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A Coq mechanised formal semantics for realistic SQL queries*

Formally reconciling SQL and bag relational algebra

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Abstract
In this article, we provide a Coq mechanised, executable, formal semantics for realistic SQL queries consisting of select [distinct] from where group by having queries with NULL values, functions, aggregates, quantifiers and nested potentially correlated sub-queries. We then relate this fragment to a Coq formalised (extended) relational algebra that enjoys a bag semantics. Doing so we provide the first formally mechanised proof of the equivalence of SQL and extended relational algebra and, from a compilation perspective, thanks to the Coq extraction mechanism to Ocaml, a Coq certified semantic analyser for a SQL compiler.

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1 Introduction
In the area of programming languages, providing a formal semantics for a language is a tricky but crucial task as it allows compilers to rigorously reason about program behaviours and to verify the correctness of optimisations [17, 23]. When considering real-life programming languages the task becomes even harder as it happens that the specifications of the language are often written in natural language. Even when they are formal, they only account for a limited subset of the considered language and are, most of the time, human-checked proven correct. In all cases, there are few strong guarantees that the whole faithfully accounts for the exact semantics and correctness of performed optimisations. To obtain such high level guarantees, a promising approach consists in using proof assistants such as Coq [25] or Isabelle [26] to define mechanised, executable semantics whose correctness is machine-checkable.

A shining demonstration of the viability of this approach for real systems is Leroy’s CompCert project [20], which specified, implemented, and proved the correctness of an optimising C compiler. This compiler is not a toy: it compiles essentially the whole ISO C99 language, targets several architectures, and achieves 90% of the performance of GCC’s optimisation level 1. The value of CompCert’s correctness proofs has surprised some observers. Quoting [27] that used random testing to assess all popular C compilers: “The striking thing about our CompCert results is that the middle-end bugs we found in all other compilers are absent. As of early 2011, the under-development version of CompCert is the only compiler we have tested for which Csmith cannot find wrong-code errors. […] The apparent unbreakability of CompCert supports a strong argument that developing compiler optimizations within a proof framework, where safety checks are explicit and machine-checked, has tangible benefits for compiler users.”

Our long-term goal, based on the same approach, aims at providing a Coq verified compiler for SQL, the standard in terms of programming languages for relational database systems. In this article we focus on semantical issues and define, using Coq, a formal semantics for SQL. More precisely, we define SQLCoq (syntax and semantics), a Coq formalisation of SQL accounting for select [distinct] from where group by having queries with NULL values, functions, aggregates, quantifiers and nested potentially correlated sub-queries. In order to convince ourselves that SQLCoq fairly reflects SQL’s semantics we developed a query generator that serves to automatically generate and execute queries against mainstream systems such as Postgresql, Oracle™, and also against SQLCoq. Obtaining the same results we hence give strong credibility of the (semantic) relevance of SQLCoq. Doing so, we provide a Coq
were denoted by positions. They did not consider group by having aggregation nor complex expressions. As will be shown in Section 2 and 3, the treatment of complex expressions is very subtle.

On the proof assistant side, the first attempt to formalise the (unnamed version of) the relational data model, using the Agda proof assistant [24], is found in [13, 14] while the first, almost complete, Coq formalisation of the relational model is found in [4] where the data model, algebra, tableaux queries, the chase as well as integrity constraints aspects were modelled. A convincing mechanisation, based on nominal Isabelle, of a subset of XQuery [19] is given in [8]. Recently, an SSreflect-based mechanisation of the Datalog language has been proposed in [5].

The very first attempt to verify, using Coq, a RDBMS is presented in [21]. However the SQL fragment they addressed was rather unrealistic as, probably for the sake of simplicity, they placed themselves in the context of an unnamed version of the language in which attributes were denoted by positions. They did not consider group by having clause, neither NULL’s nor aggregates.

More recently, a Coq modelisation of the nested relational algebra (NRA [10]) which directly serves as a semantics for SQL is provided in [2]. Finally, the closest proposal in terms of mechanised semantics for SQL is addressed in [9]. The authors describe a tool to decide whether two SQL queries are equivalent. To do so, they defined HotSQL, a K-relation [15] based semantics for SQL. Unlike ours, the considered fragment does not handle NULL values nor having clause and they used a reconstruction of the language thus not accounting for the trickier aspects of variable binding. As we shall see in Section 2 and 3 the treatment of attributes’ names and more generally environments is particularly a tough task. Furthermore, they relaxed the finite support constraint imposed to K-relations hence possibly yielding infinite query results as well as potentially infinitely many occurrences of tuples in queries’ results. Last, and more importantly, their semantics is not executable hence it is impossible to verify whether they do implement the correct SQL’s semantics.

Unlike those works we propose (i) a bag mechanised executable semantics for the realistic subset of SQL previously mentioned. (ii) By relating this fragment to a Coq formalised relational algebra that enjoys a bag semantics we provide (iii) the first formally mechanised proof of the equivalence of SQL and algebra and, from a compilation perspective, thanks to the Coq extraction mechanism to Ocaml, (iv) a Coq certified semantic analyser for a SQL compiler.

Organisation In Section 2, we first present SQL’s subtleties that are mandatory to be taken into account to provide a realistic semantics. Then we detail, in Section 3, SQL\textsubscript{Coq}’s syntax and semantics and comment on our experimental assessment thanks to our random query generator. Section 4 is devoted to the mechanisation of SQL\textsubscript{Alg}, the bag algebra. Then the translations between SQL\textsubscript{Coq} and SQL\textsubscript{Alg} as well as the equivalence theorem are presented. We conclude, draw lessons and give perspectives in Section 5.

2 SQL: simple and subtle

SQL is a declarative language. As such it is often considered simple. However, its semantics is more subtle than appears at first sight. SQL’s semantics is described by the ISO Standard [18] which consists of thousand pages written in natural language. It is often unclear and, thus, cannot serve as a formal semantics. This explains why many vendors implement various aspects of it in their own way as witnessed by [1]. Although the Standard cannot serve as a formal specification, we relied on it and, meanwhile, we tested our development against systems like Postgresql and Oracle. It is well known that SQL’s select from where construct enjoys a bag semantics: the same tuple can occur several times in the result. Purely set-theoretic operators such as $\cup$ (union), $\cap$ (intersect) and \setminus (except) have a set-theoretic one. Therefore any formal semantics must account for both sets and bags. As pointed out in Guagliardo et al., in [16], SQL deals with NULL values that are intended to represent unknown information. A 3-valued logic combined with the classical Boolean logic is used to handle them (even if 3-valued logic is not necessary as shown in [16] but in the absence of quantifiers).
However NULL's are not treated in a uniform way according to the context as illustrated in Section 2.1. Last, and more importantly, as illustrated in Section 2.2, the way SQL manages environments and expressions is complex and represent the most tricky aspect to address in order to obtain a formal semantics for realistic SQL queries. In the sequel of this section, we note \([q]\) the result of the evaluation of query \(q\), (,), the tuple constructor, \{\}\ the list constructor, \{\}\ the set constructor and \{\}\ the bag constructor. Figure 1 gathers a bunch of queries that will illustrate SQL’s most subtle aspects.

### 2.1 NULL values

The three first queries are borrowed from [16]. They exemplify the fact that NULL is neither equal to nor different from any other value (including itself): comparing NULL with any expression always yields unknown.

Query \(q1\) returns an empty result. This is explained by the fact that \([\text{select } s.a \text{ from } s] = \{\text{s.a=NULL}\}\), hence over all tuples \((r.a=x)\), in particular over \((r.a=1)\) and \((r.a=NULL)\), \([r.a \not\in \text{select } s.a \text{ from } s]\) yields not unknown, which is eventually considered as false, as remarked by Guagliardo et al., in [16].

Neither tuple belongs to the result of the first query.

Query \(q2\) returns \(\{(r.a=1); (r.a=NULL)\}\). Let subq2 be \((\text{select } * \text{ from } s \text{ where } s.a = r.a)\), it yields an empty result over all tuples \((r.a=x)\), hence \([\text{exists } \text{ (subq2)}] = \text{true}\) always false and \([\text{not exists } \text{ (subq2)}] = \text{true}\) is always true, thus \((r.a=1)\) and \((r.a=NULL)\) are in the result of \(q2\).

Query \(q3\) returns \(\{(t.a=NULL,c=2); (t.a=1,c=1)\}\). This illustrates the fact that NULL, which is neither equal nor different from NULL in a 3-valued logic, is indeed equal to NULL in the context of grouping. The semantics proposed in Section 3.2 will account for such behaviours.

### 2.2 Expressions in environments

Let us now address the way SQL manages evaluation environments in presence of aggregates and nested correlated queries. In order to evaluate simple (without aggregates) expressions, it is enough to have a single environment, containing information about the bounded attributes and the values for them. In this simple case (e.g., \(\text{select } a1, b1 \text{ from } t1\);) such an environment corresponds to a unique tuple \((a1=x, b1=y)\) where \(x\) and \(y\) range in the active domains of \(a1\) and \(b1\) respectively.

Evaluating expressions with aggregates is more involved, since an aggregate operates over a list of values, each one corresponding to a tuple. The crucial point is to understand how such a list of tuples is produced. Section 10.9 of [18] (< aggregate functions >, how to
retrieve the rows – page 545) should provide some guidance in answering this question. Unfortunately it was of no help. We thus proceeded by testing many queries over Postgresql and Oracle™. It took us significant effort to reach the semantically relevant set of queries which are given in Figure 1. For all of them, we obtained the same results on both systems. Let us comment on these queries. For $q_5$ the result is:

$$
\begin{align*}
(\{ (a_1=1, \text{max}=10); (a_1=2, \text{max}=10) \\
(a_1=3, \text{max}=5); (a_1=4, \text{max}=10) \}
\end{align*}
$$

It is easy to understand what happens when evaluating $\text{max}(b_1)$ in $q_5$: each group (where $a_1$ is fixed) contains some tuples, each of them yielding a value for $b_1$. Then $\text{max}$ is computed over this list of values. For instance, the group $T_1$ where $a_1=1$ contains exactly one occurrence of tuples of the form $(a_1=1, b_1=1)$, where $i$ ranges from 1 to 10, hence $b_1$ ranges from 1 to 10, and $\text{max}(b_1)$ is equal to 10, whereas the group where $a_1=3$ contains tuples $(a_1=3, b_1=1)$, where $i=1, \ldots , 5$, and $\text{max}(b_1)$ is equal to 5. In this simple case a group of $n$ tuples merely yields $n$ simple environments each of them consisting of a single tuple – we say that the group has been split into individual tuples.

The situation gets more complex when evaluating an aggregate expression in a nested sub-query. How to build, in that case, the suitable list of environments (tuples) in order to get the needed list of values as arguments of the aggregate itself? Assuming that an aggregate expression occurs in a subquery under more than two grouping levels, there are several groups in the evaluation context. How to combine them in order to obtain the correct list of tuples? Which groups have to be split into and which have not to? Queries $q_6$, $q_7$, $q_8$, $q_9$, $q_{10}$, $q_{11}$, $q_{12}$, $q_{13}$ and table $t_2$ have been designed to answer these questions.

Query $q_6$‘s result is $\{(a_1=1); (a_1=2)\}$. This means that subquery select $a_2$ from $t_2$ having $\text{sum}(1+0*a_1)=10$ is not empty when $a_1=1$, $a_2=2$, and is empty when $a_1=3$ and $a_1=4$. For $a_1=1$, this subquery is evaluated under the context of group $T_1$ seen above for $q_5$. This indicates that expression $\text{sum}(1+0*a_1)$ is evaluated to 10 in the context of both outer group $T_1$ and inner group $T_2 = \{(a_2=7, b_2=7); (a_2=7, b_2=7)\}$. Hence, the relevant evaluation context of $\text{sum}(1+0*a_1)$ in $[T_2;T_1]$ has to contain 10 tuples, each of them contributing by $1+0*a_1$, that is 1, to the $\text{sum}$. A simple reasoning about groups’ cardinality allows to conclude that group $T_1$ has been split into, and not $T_2$.

Let us now consider $q_7$. While very similar to $q_6$, except that the $\text{sum}$ is over $1+0*a_2$, $q_7$ is empty, meaning that the evaluation context of $\text{sum}(1+0*a_2)$ in $[T_2;T_1]$ does not contain 10 tuples. How many tuples does it actually contain? An educated guess is 2 (that is $T_2$‘s cardinality), which is confirmed by the fact that $[q_8]$ indeed contains $(a_1=1)$.

So, in the same context, $[T_2;T_1]$, SQL computes 10 values for $1+0*a_1$, from 10 tuples, and 2 values for $1+0*a_2$, from 2 tuples. The only sensible solution is that the values, and their corresponding tuples depend not only from the group context, but also from the expression to be evaluated. Given a context and an expression, there is a single relevant group, which is split into: $T_1$ for $1+0*a_1$, and $T_2$ for $1+0*a_2$ in the context $[T_2;T_1]$.

Another interesting case is when the expression under the aggregate is a constant value $k$, as in $q_9(k)$. What should be the relevant group to be split into? Is there even such a relevant group corresponding to the set of all attributes of $\text{sum}(1)$, that is the empty set?

Actually $q_9(2)$ yields the same result as $q_8$, meaning that the relevant group for a constant is the innermost group $T_2$. Surprisingly, compared to $q_7$, $q_9(10)$ is empty, which means that usual arithmetic equalities, such as $1+0*a_1 = 1$ are no longer valid in SQL, under aggregates.

At that point, what happens if both expressions $1+0*a_1$ and $1+0*a_2$ have to be evaluated in the same group context as it is the case for $q_{10}$, where $1+0*a_1$ and $1+0*a_2$ occur under distinct aggregates. There is no single obvious relevant group anymore. $q_{10}$‘s result contains $(a_1=1)$, meaning that both expressions $1+0*a_1$ and $1+0*a_2$ have been evaluated independently, the first in a context where $T_1$ has been split into, and the second where the splitted group is $T_2$. This makes clear that SQL allows two sub-expressions of a given expression to be evaluated in different contexts which is definitely contrary to what is done in other mainstream programming languages!

What if $1+0*a_1$ and $1+0*a_2$ occur under the same aggregate, as in $q_{11}(k)$? When $k=2$, $[q_{11}(2)]$ is $\bigcup_{i=1}^{4} \{(a_1=i)\}$, otherwise $[q_{11}(k)]$ is empty. Therefore $T_2$, the innermost relevant group, has been split into.

As the reader may have noticed, all expressions under the aggregates were built upon grouping attributes. What happens when such is not the case? Query $q_{12}$ contains $\text{sum}(1+0*b_1+0+b_2)$ and is not well formed according to the Standard, thus, is not evaluated. Next query $q_{13}$ ($k$) contains $\text{sum}(1+0*a_1+0*b_2)$ and behaves exactly the same as $q_{11}(k)$ does. The last query, $q_{14}$, which contains $\text{sum}(1+0*b_1+0*a_2)$, is ill-formed and not evaluated.

At that point, we are able to sum up the above lessons and precisely explain how SQL manages environments.

First, when evaluating an expression with aggregates where the top operand is a function (for instance $\ast$, as in $q_{10}$), each argument is evaluated separately.

Second, when evaluating an expression $q_9(e)$ where the top operand $q_9$ is an aggregate, this aggregate is evaluated against a list of values, each of them coming from the evaluation of $e$ over a tuple. The subtle point is to understand how to build the corresponding list of
tuples. Let us introduce a few definitions that will be helpful.

An environment, \( E = [S_n; \ldots ; S_1] \), is a stack of slices: one slice per nesting level \( i \), the innermost level being on the top. When necessary, we shall equally adopt the following, OCaml-like, notation for environments \( E = (A, G, T) :: E' \) in order to highlight the list’s head. Slices are of the form \( S = (A, G, T) \), where \( A \) (also noted \( A(S) \)) contains the relevant attributes for that level of nesting, i.e., the names introduced in the subquery at this level\(^1\); \( G \) the grouping expressions appearing in the group by (also noted \( G(S) \)); and \( T \) a non empty list of tuples\(^2\) (also noted \( T(S) \)).

When \( e \) is a constant expression, the list of tuples \( T(S_n) \) comes from the innermost slice of environment \( E = [S_n; \ldots ; S_1] \). In the simple case where all attributes of \( e \) are introduced at the same level \( i \), the relevant list is simply \( T(S_i) \). Otherwise, when attributes of \( e \) belong to at least two different levels among \( i_k \ldots i_1 \) where \( i_{j+1} > i_j \), there are two cases:

- either the expression is not well-formed (cf q12 and q14), because \( e \) contains an homogeneous expression of level \( i_j \), \( j < k \) which is not grouped.
- or the expression \( e \) is exactly built upon the attributes corresponding to the \( k \)th level \( i_k \) and the grouping expressions\(^3\) of outermost levels \( i_{k - 1} \ldots i_1 \). In this case, let \( t_{ij} \) be a fixed tuple chosen in each \( T(S_j) \) for \( j < k \), then the list of relevant tuples is made of \( t \ni t_{i_{k-1}} \ni \ldots \ni t_{i_1} \), where \( t \) ranges over \( T(S_{i_k}) \).

We are now able to present our Coq mechanised formal semantics for a realistic fragment of SQL.

3 A formal Coq mechanised semantics for SQL

\( SQL_{Coq} \) addresses the fragment consisting of select [distinct] from where group by having queries with NULL values, functions, aggregates, quantifiers and nested potentially correlated (in from, where and having clauses) sub-queries. It accounts for in, any, all and exists constructs and assigns queries a Coq mechanised (bag) semantics that complies with the Standard.

3.1 SQL\(_{Coq}\) syntax

\( SQL_{Coq} \)'s syntax is given on Figure 2, Figure 3 and Figure 4 where the left part of figures represents SQL’s abstract syntax and the right part the corresponding Coq syntax. We assume that we are given attributes, functions and aggregates. We shall allow strings, integers and booleans to be values, as well as the special NULL. On the top of them, we define usual expressions, first without aggregates \( e_1 \), and then with aggregates \( e_2 \). SQL formulas are similar to first order formulas except they are always interpreted in a finite domain, which is syntactically referred to as \( dom \) in Figure 3. Such formulas will then be used in the context of \( SQL_{Alg} \).

\( SQL_{Coq} \) sticks, syntactically, as much as possible, to SQL’s syntax but the SQL-aware reader shall notice that \( SQL_{Coq} \) slightly differs from SQL in different ways. First, for the sake of uniformity, we impose to have the whole select from where group by having construct (no optional where and group by having clauses). When the where clause is empty, it is forced to true. Similarly, as the group by clause partitions the collection of tuples obtained evaluating the from clause, when no group by is present in SQL, we force \( SQL_{Coq} \) to work with the finest partition\(^4\) which corresponds to the Group_Fine case. We also force explicit and mandatory renaming of attributes, when * is not used. In our syntax, select \( a, b \) from \( t \); is expressed by select \( a \) as \( a \), \( b \) as \( b \) from (table \( t \)) where true group by Group_Fine having true. A further, more subtle, point worth to mention is the distinction we make between \( e_1 \) and \( e_2 \). Both are expressions but the former are built only with functions (\( fn \)) and are evaluated on tuples while the latter also allow unested\(^5\) aggregates (\( ag \)) and are, in that case, evaluated on collections of tuples. In the same line, we used the same language for formulæ either occurring in the where (dealing with a single tuple) or in the having clause (dealing with collections of tuples) simply by identifying each tuple with its corresponding singleton. Also, no aliases for queries are allowed.

3.2 SQL\(_{Coq}\) semantics

Given a tuple \( t \) we note \( \ell(t) \) the attributes occurring in \( t \). We assume that we are given a database instance \( [[-]]_db \) defined as a function from relation names to bags of tuples\(^6\) as well as predefined, fixed interpretations, \( [[\cdot]]_p \), for predicates \( pr \), i.e., a function from vectors of values to Booleans, \( [[\cdot]]_a \) and \( [[\cdot]]_f \) for aggregates \( ag \) and functions \( fn \) respectively\(^7\). As established in Section 2, (complex) expressions occurring in (possibly correlated sub-) queries, are evaluated under a sliced environment, \( E = [S_n; \ldots ; S_1] \) (or \( E = (A, G, T) :: E' \)), the innermost

\(^1\)if this subquery is a select from ... these are the names in the select.

\(^2\)When there is a grouping clause at this level, it is an homogeneous group, otherwise it is a single tuple.

\(^3\)Those appearing in the group by clause of the level ; when there are no such grouping expressions, all attributes of the level are allowed.

\(^4\)The partition consisting of the collection of singletons, one singleton for each tuple in the result of the from

\(^5\)\( e_2 \) is of the form: \( \text{avg}(a) \), \( \text{sum}(a+b) \), \( \text{sum}(a+b)+3 \), \( \text{sum}(a+b)+\text{avg}(c+3) \) but not of \( \text{avg}(\text{sum}(c)+a) \)

\(^6\)These multisets enjoy some list-like operators such as \( \text{empty}, \text{map}, \text{filter}, \text{etc.} \)

\(^7\)\( pr \) is \( <, \leq \) etc.
function ::= * | - | * | / | ... | user defined fun
aggregate ::= sum | avg | min | ... | user defined ag
value ::= string val | integer val | bool val | NULL
e \(E\) ::= value | attribute | function(\(e\))
e \(A\) ::= \(e\) | aggregate(\(e\)) | function(\(e\))

Figure 2. Expressions.

formula ::= formula (and | or) formula
| not formula
| true
| \(p(\(e\))\) | \(p \in\) predicate
| \(p(\(e\)), (all | any) dom\) | \(p \in\) predicate
| \(e\) as attribute in dom
| exists dom

Figure 3. Formulas, parameterized by a finite domain of interpretation \(dom\).

select_item ::= * | \(e\) as attribute
query ::= table
| query (union | intersect | except) query
| select select_item
| from from_item
| (where formula)?
| (group by \(e\) (having formula)?)?

Figure 4. SQL and SQL\(_{\text{Coq}}\) syntax

level, \(n\), corresponding to the first slice. The evaluation of a syntactic entity \(e\) of type \(x\) in environment \(E\) will be denoted by \([e]_E^{\langle x, n \rangle}\) (where \(x\) is \(f\) for expressions built only with functions, \(a\) for expressions built also with aggregates, \(b\) for formulas and \(q\) for queries).

The semantics of simple expressions, which poses no difficulties, is given in Figure 5. The semantics of complex expressions detailed in Figure 6, deserves comments.

When the complex expression is headed by a function, \(f_{\text{fun}}(\tau)\), it simply amounts to a recursive call. When the complex expression is of the form \(a q(e)\), according to Section 2, one has first to find the suitable level of nesting for getting the group to be split into. Then, produce the list of values by evaluating \(e\), and then compute the evaluation of \(a q\) against this list of values. In environment \(E=\{S_0; \ldots; S_1\}\), level \(i\) is a suitable candidate expressed...
Figure 5. Simple expressions’ semantics.

\[
\begin{align*}
[c]_E' &= c & \text{if } c \text{ is a value} \\
[a]_E &= \text{default} & \text{if } a \text{ is an attribute} \\
[a|(A,G,[]):E] &= [a]_E & \text{if } a \in \ell(t) \\
[a|(A,G,t;T):E] &= t.a & \text{if } a \notin \ell(t) \\
[f|(\tau)]_E &= [fn]_f([c]_E) & \text{if } fn \text{ is a function,} \\
& & \text{and } \tau \text{ is a list of simple expressions}
\end{align*}
\]

\textbf{Figure 6.} Complex (with aggregates) expressions’ semantics.

\[
\begin{align*}
\frac{c \in V}{E(u(G,c))} & \quad e \in G \quad \bigwedge_{\tau} \frac{E(u(A,G,[]):E)}{E(u(A,G,t;T):E)} \quad \frac{E((A,G,T) :: \tau)}{E((A,G,T) :: \tau)} \\
\frac{c \in V}{E_e(c) = \tau} & \quad e \notin V \\
\frac{E_e(c) = \text{undefined}}{E_e((A,G,T) :: \tau), e) = (A,G,T) :: \tau} \\
\frac{E_e((A,G,T) :: \tau), e) = (A,G,T) :: \tau}{[\tau]_E} = [fn]_f([c]_E) \\
[a(q)]_E & = [a]_a \left([c]_E | (A,G,[t];E') \right)_{t \in T} \\
& \text{iff } E_e(c, e) = (A,G,T) :: \tau'
\end{align*}
\]
This means that w single environment, w condition and then filtering it thanks to the.
informally, a first step consists in evaluating the apply duplicate elimination thanks to sq do not explicitly mention all to assign them a bag semantics even if our notations interpretation, \[ [ \_ ] \] : env_type H. In our formalisation this is expressed as constraints over \[ [ ] \] and \[ [ ] \]. For formulae, we used a 3-valued logic. The evaluation of pr(\mathcal{E}) in environment \mathcal{E} is equal to unknown iff there exists \epsilon_i in \mathcal{F} such that \[ [ \epsilon_i ] \] = NULL. As usual, unknown distributes according to well-known 3-valued logic rules. Quantifiers all and any are respectively seen as a finite conjunct and a finite thanks to \bigcup_{S \in \mathcal{E}} A(T). This is exactly what is done when \mathcal{E}' = (\ell(T), [\_], [\_]) :: \mathcal{E}.

Then the (intermediate) collection of tuples obtained is partitioned according to the grouping expressions in the group by G, yielding a collection of collections of tuples: the groups. When there is no grouping clause, the finest partition denoted Group_Fine in the Coq development is used.

The way groups are further filtered w.r.t. the having condition \mathcal{H} follows the same pattern as where, except that some complex expressions may occur in \mathcal{H}. When evaluating an expression of the form \text{ag}(e) for a group T, all tuples of the group are needed; when evaluating a simple expression, any tuple of T yields the same result, T being homogeneous w.r.t. the grouping criterion G. Hence the proper evaluation environment for filtering the group \mathcal{T} w.r.t. \mathcal{H} in environment \mathcal{E} is \( (\ell(T), G, T) :: \mathcal{E} \).

Last, the select clause is applied yielding again a collection of tuples as a result.

### About NULL’s
At the expression level, NULL’s are simply handled by the fact that they behave as an absorbing element w.r.t. functions and are simply discarded for aggregates except for count(*) where they contribute as 1. In our formalisation this is expressed as constraints over \[ [ ] \] a and \[ [ ] \]. For formulae, we used a 3-valued logic. The evaluation of pr(\mathcal{E}) in environment \mathcal{E} is equal to unknown iff there exists \epsilon_i in \mathcal{F} such that \[ [ \epsilon_i ] \] = NULL. As usual, unknown distributes according to well-known 3-valued logic rules. Quantifiers all and any are respectively seen as a finite conjunct and a finite.

Figure 7. Formulas’ semantics.
As we needed to generate queries that were to be accepted by Postgresql and Oracle™, we not only relied on SQLCoq’s grammar but also imposed well-formed conditions to our generator. We ran our experiment on 10,000 queries. In all cases we observed the same

Figure 8. SQL queries’ semantics.
results for Postgresql (Version 9.5.12), Oracle™(https://livesql.oracle.com/apex/livesql/file/index.html running Oracle Database 18c) and SQLCoq. At that point we strongly believe that SQLCoq faithfully reflects SQL’s semantics.

4 SQLAlg: a Coq mechanised algebra for SQL

We now present SQLAlg, our Coq formalisation of an algebra that hosts SQLCoq. SQLAlg borrows from the extended relational algebra presented in [12] which consists of the well-known relational operators $\pi$ (projection) which corresponds to $\text{select}$, $\sigma$ (selection) corresponding to $\text{where}$ and $\bowtie$ (join) to $\text{from}$ together with the set theoretic operators. To account for SQL, the algebra in [12] is extended with the $\gamma$ (grouping) operator.

4.1 Syntax and semantics

However, as it is presented the algebra in [12] does not account for $\text{having}$ conditions neither for complex expressions (grouping is only possible over attributes and aggregates are computed over single attributes) nor for environments. Unlike this proposal, ours is far more expressive as it allows for grouping over simple expressions and allows complex expressions $e^\sigma$ in projections.

So as to deal with $\text{having}$ conditions, that directly operate on groups that carry more information than single tuples, SQLAlg extends what is presented in [12] by adding an extra parameter to $\gamma$: the $\text{having}$ condition. As pointed in [12], one should notice that the $\delta$ (duplicate elimination) operator is absent as it is a special case of $\gamma^{10}$.

$$Q ::= \text{table} \mid Q \ (\text{union} \ | \ \text{intersect} \ | \ \text{except}) \ Q \mid Q \bowtie Q \mid \pi(e^\sigma \text{ as attribute})(Q) \mid \sigma^\text{formula}(Q) \mid \gamma(e^\sigma \text{ as attribute, } \delta^\gamma, \text{ formula})(Q)$$

Figure 9. SQLAlg syntax

Expressions (simple and complex ones) as well as formulas$^{11}$ are shared with SQLCoq. In order to define the semantics of SQLAlg expressions, environments are needed, for the same reasons as for SQLCoq: accounting for nesting. SQLAlg environments are exactly the same as for SQLCoq. What should be noticed is that $\bowtie$ is the true natural join, and that $\gamma$ can be seen as a degenerated case of $\text{select}$ from $\text{group}$ where $\text{having}$, where the $\text{where}$ condition is absent (or set to $\text{true}$).

$^{10}\delta(Q) = \gamma(\vec{a} \bowtie \vec{a}, \pi, \text{true})(Q)$ where $\vec{a}$ spans over the labels of query $Q$.

$^{11}$For algebraic formulas, the domain parameter $\text{dom}$ is actually algebraic queries.

Let us at that point formally relate SQLCoq and SQLAlg.

4.2 SQLCoq and SQLAlg are equivalent

On Figure 11, we give $\text{Alg}^\text{T}$ a translation from SQLCoq to SQLAlg, and its back translation $\text{T Alg}$ (l). Both use auxiliary translations ($\text{T}^\text{sq}$ (l), resp. $\text{T}^\text{sq}$ (r)) which simply traverse formulas in order to translate the queries they contain. Since simple and complex expressions are shared, they are left unchanged by these translations. These translations are sound, provided that they are applied on “reasonable” database instances and queries.

Definition 4.1. A database instance $[\cdot \bowtie \cdot]_{\text{db}}$ is well-sorted if and only if all tuples in the same table have the same labels:

$\forall r, t_1, t_2, \ t_1 \in [\cdot]_{\text{db}} \land t_2 \in [\cdot]_{\text{db}} \Rightarrow (t_1 = t_2)$.

Definition 4.2. A SQLCoq query $sq$ is well-formed if and only of all labels in its $\text{from}$ clauses are pairwise disjoint and its sub-queries are well-formed:

If $t$ is a table

- $\text{Alg}^\text{T}(t)$ if $t$ is a table

- $\text{Alg}^\text{T}(\text{union})$ $\text{Alg}^\text{T}(\pi^\text{formula}(Q))$ $\text{Alg}^\text{T}(\pi^\text{formula}(Q))$ $\text{Alg}^\text{T}(\pi^\text{formula}(Q))$ $\text{Alg}^\text{T}(\pi^\text{formula}(Q))$ $\text{Alg}^\text{T}(\pi^\text{formula}(Q))$ $\text{Alg}^\text{T}(\pi^\text{formula}(Q))$

- $\text{Alg}^\text{T}(\text{select} s \text{ from } q_i \text{ as } b_j)$ where $u$ group by $G$ having $h$
Figure 11. Translations between SQL$_{\text{Coq}}$ and SQL$_{\text{Alg}}$.

| $\mathcal{T}^q(\text{tbl}) = \text{tbl}$ | $\mathcal{T}^q(\text{q}_1 \text{ union } \text{q}_2) = \mathcal{T}^q(\text{q}_1) \text{ union } \mathcal{T}^q(\text{q}_2)$ |
| $\mathcal{T}^q(\text{q}_1 \text{ intersect } \text{q}_2) = \mathcal{T}^q(\text{q}_1) \text{ intersect } \mathcal{T}^q(\text{q}_2)$ | $\mathcal{T}^q(\text{q}_1 \text{ except } \text{q}_2) = \mathcal{T}^q(\text{q}_1) \text{ except } \mathcal{T}^q(\text{q}_2)$ |

| $\mathcal{T}^q(\text{select } e_i \text{ as } a_i \text{ from } \text{f}_i \text{ where } w) = \pi_{(\ell_i \text{ as } a_i)}(\pi_{(\ell_i \text{ as } \tau_i)}(\pi_{(\tau_i \text{ as } f_i)}(f_i)))$ | $\mathcal{T}^q(\text{select } e_i \text{ as } a_i \text{ from } \text{f}_i \text{ where } w \text{ group by G having h}) = \gamma_{(\tau_i \text{ as } \pi_i, \tau_i), \mathcal{T}^h(\text{h})}(\pi_{(\tau_i \text{ as } f_i)}(f_i))$ |

| $\mathcal{T}^\text{from}(q(a_i \text{ as } b_i)) = \pi_{(\ell_i \text{ as } b_i)}(\mathcal{T}^q(\text{q}))$ | $\mathcal{T}^\text{from}(q(\text{all } q)) = \mathcal{T}^\text{from}(\text{pr}(\text{true}))$ |

Provided that those conditions be fulfilled we can state that the following equivalence Theorem.

**Theorem 4.3.** Let $[\cdot]_d$ be a well-sorted database instance and $sq$ be a SQL$_{\text{Coq}}$ query, $aq$ a SQL$_{\text{Alg}}$ query then:

$$\forall \mathcal{E}, sq, [\mathcal{W}(sq)]_d = [sq]_E$$

$$\forall \mathcal{E}, aq, [\mathcal{T}^q(aq)]_E = [aq]_E$$

The proof proceeds by (mutual) structural induction over queries and formulas. Actually the proof is made by induction over the sizes of queries and formulas. It consists of 500 lines of Coq code and heavily rely on a tactic which allows to automate the proofs that size for sub-objects is decreasing. For the correctness of $\mathcal{T}^q(\_)$, well-formedness hypothesis of the theorem essentially ensures that Cartesian product and natural join coincide. What was interesting is that the well-formedness hypothesis was mandatory and this shed light on the fact that, indeed, SQL from behaves as a cross product. For both translations, well-sortedness ensures that reasoning over tuples’ labels in the evaluation of a query can be made globally, by ”statically” computing the labels over a query.

5 Conclusions

Seeking for a formal semantics for SQL has been a long-standing quest for the database community. In this article, we presented a formal, Coq mechanised, executable semantics for a large realistic fragment of SQL. Both aspects, mechanised and executable, are of paramount importance.

Mechanisation inside a proof assistant, such as Coq, raises some relevant issues and enlightening questions. Indeed, as for theoretical reasons Coq requires (recursive) functions to be total, once the syntax is fixed, the semantics has to be ”totally” defined. This implies that no details can be swept under the carpet. All cases must be considered. This led us to not only discover a bunch of weird queries, the ones of Figure 1, but also to discover strange boundary conditions. For instance, provided empty be the empty relation, query select 1 from empty group by 1+1 having 2=2; returns empty on both Postgresql and Oracle$^\text{TM}$. However, query select 1 from empty having 2=2; yields 1 in Postgresql while Oracle$^\text{TM}$ answers empty set. In that case, Postgresql consider that the partition of the emptyset $\emptyset$ is $\{\emptyset\}$ whereas Oracle$^\text{TM}$ uses the true mathematical definition: $\emptyset$. We implemented Oracle$^\text{TM}$’s semantics for this case.

Providing an executable semantics allows one to be convinced that the semantics is correct as it can be confronted to real systems. This is what we achieved thanks to our query generator.

Last combining mechanisation and execution in the same framework, namely Coq, provides the strongest possible guarantees that there are no gaps between the definition and the execution: no transcription error between a pen and paper definition and the corresponding program may occur.

Thanks to our formal semantics we have been able to relate SQL$_{\text{Coq}}$ and SQL$_{\text{Alg}}$ establishing, the first, to our best knowledge, equivalence result for that SQL fragment. Moreover, by doing so, we can recover the well-known algebraic equivalences presented in textbooks. Such equivalences are proven, using Coq, in [3].

In an early version of the development, we defined a pure set-theoretic semantics and only addressed the
SQL’s fragment with no duplicates. Then we addressed
the bag aspects of SQL and were pleasantly surprised to
discover that it was not so dramatic. Therefore, the wide-
spread belief that the problem for SQL is to assign it a
bag semantics is not as crucial as it seemed to be. What
was really challenging was to accurately and faithfully
grasp SQL’s management of expressions and environ-
ments in the presence of nested queries. The ISO/IEC
document was of little help along this path. On the con-
trary, Coq was an enlightening, very demanding master
of invaluable help. Even if we knew it, it confirmed us
that SQL having initially been designed as a domain
specific language intended not to be Turing-complete,
the fact of adding more features along the time in the
standardisation process, seriously, and sadly, departed
it from its original elegant foundations. By formally
relating SQL and an extended relational algebra, we,
humbly, wanted to pay tribute to the pioneers that
designed the foundational aspects of RDBMs.

Our long term goal is to provide a Coq verified SQL’s
compiler. The work presented in this article allows to
obtain a mechanised semantic analyser that we plan
to extend to features like order by. In [6] we pro-
vided a certification of the physical layer of a SQL
ingine where mainstream physical operators such as
sequential scans, nested loop joins, index joins or bitmap index joins are formally specified and im-
plemented. What remains to be done is to address the
logical optimisation part of the compiler.

References

tions. http://troels.arvin.dk/db/rdms
Siméon. 2017. Handling Environments in a Nested Relational
Algebra with Combinators and an Implementation in a Veri-
ﬁed Query Compiler. In SIGMOD Conference, Chicago, IL,
USA, May 14-19, 2017, S. Salihog˘lu, W. Zhou, R. Chirkova,
J. Yang, and D. Suciu (Eds.). ACM, 1555–1569.
[3] Véronique Benzaken and Evelyne Contejean. 2016. SQLC-
Certi: Coq mechanisation of SQL’s compilation: Formally
hal.archives-ouvertes.fr/hal-01487062
Formalization of the Relational Data Model. In 23rd European
Symposium on Programming (ESOP).
fying Standard and Stratified Datalog Inference Engines in
SSReflect. In 8th International Conference on Interactive
Theorem Proving (ITP 2017), M. Ayala-Rincon and C. Munoz
(Eds.), Vol. 10499. Springer.
A Coq formalisation of SQL’s execution engines. In Inter-
national Conference on Interactive Theorem Proving (ITP
2018), Oxford, United Kingdom.
Algebra: Optimisation, Semantics, and Equivalence of SQL
Queries. IEEE Trans., on Software Engineering SE-11 (April