):=-(\partial_t^2-\Delta_g+q_j)a_{j,1,\rho}(t,x',x_n).$$

In particular, we have

$$a_{1,2,\rho}(t,x)=a_{2,2,\rho}(t,x),\quad (t,x)\in(0,t_0+\delta)\times\partial M,\ \rho>1.\tag{3.19}$$

Combining these properties with the fact that, for $t\in[0,t_0+\delta]$, the function $x_n\mapsto\chi((t-t_0)-x_n)$ is supported on $[-t_0-2\delta,3\delta]$ we deduce that

$$\mathrm{supp}(a_0(t,\cdot))\cup\mathrm{supp}(a_{j,1}(t,\cdot))\cup\mathrm{supp}(a_{j,2,\rho}(t,\cdot))\subset M_\delta\subset M_\varepsilon,\quad j=1,2.$$

and we can extend the map

$$G_{j,\rho}:(t,x)\longmapsto e^{i\rho(t-\psi(x))}\left(a_0(t,x)+\frac{a_{j,1}(t,x)}{\rho}+\frac{a_{j,2,\rho}(t,x)}{\rho^2}\right)$$

by zero to a function lying in $H^2((0,t_0+\delta)\times M)$. Moreover, (3.16)-(3.17) imply that

$$\begin{aligned}&\left\|\partial_t^2G_{j,\rho}-\Delta_gG_{j,\rho}+q_jG_{j,\rho}\right\|_{L^2((0,t_0+\delta)\times M)}\\&=\left\|\frac{(\partial_t^2-\Delta_g+q_j)(a_{j,1}-a_{j,1,\rho})}{\rho}+\frac{\partial_t^2a_{j,2,\rho}-\Delta_ga_{j,2,\rho}+q_ja_{j,2,\rho}}{\rho^2}\right\|_{L^2((0,t_0+\delta)\times M)}\\&\leqslant C\left(\rho^{-1}\|a_{j,1}-a_{j,1,\rho}\|_{H^2((0,t_0+\delta)\times M)}+\frac{1+\|a_{j,1,\rho}\|_{H^4((0,t_0+\delta)\times M)}}{\rho^2}\right).\end{aligned}$$

Here we have exploited the explicit expression of $a_{j,2,\rho}$. Combining this with (3.14)-(3.15), we find

$$\lim_{\rho\to+\infty}\rho\left\|\partial_t^2G_{j,\rho}-\Delta_gG_{j,\rho}+q_jG_{j,\rho}\right\|_{L^2((0,t_0+\delta)\times M)}=0.\tag{3.20}$$

We choose $R_{j,\rho} \in \mathcal{C}([0,t_0+\delta];H^1(M)) \cap \mathcal{C}^1([0,t_0+\delta];L^2(M))$ to be the unique solution of the IBVP

$$\begin{cases} \partial_t^2 R_{j,\rho} - \Delta_g R_{j,\rho} + q_j R_{j,\rho} = -(\partial_t^2 G_{j,\rho} - \Delta_g G_{j,\rho} + q_j G_{j,\rho}), & \text{in } (0,t_0+\delta)\times M, \\ R_{j,\rho} = 0, & \text{on } (0,t_0+\delta)\times \partial M, \\ R_{j,\rho}(0,\cdot) = 0, \quad \partial_t R_{j,\rho}(0,\cdot) = 0 & \text{in } M. \end{cases} \tag{3.21}$$

Applying [26, Theorem 2.1], we obtain

$$\begin{aligned} \|\partial_\nu R_{j,\rho}\|_{L^2((0,t_0+\delta)\times\partial M)} &\leqslant C\left(\left\|\partial_t^2 G_{j,\rho} - \Delta_g G_{j,\rho} + q_j G_{j,\rho}\right\|_{L^2((0,t_0+\delta)\times M)} + \|q_j R_{j,\rho}\|_{L^2((0,t_0+\delta)\times M)}\right) \\ &\leqslant C\left(\left\|\partial_t^2 G_{j,\rho} - \Delta_g G_{j,\rho} + q_j G_{j,\rho}\right\|_{L^2((0,t_0+\delta)\times M)} + \|R_{j,\rho}\|_{\mathcal{C}([0,t_0+\delta];H^1(M))}\right) \\ &\leqslant C\left\|\partial_t^2 G_{j,\rho} - \Delta_g G_{j,\rho} + q_j G_{j,\rho}\right\|_{L^2((0,t_0+\delta)\times M)}. \end{aligned}$$

and (3.20) implies (3.5).

We are now in position to complete the proof of Theorem 3.1.

3.2. **Proof of Theorem 3.1.** Note first that, according to (3.13) and (3.19), we have

$$G_{1,\rho}(t,x) = G_{1,\rho}(t,x) := f(t,x), \quad (t,x)\in[0,t_0+\delta]\times\partial M.$$

Using the fact that $f\in H^1((0,t_0+\delta)\times\partial M)$, satisfies $f_{|(0,t_0)\times\partial M}=0$, we extend f by symmetry in t to an element of $H^1((0,T)\times\partial M)$ satisfying $f_{|t=0}=0$. Then, we fix u_j, $j=1,2$, respectively the solution of the initial boundary problem

$$\begin{cases} \partial_t^2 u_j - \Delta_g u_j + q_j u_j = 0, & \text{in } (0,T)\times M, \\ u_j = f, & \text{on } (0,T)\times\partial M, \\ u_j(0,\cdot)=0, \quad u_j(0,\cdot)=0 & \text{in } M. \end{cases} \tag{3.22}$$

Since the restriction of u_j to $(0,t_0+\delta)\times M$ solves the initial boundary value problem

$$\begin{cases} \partial_t^2 u_j - \Delta_g u_j + q_j u_j = 0, & \text{in } (0,t_0+\delta)\times M, \\ u_j = G_{j,\rho}, & \text{on } (0,t_0+\delta)\times\partial M, \\ u_j(0,\cdot)=0, \quad u_j(0,\cdot)=0 & \text{in } M. \end{cases}$$

by the uniqueness of the solution of this problem we deduce that u_j takes the form (3.3) on $(0,t_0+\delta)\times M$. Moreover, due to the expression involving χ_1 in (3.11),(3.12), (3.18), on can check that $\mathrm{supp}(f)\subset(0,T]\times\gamma$. Combining this with the condition $\Lambda_{q_1,\gamma,*}=\Lambda_{q_2,\gamma,*}$, we get

$$(\partial_\nu u_1 - \partial_\nu u_2)(t,x) = 0, \quad (t,x)\in(0,T)\times\gamma.$$

On the other hand, applying (3.13) and (3.19), for all $(t,x)\in(0,t_0+\delta)\times\gamma$, we obtain

$$\begin{aligned} 0 &= \rho(\partial_\nu u_1 - \partial_\nu u_2) \\ &= e^{i\rho(\psi(x)+t)}(\partial_\nu a_{1,1} - \partial_\nu a_{2,1}) + \frac{e^{i\rho(\psi(x)+t)}(\partial_\nu a_{1,2,\rho} - \partial_\nu a_{2,2,\rho})}{\rho} \\ &\quad \rho(\partial_\nu R_{1,\rho} - \partial_\nu R_{2,\rho}). \end{aligned} \tag{3.23}$$

Applying (3.17) and using the form of $a_{j,2,\rho}$, $j=1,2$, we find

$$\begin{aligned} \|\partial_\nu a_{1,2,\rho} - \partial_\nu a_{2,2,\rho}\|_{L^2((0,t_0+\delta)\times\partial M)} &\leqslant C(\|a_{1,2,\rho}\|_{H^2((0,t_0+\delta)\times M)} + \|a_{2,2,\rho}\|_{H^2((0,t_0+\delta)\times M)}) \\ &\leqslant C(1 + \|a_{1,1,\rho}\|_{H^4((0,t_0+\delta)\times M)} + \|a_{2,1,\rho}\|_{H^4((0,t_0+\delta)\times M)}) \\ &\leqslant C\rho^{\frac{1}{2}}. \end{aligned}$$

Combining this with (3.5) and sending $\rho\to+\infty$ in (3.23), we obtain

$$\|\partial_\nu a_{1,1} - \partial_\nu a_{2,1}\|_{L^2((0,t_0+\delta)\times\gamma)} = 0.$$

It follows that

$$\partial_\nu a_{1,1}(t,x) - \partial_\nu a_{2,1}(t,x) = 0, \quad (t,x)\in(0,T)\times\gamma.$$

Passing to boundary coordinates, this condition becomes

$$\partial_{x_n} a_{1,1}(t,x',0) - \partial_{x_n} a_{2,1}(t,x',0) = 0, \quad (t,x') \in (0,T)\times\gamma.$$

and, fixing $a_1 = a_{1,1} - a_{2,1}$ we deduce that

$$\partial_{s_1}\widetilde{a_1}(0,0,x_0) - \partial_{s_2}\widetilde{a_1}(0,0,x_0) = 0.$$

On the other hand, fixing $q = q_1 - q_2$, one can check that

$$\widetilde{a_1}(s_1,s_2,x') = \frac{i}{4}\chi(s_1)\chi_1(x')\beta\left(x',\frac{s_1-s_2}{2}\right)^{-1/4}\left(\int_{s_2}^{s_1}\beta\left(x',\frac{\tau-s_2}{2}\right)^{1/4}\widetilde{q}(\tau,s_2,x')d\tau\right)$$

and we deduce that

$$\partial_{s_1}\widetilde{a_1}(0,0,x_0) - \partial_{s_2}\widetilde{a_1}(0,0,x_0) = \frac{i\tilde{q}(0,0,x_0)}{2} = \frac{iq(t_0,x_0,0)}{2}.$$

This proves that $q(t_0,x_0)=0$ and we deduce that $q_1(t_0,x_0) = q_2(t_0,x_0)$. □

4. Recovery of time-dependent coefficients inside the domain

In this section we will recall some results related to the recovery of coefficients inside the domain. Our first result is stated in a simple manifold and concerns recovery of time-dependent coefficients inside the manifold with restriction of the data on the bottom $t=0$ and the top $t=T$ of the time-space manifold $(0,T)\times M$.

Theorem 4.1. *Assume that (M,g) is a simple manifold. Let $T>0$ and let $q_1,\ q_2 \in L^\infty((0,T)\times M)$. Then the condition*

$$D_{q_1,\sharp}H = D_{q_2,\sharp}H, \quad H\in\mathcal{H}$$

implies that $q_1 = q_2$.

This result follows from [20, Theorem 1.2].

Now let us recall an improvement of this result in the Euclidean case. More precisely, let $M=\overline{\Omega}$ with Ω an open bounded, connected and smooth open subset of $\mathbb{R}^n$.

We introduce also the operator $\mathcal{D}_{q,U} : \mathcal{H}_U \ni H \mapsto (\partial_\nu w_{|V}, w(T,\cdot))$, with w solving (2.16).

Theorem 4.2. *For $q_1,\ q_2\in L^\infty((0,T)\times\Omega)$, the condition $\mathcal{D}_{q_1,U} = \mathcal{D}_{q_2,U}$ implies $q_1=q_2$.*

This result follows from [19, Theorem 1.1] combined with the definition of the trace map given in [19, Proposition A.1].

Armed with these results and the one of Theorem 3.1, we will complete the proof of Theorem 1.1, 1.2 and 1.3.

5. Recovery of the nonlinear terms

The goal of this section is to combine all the tools of the preceding sections in order to complete the proof of Theorem 1.1, 1.2 and 1.3.

Proof of Theorem 1.1. In view of Theorem 2.1, for any $\lambda\in[-L_1,L_1]$ we have

$$\mathcal{N}'_{F_j,\gamma}(\lambda\chi)h = \Lambda_{q_{F_j,\lambda\chi},\gamma}h, \quad h\in\mathcal{H}_{*,\gamma},$$

where we recall that $q_{F_j,\lambda\chi}(t,x) := \partial_u F_j(t,x,u_{F_j,\lambda\chi}(t,x))$. Thus, condition (1.8) implies that $\Lambda_{q_{F_1,\lambda\chi},\gamma,*} = \Lambda_{q_{F_2,\lambda\chi},\gamma,*}$. Moreover, by the Sobolev embedding theorem and Lemma 2.2, we find

$$u_{F_j,\lambda\chi} \in (\mathcal{C}([0,T];H^2(M))\cap\mathcal{C}^2([0,T];L^2(M))) \subset H^2((0,T)\times M)\cap\mathcal{C}([0,T]\times M).$$

Combining this with the fact that $F_j\in\mathcal{C}^3(\mathbb{R}_+\times M\times\mathbb{R})$, we deduce that $q_{F_j,\lambda\chi}\in H^2((0,T)\times M)\cap\mathcal{C}([0,T]\times M)$ and applying Theorem 3.1, we obtain

$$q_{F_1,\lambda\chi}(t,x) = q_{F_2,\lambda\chi}(t,x), \quad (t,x,\lambda)\in(0,T)\times\gamma\times[-L_1,L_1].$$

Therefore, using the fact that $\chi = 1$ on $[\delta, T] \times \gamma$, we obtain

$$\partial_u F_1(t, x, \lambda) = q_{F_1,\lambda\chi}(t, x) = q_{F_2,\lambda\chi}(t, x) = \partial_u F_2(t, x, \lambda), \quad (t, x, \lambda) \in [\delta, T] \times \gamma \times [-L_1, L_1].$$

Finally, applying (1.7), we obtain (1.9). □

Proof of Theorem 1.2 and 1.3. Let us first fix

$$q_{F_j,\lambda}(t, x) := \partial_u F_j(t, x, u_{F_j,(\lambda,\lambda,0)}(t, x)).$$

By the Sobolev embedding theorem, we have $u_{F_j,(\lambda,\lambda,0)} \in \mathcal{C}([0, T] \times M)$ and we deduce that $q_{F_j,\lambda} \in \mathcal{C}([0, T] \times M)$. Then, according to Theorem 2.1, condition (1.11) implies that

$$\mathcal{D}_{q_{F_1,\lambda},\partial M} = \mathcal{D}_{q_{F_2,\lambda},\partial M}, \quad \lambda \in \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right].$$

Therefore, applying Theorem 4.1, we obtain

$$q_{F_1,\lambda}(t, x) = q_{F_2,\lambda}(t, x), \quad (t, x, \lambda) \in (0, T) \times M \times \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right].$$

It follows that

$$\partial_u F_1(0, x, \lambda) = q_{F_1,\lambda}(0, x) = q_{F_2,\lambda}(0, x) = \partial_u F_2(0, x, \lambda), \quad (x, \lambda) \in M \times \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right], \tag{5.1}$$

$$\partial_u F_1(t, x, \lambda) = q_{F_1,\lambda}(t, x) = q_{F_2,\lambda}(t, x) = \partial_u F_2(t, x, \lambda), \quad (t, x, \lambda) \in [0, T] \times \partial M \times \left[-\frac{L}{(2T|M|)^{\frac{1}{2}}}, \frac{L}{(2T|M|)^{\frac{1}{2}}}\right]. \tag{5.2}$$

Combining this with (1.10) we deduce (1.12)-(1.13). This proves Theorem 1.2. In a similar way, Theorem 1.3 can be deduced by Combining Theorem 2.1 with Theorem 4.2. □

Acknowledgements

The author would like to thank Lauri Oksanen for fruitful discussions about this problem.

References

[1] M. Bellassoued, M. Choulli, M. Yamamoto, *Stability estimate for an inverse wave equation and a multidimensional Borg-Levinson theorem*, J. Diff. Equat., **247** (2) (2009), 465-494.

[2] M. D. Blair, H. F. Smith , C. D. Sogge, *Strichartz estimates for the wave equation on manifolds with boundary*, Annales de l'IHP (C) Nonlinear Analysis, **26** (2009), 1817-1829.

[3] N. Burq, G. Lebeau, F. Planchon, *Global existence for energy critical waves in 3-D domains*, J. Amer. Math. Soc, **21** (3) (2008), 831-845.

[4] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations*, Clarenson press-Oxford, 1998.

[5] M. Choulli and Y. Kian, *Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term*, J. Math. Pures Appl., **114** (2018), 235-261.

[6] M. Choulli, E. M. Ouhabaz, M. Yamamoto, *Stable determination of a semilinear term in a parabolic equation*, Commun. Pure Appl. Anal. **5** (3) (2006), 447-462.

[7] M. Christ and A. Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal., **179(2)** (2001), 409-425.

[8] M. Cristofol, J. Garnier, F. Hamel, R. Lionel, *Uniqueness from pointwise observations in a multi-parameter inverse problem*, Comm. Pure Appl. Anal., **11** (2012), 173-188.

[9] M. de Hoop, G. Uhlmann, Y. Wang, *Nonlinear responses from the interaction of two progressing waves at an interface*, to appear in Annales de l'IHP (C) Nonlinear Analysis, https://doi.org/10.1016/j.anihpc.2018.04.005.

[10] G. Hu and Y. Kian, *Determination of singular time-dependent coefficients for wave equations from full and partial data*, Inverse Probl. Imaging, **12** (2018), 745-772.

[11] S. Ibrahim and R. Jrad, *Strichartz type estimates and the well-posedness of an energy critical 2D wave equation in a bounded domain*, J. Differ. Equ., **250** (9) (2011), 3740-3771.

[12] V. Isakov, *On uniqueness in inverse problems for semilinear parabolic equations*, Arch. Rat. Mech. Anal., **124** (1993), 1-12.

[13] V. Isakov, *Uniqueness of recovery of some systems of semilinear partial differential equations*, Inverse Problems, **17** (2001), 607-618.
[14] V. Isakov, *Uniqueness of recovery of some quasilinear Partial differential equations*, Commun. Partial Diff. Eqns., **26** (2001), 1947-1973.
[15] V. Isakov and A. Nachman, *Global Uniqueness for a two-dimensional elliptic inverse problem*, Trans. of AMS, **347** (1995), 3375-3391.
[16] L. Kapitanski, *Cauchy problem for a semilinear wave equation. II*, Journal of Soviet Mathematics, **62** (1992), 2746-2777.
[17] A. Katchalov, Y. Kurylev, M. Lassas, *Inverse boundary spectral problems,* Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. **123** Chapman & Hall/CRC, Boca Raton, FL, 2001.
[18] Y. Kian, *Cauchy problem for semilinear wave equation with time-dependent metrics*, Nonlinear Analysis: Theory, Methods & Applications, **73** (2010), 2204-2212.
[19] Y. Kian, *Unique determination of a time-dependent potential for wave equations from partial data*, Annales de l'IHP (C) Nonlinear Analysis, **34** (2017), 973-990.
[20] Y. Kian and L. Oksanen, *Recovery of time-dependent coefficient on Riemanian manifold for hyperbolic equations*, to appear in IMRN, https://doi.org/10.1093/imrn/rnx263.
[21] Y. Kian and M. Yamamoto, *On existence and uniqueness of solutions for semilinear fractional wave equations*, Fractional Calculus and Applied Analysis, **20** (1) (2017), 117-138.
[22] M. Klibanov, *Global uniqueness of a multidimensional inverse problem for a nonlinear parabolic equation by a Carleman estimate*, Inverse Problems, **20** (2004), 1003.
[23] Y. Kurylev, M. Lassas, G. Uhlmann, *Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations*, Inventiones mathematicae, **212** (2018), 781-857.
[24] Y. Kurylev, M. Lassas, L. Oksanen, G. Uhlmann, *Inverse problem for Einstein-scalar field equations*, preprint, arXiv:1406.4776.
[25] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Ural'tzeva, *Linear and quasilinear equations of parabolic type*, Nauka, Moscow, 1967 in Russian ; English translation : American Math. Soc., Providence, RI, 1968.
[26] I. Lasiecka, J-L. Lions, R. Triggiani, *Non homogeneous boundary value problems for second order hyperbolic operators* J. Math. Pures Appl., **65** (1986), 149-192.
[27] J-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris, 1968.
[28] J-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. II, Dunod, Paris, 1968.
[29] G. Nakamura and M. Watanabe, *An inverse boundary value problem for a nonlinear wave equation*, Inverse Probl. Imaging, **2** (2008), no. 1, 121-131.
[30] G. Nakamura and M. Vashisth, *Inverse boundary value problem for non-linear hyperbolic partial differential Equations*, arXiv:1712.09945.
[31] P. Stefanov and G. Uhlmann, *Stable determination of the hyperbolic Dirichlet-to-Neumann map for generic simple metrics*, IMRN, **17** (2005), 1047-1061.
[32] P. Stefanov and Y. Yang, *The Inverse Problem for the Dirichlet-to-Neumann map on Lorentzian manifolds*, Anal. PDE, **11** (2018), 1381-1414.
[33] Z. Sun and G. Uhlmann, *Inverse problems in quasilinear anisotropic media*, Amer. J. Math. 1997, 119, 771-799.

Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France.
E-mail address: yavar.kian@univ-amu.fr