Operational Modal Analysis in Frequency Domain using Gaussian Mixture Models

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ABSTRACT

Operational Modal Analysis is widely gaining popularity as a means to perform system identification of a structure. Instead of using a detailed experimental setup Operational Modal Analysis relies on measurement of ambient displacements to identify the system. Due to the random nature of ambient excitations and their output responses, various statistical methods have been developed throughout the literature both in the time-domain and the frequency-domain. The most popular of these algorithms rely on the assumption that the structure can be modelled as a multi degree of freedom second order differential system. In this paper we drop the second order differential assumption and treat the identification problem as a curve-fitting problem, by fitting a Gaussian Mixture Model in the frequency domain. We further derive equivalent models for the covariance-driven and the data-driven algorithms. Moreover, we introduce a model comparison criterion to automatically choose the optimum number of Gaussian's. Later the algorithm is used to predict modal frequencies on a simulated problem.

NOMENCLATURE

$[M]$	Mass Matrix
$[C]$	Damping Matrix
$[K]$	Stiffness Matrix
$\{x(t)\}$	Displacement vector at time t
$\{f(t)\}$	Force vector at time t
$k(\tau)$	Auto-correlation function at time-lag τ
$S(s)$	Spectral Density at s
GP	Gaussian Process
Q	Number Of Gaussian's
GMM	Gaussian Mixture Model
OMA	Operational Modal Analysis
BIC	Bayesian Information Criterion
MLE	Maximum Likelihood Estimate
k	Number Of free parameters in an algorithm

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1 INTRODUCTION

Modal analysis has been widely used as a means of identifying dynamic properties such as modal frequencies, damping ratios and mode shapes of a structural system. Traditionally, the system is subjected to artificial input excitations and output deformations (displacements, velocities or accelerations) are measured. These later help in identifying the modal parameters of the system, this process is called Experimental Modal Analysis (EMA). In the last few decades several algorithms primarily using the assumption of second order differential, Multi Degree Of Freedom (MDOF) system (equation 1) have been developed to find modal parameters in EMA [1] [2].

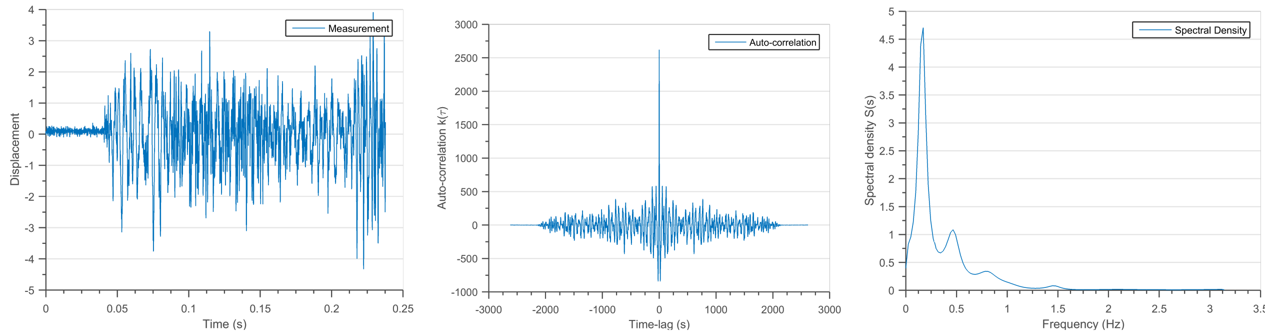
$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{f(t)\} \tag{1}$$

Here, $[M]$, $[C]$ and $[K]$ denote the mass, damping and stiffness matrices respectively. While, $\{x(t)\}$ and $\{f(t)\}$ denote the displacement and force vectors at the time t.

Since the last decade Operational Modal Analysis (OMA) has gained considerable interest in the community. OMA identifies the modal parameters only from the output measurements while assuming ambient excitations as random noise. OMA is cheaper because it does not require expensive experimental setup and and can be used in real time operational use cases such as health monitoring [3] [4] [5]. Several algorithms in OMA can be seen as extensions of EMA algorithms based on the similar assumption of second order MDOF system.

In this paper we approach the problem of finding modal parameters as a problem of curve fitting. We drop the assumption of second order differential MDOF system and use a Gaussian Mixture Model (GMM) [6] [7] to fit the spectral density. Moreover we introduce a criteria called Bayesian Information criteria (BIC) which performs a trade-off on the accuracy of the fit and complexity of the model to estimate the modal order [8] [9] [10].

The remaining paper proceeds as follows, section 2 gives an overview of the traditional operational modal analysis. Sec-



(a) Measured output on accelerometers $x(t)$

(b) Auto-correlation $k(\tau)$ of the measured output fig: 1(a)

(c) Power spectrum density $S(s)$ of the output fig: 1(a)

Figure 1: Different types of measurements for estimation of Modal parameters in OMA

tion 3 details the changes made to current algorithms and introduces the BIC. Section 4 demonstrates the capabilities of the algorithm on a simulated dataset and finally section 5 concludes the paper with future outlook.

2 OPERATIONAL MODAL ANALYSIS

As stated earlier the operational modal analysis is an output dependent modal identification technique. The only thing required is the measurement from the accelerometers placed on the structure. Figure 1(a) shows an example of ambient measurements $x(t)$ on a structure. In almost all OMA algorithms the measurement $x(t)$ is assumed to be generated from a random force excitation.

The following subsections describe the various time-domain subsection 2.1 and frequency-domain algorithms subsection 2.2 for performing OMA.

2.1 Time-domain OMA

In the time-domain a general auto-regression moving average (ARMA) model can be applied to the measurement $x(t)$ [11]. Here, the modal parameters can be computed from the coefficients of polynomials in ARMA models [12].

If we assume that a second order differential (equation 1) completely describes the system dynamics. Then Natural Excitation Technique [13] proves that the auto-correlation function $k(\tau)$ in equation 2 can be written as sum of decaying sinusoid's as described by equation 3. The auto-correlation describes the similarity between measurement as a function of time lag τ between them figure 1(b).

$$k(\tau) = \int x(t)x(t-\tau)dt \tag{2}$$

Here, $k(\tau)$ denotes the auto-correlation for random vector $x(t)$ as a function of time lag τ.

$$k(\tau) = \sum A_i exp(-\lambda_i \tau) sin(B_i \tau) \tag{3}$$

Here, λ_i and A_i denotes the modal frequency and mode shapes for the i^{th} mode. The above coefficients are found by minimizing the least square error between the measured $k(\tau)$ from equation 2 and the predicted $k(\tau)$ from equation 3. This process is very similar to the Least Square Complex Exponential (LSCE) [14] [15] [1] algorithm developed for time-domain EMA.

2.2 Frequency-domain OMA

If we assume the measurement $x(t)$ to be a stationary random process, then according to bochner's theorem [16] the spectral density or power spectrum $S(s)$ can be represented as equation 4.

$$S(s) = \int k(\tau) exp(-2\pi i s^T \tau) d\tau \tag{4}$$

Here, $S(s)$ is the power spectrum for the measurement $x(t)$, where s lies in the frequency-damping plane. Figure 1(c) shows the power spectrum calculated for the measurement $x(t)$ shown in figure 1(a).

Initially the Peak Picking technique (PP) [17] was used in the frequency-domain to identify modal frequencies and shapes. The PP technique is a very easy way to identify modes but

Measurement; eg. figure 1(a)	Auto-correlation; eg. figure 1(b)	Power Spectrum; eg. figure 1(c)
$x(t)$	$k(\tau) = \int x(t)x(t-\tau)dt$	$S(s) = \int k(\tau)exp(-2\pi i s^T \tau)d\tau$
Assumption: Second Order Differential		
	$k(\tau) = \sum A_i exp(-\lambda_i \tau) sin(B_i \tau)$	$S(j\omega) = \frac{\sum a_k (j\omega)^k}{\sum b_l (j\omega)^l}$
Assumption: Gaussian Mixture Model		
$x(t) = GP(0, cov_{SM})$	$k(\tau) = \sum w_i cos(2\pi\mu_i\tau)exp\{-2\pi^2\sigma_i^2\tau^2\}$	$S(s) = \sum w_i \frac{1}{\sqrt{2\pi\sigma_i^2}} exp\{\frac{1}{2\sigma_i^2}(s-\mu_i)^2\}$

TABLE 1: Comparison of fitting functions

becomes inefficient for complex structures [18]. This gave rise to the Frequency Domain Decomposition (FDD) [19] where modal frequency are denoted as the eigenvalues of spectral density matrix equation 5.

$$S(j\omega) = U\Sigma U^H \tag{5}$$

Here, modal frequencies and mode shapes can be derived from Σ and U respectively using FDD [19] or Enhanced-FDD [20].

Majority of frequency-domain algorithms in EMA fit a Rational Fractional Polynomial (RFP) [2] in the frequency domain for modal identification [21] [22]. The Rational Fractional Polynomial equation 6 form can be derived if we assume the system to be second order differential equation 1.

$$S(j\omega) = \frac{\sum a_k (j\omega)^k}{\sum b_l (j\omega)^l} \tag{6}$$

Here, the poles of the polynomial denote the modal frequencies, while other modal parameters can be derived from the coefficients a_k and b_l. The coefficients of the polynomial can be found by minimizing the least squared error. RFP based algorithms face problems since as the number of modes increase the matrix becomes ill-conditioned which gives rise to stability issues in prediction of modal parameters. In the next section we will drop the assumption of second order differential system and treat the modal identification as a purely curve-fitting problem.

3 GAUSSIAN MIXTURE MODELS (GMM)

Two of the above mentioned OMA algorithms ”Natural Excitation Technique” in the time domain and ”Rational Fractional Polynomial” in the frequency domain, have a core assumption of second order differential system. This assumption fails for non-linear systems and for cases where modal frequencies are very close. In the following section we propose to use Gaussian Mixture Models to fit the power spectrum curve.

Scale location mixtures of Gaussian’s can approximate a curve to arbitrary precision with enough components [23]. Due to the above property GMM’s are widely used in machine learning tasks such as speech recognition [24], financial modelling [25], handwriting recognition [26] and many more.

Due to the formulation of GMM, the mean, standard deviation and weight information of the gaussian’s can be used to derive the modal frequency, damping and mode shape of the system respectively. For a positive half power spectrum the GMM will be equivalent to equation 7.

$$S(s) = \sum_i^Q w_i \frac{1}{\sqrt{2\pi\sigma_i^2}} exp\{\frac{1}{2\sigma_i^2}(s-\mu_i)^2\} \tag{7}$$

Here, μ_i, σ_i and w_i are the mean, standard deviation and weight respectively of the i^{th} gaussian. While, Q denotes the number of gaussians used in the GMM. The mean, standard deviation and weight can be found by minimizing the least square error between measured power spectrum and predicted power spectrum $S(s)$. The method to estimate Q will be explained in more detail in subsection 3.1.

The GMM model in the frequency-domain can be transformed to perform covariance-driven modal identification using the equation 4. If we assume $x(t)$ to be a stationary random process then using to equation 7 and equation 4 we can get equation 8 in the time domain [27].

$$k(\tau) = \sum_i^Q w_i cos(2\pi\mu_i\tau)exp\{-2\pi^2\sigma_i^2\tau^2\} \tag{8}$$

Here, μ_i, σ_i and w_i are the mean, standard deviation and weight respectively of the i^{th} gaussian. While, Q denotes the number of gaussians used in the GMM, τ is the time lag between two measurement instances. The parameters can be found by minimizing the least squared error.

Moreover, if we assume that $x(t)$ is a zero-mean gaussian process, then we can transform GMM in frequency-domain to time-domain. The equation 7 and equation 8 are equivalent to fitting a zero-mean gaussian process with a spectral mixture covariance function [28].

$$x(t) = GP(0, cov_{SM}(t, t')) \tag{9}$$

Here, GP denotes a gaussian process [29], while cov_{SM} represents a spectral mixture covariance function which resembles equation 8 [28].

We would like to emphasize that keeping the computational complexities aside, fitting a spectral mixture gaussian process in time-domain equation 7, fitting equation 8 for covariance-driven modal identification and fitting a GMM equation 7 in the frequency-domain are equivalent. In fact the initial idea of this paper was to fit a Gaussian Process (GP) in the data domain, but GP's are computationally heavy and we achieved a good accuracy by fitting the GMM in frequency domain. Refer to table 1 for a more comprehensive view at various fitting functions.

3.1 Bayesian Information Criteria (BIC)

While the modal parameters can be chosen by minimizing the least square error, how to choose the number of modes is a recurring question in several OMA algorithms. This problem is partially resolved by using stabilization diagrams or mode identification functions [21] [30] [31]. But in practical situations engineering judgement is required to estimate the optimal modal order.

Here, we use the Bayesian Information Criteria (BIC) [32] which penalises more complex models to estimate the parameter Q in equation 7. It has been shown earlier that the BIC when applied to GMM's does not underestimate the true number of components [33].

$$BIC(Q) = n \ln(MLE) + k \ln(n) \tag{10}$$

Here, n denotes the number of data-points to fit, MLE denotes the maximum likelihood estimation of the fit and k denote the number of free parameters to fit. The BIC performs a trade-off between the data-fit term $n \ln(MLE)$ and the complexity penalty term $k \ln(n)$, basically penalizing for over-fitting. Lowest value of BIC is preferred.

4 RESULTS

In this section we conduct experiments, applying our approach on a simple 3 degree of freedom system with close by modes. As stated earlier in section 3 we fit a Gaussian Mixture Model (GMM) on the spectral density. Later we will compare the Bayesian Information Criteria to find the optimal value of number of gaussians for the measurement.

The toolbox used for this paper is Matlab's Curve Fitting Toolbox [34]. All experiments were performed on an Intel quad-core processor with 4Gb RAM. Using the curve fitting toolbox the fitting can be performed by a few lines of code. When compared to other frequency-domain techniques like RFP which suffer from ill-conditioned matrices, the GMM technique is highly stable and finds the coefficient's in seconds.

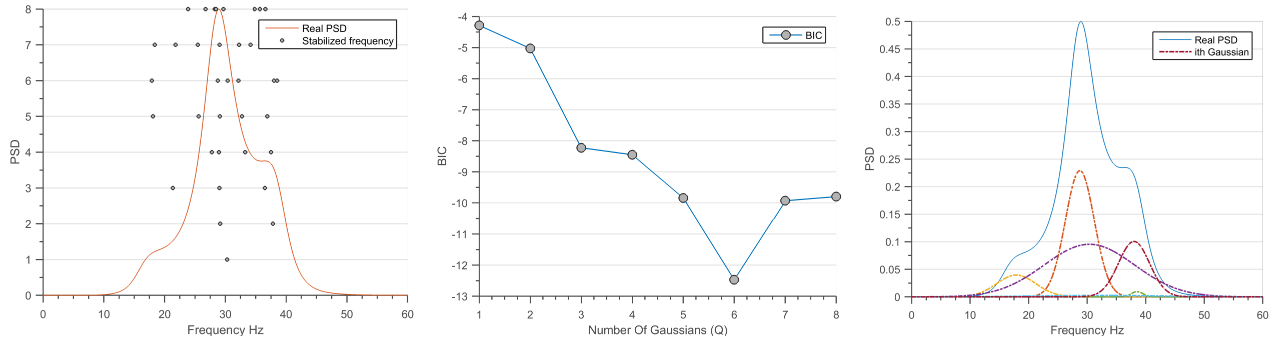
Figure 2(a) shows the stabilization diagram with increasing number of gaussians Q. We can observe that as the number of Q increases the algorithm starts finding better and better modes. We can also observe that there are three modes which start stabilizing from $Q = 5$. The, figure 2(b) shows the BIC criterion with increasing number of gaussian's Q. We can see that that the BIC is minimum for $Q = 6$ and hence if we add anymore gaussian's for our dataset we will be performing over-fitting.

Figure 2(c) shows the 6 constituent gaussians which represent the $Q = 6$ case. The three principal peaks represent the modal frequencies of the system, these correspond to the stabilized frequencies from figure 2(a). The remaining three peaks are there to compensate for the spectral density not explained by the three principal peaks.

In the current setting of the GMM model we only propose a quick and easy way to identify the most important frequencies of a structural system. Neither the mode shapes nor the damping ratios are estimated in the current format. As can be observed from figure 2(c) the mode shapes are not only dependent on the principal gaussian's but also on the neighbouring gaussian's. Since some part of the spectral density is defined by non-stabilized gaussian's, in future we would like to derive a method to estimate mode-shape and damping ratio such that the contributions of neighbouring gaussian's are also taken into account.

5 CONCLUSION

In this paper we have proposed to identify model frequencies of a system by curve-fitting a mixture of gaussians in the frequency domain. While the common assumption that the structure can be modelled as a MDOF second order differential system causes stability issues in presence of non-linear systems. The GMM model is mathematically stable, gives results in seconds and can fit a function upto arbitrary accuracy. Moreover we introduce the BIC to identify the optimum num-



(a) Stabilization diagram with increasing number of gaussians Q, the dots denote the stabilized frequencies. We can observe that as the number of Q increases the algorithm starts finding better and better modes.

(b) The BIC criterion with increasing number of gaussian's Q. We can see that that the BIC is minimum for $Q = 6$ and hence if we add anymore gaussian's for our dataset we will be performing over-fitting

(c) Distribution of gaussians for $Q = 6$. We can see that the three modal frequencies have majority of the participation in representing the spectral density.

Figure 2: Results of applying GMM on a Spectral density

ber of gaussians and perform a trade-off between accuracy of fit and over-fitting.

Without doubt this is very nascent stage of application of GMM for system identification and there remains problems such as identification of mode-shape and damping ratio in this algorithm. We wish to tackle these problems in the future. We also wish to apply the algorithm on a real world dataset and compare with respect to other time domain and frequency domain techniques.

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