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PI controllers for the general Saint-Venant equations

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Abstract

We study the exponential stability in the $H^2$ norm of the nonlinear Saint-Venant (or shallow water) equations with arbitrary friction and slope using a single Proportional-Integral (PI) control at one end of the channel. Using a local dissipative entropy we find a simple and explicit condition on the gain the PI control to ensure the exponential stability of any steady-states. This condition is independent of the slope, the friction coefficient, the length of the river, the inflow disturbance and, more surprisingly, can be made independent of the steady-state considered. When the inflow disturbance is time-dependent and no steady-state exist, we still have the Input-to-State stability of the system, and we show that changing slightly the PI control enables to recover the exponential stability of slowly varying trajectories.

Introduction

Discovered in 1871, the Saint-Venant equations \cite{2} (or 1-D shallow water equations) are among the most famous equations in fluid dynamics and have been investigated in hundreds of studies. Their richness, although being quite simple, has made them become a major tool in practice for many industrial goal, the most famous being probably the regulation of navigable rivers. They are the ground model for such purpose in France and Belgium. Regulation of rivers is a major issue, for navigation, freight transport, renewable energy production, but also for safety reasons, especially as several nuclear plants all around the world are implanted close to rivers. For these reasons, the stability of the steady-states of the Saint-Venant equations has been, and is still, a major issue.

Many results were obtained in the last decades. In 1999, the robust stability of the homogeneous linearized Saint-Venant equations was shown using a Lyapunov approach and proportional feedback controller \cite{9}. Later the stability of the homogeneous nonlinear Saint-Venant equations was achieved, still using proportional feedback controller. In 2008, through a semi-group approach \cite{15}, the stability of the inhomogeneous nonlinear Saint-Venant equation was shown for sufficiently small friction and slope (or equivalently sufficiently small canal), and these results were successfully applied to real dataset from the Sambre river in Belgium. More recently, in \cite{5} the authors give sufficient conditions to stabilize the nonlinear Saint-Venant equations with arbitrary friction for the $H^2$ norm but no slope using again proportional feedback controllers, and in \cite{19} with both arbitrary friction and slope, this last result being proved by exhibiting an explicit local entropy for the nonlinear inhomogeneous Saint-Venant equations.

It is worth mentioning that other stability results have also been obtained in less classical cases or with less classical feedbacks. For instance in \cite{7} was shown the rapid stabilization of the homogeneous nonlinear Saint-Venant equations when a shock (e.g. a hydraulic jump) occurs in the target steady-state. Also, several results (e.g. \cite{13}) were obtained using a backstepping approach, a very powerful method based on a Volterra transformation, developed mainly for PDE in \cite{21}, and generalized recently with a Fredholm transformation for hyperbolic systems \cite{11,31,32}. One may look at \cite{19} for a more detailed survey about this method and its use for the Saint-Venant equations. However, backstepping gives rise to non-local and non-static feedback laws that are likely to be harder to implement, and, to our knowledge, have not been implemented yet.
Most of the previous results were performed with static proportional feedback controllers. When it comes to industrial applications, however, the proportional integral (PI) control is by far the most popular regulator. It is used for instance for the regulation of the Sambre and Meuse river in Belgium [4, Chapter 8]. The reason behind such preference is the robustness of the PI control with off-set errors [1, Chap. 11.3]. An example can be found in [14] where the authors show the interest of adding an integral term to a proportional control on a linear and homogeneous system, and exhibit coherent experimental result.

For these reasons, the PI controller has fed a wide literature, at least when used on finite dimensional systems. However, despite their undisputable practical interest, PI controllers for nonlinear infinite dimensional systems have shown hard to handle mathematically and even studying simple systems give sometimes rise to lengthy proofs with relatively sophisticated tools [10]. While the behaviour and the stability of linearized equations with PI controller has been well understood in the past, partly thanks to spectral tools like the spectral mapping theorem (e.g. [23, 22] for hyperbolic systems), no such tools exist for nonlinear systems and the stability of the nonlinear Saint-Venant equations has remained a challenge until today. Among the existing linear result using a spectral approach on can refer to [29, 30] where the authors finds a sufficient condition for the stabilization of the linearized inhomogeneous Saint-Venant equations. Necessary and sufficient conditions for the linearized homogeneous Saint-Venant equations are given in [4] Section 2.2.4.1, 3.4.4]. In [12] the authors find a necessary and sufficient condition for a linear scalar equation and show the difficulty of finding good conditions for the nonlinear equation, while in [9] the authors deal with 2 × 2 systems. Among the existing nonlinear results one can refer to [26] in the case where the operator without PI control generates an exponentially stable semi-group, [27] where the authors finds a sufficient condition for the nonlinear homogeneous Saint-Venant equations, [4] 2.2.4.2 where the authors find a necessary and sufficient condition also for the nonlinear homogeneous Saint-Venant equations, while [4] 5.4.4.5] give a sufficient condition for the inhomogeneous Saint-Venant equations for a single channel or a network, but in the particular case of constant steady-states only, which simplifies their analysis [17]. Strictly speaking this last result was derived for the linearized system but with a Lyapunov approach which can easily be generalized to the nonlinear system. More recently, and to our knowledge this is the most advanced result, [6] gave a sufficient condition of stability for the inhomogeneous Saint-Venant equations with an arbitrary friction and river length but only in the absence of slope, using a Lyapunov approach.

In this paper, we consider the stabilization of the general nonlinear Saint-Venant equations with a single boundary PI control. We give a simple and explicit condition on the parameters of the PI controller such that any steady-state is exponentially stable for the $H^2$ norm. While stability results in inhomogeneous and nonlinear systems often raise to a limit length for the domain, depending on the source term, above with we are unable to guaranty any stability ([17, 18, 3, 15] or [4, Chap. 6]), this result holds whatever the friction, the slope, and the length of the channel. Besides, our condition is independent of the slope, the friction coefficient, the river length, and, more surprisingly, can be made independent of the steady-state considered. Finally, when there is no slope this condition is less restrictive that the condition obtained in [6] and when there is no friction or slope this condition coincides with the necessary and sufficient spectral condition of stability for the linearized system given in [5] and [4] Theorem 2.7]. The case where the inflow disturbances are time dependant and no steady-states exists was seldom considered in the literature. However, it is in fact unlikely that the industrial target state is a real steady-state as the inflow disturbance often depends on time in practice, even though only slowly. Therefore, in the more general framework of slowly time-varying target states, we show the Input-to-State Stability (ISS) of the system with respect to the variation of the inflow disturbance. Finally, we show that if we allow the controller to depend on the target state, by changing slightly the PI controller, we can ensure the exponential stability of slowly-varying target trajectories that are the natural target trajectories to consider when there is no steady-state to the system.

This paper is organised as follows: in Section 1 we give a description of the nonlinear Saint-Venant equations, we introduce the time-varying target trajectories together with some definitions and existence results, then we state our main results. In Section 2 we prove our main result Theorem 1.3 that deals with the exponential stability of time-varying state. In Appendix, we show that Corollary 1 dealing with the exponential stability of steady-states, and Theorem 1.4 showing the ISS of the system with respect to the variation of the inflow disturbance, are both deduced from the proof of Theorem 1.3.
1 Model description

We consider the following nonlinear Saint-Venant equations for a rectangular channel with arbitrary slope and friction.

\[
\begin{align*}
\partial_t H + \partial_x(HV) &= 0, \\
\partial_t V + V \partial_x V + g \partial_x H + \left( \frac{k V^2}{H} - C(x) \right) &= 0.
\end{align*}
\] (1.1)

Here, \( k \) is an arbitrary nonnegative friction coefficient and \( C \) denotes the slope, which is assumed to be a \( C^2 \) function, with \( C(x) := -g dB/dx \) where \( B \) is the bathymetry and \( g \) the acceleration of gravity. We are interested in systems where the water flow uphill is a given function, unknown and imposed by external conditions, for instance a flow coming from another country, while the water flow downhill is controlled through a hydraulic installation. Therefore we have the following boundary conditions,

\[
\begin{align*}
H(t, 0) &= V(t, 0) = Q_0(t), \\
H(t, L) &= V(t, L) = U(t),
\end{align*}
\] (1.2)

where \( U(t) \) is a control feedback and \( Q_0(t) \) is the incoming flow, which is a given (and unknown) function. Here \( L \) denotes the length of the water channel. In practical situations, the formal control \( U(t) \) can be expressed by a simple linear model \[6\]

\[
U(t) = v_G(H(t, L) - U_1(t)),
\] (1.3)

where \( U_1(t) \) is the elevation of the gate of the dam, which is the real control input that can be chosen, while \( v_G \) is a constant depending on the parameters of the gate (potentially unknown as well).

Usually, the industrial goal of such system is to stabilize the level of the water at the end point \( H(t, L) \), called control point, to a target value \( H_c > 0 \). On the other hand, the usual mathematical goal in such problem is to stabilize a target steady-state \( (H^*, V^*) \), potentially nonuniform \[4\][Preface]. However, in the present problem (1.1)–(1.2), it is clear that, when \( Q_0 \) is not constant, it is impossible to aim at stabilizing any steady-state and one needs to aim at stabilizing other target trajectories. Therefore, we define the following target trajectory \((H_1, V_1)\) that we aim stabilizing as the solution of

\[
\begin{align*}
\partial_t H_1 + \partial_x(H_1 V_1) &= 0, \\
\partial_t V_1 + V_1 \partial_x V_1 + g \partial_x H_1 + \left( \frac{k V_1^2}{H_1} - C(x) \right) &= 0, \\
H_1(t, 0) &= V_1(t, 0) = Q_0(t), \\
H_1(t, L) &= H_c,
\end{align*}
\] (1.4)

with the initial condition

\[
H_1(0, \cdot) = H^*(\cdot) \text{ and } V_1(0, \cdot) = V^*(\cdot),
\] (1.5)

where \((H^*, V^*)\) is the (unique) steady-state solution of the system when \( Q_0 \) is constant, equal to \( Q_0(0) \). Namely \((H^*, V^*)\) is the solution of

\[
\begin{align*}
\partial_x(HV) &= 0, \\
V \partial_x V + g \partial_x H + \left( \frac{k V^2}{H} - C(x) \right) &= 0, \\
H(L) &= H_c,
\end{align*}
\] (1.6)

with condition at \( x = 0 \)

\[
H^*(0)V^*(0) = Q_0(0).
\] (1.7)

We are now going to show that the trajectory \((H_1, V_1)\) exists for any time and satisfies some bounds.
Existence and bounds of the target trajectory Instead of studying directly our target trajectory \((H_1, V_1)\) we first construct an intermediary family of functions \((H_0, V_0)\). We defined previously \((H^*, V^*)\) as the steady-state associated to a constant flux \(Q_0 \equiv Q_0(0)\), i.e. \((H^*, V^*)\) is the solution of the ODE problem \((1.6)\) with initial condition \((0^+V^*)(0) = Q_0(0)\). But in fact at each time \(t^* \in \mathbb{R}^*_+\), we can define a steady-state \((H^*_{t^*}, V^*_{t^*})\) associated to a constant flux \(Q_0 \equiv Q_0(t^*)\), i.e. \((H^*_{t^*}, V^*_{t^*})\) is the solution of the ODE problem \((1.6)\) with initial condition satisfying

\[
H^*_{t^*}(0)V^*_{t^*}(0) = Q_0(t^*). \tag{1.8}
\]

This problem could seem peculiar as all conditions should be imposed exclusively in 0 or in \(L\) to ensure the well-posedness. However looking at the first equation of \((1.6)\), the problem \((1.6), (1.8)\) is in fact equivalent to a single ODE on \(H^*_{t^*}\) with boundary condition \(H^*_{t^*}(L) = H_c\) and \(V^*_{t^*}\) defined by \(V^*_{t^*} = Q_0(t)/H^*_{t^*}\). Thus for each \(t^* \in [0, +\infty)\) such function exists on \([0, L]\), is unique and \(C^3\) provided that the state stays in the fluvial regime (or subcritical regime), i.e. \(gH^*_{t^*} > V^*_{t^*}e^2\) on \([0, L]\), which, for a given \(H_c\), is equivalent to a bound on \(Q_0(t^*)\) (see [19] for more details). As we are interested in stabilizing physical trajectories in the fluvial regime, we assume that this assumption is satisfied in the following and that there exist \(\alpha > 0\) and \(H_\infty > 0\) independent of \(t^* \in [0, \infty)\) such that

\[
H^*_{t^*} < \frac{1}{2}H_\infty \text{ on } [0, L],
\]

\[
gH^*_{t^*} - V^*_{t^*} > 2\alpha \text{ on } [0, L]. \tag{1.9}
\]

For a given \(H_c\), this is again equivalent to imposing a bound \(Q_\infty\) on \(\|Q_0\|_{L^\infty([0,\infty))}\) from \((1.6)\) and \((1.8)\), which would be more logical. However, for convenience, we will still use \(H_\infty\) and \(\alpha\) in the following. This assumptions is quite physical, especially as in practical situation the river is in fluvial regime and \(Q_0(t)\) is often periodic or quasi-periodic. This gives a family of one-variable functions indexed by a parameter \(t^*\), which can also be seen as the two-variable functions \((H_0, V_0) : (t, x) \rightarrow (H^*_{t^*}(x), V^*_{t^*}(x))\). Besides, from \((1.7)\), as \((H^*_{t^*}, V^*_{t^*})\) is the solution of a system of ODE with a parameter \(t\), the two variable functions \((H_0, V_0)\) therefore belongs to \(C^3([0, +\infty) \times (0, L))\) (see [19][Chap. 5, Cor. 4.1]). And from its definition, one can note that \((H_0(0, \cdot), V_0(0, \cdot)) = (H^*, V^*)\). Now that we have introduced this intermediary family of functions, we can show the existence of the target trajectory \((H_1, V_1)\) and we have the following Input-to-State Stability (ISS) result (see [23] for a definition of ISS for finite dimensional systems, [20] Chap 1, Chap 3 for a generalization to first-order hyperbolic PDE and [24] for the use of Lyapunov function to achieve ISS on time-varying hyperbolic systems),

**Proposition 1.1.** There exist positive constants \(c_1, c_2\) such that if \(\partial_t Q_0 \in C^2([0, \infty))\), there exist \(\mu > 0\), \(\nu > 0\) and \(\delta > 0\) such that if \(\|\partial_t Q_0\|_{C^2([0, +\infty))} \leq \delta\), then for any \((H_1^0, V_1^0) \in H^2(0, L, \mathbb{R}^2)\) such that

\[
\|H_1^0 - H^*\|_{H^2(0, L)} + \|V_1^0 - V^*\|_{H^2(0, L)} \leq \nu,
\]

the system \((1.4)\) with initial condition \((H_1^0, V_1^0)\) has a unique solution \((H_1, V_1) \in C^0([0, +\infty), H^2(0, L))\) which satisfies the following ISS inequality

\[
\|H_1(t, \cdot) - H_0(t, \cdot)\|_{H^2(0, L)} + \|V_1(t, \cdot) - V_0(t, \cdot)\|_{H^2(0, L)} \leq c_1(\|H_1^0 - H^*\|_{H^2(0, L)} + \|V_1^0 - V^*\|_{H^2(0, L)} e^{-\frac{\mu}{2}} + c_2 \left( \int_0^t \|\partial_t Q_0(s) + \partial_{tt} Q_0(s) + \partial_{ttt} Q_0(s) e^{\frac{\mu}{2}} ds \right) e^{-\frac{\mu}{2}}. \tag{1.10}
\]

This result is shown in Appendix A and a definition of the \(C^2\) norm is recalled in Remark 1.1. Note that \(Q_0\) is supposed to be bounded, which is quite physical, but there is no additional requirement on this bound besides the physical assumption given by \(Q_\infty\) of remaining in the fluvial regime. This is important as in practical situations the value of the incoming flow can change a lot, even though slowly.
Here, we choose to stabilize the trajectory \((H_1, V_1)\) associated to \(H_0^0 = H^*\) and \(V_0^0 = V^*\). As we will see, this target trajectory can be seen as the natural trajectory to stabilize as it satisfies the industrial goal \(H(t, L) = H_c\) and it coincides with the steady-state solution when \(Q_0\) is a constant. In this last case \(Q_0\) and \(H_c\) are imposed and \(H^*\) and \(V^* = Q_0/H^*\) are thus fully determined using \((1.6)\). But one can note from \((1.10)\) that, in fact, the behaviour of \((H_1, V_1)\) at large time does not depend on the initial condition \((H_0^0, V_0^0)\) in \((1.5)\), provided that it is close in \(H^2\) norm to \((H^*, V^*)\).

**Remark 1.1.** The same ISS result can be shown replacing the \(H^2\) norm in Proposition 1.1 by the \(H^p\) norm where \(p \in \mathbb{N}^* \setminus \{1\}\), with the condition \(\|\partial_t Q_0\|_{C^p([0, +\infty))} \leq \delta\) instead of \(\|\partial_t Q_0\|_{C^2([0, +\infty))} \leq \delta\). This is shown in Appendix A. We define here the \(C^p\) norm for a function \(U \in C^p(I)\), where \(I\) is an interval, as

\[
\|U\|_{C^p(I)} = \max_{i \in [0, p]} \|\partial_t^i U\|_{L^\infty(I)}
\]

Thus, from Proposition 1.1 and (1.9), there exists a constant \(\delta > 0\) such that, if \(\|\partial_t Q_0\|_{C^2([0, +\infty))} < \delta\), then \((H_1, V_1) \in C^0([0, +\infty), \mathbb{H}^2(0, L))\) and

\[
H_1(t, x) < H_\infty, \quad \forall (t, x) \in [0, +\infty) \times [0, L],
\]

(1.12)

\[
gH_1(t, x) - V_1^2(t, x) > \alpha, \quad \forall (t, x) \in [0, +\infty) \times [0, L].
\]

(1.13)

Besides, when \(Q_0\) is a constant, it is easy to check that \((H_0, V_0) = (H^*, V^*)\) is also solution of \((1.4), (1.5)\). Thus, from the uniqueness of the solution of \((1.4), (1.5)\), \((H_1, V_1) = (H^*, V^*)\) and therefore we recover a steady-state. This illustrates that \((H_1, V_1)\) can be seen as the natural target state when \(Q_0\) is not a constant anymore. Moreover, from \((1.4)\), stabilizing \((H_1, V_1)\) also satisfies the industrial goal by stabilizing \(H(t, L)\) on the value \(H_c\).

**Control design and main result** As mentioned in the introduction, a usual type of controller used in practice to reach this aim is the proportional-integral (PI) controller. It has the advantage of eliminating the offset coming from constant load disturbances, which can usually appear in these systems as the command on the gate’s level are only known up to some constant incertainties. A generic PI controller is given by

\[
U_1(t) = k_p(H_c - H(t, L)) + k_I Z,
\]

(1.14)

where \(k_p\) and \(k_I\) are coefficients that can be designed and \(Z\) accounts for the integral term, i.e.

\[
\dot{Z} = H_c - H(t, L).
\]

(1.15)

With such controller, and using (1.3), the boundary conditions (1.2) become (1.15) and

\[
H(t, 0)V(t, 0) = Q_0(t),
\]

\[
H(t, L)V(t, L) = v_G(1 + k_p)H(t, L) - v_G k_p H_c - v_G k_I Z,
\]

(1.16)

In Corollary 1, we show that this boundary control can be used to stabilize exponentially a steady-state when \(Q_0\) is a constant. In Theorem 1.4, we show that this control can also provide an Input-to-State Stability property with respect to \(\partial_t Q_0\). However, this control (1.14) cannot be used to stabilize a dynamic target trajectory \((H_1, V_1)\), as there is no function \(Z_t \in C^1([0, +\infty))\) such that \((H_1, V_1, Z_t)\) is a solution of (1.1), (1.15), (1.16) while \((H_1, V_1)\) is a solution of (1.4). Therefore, when stabilizing a dynamic target trajectory, one has to add an additional term and use

\[
U_1(t) = k_p(H_c - H(t, L)) + k_I Z - f(t),
\]

(1.17)

where \(f(t) := H_1(t, L)V_1(t, L)/v_G\). The boundary conditions (1.2) become then

\[
H(t, 0)V(t, 0) = Q_0(t),
\]

\[
H(t, L)V(t, L) = H_1V_1(t, L) + v_G(1 + k_p)(H(t, L) - H_c) - v_G k_I Z,
\]

(1.18)
where we have actually changed \(Z\) and re-define \(Z := Z - k_p/k_I\), which still satisfies the equation (1.15). This new control (1.17) assumes that \(V_1(t, L)\) is known at least up to a constant, as \(H_1(t, L) = H_c\) and additional constants can be incorporated into \(Z\). When no knowledge on the target state is available besides \(H_c\), it is impossible to stabilize exponentially the system, and the best one can get is the Input-to-State Stability which is given by Theorem 1.4. However in the following we will keep working with (1.17) and (1.18) to show Theorem 1.3 and the exponential stability of the system, as the proof of Theorem 1.4 and Corollary 1 which uses only the control (1.14) and (1.16) are easily deduced from the proof of Theorem 1.3.

We introduce the first-order compatibility conditions associated to the boundary conditions (1.18) for an initial condition \((H^0, V^0, Z^0)\)

\[ H^0(0)V^0(0) = Q_0(0), \]
\[ H^0(L)V^0(L) = H_1V_1(0, L) + v_G(1 + k_p)(H^0(L) - H_c) - k_f Z^0, \]
\[ - \partial_x(H^0(0)V^0(0) + g\frac{H^0(0)}{2}) - (k(V^0)^2(0) - CH^0(0)) = Q'_0(0), \]
\[ - \partial_x(H^0(L)V^0(L) + g\frac{H^0(L)}{2}) - (k(V^0)^2(L) - CH^0(L)) = \partial_t(H_1V_1)(0, L) - v_G(1 + k_p)\partial_x(H^0(L)V^0(L)) + k_f(H^0(L) - H_c). \]

With such compatibility conditions the system (1.1), (1.15), (1.18) is well-posed and we have the following theorem due to Wang [28][Theorem 2.1]:

**Theorem 1.2.** Let \(T > 0\), and assume that \(\|\partial_t Q_0\|_{C^2([0, +\infty))} \leq \delta(T)\), such that \((H_1, V_1)\) is well-defined and belongs to \(C^2([0, T], H^2(0, L))\). There exists \(\nu(T) > 0\) such that for any \((H^0, V^0, Z^0) \in (H^2((0, L))))^2 \times \mathbb{R}\)

\[ \|H^0(\cdot) - H_1(0, \cdot)\|_{H^2((0, L))} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2((0, L))} + |Z^0| \leq \nu(T), \]

And satisfying the compatibility conditions (1.19), the system (1.1), (1.15), (1.18) has a unique solution \((H, V, Z) \in (C^2([0, T], H^2((0, L))))^2 \times C^1([0, T])\). Moreover there exists a positive constant \(C(T)\) such that

\[ \|H(t, \cdot) - H_1(t, \cdot)\|_{H^2((0, L))} + \|V(t, \cdot) - V_1(t, \cdot)\|_{H^2((0, L))} + |Z| \leq C(T) \left( \|H^0(\cdot) - H_1(0, \cdot)\|_{H^2((0, L))} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2((0, L))} + |Z^0| \right). \]

To apply the result from [28], note that \(Z\) can be seen as a third component of the hyperbolic system with a null propagation speed, a constant initial condition \(Z^0\) and \(Z(t)\) being itself zero everywhere on \([0, L]\) including at the boundaries.

**Remark 1.2.** If, in addition, \((H^0, V^0) \in H^2((0, L) ; \mathbb{R}^2)\), then the unique solution \((H, V, Z)\) given by Theorem 1.2 belongs to \(C^2([0, T], H^3((0, L)) ; \mathbb{R}^2) \times C^1([0, T])\) and there exists a constant \(C(T)\) such that

\[ \|H(t, \cdot) - H_1(t, \cdot)\|_{H^3((0, L))} + \|V(t, \cdot) - V_1(t, \cdot)\|_{H^2((0, L))} + |Z| \leq C(T) \left( \|H^0(\cdot) - H_1(0, \cdot)\|_{H^2((0, L))} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2((0, L))} + |Z^0| \right). \]

We recall the definition of exponential stability

**Definition 1.1.** We say that a trajectory \((H_1, V_1)\) is exponentially stable for the \(H^2\) norm if there exists \(\nu > 0, C > 0, \gamma > 0\) such that for any \(T > t_0 \geq 0\) and any \((H^0, V^0, Z^0)\) satisfying

\[ \|H^0(\cdot) - H_1(t_0, \cdot)\|_{H^2((0, L))} + \|V^0(\cdot) - V_1(t_0, \cdot)\|_{H^2((0, L))} + |Z^0| \leq \nu, \]

and the compatibility conditions (1.19), the system (1.1), (1.15), (1.18) with initial condition \((H^0, V^0, Z^0)\) at \(t_0\) has a unique solution \((H, V, Z) \in (C^2([t_0, T], H^2((0, L))))^2 \times C^1([t_0, T])\) and,

\[ \|H(t, \cdot) - H_1(t, \cdot)\|_{H^2((0, L))} + \|V(t, \cdot) - V_1(t, \cdot)\|_{H^2((0, L))} + |Z| \leq Ce^{-\gamma t} \left( \|H^0(\cdot) - H_1(t_0, \cdot)\|_{H^2((0, L))} + \|V^0(\cdot) - V_1(t_0, \cdot)\|_{H^2((0, L))} + |Z^0| \right), \forall t \in [t_0, +\infty). \]
Remark 1.3. From (1.4) and Sobolev inequality, this exponential stability implies in particular the exponential convergence of $H(t, L)$ to $H_c$.

We can now state the main results of this article.

**Theorem 1.3.** There exists $\delta > 0$ such that if $\| \partial_t Q_0 \|_{C^2((0, +\infty))} \leq \delta$, then the trajectory $(H_1, V_1)$ given by (1.4) of system (1.1), (1.15), (1.18) is exponentially stable for the $H^2$ norm if:

$$k_p > -1 \text{ and } k_I > 0,$$

or $k_p < -1 - \frac{gH_1(t, L) - V_1^2(t, L)}{v_G V_1(t, L)}$ and $k_I < 0$. (1.25)

This result is proved in Section 2. The main idea of the proof consists in finding a local convex and dissipative entropy for the system (1.1), (1.15), (1.18).

In particular, in the case where $Q_0$ is constant, we can use the static boundary control (1.14), and we have the following corollary:

**Corollary 1.** If $Q_0$ is constant, then the steady-state $(H^*, V^*)$ of the system (1.1), (1.15), (1.16) given by (1.6)–(1.7) is exponentially stable for the $H^2$ norm if:

$$k_p > -1 \text{ and } k_I > 0,$$

or $k_p < -1 - \frac{gH^*(L) - V^{*2}(L)}{v_G V^*(L)}$ and $k_I < 0$. (1.26)

**Proof.** This is a particular case of Theorem 1.3. To see this, note, as mentioned earlier, that when $Q_0$ is constant, then $(H_1, V_1) = (H^*, V^*)$. Then, observe that $f(t)$ given in (1.17) is a constant that can be added in $Z$ (i.e. we can re-define $Z := Z - f(t)$, which still satisfies (1.15)).

**Remark 1.4.** In the literature, results about PI control of the Saint-Venant equations sometimes leave the step of modeling the spillway and use a generic formulation of the PI control on the outflow rate of the form

$$H(t, L)V(t, L) = k_1(H(t, L) - H_c) - k_2 Z,$$

where $Z$ is the integral term, still given by (1.15). Note that, with these notations, the sufficient condition of Corollary 1 becomes

$$k_p > 0 \text{ and } k_I > 0,$$

or $k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)}$ and $k_I < 0$. (1.28)

which is a known result in the linear case using a spectral approach. Theorem 1.3 and Corollary 1 show that this result remains true when the system is nonlinear, using a Lyapunov approach.

**Remark 1.5.** When the system is homogeneous, conditions (1.26) are optimal (necessary and sufficient) [3], [4, Section 2.2.4.1].

This approach uses very little knowledge of the state of the system, as we only measure the height at the boundary $x = L$. In practical situations, however, we may have also little knowledge of the target trajectory $(H_1, V_1)$ or the input disturbance $Q_0(t)$ and we only know $H_c$. In this case we cannot use a controller of the form (1.18), but only a static controller of the form (1.16), namely

$$H(t, L)V(t, L) = v_G(1 + k_p)H(t, L) - v_G k_p H_c - v_G k_I Z.$$ (1.29)

In this case, it is impossible to aim at stabilizing the target trajectory $(H_1, V_1)$, but we still have the Input-to-state Stability with respect to the input disturbance $\partial_t Q_0$. 7
**Theorem 1.4.** There exists $\nu > 0$, $\delta > 0$, $\gamma > 0$ and $C$, such that if $\|\partial_t Q_0\|_{C^2([0, +\infty))} \leq \delta$, then for any $T > 0$ and $(H^0, V^0) \in (H^2(0, L))^2$ such that

$$
\|H^0 - H^*\|_{H^2(0, L)} + \|V^0 - V^*\|_{H^2(0, L)} \leq \nu,
$$

the system (1.1), (1.15), (1.16) with initial condition $(H^0, V^0)$ has a unique solution $(H, V) \in C^0([0, T], H^2(0, L))$ which satisfies the following ISS inequality

$$
\|H(t, \cdot) - H_0(t, \cdot)\|_{H^2(0, L)} + \|V(t, \cdot) - V_0(t, \cdot)\|_{H^2(0, L)} 
\leq Ce^{-\gamma t} \left(\|H^0 - H^*, V^0 - V^*\|_{H^2(0, L)} + \int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)| e^{\gamma s} ds\right).
$$

The proof is given in Appendix B and is a consequence from the proof of Theorem 1.3.

In Section 2 we prove Theorem 1.3.

2 Exponential stability for the $H^2$ norm

This section is divided in three parts. First we transform the system through a change of variables. Then we state three lemma, useful for the analysis. Finally we prove Theorem 1.3.

2.1 A change of variables

For any solution of (1.1), (1.15), (1.18) we define the perturbation as

$$
\begin{pmatrix}
h_v
\end{pmatrix} = \begin{pmatrix}
H - H_1
V - V_1
\end{pmatrix}.
$$

(2.1)

Let us assume that there exists $\nu \in (0, \nu_0)$ to be selected later on, such that

$$
\|H^0(\cdot) - H_1(0, \cdot)\|_{H^2(0, L)} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2(0, L)} + |Z^0| \leq \nu.
$$

(2.2)

The boundary conditions (1.18) can be written in the following form

$$
v(t, 0) = B_1(h(t, 0), t),
$$

$$
v(t, L) = B_2(h(t, L), Z, t),
$$

(2.3)

with

$$
\partial_t B_1(0, t) = -\frac{V_1(t, 0)}{H_1(t, 0)},
$$

$$
\partial_t B_2(0, 0, t) = \frac{V_1(1 + k_p) - V_1(t, L)}{H_1(t, L)},
$$

$$
\partial_B B_2(0, 0, t) = -\frac{V_1 k_l}{H_1(t, L)}.
$$

(2.4)

We introduce the following change of variables:

$$
\begin{pmatrix}
u_1
\nu_2
\end{pmatrix} = \begin{pmatrix}
v + \sqrt{\frac{\nu}{H_1^0}} h
v - \sqrt{\frac{\nu}{H_1^0}} h
\end{pmatrix}.
$$

(2.5)
Note that this change of variables is very similar to the change of variables used in [3, 19] with the only difference that \( (H_1, V_1) \) is not a steady-state anymore. It corresponds to the transformation in Riemann coordinates for the perturbations. Indeed, denoting \[ S = \frac{g}{H_{1(t,x)}}, \]
and using (1.1), (1.15), (1.18), (1.4), (2.1)–(2.5), one has
\[ A(u, x, t) = \left( \frac{A_1(u, x, t)}{A_2(u, x, t)} \right) = S(x, t) F \left( \frac{H_{1(t,x)}}{V_{1(t,x)}} \right) S^{-1}(x, t), \]
\[ B(u, x, t) = \left( \frac{B_1(u, x, t)}{B_2(u, x, t)} \right) = S(x, t) F \left( \frac{H_{1(t,x)}}{V_{1(t,x)}} \right) \left( \frac{\partial_t H_{1(t,x)}}{\partial_x V_{1(t,x)}} + \partial_x (S^{-1}(x,t)) \right) \]
and thus
\[ A_1(0, x, t) = V_1 + \sqrt{gH_1}, \quad A_2(0, x, t) = V_1 - \sqrt{gH_1}, \]
\[ l_1(0, x, t) = B_1(0, x, t) = B_2(0, x, t) = 0, \]
\[ \frac{\partial B_1}{\partial u}(0, x, t) = \gamma_1(t, x)u_1(t, x) + \gamma_2(t, x)u_2(t, x), \]
\[ \frac{\partial B_2}{\partial u}(0, x, t) = \delta_1(t, x)u_1(t, x) + \delta_2(t, x)u_2(t, x). \]

where
\[
\gamma_1 = \frac{3}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{3}{4} V_{1x} + \frac{kV_1}{H_1} - \frac{kV_1^2}{2H_1^2} \sqrt{\frac{H_1}{g}}
\]
\[
\gamma_2 = \frac{1}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{1}{4} V_{1x} + \frac{kV_1}{H_1} + \frac{kV_1^2}{2H_1^2} \sqrt{\frac{H_1}{g}}
\]
\[
\delta_1 = -\frac{1}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{1}{4} V_{1x} + \frac{kV_1}{H_1} - \frac{kV_1^2}{2H_1^2} \sqrt{\frac{H_1}{g}}
\]
\[
\delta_2 = -\frac{3}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{3}{4} V_{1x} + \frac{kV_1}{H_1} + \frac{kV_1^2}{2H_1^2} \sqrt{\frac{H_1}{g}}
\]

And for the boundary conditions, there exists \( \nu_1 \in (0, \nu_0) \) such that for any \( \nu \in (0, \nu_1) \), one has:
\[
u_1(t, 0) = D_1(u_2(t, 0), t),
\]
\[
u_2(t, L) = D_2(u_1(t, L), Z, t),
\]
\[ \frac{\partial Z}{\partial t} = \left( \frac{u_1(t, L) - u_2(t, L)}{2} \right) \sqrt{\frac{H_{1(t,L)}}{g}}, \]
where $\mathcal{D}_1$ and $\mathcal{D}_2$ are $C^2$ functions and

$$
\partial_1 \mathcal{D}_1(0, t) = \frac{\lambda_2(0)}{\lambda_1(0)},
$$
$$
\partial_1 \mathcal{D}_2(0, 0, t) = -\frac{\lambda_1(L) - \nu G(1 + k_p)}{\lambda_2(L) + \nu G(1 + k_p)},
$$
$$
\partial_2 \mathcal{D}_2(0, 0, t) = -2 \frac{\nu G k_1}{\nu G(1 + k_p) + \lambda_2(t, L)}.
$$

Expression (2.14) is simply a computation, very similar to what it done in [19] for instance, while the derivation of (2.15) and (2.16) are detailed in the appendix.

**Remark 2.1.** Obviously, from the change of variables (2.1), (2.5), the exponential stability of the system (1.1), (1.15), (1.18) is equivalent to the exponential stability of the steady-state $u^* = 0$ for the system (2.8), (2.15).

As the operator $A$, given by (2.9), is a $C^2$ function in $u, t$ and $x$ (and in particular $C^1$) and as, from (2.13) and (1.13), $\lambda_1(0, t) > 0 > \lambda_2(0, x, t)$, there exists $\nu_2 \in (0, \nu_1)$ and $E \in C^1(\mathcal{B}_{\nu_2} \times (0, L) \times [0, +\infty); \mathcal{M}_2(\mathbb{R}))$, where $\mathcal{B}_{\nu_2} \subset \mathbb{R}^2$ is the disc of radius $\nu_2$, such that for any $\|u(t, \cdot)\|_{H^2(0, L)} \leq \nu_2$,

$$
E(u(t, x, t), x, t) A(u(t, x, t), x, t) = D(u(t, x), x, t) E(u(t, x), x, t),
$$
$$
E(0, x, t) = I_d,
$$

where $D(u(t, x), x, t) = (D_i(u(t, x), x, t))_{i \in 1, 2}$ is a diagonal matrix and $I_d$ is the identity matrix. Before going any further, let us note a few useful properties of these functions. For simplicity in the following we will denote for any $n \in \mathbb{N}^+$ and any function $U \in L^\infty((0, T) \times (0, L); \mathbb{R}^n)$ (resp. $L^\infty((0, L); \mathbb{R}^n)$)

$$
\|U\|_{\infty} := \|U\|_{L^\infty((0, T) \times (0, L); \mathbb{R}^n)},
$$
$$
\text{(resp.)}\|U\|_{\infty} := \|U\|_{L^\infty((0, L); \mathbb{R}^n)}.
$$

We may also denote $\|u\|_{H^2(0, L)}$ instead of $\|u(t, \cdot)\|_{H^2(0, L)}$ to lighten the expressions. From the definition of $A$ given in (2.9), and from (1.13), for $\|u\|_{H^2(0, L)} \leq \nu_2$, there exists a constant $C_1$ depending only on $H_\infty, \alpha$ and $\nu_2$ such that we have the following estimates

$$
\max (\|\partial_t(A(u(t, x, t), x, t) - A(0, x, t))\|_{\infty}, \|\partial_t(D(u(t, x), x, t) - D(0, x, t))\|_{\infty}, \|\partial_t(E(u(t, x), x, t))\|_{\infty})
$$
$$
\leq C_1 (\|u\|_{\infty}(\|\partial_t H_1\|_{\infty} + \|\partial_t V_1\|_{\infty} + \|\partial_t u\|_{\infty}),
$$
$$
\max (\|\partial_t(A(u(t, x, t), x, t) - A(0, x, t))\|_{\infty}, \|\partial_t(D(u(t, x), x, t) - D(0, x, t))\|_{\infty}, \|\partial_t(E(u(t, x), x, t))\|_{\infty})
$$
$$
\leq C_1 (\|u\|_{\infty}(\|\partial_t H_1\|_{\infty} + \|\partial_t V_1\|_{\infty} + \|\partial_t u\|_{\infty})).
$$

For $E$ and $D$, this comes from the fact that $E$ and $D$ are $C^\infty$ functions with respect to the coefficients of $A$ (note that $D$ is the matrix of eigenvalues of $A$), and that $A \in C^2(\mathcal{B}_{\nu_2}; C^1([0, +\infty) \times [0, L]))$.

### 2.2 Three useful lemma

We introduce now three lemma, which will be useful in the following analysis. The first one is a classical result about Lyapunov functions,

**Lemma 2.1.** Let $V : (H^2(0, L))^2 \times \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that there exists a constant $c > 0$ such that

$$
c (\|U\|_{H^2(0, L)} + |z|) \leq V(U, z, t) \leq \frac{1}{c} (\|U\|_{H^2(0, L)} + |z|), \quad \forall (U, z, t) \in (H^2(0, L))^2 \times \mathbb{R} \times \mathbb{R}_+.
$$

If there exists $\gamma > 0$ and $\delta > 0$ such that, for any solution $(u, Z)$ of the system (2.8), (2.15) with initial conditions satisfying $\|u(0, \cdot)\|_{H^2(0, L)} + |Z(0)| \leq \delta$,

$$
\frac{d}{dt}[V(u(t, \cdot), t)] < -\gamma V(u(t, \cdot), t)
$$

(2.21)
in a distribution sense, then the system (2.8), (2.15) is exponentially stable for the \( H^2 \) norm and \( V \) is called a Lyapunov function for the system (2.8), (2.15).

This first lemma reduces the problem of proving the exponential stability to finding a Lyapunov function \( V \) for the system (2.8), (2.15). A proper definition of a differential inequality in a distribution sense as in (2.21) can be found in [17]. To lighten this article we do not give a proof of this classical lemma, although a proof for a very similar case (Lyapunov function that does not depend explicitly on time and for the \( C^1 \) norm instead) can be found for instance in [17][Proposition 2.1], and is easily extended to this case.

The second Lemma is a variation of a result shown in [19] that gives a local entropy of the Saint-Venant equations. Let us first introduce the following function \( \phi \) defined by

\[
\phi_1(t, x) = \exp \left( \int_0^x \frac{\gamma_1}{\lambda_1} \, dx \right), \\
\phi_2(t, x) = \exp \left( -\int_0^x \frac{\delta_2}{\lambda_2} \, dx \right), \\
\phi(t, x) = \frac{\phi_1(t, x)}{\phi_2(t, x)},
\]

where \( \lambda_1 \) and \( \lambda_2 \) are defined by

\[
\lambda_1(t, x) := \Lambda_1(0, x, t) > 0, \quad \lambda_2(t, x) := -\Lambda_2(0, x, t) > 0.
\]

We can now state the following lemma

**Lemma 2.2.** There exists \( \delta_0 > 0 \) such that if \( \| \partial_t H_1 \|_{C^1([0, +\infty) \times (0, L])} \leq \delta_0 \), the function \( \lambda_2 \phi / \lambda_1 \) is solution on \([0, L]\) to the following equation

\[
\partial_x f = \frac{\phi \gamma_2}{\lambda_1} + \frac{\phi^{-1} \delta_1}{\lambda_2} f^2 + \sqrt{\frac{g}{H_1}} \partial_x H_1, \quad \forall \, x \in [0, L], \ t \in [0, +\infty),
\]

and for any \( x \in [0, L] \) and any \( t \in [0, +\infty) \),

\[
\left( \frac{\phi \gamma_2}{\lambda_1} + \frac{\phi^{-1} \delta_1}{\lambda_2} f^2 + \sqrt{\frac{g}{H_1}} \partial_x H_1 \right) > 0.
\]

The proof is given in the Appendix.

Eventually, we introduce our last Lemma, which seems very natural and is stated here to lighten the proof of Theorem 1.3

**Lemma 2.3.** There exists \( l > 0 \) and \( C > 0 \) such that if \( \| \partial_t Q_0 \|_{C^3([0, +\infty))} \leq l \), then

\[
\max \left( \| \partial_t H_1 \|_{C^1([0, +\infty), L^\infty(0, L))}, \| \partial_t V_1 \|_{C^1([0, +\infty), L^\infty(0, L))} \right) < C \| \partial_t Q_0 \|_{C^3([0, +\infty))}.
\]

This is a consequence of the ISS property (Proposition 1.1) and Remark 1.1 with \( p = 3 \), the relations (1.4), and Sobolev inequality. Thanks to this Lemma, we now only need to find a bound on \( \partial_t H_1 \) and \( \partial_t V_1 \) instead of a bound on \( \partial_t Q_0 \) in the proof of Theorem 1.3

**2.3 Proof of Theorem 1.3**

We can now prove Theorem 1.3
Proof of Theorem 1.3. From Theorem (1.2), Remark 2.1, and Lemma 2.1, one only needs to find a Lyapunov function \(V\) where \(\nu\) where

\[
f_1(t, x)e^{-\mu x}(E(U(x), x, t)U_1^2(t, x) + f_2(t, x)e^{\mu x}(E(U(x), x, t)U_2^2(t, x)dx + qz^2),
\]

(2.27)

where \(f_1, f_2\) are positive and bounded functions which will be defined later on, and \(\mu\) and \(q\) are positive constants which will also be defined later on. Recall that \(E\) is still given by (2.17). Let \(T > t_0 \geq 0\) and \((u^0, Z^0) \in H^2(0, L) \times \mathbb{R}\) satisfying the compatibility condition (1.19) and such that

\[
\langle \|u^0\|_{H^2(0, L)} + |Z^0| \rangle < \nu,
\]

(2.28)

where \(\nu\) is a constant to be chosen later on but such that \(\nu < \min(\nu_2, \nu(T))\). Recall that \(\nu(T)\) is given by Theorem 1.2. From Theorem 1.2 there exists a unique solution \(u \in C^0([t_0, T], H^2(0, L))\). We suppose in addition that \((u^0, Z^0) \in H^2(0, L)\), and that (2.28) also hold for the \(H^3\) norm instead of the \(H^2\) norm in \(u\). From Remark 1.2, \((u, Z) \in C^0([t_0, T] \times H^2(0, L) \times C^4([t_0, T])\). This assumption is here to allow us to compute easily the derivative of \(u\) but will be relaxed later on by density.

Let \(\delta \in (0, \delta_0)\) to be chosen later on, with \(\delta_0\) is given by Lemma 2.2 and assume that

\[
\max(||\partial_t H_1||_{C^1([t_0, \infty) ; L^\infty(0, L))}, ||\partial_t V_1||_{C^1([t_0, \infty) ; L^\infty(0, L))}) < \delta.
\]

(2.29)

As this is the only assumption on \(H_1\) and \(V_1\), we can assume from now on that \(t_0 = 0\) without loss of generality.

Looking at (2.27), \(V_a\) is indeed a function defined on \(H^2(0, L) \times \mathbb{R} \times \mathbb{R}_+\), but for notational ease we will denote \(V_a(t) := V_a(u(t, \cdot), Z(t), t)\), where \(Z(t)\) is given by (1.15), and \(E := E(u(t, x, x, t))\). Similarly we introduce

\[
V_b(U, t) := \int_0^L f_1(t, x)e^{-\mu x}(E(U(x), x, t)I(U(x, x, t))I_1^2(t, x) + f_2(t, x)e^{\mu x}(E(U(x), x, t)I(U, x, t))I_2^2(t, x)dx + q \left( \frac{H_1(t, L)}{4g} (U_1(L) - U_2(L))^2, \right.
\]

\[
V_c(U, t) := \int_0^L f_1(t, x)e^{-\mu x}(E(U(x), x, t)J(U(x, x, t))J_1^2(t, x) + f_2(t, x)e^{\mu x}(E(U(x), x, t)J(U, x, t))J_2^2(t, x)dx
\]

\[
+ q \left( \frac{H_1(t, L)}{4g} (I_1(t, L) - I_2(t, L)) + \frac{\partial_t H_1(t, L)}{4} \sqrt{\frac{1}{gH_1(t, L)}} (U_1(L) - U_2(L)) \right)^2, \]

(2.30)

where

\[
I(U, x, t) := (A(U, x, t)\partial_x(U) + (\partial_x A(U, x, t) + \partial_t A(U, x, t))\partial_t U)\partial_x U + \partial_t B(U, x, t) + (\partial_t B(U, x, t))\partial_t U),
\]

\[
J(U, x, t) := (A(U, x, t)\partial_x(U^2) + (\partial_x A(U, x, t)\partial_x U + (\partial_t A(U, x, t))\partial_t U)
\]

\[
+ (\partial_t A(U, x, t) + \partial_x (\partial_t A(U, x, t))\partial_x U)
\]

\[
+ 2\partial_t A(U, x, t)\partial_x U + 2\partial_t A(U, x, t)\partial_x U + ((\partial_t^2 A(U, x, t))\partial_t U)\partial_t U
\]

\[
+ \partial_t B(U, x) + 2\partial_t B(U, x)\partial_t U + (\partial_t^2 B(U, x, \partial_t U))\partial_t U). \]

(2.31)

Observe that for a solution \(u\) of (2.8), and using the expression of \(Z\) given by (1.15), the expressions of \(V_b(u(t, \cdot), t)\) and \(V_c(u(t, \cdot), t)\) become

\[
V_b(u(t, \cdot), t) := \int_0^L f_1(t, x)e^{-\mu x}(E\partial_t u_1^2(t, x) + f_2(t, x)e^{\mu x}(E\partial_t u_2^2(t, x)dx + q(\dot{Z}(t))^2,
\]

(2.32)

\[
V_c(u(t, \cdot), t) := \int_0^L f_1(t, x)e^{-\mu x}(E\partial_t^2 u_1^2(t, x) + f_2(t, x)e^{\mu x}(E\partial_t^2 u_2^2(t, x)dx + q(\dot{Z}(t))^2,
\]

(2.33)
which justifies the expression chosen for (2.30) and (2.31). We also note for notational ease
depending only on $a$

\[ V = -2 \int_0^L f_1(t, x)e^{-\mu x}(Eu)_1^2 + (EB)_1(u, x, t) \]
\[ + f_2(t, x)e^{\mu x}(Eu)_2^2 + (EB)_2(u, x, t) \]
\[ + \int_0^L \partial_t(f_1)e^{-\mu x}(Eu)_1^2 + \partial_t(f_2)e^{\mu x}(Eu)_2^2 \]
\[ + 2 \int_0^L f_1e^{-\mu x}(Eu)_1 \left( \partial_t E + a_1E, \partial_x u \right) u_1 + f_2e^{\mu x} \left( \partial_t E + a_1E, \partial_x u \right) u_2 \]
\[ = -2 \int_0^L f_1(t, x)e^{-\mu x}(Eu)_1 \left[ D_1(u, x, t) \partial_t E - (\partial_t E + \partial_1E, \partial_x u) u_1 \right] \]
\[ + f_2(t, x)e^{\mu x}(Eu)_2 \left[ D_2(u, x, t) \partial_t E - (\partial_t E + \partial_1E, \partial_x u) u_2 \right] \]
\[ + \int_0^L \partial_t(f_1)e^{-\mu x}(Eu)_1^2 + \partial_t(f_2)e^{\mu x}(Eu)_2^2 - 2 \int_0^L f_1e^{-\mu x}(Eu)_1(EB)_1(u, x, t) + f_2e^{\mu x}(Eu)_2(EB)_2(u, x, t) \]
\[ + 2 \int_0^L f_1e^{-\mu x}(Eu)_1 \left( \partial_t E + a_1E, \partial_x u \right) u_1 + f_2e^{\mu x} \left( \partial_t E + a_1E, \partial_x u \right) u_2 \]
\[ = -2 \int_0^L f_1e^{-\mu x}D_1(Eu)_1^2 + D_2f_2e^{\mu x}(Eu)_2^2 \]
\[ - \int_0^L (Eu)_1e^{-\mu x}\left[ -\partial_1(D_1f_1) - f_1\partial_u(D_1)\partial_x u(Eu)_1 - 2f_1D_1 \left( \partial_t E + \partial_1E, \partial_x u \right) u_1 \right] \]
\[ + (Eu)_2e^{\mu x}\left[ -\partial_1(D_2f_2) - f_2\partial_u(D_2)\partial_x u(Eu)_2 - 2f_2D_2 \left( \partial_t E + \partial_1E, \partial_x u \right) u_2 \right] \]
\[ + \int_0^L \partial_t(f_1)e^{-\mu x}(Eu)_1^2 + \partial_t(f_2)e^{\mu x}(Eu)_2^2 - 2 \int_0^L f_1e^{-\mu x}(Eu)_1(EB)_1(u, x, t) + f_2e^{\mu x}(Eu)_2(EB)_2(u, x, t) \]
\[ + 2 \int_0^L f_1e^{-\mu x}(Eu)_1 \left( \partial_t E + a_1E, \partial_x u \right) u_1 + f_2e^{\mu x} \left( \partial_t E + a_1E, \partial_x u \right) u_2 \]
\[ - \mu \int_0^L D_1f_1e^{-\mu x}(Eu)_1^2 - D_2f_2e^{\mu x}(Eu)_2^2 \]
\[ + 2Q(t)\tilde{Z}(t) \]

(2.33)

In order to simplify this expression, observe that from (2.17), (2.23), $D_1(0, x, t) = \lambda_1(t, x)$ and
\[ D_2(0, x, t) = -\lambda_2(t, x), \]
using the fact that $D$ is $C^1$ in $u$, and using (2.19) and (2.29) there exists $C > 0$
depending only on $H_\infty$ and $\alpha, \nu$ and $\delta$ such that
\[
\|D_i - \text{sgn}(D_i(0, x, t))\lambda_1\|_\infty \leq \|Cu\|_\infty, \quad i = 1, 2,
\]
\[
\|\partial_x D_i + a_1D_i\partial_x u - \text{sgn}(D_i(0, x, t))\partial_x \lambda_1\|_\infty \leq C (\|\partial_x u\|_\infty + \|u\|_\infty), \quad \|\partial_x E\|_\infty \leq C (\|u\|_\infty + \|\partial_x u\|_\infty).
\]
and
\[
\|\partial_t E + a_1E\partial_t u\|_\infty \leq C (\|u\|_\infty + \|\partial_x u\|_\infty).
\]
Thus, using this together with (2.34)

\[
\dot{V}_a \leq -\left[ f_1 e^{−μx} D_1 (Eu)^2_t + D_2 f_2 e^{μx} (Eu)^2_t \right]_0^{L} \\
- \int_0^L (Eu)^2_t e^{−μx} (−∂_x (λ_1 f_1) − ∂_t (f_1)) + (Eu)^2_t e^{μx} (∂_x (λ_2 f_2) − ∂_t (f_2)) dx \\
- 2 \int_0^L f_1 e^{−μx} (Eu)_1 (EB)_1 (u, x, t) + f_2 e^{μx} (Eu)_2 (EB)_2 (u, x, t) dx \\
- μ \min_{x \in [0, L]} (λ_1, λ_2) V_a + μ \min_{x \in [0, L]} (λ_1, λ_2) q Z^2 (t) + 2q Z(t) \dot{Z}(t) \\
+ C (||u||_∞ + ||∂_x u||_∞) \int_0^L (Eu)^2_t + (Eu)^2_t dx \\
+ C (||u||_∞ + ||∂_x u||_∞)^2 \int_0^L ||(Eu)_1|| + ||(Eu)_2|| dx,
\]

where C is a constant that may change between lines but only depends on ν, an upper bound of δ (for instance δ_0), μ, H_∞ and α. Note that C is continuous in μ ∈ [0, ∞), thus it can be made independent of μ by imposing an upper bound on μ, for instance μ ∈ (0, 1]. Finally, from the second equation of (2.17), and the fact that E is C^1 in u, there exists a continuous function r_1 such that, for any vector v ∈ R^2

\[
E(u(t, x), v) = (u(t, x), r_1(u(t, x), x, t))v, \ \forall (t, x) \in [0, T] \times [0, L].
\]

As E(u(t, x), x, t) is locally a C^∞ function of the coefficients of A, r_1 is bounded on B_{ν_2} × [0, L] × [0, T] by a bound that only depends on ν_2, H_∞ and α. Thus there exists a constant C depending only on ν_2, H_∞ and α such that

\[
\frac{1}{C} ||v||_{L^2([0, L]; R^2)} \leq ||E(v)||_{L^2([0, L]; R^2)} \leq C ||v||_{L^2([0, L]; R^2)}.
\]

Thus, using this together with the fact that D_1 and D_2 are C^1 with u, (2.27), and Young’s inequality and then Cauchy-Schwarz inequality on the last integral term,

\[
\dot{V}_a \leq -\left[ f_1 e^{−μx} λ_1 (Eu)^2_t - λ_2 f_2 e^{μx} (Eu)^2_t \right]_0^{L} \\
- \int_0^L (Eu)^2_t e^{−μx} (−∂_x (λ_1 f_1) − ∂_t (f_1)) + (Eu)^2_t e^{μx} (∂_x (λ_2 f_2) − ∂_t (f_2)) dx \\
- 2 \int_0^L f_1 e^{−μx} (Eu)_1 (EB)_1 (u, x, t) + f_2 e^{μx} (Eu)_2 (EB)_2 (u, x, t) dx \\
- μ \min_{x \in [0, L]} (λ_1, λ_2) V_a + μ \min_{x \in [0, L]} (λ_1, λ_2) q Z^2 (t) + 2q Z(t) \dot{Z}(t) \\
+ C (||u||_∞ + ||∂_x u||_∞) ||u||^2_{L^2(0, L)} \\
+ C (||u||_∞ + ||∂_x u||_∞)^2 ||u(t, 0)||^2 + ||u(t, L)||^2),
\]

Now, as E and B are C^2 with u and continuous with x and t, and as B(0, x, t) = 0, there exists a continuous function r_2 ∈ C^0(B_{ν_2} × [0, T] × [0, L]; R^{n×n}) such that,

\[
(EB)(u(t, x), x, t) = ∂_u (EB)(0, x, t) u(t, x) + (r_2(u(t, x), x, t)) u(t, x), \ \forall t \in [0, T] \times [0, L].
\]

Note that from (2.10), r_2 is bounded on B_{ν_2} × [0, L] × [0, T] by a constant that only depends on ν_2, δ, H_∞ and α. From (2.10) and (2.17) \dot{∂}_u (EB)(0, x, t) = ∂_u B(0, x, t). Besides, from (2.17), E is invertible and C^1, thus an inequality similar to (2.17) holds for E^1, and u = E^{-1}(Eu). Therefore, using (2.43) together with
\( r_1 \) and \( r_2 \) are bounded, and the expression of \( \partial u B(0, x, t) \) given in (2.13)–(2.14), one has

\[
\dot{V}_a \leq - \left[ f_1 e^{-\mu x} \lambda_1 u_1^2 - \lambda_2 f_2 e^{\mu x} u_2^2 \right] L_v
- \int_0^L (E u)^2 e^{-\mu x} (-\partial_x(f_1) - \delta_t(f_1)) + (E u)^2 e^{\mu x} (\partial_x(f_2) - \delta_t(f_2)) \, dx
- 2 \int_0^L f_1 e^{-\mu x} \gamma_1 (E u)^2_1 + f_2 e^{\mu x} \delta_2(E u)^2_1 + (\gamma_2 f_1 e^{-\mu x} + \delta_1 f_2 e^{\mu x}) (E u)_1(E u)_2 \, dx
- \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V_a + \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) qZ^2(t) + 2qZ(t) \dot{Z}(t)
+ C \left( \|u\|_\infty + \|\partial_x u\|_\infty + \|u\|_2(0, L) \right) \|u\|_\infty^2 + C \|u\|_\infty^3 + C \|u\|_{\infty} (|u(t, 0)^2| + |u(t, L)|).
\]

As \( D_1 \) and \( D_2 \) are of class \( C^2 \), denoting for simplicity \( k_2 := \partial D_1(0, t), k_1 := \partial D_2(0, 0, t) \) and \( k_3 := -\partial_2 D_2(0, 0, t) \), and using (2.15)

\[
\dot{V}_a \leq - \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V_a + \left[ f_1 \lambda_1 k_2^2 - \lambda_2 f_2 u_2^2 \right] (t, 0)
- I_1(u_1(t, L), Z(t)) - \int_0^L I_2((E u)_1, (E u)_2) \, dx
+ C \left( \|u\|_\infty + \|\partial_x u\|_\infty \right) \left( \|u\|_2(0, L) + \|u\|_\infty + \|\partial_x u\|_\infty \right) + (|u(t, 0)|^2 + |u(t, L)|).
\]

where \( I_1 \) and \( I_2 \) denote the following quadratic forms

\[
I_1(x, y) = (\lambda_1 f_1(L)e^{-\mu L} - \lambda_2 f_2(L)e^{\mu L} k_3^2) x^2 + \left( q \sqrt{\frac{H_1}{g}} - \lambda_2 f_2(L)e^{\mu L} k_3^2 - \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) q \right) y^2
+ (2\lambda_2 f_2(L)e^{\mu L} k_3 k_1 - q \sqrt{\frac{H_1}{g}} (k_1 - 1)) xy,
\]

\[
I_2(x, y) = ((-\lambda_1 f_1)_x + 2 f_1 \gamma_1(t, x) - \partial_t f_1) e^{-\mu x} x^2 + ((\lambda_2 f_2)_x + 2 f_2 \delta_2(t, x) - \partial_t f_2) e^{\mu x} y^2
+ 2 (\gamma_2 f_1 e^{-\mu x} + \delta_1 f_2 e^{\mu x}) xy.
\]

We can perform similarly with \( V_b \) and \( V_c \), to do this observe that \( \partial_1 u \) and \( \partial_2^2 u \) are respectively solutions of

\[
\partial_t(\partial_1 u) + A(u, x, t) \partial_x(\partial_1 u) + (\partial_1 B(u, x, t))(\partial_1 u) + (\partial_t A(u, x, t) + \partial_1 A(u, x, t) \partial_1 u) \partial_x u + \partial_1 B(u, x, t) = 0
\]

(2.47)

\[
\partial_t(\partial_2^2 u) + A(u, x, t) \partial_x(\partial_2^2 u) + (\partial_2^2 A(u, x, t) \partial_2^2 u) \partial_x u + (\partial_2 A(u, x, t))(\partial_2^2 u) = 0
\]

(2.48)

which are very similar to (2.8), as they only differ by quadratic perturbations or terms involving a time
derivative of \((H_1, V_1)\). We get then
\[
\dot{V} = \dot{V}_a + \dot{V}_b + \dot{V}_c \leq -\mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V + [f_1 \lambda_1 k_2^2 - \lambda_2 f_2] (u_2^2(t, 0) + (\partial_t u_2(t, 0))^2 + (\partial_t^2 u_2(t, 0))^2)
- I_1(u_1(t, L), Z) - I_1(\partial_t u_1(t, L), \dot{Z}) - I_1(\partial_t^2 u_1(t, L), \dot{Z})
- \int_0^L I_2((E u)_1, (E u)_2) + I_2((E \partial_t u)_1, (E \partial_t u)_2) + I_2((E \partial_t^2 u)_1, (E \partial_t^2 u)_2) dx
+ C(\|u\|_\infty + \|\partial_x u\|_\infty) \left(\|u\|_{L^2(0, L)}^2 + \|\partial_t u\|_{L^2(0, L)}^2 + \|\partial_t^2 u\|_{L^2(0, L)}^2 + \|u\|_\infty + \|\partial_x u\|_\infty)^2
+ |u_2(t, 0)|^2 + (|u_1(t, L)| + |Z|)^2 + |\partial_t u_2(t, 0)|^2 + (|\partial_t u_1(t, L)| + |\dot{Z}|)^2 + |\partial_t^2 u_2(t, 0)|^2 + (|\partial_t^2 u_1(t, L)| + |\dot{Z}|)^2\right)
+ C \delta \left(|u_2(t, 0)|^2 + (|u_1(t, L)| + |Z|)^2 + |\partial_t u_2(t, 0)|^2 + (|\partial_t u_1(t, L)| + |\dot{Z}|)^2\right) + C V.
\]

The two last terms come from the successive differentiations of the boundary conditions (2.15), together with (2.29), or the terms in (2.47)–(2.48) involving a time derivative of \(A\) or \(B\). One can see that three identical quadratic form appears in the integral in \((E \partial_t u)_1, (E \partial_t^2 u)_2\), \(i = 0, 1, 2\), as well as three identical quadratic form at the boundaries in \((\partial_t^2 u_1(t, L), \partial_t^2 Z), i = 0, 1, 2\), and three identical terms proportional respectively to \((\partial_t^2 u_2(t, 0)), i = 0, 1, 2\). Thus a sufficient condition to have \(V\) decreasing strictly would be that the square terms and the forms that appear at the boundaries are negative-definite and the quadratic form in the integral is negative, i.e. the three following conditions:

1. Condition at 0
\[
\frac{\lambda_2 f_2(0)}{\lambda_1 f_1(0)} > k_2^2.
\]

2. Condition at \(L\)
\[
\frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} > k_1^2,
\]
\[
(\lambda_1 f_1(L) - \lambda_2 f_2(L)k_2^2) \left(q \sqrt{\frac{H_1}{g}} - \lambda_2 f_2(L)k_3\right) k_3 - \left(\lambda_2 f_2(L)k_3k_1 - \frac{1}{2} q \sqrt{\frac{H_1}{g}} (k_1 - 1)\right)^2 > 0.
\]

3. Condition from the integral
\[
((-\lambda_1 f_1)_x + 2 f_1 \gamma_1(t, x) - \partial_t f_1) > 0,
\]
\[
((-\lambda_1 f_1)_x + 2 f_1 \gamma_1(t, x) - \partial_t f_1) (\lambda_2 f_2)_x + 2 f_2 \delta_2(t, x) - \partial_t f_2) - (\gamma_2 f_1 + \delta_1 f_2)^2 > 0, \forall (t, x) \in [0, T] \times (0, L).
\]

Let assume for the moment that (2.50)–(2.52) are satisfied for any \(\delta \in (0, \delta_3)\) where \(\delta_3\) is a positive constant. Then, as the inequalities (2.50)–(2.52) are strict, by continuity there exist \(\mu > 0\) such that the square terms and the quadratic forms \(I_1\) at the boundaries and the quadratic forms \(I_2\) in the integral are positive definite. And there exists \(\nu_3 \in (0, \nu_2)\) and \(\delta_4 \in (0, \delta_3)\) such that, for any \(\nu \in (0, \nu_2)\), and any \(\delta \in (0, \delta_4)\),
\[
\dot{V} \leq -\mu \min_{[0, L]} (\lambda_1, \lambda_2) V + C \delta V + C \left(\|u\|_\infty + \|\partial_x u\|_\infty\right)^2,
\]
where \(C\) is a positive constant depending only on the system. Note that here, the cubic boundary terms that appeared in (2.49) have been compensated by the strictly negative quadratic boundary terms, taking
\( \nu \) sufficiently small and using (1.21). Choosing \( \delta_5 \in (0, \delta_4) \) such that \( \delta_5 < \mu \min_{[0,L]}(\lambda_1, \lambda_2)/4C \), for any \( \delta \in (0, \delta_5) \) one has

\[
\dot{V} \leq -\frac{3}{4} \mu \min_{[0,L]}(\lambda_1, \lambda_2)V + C \left( \|u\|_{\infty} + \|\partial_x u\|_{\infty} \right)^2.
\]  

(2.54)

Now, if we assume in addition that (2.20) hold, using (1.21), and Sobolev inequality, there exists \( \nu_4 \in (0, \nu_3) \) such that, for any \( \nu \in (0, \nu_4) \),

\[
C \left( \|u\|_{\infty} + \|\partial_x u\|_{\infty} \right)^2 \leq \frac{\mu}{4} \min_{[0,L]}(\lambda_1, \lambda_2)V,
\]  

(2.55)

thus, setting \( \gamma = \min_{[0,L]}(\lambda_1, \lambda_2) \),

\[
\dot{V} \leq -\frac{\gamma}{2}V.
\]  

(2.56)

which shows the exponential decay of \( V \) and ends the proof of Theorem 1.3.

In other words, all that remains to do is to find \( f_1, f_2 \) and \( q \) such that (2.50)–(2.52) are satisfied and such that \( V \) satisfies (2.20). In order to find such function we are now going to use Lemma 2.2. To understand the link between Lemma 2.2 and the three conditions (2.50)–(2.52), observe that the condition (2.52) give rise to a differential inequation, which, as it will appear later on, is linked to the differential equation solved by Lemma 2.2. Then (2.50) and (2.51) can be seen as boundary conditions/values of the solution of this differential inequation.

From Lemma 2.2 we know that there exists a solution on \([0, L]\) to equation (2.21), namely \( \lambda_1 \phi/\lambda_1 \). Therefore, as \([0, L]\) is a compact set, there exists \( \varepsilon_1 \) such that for any \( \varepsilon \in [0, \varepsilon_1] \) there exists a solution \( f_\varepsilon(t, x) \) to the following system

\[
\partial_t f_\varepsilon(t, x) = \left( \frac{\phi \gamma_2}{\lambda_1} + \frac{\delta_1}{\phi \lambda_2} (f_\varepsilon)^2 + \sqrt{g} \frac{H_1}{H} \partial_t H_1 \right) + \varepsilon,
\]  

(2.57)

and moreover \((t, x, \varepsilon) \to f_\varepsilon(t, x)\) is of class \( C^0 \) and \( \partial_x f_\varepsilon(t, x) \) as well. This is a classical result on ODE due to Peano (see e.g. [16][Chap. 5, Th 3.1]). From (2.57), \( \partial_t f_\varepsilon \) satisfies the following equation

\[
\partial_t \partial_x f_\varepsilon = 2 \frac{\delta_1}{\phi \lambda_2} f_\varepsilon \partial_t f_\varepsilon + \left( \frac{\phi \gamma_2}{\lambda_1} \right)_t + \left( \frac{\delta_1}{\phi \lambda_2} \right)_t f_\varepsilon^2 + \sqrt{g} \frac{H_1}{H} \partial_t^2 H_1 - \frac{1}{2} \sqrt{g} \frac{H_1}{H} (\partial_x H_1)^2.
\]  

(2.58)

We used here that, from Proposition 1.1 and Remark 1.1 \((H_1, V_1) \in C^0([0, +\infty); H^3(0, L))\), and from (1.4), \( \partial_t \partial_x H_1 = -\partial_x^2 (HV) \) and \( \partial_t \partial_x V_1 = \partial_x (-V_1 \partial_x V_1 - g \partial_x^2 H_1 - (kV_2^2/H_1 - qC)) \). Thus \( \partial_x^2 H_1 \) belongs to \( C^0([0, T] \times H^1(0, L)) \), and \((\gamma_1, \gamma_2, \delta_1, \delta_2)\) belong to \( C^1([0, T] \times H^1(0, L)) \). Using (2.58), we have

\[
\partial_t f_\varepsilon(t, x) = \partial_t f_\varepsilon(t, 0) \exp \left( \int_0^t 2 \frac{\delta_1}{\phi \lambda_2} f_\varepsilon(t, y) dy \right)
\]  

\[+ \int_0^t \exp \left( \int_y^t 2 \frac{\delta_1}{\phi \lambda_2} f_\varepsilon(t, \omega) d\omega \right) \left( \frac{\phi \gamma_2}{\lambda_1} \right)_t + \left( \frac{\delta_1}{\phi \lambda_2} \right)_t f_\varepsilon^2 + \sqrt{g} \frac{H_1}{H} \partial_t^2 H_1 - \frac{1}{2} \sqrt{g} \frac{H_1}{H} (\partial_x H_1)^2 \right) dy.
\]

(2.59)

Instead of seeing the function \( f_\varepsilon \) as a solution of an ODE with a parameter \( t \), one can see it as a solution of an ODE with parameters \( \lambda_1, \lambda_2, \gamma_2, \delta_1, \partial_t H_1 \) and \( \varepsilon \) that we denote \( g_\varepsilon(x, \lambda_1, \lambda_2, \gamma_2, \delta_1, \partial_t H_1) \). From [16][Theorem 2.1] \( g_\varepsilon \) is continuous with these parameters and with \( \varepsilon \). But from (1.12), (1.13), and (2.29), all these parameters are bounded and therefore belong to a compact set when \( t \in [0, +\infty) \). Thus,

\[
\varepsilon \to g_\varepsilon(x, \lambda_1(t), \lambda_2(t), \gamma_1(t), \delta_1(t), \partial_t H_1(t)) = f_\varepsilon(t, x)
\]  

(2.60)

is uniformly continuous in \( \varepsilon \) for \((t, x) \in [0, \infty) \times [0, L]\). This, together with (2.59) implies that
\[
\left| \int_0^x \exp \left( \int_y^x 2 \frac{\delta_1}{\phi \lambda_2} f_\varepsilon(t, \omega) d\omega \right) \partial_t (\partial_\varepsilon (H_1 V_1)) dy \right| \leq C_0 \max \left( \|\partial_t H_1\|_{C^1([0, \infty); L^\infty(0, L))}, \|\partial_t V_1\|_{C^1([0, \infty); L^\infty(0, L))} \right),
\]
where \(C_0\) is a constant that only depends on \(L, H_\infty, \alpha\), and is continuous with \(\varepsilon \in [0, \varepsilon_1]\). Similarly there exists a constant \(C_1 > 0\) depending only on \(L, H_\infty\) and \(\alpha\) such that
\[
\|\partial_t \partial_\varepsilon\|_{L^\infty([0, \infty) \times (0, L)} \leq C_1 \max \left( \|\partial_t H_1\|_{C^1([0, \infty); L^\infty(0, L))}, \|\partial_t V_1\|_{C^1([0, \infty); L^\infty(0, L))} \right),
\]
and similarly for \(\phi_2\). This, together with the definition of \(\lambda_1\) and \(\lambda_2\) given by (2.23), (2.59), and using the continuity of \(\varepsilon \to f_\varepsilon\) on \([0, \varepsilon_1]\) (recall that this continuity is uniform with respect to \((t,x) \in [0, +\infty) \times [0, L]\)), we get that there exists \(C > 0\) depending only on \(H_\infty, \alpha\) and \(\varepsilon\) and continuous with \(\varepsilon\) on \([0, \varepsilon_1]\) such that
\[
|\partial_t f_\varepsilon(t,x)| \leq (|\partial_t f_\varepsilon(t,0)| + \max \left( \|\partial_t H_1\|_{C^1([0, \infty); L^\infty(0, L))}, \|\partial_t V_1\|_{C^1([0, \infty); L^\infty(0, L))} \right)) C(\varepsilon).
\]
But, from (2.57) \(\partial_t f_\varepsilon(t,0) = (\lambda_2/\lambda_1) \eta_1\), thus using (2.29) we obtain
\[
|\partial_t f_\varepsilon(t,x)| \leq \delta C_2(\varepsilon),
\]
where \(C_2\) is again a constant that only depends on \(\varepsilon, \alpha\) and \(H_\infty\) and is continuous with \(\varepsilon\) on \([0, \varepsilon_1]\). We can now restrict ourselves to \(\varepsilon \in [0, \varepsilon_1/2]\) and then \(C_2\) can be chosen independent of \(\varepsilon\) by simply taking its maximum on \([0, \varepsilon_1/2]\). Recall that from Lemma 2.2 we have, \(f_0 = \phi \lambda_2/\lambda_1\), and
\[
\left( \frac{\phi \lambda_2}{\lambda_1} + \frac{\delta_1}{\phi \lambda_2} f_0^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) > 0.
\]
Recall that we still have not chosen the bound \(\delta \in (0, \delta_0)\) on \(\|\partial_t H_1\|_{C^1([0, \infty); L^\infty(0, L))}\) and \(\|\partial_t V_1\|_{C^1([0, \infty); L^\infty(0, L))}\) given in (2.20). From the assumptions on \(k_2\) and \(k_1\), i.e. (1.25), and (2.16), and recalling that \(k_2 = \partial_t D_2(0,0,t)\) and \(k_3 = -\partial_2 D_2(0,0,t)\), one has
\[
k_2^2 < \left( \frac{\lambda_2(L)}{\lambda_1(L)} \right)^2, \quad k_3 > 0.
\]
Thus, using (2.23),
\[
\eta_1 := \min \left( \left[ \frac{1}{|k_3|} - \frac{\lambda_2(L)}{\lambda_1(L)} \right], \frac{\lambda_2(L)}{\lambda_1(L)} \right) > 0.
\]
As \(\varepsilon \to f_\varepsilon(t,x)\) is uniformly continuous with \(\varepsilon\) for \((t,x) \in [0, \infty) \times [0, L]\), there exists \(\varepsilon_2 \in (0, \varepsilon_1/2)\) such that for any \((t,x) \in [0, \infty) \times [0, L]\)
\[
|f_{\varepsilon_2}(t,x) - f_0(t,x)| \leq \phi(t,L) \eta_1,
\]
and
\[
\left( \frac{\phi \gamma_2}{\lambda_1} + \frac{\delta_1}{\phi \lambda_2} f_{\varepsilon_2}^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) > 0.
\]
Note that \(\varepsilon_2\) depends \textit{a priori} on \(\delta\) from (2.69). However, from Lemma 2.2 we can in fact choose \(\varepsilon_2\) independent of \(\delta\) and depending only on an upper bound of \(\delta\) (for instance \(\delta_0\) given by Lemma 2.2). This is important as, in the following, we will choose a \(\delta\) that may depends on \(\varepsilon\).

We select \(f_1\) and \(f_2\) in the following way:
\[
f_1(t,x) = \frac{\phi_2^2}{\lambda_1 f_{\varepsilon_2}(t,x)} > 0,
\]
\[
f_2(t,x) = \frac{\phi_2^2 f_{\varepsilon_2}(t,x)}{\lambda_2} > 0.
\]
and we can now check that the condition (2.52) is verified for δ small enough as
\[
(-\lambda_1 f_1)_x = -2 \frac{(\phi_1 \phi_2)_{x,1} f_1}{\phi_1} + \phi_1^2 \frac{\partial_x f_{\varepsilon_2}}{f_{\varepsilon_2}}(t, x).
\] (2.71)

Thus from (2.22)
\[
- (\lambda_1 f_1)_x + 2\gamma_1 f_1 = \phi_1^2 \frac{\partial_x f_{\varepsilon_2}}{f_{\varepsilon_2}}
\] (2.72)
and similarly
\[
(\lambda_2 f_2)_x + 2\delta_2 f_2 = (\phi_2^2 f_{\varepsilon_2}(t, x))_x - (\phi_2^2 f_{\varepsilon_2}(t, x)) = \phi_2^2 \partial_x f_{\varepsilon_2}.
\] (2.73)

Therefore, from (2.57), (2.72), and (2.73), one has
\[
(- (\lambda_1 f_1)_x + 2\gamma_1 f_1 - \partial_t f_1)(\lambda_2 f_2)_x + 2\delta_2 f_2 - \partial_t f_2 = \left( \frac{\phi_1 \phi_2}{f_{\varepsilon_2}} \right)^2 \left( \left( \frac{\phi_2^2}{\lambda_1} + \frac{\delta_1}{\phi_2} f_{\varepsilon_2} \phi_1 \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) + \varepsilon_2 \right)^2 - \partial_x f_{\varepsilon_2} \left( \frac{\phi_1^2}{f_{\varepsilon_2}} \partial_t f_2 + \phi_2^2 \partial_t f_1 \right) + (\partial_t f_1)(\partial_t f_2).
\] (2.74)

But we have
\[
\partial_t f_1 = 2 \left( \frac{\partial_t \phi_1}{\phi_1} \frac{\lambda_1 f_{\varepsilon_2}}{\lambda_1 f_{\varepsilon_2}} - \frac{\partial_t \lambda_1}{\lambda_1 f_{\varepsilon_2}} + \frac{\partial_t f_{\varepsilon_2}}{\lambda_1 f_{\varepsilon_2}} \phi_1^2, \right.
\] (2.75)
and besides, from (1.4) and (2.29), there exists \( C_3 > 0 \) depending only on \( \alpha_1 \) and \( H_\infty \), and an upper bound of \( \delta \) (for instance \( \delta_0 \)), such that
\[
\max(\|H_{1x}\|_{L^\infty((0, +\infty)\times(0, L))}, \|V_{1x}\|_{L^\infty((0, +\infty)\times(0, L))}) \leq C_3.
\] (2.76)

Thus, using (2.14) and (2.23), there exists \( C_4 > 0 \) depending only on \( \alpha_1 \) and \( H_\infty \), and \( \delta_0 \) (but not on \( \delta \)) such that
\[
\max(\|\phi_1\|_{L^\infty((0, +\infty)\times(0, L))}, \|\phi^{-1}_1\|_{L^\infty((0, +\infty)\times(0, L))}) < C_4,
\] (2.77)
and similarly for \( \phi_2 \). Observe now that, from \( f_0 = \lambda_2 \phi_1/\lambda_1 \) and (2.77), \( |f_0| \) and \( 1/|f_0| \) can be bounded by a constant depending only on \( \alpha_1 \), \( H_\infty \), and \( \delta_0 \). Thus from (2.68)
\[
1/C_5 \leq \|f_{e_2}\|_{L^\infty((0, +\infty)\times(0, L))} \leq C_5,
\] (2.78)
where \( C_5 \) only depends on \( \alpha_1 \), \( H_\infty \) and \( \delta_0 \). And therefore, from (2.23), (2.64), (2.62), and (2.78) one has
\[
|\partial_t f_1| \leq C_6 \delta,
\] (2.79)
and similarly
\[
|\partial_t f_2| \leq C_7 \delta,
\] (2.80)
where \( C_6 \) and \( C_7 \) are constants that only depend on \( \alpha_1 \), \( H_\infty \) (and \( \delta_0 \)). We now select the bound on \( \max(|\partial_t H_1|, |\partial_t V_1|) \): we select \( \delta_3 \in (0, \delta_0) \) such that, for any \( \delta \in [0, \delta_3] \) and any \( (t, x) \in [0, \infty) \times [0, L] \),
\[
C_6 C_6^2 C_7^2 \delta \leq \varepsilon_2,
\] (2.81)
for any \( x \in [0, L] \) and any \( X \in [1/C_5, C_5] \). Observe that this is obviously possible as \( \varepsilon_2 > 0 \) and, when \( \delta_1 = 0 \), \eqref{2.82} is verified and the inequality is strict. Then, from \eqref{2.22}, \eqref{2.77}, \eqref{2.74}, \eqref{2.78}–\eqref{2.82},

\[
(-\lambda_1 f_1)_x + 2\gamma_1 f_1 - \partial_t f_1) (\lambda_2 f_2)_x + 2\delta_2 f_2 - \partial_t f_2 > \left( \frac{\phi_1\phi_2}{f_{\varepsilon_2}} \right)^2 \left( \frac{\phi_1}{\lambda_1} + \frac{\delta_1}{\phi\lambda_2} f_{\varepsilon_2} \right)^2 \tag{2.83}
\]

\[
= \left( \frac{\lambda_2}{\lambda_1} f_1 + \frac{\delta_1}{\lambda_2} f_2 \right)^2 ,
\]

which is exactly the second inequality of \eqref{2.52}. Besides, from \eqref{2.25} and \eqref{2.81},

\[
(-\lambda_1 f_1)_x + 2\gamma_1 f_1 - \partial_t f_1) = \phi_1^2 \frac{\partial_t f_{\varepsilon_2}}{f_{\varepsilon_2}} - \partial_t f_1
\]

\[
\quad = \phi_1^2 \left( \frac{\phi_2}{\lambda_1} + \frac{\delta_1}{\phi\lambda_2} f_{\varepsilon_2} + \frac{g}{H_1 \partial_t H_1} \right) + \varepsilon_2 - \frac{\partial_t f_1 f_{\varepsilon_2}}{\phi_1^2} \tag{2.84}
\]

\[
> 0.
\]

We can now check that \eqref{2.50} and \eqref{2.51} are also verified thanks to the choice of \( \varepsilon_2 \) and \( \eta_1 \). Indeed, using \eqref{2.77} and \eqref{2.16}, one has

\[
\frac{\lambda_2(0)f_2(t, 0)}{\lambda_1(0)f_1(t, 0)} = \frac{f_{\varepsilon_2}^2(t, 0)}{f_{\varepsilon_2}^2(0)} = \left( \frac{\lambda_2(0)}{\lambda_1(0)} + \varepsilon_2 \right)^2 > \left( \frac{\lambda_2(0)}{\lambda_1(0)} \right)^2 = k_2^2 . \tag{2.85}
\]

This explains our choice of initial condition for \( f_{\varepsilon_2} \). Now, from \eqref{2.68}, one has

\[
\frac{\lambda_1(t, L)f_1(t, L)}{\lambda_2(t, L)f_2(t, L)} = \frac{\phi_2^2(t, L)}{f_{\varepsilon_2}^2(L)} > \frac{1}{\left( \frac{\lambda_2(t, L)}{\lambda_1(t, L)} + \eta_1 \right)^2} , \tag{2.86}
\]

and from the definition of \( \eta_1 \) given by \eqref{2.67},

\[
\eta_1 + \frac{\lambda_2(L)}{\lambda_1(L)} = \min \left( \frac{1}{|k_1|}, 1 \right) . \tag{2.87}
\]

Therefore

\[
\frac{\lambda_1(t, L)f_1(t, L)}{\lambda_2(t, L)f_2(t, L)} > \max(k_1^2, 1) , \tag{2.88}
\]

and in particular the condition \eqref{2.51a} is verified. Let us now look at condition \eqref{2.51b}. So far we have not selected the positive constant \( q \). We want to show that there exists \( q > 0 \) such that the condition \eqref{2.51b} is satisfied. Observe that the left-hand side of \eqref{2.51b} can be seen as a polynomial in \( q \), and the condition \eqref{2.51b} can be rewritten as

\[
P(q) := -\frac{q^2 H_1}{4} (k_1 - 1)^2 + q \sqrt{\frac{H_1}{g} k_3 \left( \lambda_1 f_1(L) - \lambda_2 f_2(L) (k_1 - k_1(k_1 - 1)) \right)} - (\lambda_1 f_1(L)) (\lambda_2 f_2(L)) k_3^2
\]

\[
= -\frac{q^2 H_1}{4} (k_1 - 1)^2 + q \sqrt{\frac{H_1}{g} k_3 \left( \lambda_1 f_1(L) - \lambda_2 f_2(L) k_1 \right)} - (\lambda_1 f_1(L)) (\lambda_2 f_2(L)) k_3^2 > 0 .
\]

\[
(2.89)
\]

From \eqref{2.88} \( \lambda_1 f_1(t, L) > \lambda_2 f_2(t, L) k_1 \) and from \eqref{2.66} \( k_3 > 0 \). Thus the real roots of \( P \) are positive if they exist. This implies that there exists a positive constant \( q \) such that \eqref{2.51b} is satisfied if the discriminant of \( P \) is positive. Denoting its discriminant by \( \Delta \),

\[
\Delta = \frac{H_1}{g} k_3^2 \lambda_2^2 f_2(t, L)^2 \left( \left( \frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} - k_1 \right)^2 - \left( \frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} \right)^2 (k_1 - 1)^2 \right) . \tag{2.90}
\]

20
Let us introduce $h : X \rightarrow (X - k_1)^2 - X(k_1 - 1)^2$. The function $h$ is a second order polynomial with a positive dominant coefficient and observe that its roots are $k_1^2$ and 1. Thus $h$ is increasing strictly on $[\max(k_1^2, 1), +\infty)$. Hence, using (2.88),

$$
\Delta = \frac{H_1 k_2^2 h(t, L)^2(\lambda_1 f_1(L))}{\lambda_2 f_2(L)}
$$

(2.91)

This proves that there exists $q > 0$ such that (2.51b) is satisfied, and we select such $q$. All it remains to do now is to show that the function $(U, z) \rightarrow V(t, U, z)$, which is now entirely selected, satisfies (2.20).

From (1.13) and (1.12) we know that for any $(t, x) \in [0, \infty) \times [0, L],

$$
\sqrt{gH_\infty} > \lambda_2 > \alpha, \ 2\sqrt{gH_\infty} > \lambda_1 > \alpha.
$$

(2.92)

Besides, from the definition of $\phi_1$ and $\phi_2$ given by (2.22), (2.14) and the bound (1.13), (1.12), there exists a constant $C_8$ that only depends on $\delta, \alpha$ and $H_\infty$ such that

$$
\frac{1}{C_8} \leq \|\phi_1\|_\infty \leq C_8, \quad \frac{1}{C_8} \leq \|\phi_2\|_\infty \leq C_8.
$$

(2.93)

Thus, using that $f_0 = \lambda_2 \phi_1 / \phi_1$, (2.70), (2.68), (2.93) and (2.92), there exists $c_1 > 0$ constant independent of $U$ and $z$ such that, for any $(U, z) \in H^2(0, L) \times \mathbb{R},

$$
c_1 \left(\|U\|_{H^2(0, L)} + |z|\right) \leq V(t, (U, z)) \leq \frac{1}{c_1} \left(\|U\|_{H^2(0, L)} + |Z|\right),
$$

(2.94)

which is exactly (2.20). This concludes the proof of Theorem 1.3.

## 3 Conclusion

In this paper we gave simple conditions on the design of a single PI controller to ensure the exponential stability of the nonlinear Saint-Venant equations with arbitrary friction and slope in the $H^2$ norm. These conditions apply when the inflow is an unknown constant, in that case the system has steady-states and any of them are stable. But they also apply when the inflow is time-dependant and slowly variable. In that case, no steady-states exists and one has to stabilize other target states. When the values of the target state are known at end of the river, we have exponential stability of the target state. Otherwise, we have the Input-to-State stability with respect to the variation of the inflow disturbance. These sufficient conditions are found using a local quadratic entropy and, to the best of our knowledge, are less restrictive than any of the conditions that existed so far, even in the linear case. In [5] it was shown that, in absence of friction and slope, these conditions were optimal for the linear case. However, so far there is no answer when there is some slope or friction and whether these conditions are optimal or not would be a very interesting issue for a further study. Its possible application to a network of channels would also be a matter of interest. Finally, many stabilizing devices for finite dimensional systems also use a PID control with an additional derivative term. It has been shown in [12] that this control cannot ensure exponential stability for a homogeneous hyperbolic equation. It would be an interesting question to know whether a filtering on the derivative term could enable to recover the stability for infinite dimensional system and whether this would enable a faster stabilization than the PI control.

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A Proof of Proposition 1.10

This appendix uses many computations that are very similar to the computations in Section 2 but in a simpler way. Thus, in order to avoid writing two times the same thing and to keep the proof relatively short, some steps might be quicker in this appendix. Let $T_1 > 0$ and to be chosen later on. As $(H_0(0), V_0(0))$ satisfies (1.9), there exists $\nu_0 > 0$ such that for $\nu \in (0, \nu_0)$, $F(H_1^0, V_0^0)$ has two distinct nonzero eigenvalues. Recall that $F$ is given by (2.7) and that $\nu$ is the bound on $\|H_1^0 - H_0(0), V_1^0 - V_0(0)\|_{H^2(0,L)}$. Besides, from (1.8) $(H_0(t,\cdot), V_0(t,\cdot))$ can be seen as the solution of a system of ODE with a parameter $t$ in the initial condition. Thus, as $\delta_0Q_0 \in C^2([0, +\infty))$ and the slope $C$ satisfies $C \in C^2([0, L])$, using (1.6) and (15) Chap. 5, Theorem 3.1, $(H_0, V_0) \in C^3([0, T_1] \times C^3([0, L]))$ and there exists a constant $C$ depending only on $H_{\infty}$, $\alpha$ and an upper bound of $\delta$, such that,

$$\|\partial_t^i H_0, \partial_t^i V_0\|_{C^2([0, L])} \leq C \sum_{i=1}^3 \partial_t^i Q_0, \forall i \in [1, 3],$$

(A.1)

and in particular

$$\|\partial_t^i H_0, \partial_t^i V_0\|_{C^2([0, T_1], C^2([0, L])))} \leq C \|\partial_t^i Q_0\|_{C^2(0, +\infty)}.$$  

(A.2)

Thus [28][Theorem 2.1] can still be used on $(H_1 - H_0)$ and there exist $\delta_0(T_1) > 0$ and $\nu_0(T_1) \in (0, \nu_0)$ such that, if $\nu \in (0, \nu_0(T_1))$ and $\delta \in (0, \delta_0(T_1))$, there exists a unique solution $(H_1, V_1) \in C^3([0, T_1]; H^2(0, L))^2$ to the system (1.4) - (1.5). Besides $(H_1, V_1)$ satisfies an estimate as (1.21) but with $(H_1, V_1)$ instead of $(H, V)$ and $(H_0, V_0)$ instead of $(H_1, V_1)$. We denote by $C(T_1)$ the associated constant. Let us define $h_1 := H_1 - H_0$ and $v_1 := V_1 - V_0$. We transform $(h_1, v_1)^T$ into $w = (w_1, w_2)^T$ using the change of variables defined by (2.1) - (2.5) with $H_0$ and $V_0$ instead of $H_1$ and $V_1$. Thus we obtain

$$\partial_t w + A_0(w, x) \partial_x w + B_0(w, x) + S_0 \left( \frac{\partial H_1}{\partial t} \right) = 0,$$

$$w_1(t, 0) = H_1(w_2(t, 0), Q_0(t) - Q_0(0)),$$

$$w_2(t, L) = H_2(w_2(t, L)),$$

(A.3)

where $A_0, B_0$ and $S_0$ have the same expression as $A, B$ and $S$ (given by (2.9), (2.10), (2.6)) but with $(H_0, V_0)$ instead of $(H_1, V_1)$. Similarly we define

$$\lambda_1^0 = V_0 + \sqrt{gH_0}, \lambda_2^0 = \sqrt{gH_0} - V_0,$$

and $\phi_0$, defined as $\phi$ but with $(H_0, V_0)$ instead of $(H_1, V_1)$. Similarly as in Appendix C

$$\mathcal{K}_0^1(0) = -\lambda_0^0(L)/\lambda_0^0(L), \mathcal{K}_0^1(0) = -\lambda_0^0(0)/\lambda_0^0(0),$$

(A.4)

which is of the form (2.15) with $\nu_G = 0$ and $Z = 0$. Before going any further, note that we can perform the same computations as in Section 2 with no problem, as the proof in Section 2 only used Proposition 1.1 to get that $(H_1, V_1)$ exists for any time and that (1.13) and Lemma 2.3 hold, but we will see now that such claims are true for $H_0$ and $V_0$. The existence of $(H_0, V_0)$ was already shown in section 1 and (1.9) is exactly (1.13) with $(H_0, V_0)$ instead of $(H_1, V_1)$. Finally, (A.2) is exactly the equivalent of Lemma 2.3 for $(H_0, V_0)$. We define now the Lyapunov fonction candidate $V := V_a(w(t, x), t) + B_0(w(t, x), t) + V_c(w(t, x), t) + V_0(w(t, x), t)$ where $V_a, V_b$ and $V_c$ are defined in (2.27), (2.30) with $f_1$ and $f_2$ chosen as $f_1 := (\phi_0^2)^2/(\lambda_0^0 \eta)$ and $f_2 := (\phi_0^2)^2 \eta/(\lambda_0^0)$, where $\eta$ is a function such that there exists a constant $\epsilon > 0$ independent of $w$ such that

$$\eta' = \frac{\gamma_0^2}{\lambda_1^0} + \frac{\delta_0^2}{\lambda_2^0}t^2 + \epsilon, \forall x \in [0, L],$$

$$\eta(0) = \frac{\lambda_0^0(0)}{\lambda_0^0(0)} \phi_0(0) + \epsilon.$$

(A.5)

Note that $\eta$ exists as, for any $t \in [0, +\infty)$, $(\phi(t, \cdot)^0 \lambda_0^0(t, \cdot)/\lambda_0^0(t))$ is a solution of

$$\partial_x f = \frac{\gamma_0^2}{\lambda_1^0} + \frac{\delta_0^2}{\lambda_2^0}t^2, \forall x \in [0, L],$$

(A.6)
this can be proved as in Lemma 2.2 and this case was actually shown in [19]. Note that from (1.6), (1.8) and (A.2), \((H_0)_x\) and \((V_0)_x\) can be bounded by above and by below by constants that only depends on \(H_\infty\), \(\alpha\) and an upper bound of \(Q_0\) (which can also be expressed only with \(H_\infty\), \(\alpha\) from (1.9)). Therefore, looking at their definition, the function \(f_1\) and \(f_2\) can also be bounded by above and below by constants that only depends on \(H_\infty\), \(\alpha\) and \(\varepsilon\). Thus there exist \(c_1 > 0\) and \(c_2 > 0\) depending only on \(H_\infty\) and \(\alpha\), \(\varepsilon\) and \(\mu\) such that

\[
c_1 \|h_1(t, \cdot), v_1(t, \cdot)\|_{H^2(0,L)}^2 \leq V(t) \leq c_2 \|h_1(t, \cdot), v_1(t, \cdot)\|_{H^2(0,L)}^2 \quad \forall \ t \in [0,T_1].
\]  

(A.8)

Consequently, by differentiating \(V\) exactly as in (2.33)–(2.49) and from (A.3), we obtain that there exists \(\mu > 0\), \(v_1 \in (0, V_0(T_1))\) and \(\delta_3 > 0\) such that, for any \(\|h_1(0, \cdot), v_1(0, \cdot)\|_{H^2(0,L)} \leq v_1\), and \(\|\partial_t Q_0\|_{C^2([0,\infty))} \leq \delta\), where \(\delta \in (0, \delta_3)\),

\[
\dot{V} \leq -\mu V + \int_0^L 2f_1w_1(S_0 \left(\frac{\partial_t H_0}{\partial_t V_0}\right))_1 + 2f_2w_2(S_0 \left(\frac{\partial_t H_0}{\partial_t V_0}\right))_2 dx,
\]

\[
+ \int_0^L 2f_1\partial_t w_1(S_0 \left(\frac{\partial^2_t H_0}{\partial^2_{tt} V_0}\right))_1 + 2f_2\partial_t w_2(S_0 \left(\frac{\partial^2_t H_0}{\partial^2_{tt} V_0}\right))_2 dx,
\]

(A.9)

Thus, using Cauchy-Schwarz inequality, (A.8), and (A.1) there exists \(C_1 > 0\) depending only on \(H_\infty\), \(\alpha\) and an upper bound of \(\mu\) such that

\[
\dot{V}(t) \leq -\mu V(t) + C_1|\partial_t Q_0(t) + \partial^2_{tt} Q_0(t) + \partial^3_{ttt} Q_0(t)|V^{1/2}(t), \quad \forall \ t \in [0,T_1].
\]  

(A.10)

and in particular

\[
\dot{V}(t) \leq -\mu V(t) + C_1|\partial_t Q_0(t) + \partial^2_{tt} Q_0(t) + \partial^3_{ttt} Q_0(t)|V^{1/2}(t), \quad \forall \ t \in [0,T_1].
\]  

(A.11)

Let us define \(V_{eq} := (C_1\delta/\mu)^2\). From (A.11), if \(V(t) > 2V_{eq}\), then there exists a constant \(k > 0\) such that \(\dot{V}(t) < -kV^{1/2}(t)\). We now choose \(\delta\) such that \(\sqrt{2}C_1\delta/(\mu \sqrt{c_1}) < v_1\). Thus, from (A.11) and as \(c_1, c_2, C_1\) and \(\mu\) do not depend on \(T_1\), we can choose \(T_1\) large enough such that

\[
V(T_1) \leq 2V_{eq} \leq C_1v_1^2.
\]  

(A.12)

which implies that

\[
\|h_1(T_1, \cdot), v_1(T_1, \cdot)\|_{C^2([0,\infty))} \leq v_1
\]  

(A.13)

and therefore there exists a unique solution \((h_1, v_1) \in C^0([T_1, 2T_1], H^2(0, L))\), with initial condition \((h_1(T_1, \cdot), v_1(T_1, \cdot))\) (we use the same existence Theorem in [23][Theorem 2.1]) and, noting that \(V(T_1) \leq 2V_{eq}\) implies \(V(2T_1) \leq 2V_{eq}\), this analysis still hold. We can do similarly for any \([nT_1, (n + 1)T_1]\) with \(n \in \mathbb{N}\), thus, as \((H_0, V_0) \in C^0([0, +\infty), H^2(0, L))\), there exists a unique solution \((H_1, V_1) \in C^0([0, +\infty), H^2(0, L))\) and (A.10) holds for any \(t \in [0, +\infty)\). Therefore denoting \(g(t) = V(t)e^{at}\), we deduce from (A.10) that

\[
g'(t) \leq C_1|\partial_t Q_0(t) + \partial^2_{tt} Q_0(t) + \partial^3_{ttt} Q_0(t)| e^{\frac{a}{2} t} \sqrt{g(t)}.
\]  

(A.14)

Thus

\[
V^{1/2}(t) \leq V^{1/2}(0)e^{-\frac{a}{2} t} + \frac{C_1}{2} \left(\int_0^t |\partial_t Q_0(s) + \partial^2_{tt} Q_0(s) + \partial^3_{ttt} Q_0(s)| e^\frac{a}{2} s ds\right) e^{-\frac{\alpha}{2} t}.
\]  

(A.15)

This implies the ISS property

\[
\|h_1(t, \cdot), v_1(t, \cdot)\|_{H^p((0,L);\mathbb{R}^2)} \leq \frac{\sqrt{2\pi}}{c_1} \|h_1(0, \cdot), v_1(0, \cdot)\|_{H^2((0,L);\mathbb{R}^2)} e^{-\frac{\alpha}{2} t} + \frac{C_1}{2\sqrt{c_1}} \left(\int_0^t |\partial_t Q_0(s) + \partial^2_{tt} Q_0(s) + \partial^3_{ttt} Q_0(s)| e^\frac{a}{2} s ds\right) e^{-\frac{\alpha}{2} t}.
\]  

(A.16)

This ends the proof of Proposition 1.1. To extend this proof to the \(H^p\) norm for \(p > 2\), note that using the same argument (A.2) holds with the \(C^p([0,T_1]; C^3([0,L]))\) norm in the left-hand side and the \(C^p\) norm in the right-hand side. We can can define \(V_3, \ldots, V_p\) on \(H^p(0, L) \times \mathbb{R} \times \mathbb{R}_+\) as in (2.30) such that \(V_k(w(t,x), t) = V_n(\partial^k_t w(t,x), t)\), for any \(k \in [3,p]\). Then (A.8) holds with \(V := V_n + V_3 + V_4 + \ldots \ V_p\) and the \(H^p\) norm, and the rest can done done identically.
B Proof of Theorem 1.4

Theorem 1.4 result from the proof of Theorem 1.3. Note that the boundary conditions (1.16) can be written under the form (1.18) with \((H_0, V_0)\) instead of \((H_1, V_1)\) where the only difference is that \(Z\) satisfies now

\[
\dot{Z} = H_c - H(t, L) + \frac{f(t)}{v_G k_p},
\]

where \(f(t) = H_c \partial_t V_0(t, L)\). The rest of the proof can be conducted as in Appendix A for \((H_1, V_1)\), with \(a\) priori two differences: \((H, V)\) satisfies the boundary conditions of the form (1.18) and not of the form given in (1.4), and \(\dot{Z}\) satisfies (B.1) instead of (1.15). However, note that in Appendix A the only assumption used on the boundary conditions of the transformed system is that they are of the form (2.3), which is still the case here. Thus, the only difference with Appendix A are some additional terms when \(\dot{Z}\) is used, which is in the boundary terms in the derivative of the Lyapunov function. There exists therefore \(\delta_4 > 0\) and \(\nu_2 > 0\) such that, for any \(\|h_1(0, \cdot), v_1(0, \cdot)\|_{H^2(0,L)} \leq \nu_2\), and \(\|\partial_t Q_0\|_{C^2([0,\infty))} \leq \delta\), where \(\delta \in (0, \delta_4)\),

\[
\dot{V}(t) \leq -\gamma \frac{1}{2} V(t) + C_1 |\partial_t Q_0(t) + \partial^2_{tt} Q_0(t) + \partial^3_{ttt} Q_0(t)| V^{1/2} + 2q Z f(t) + 2q \dot{Z} f'(t) + 2q \ddot{Z} f''(t),
\]

where \(C_1\) is a constant only depending on \(H_{\infty}, \alpha, \nu_2\) and \(\delta_4\). Using Lemma 2.3 there exists a constant \(C > 0\) depending only on \(H_{\infty}, \alpha, \nu_2\) and \(\delta_4\) such that

\[
\dot{V} \leq -\gamma \frac{1}{2} V + CV^{1/2} |\partial_t Q_0(t) + \partial^2_{tt} Q_0(t) + \partial^3_{ttt} Q_0(t)|.
\]

The same argument as in Appendix A (A.14) - (A.16) implies directly the ISS property (1.30).

C Boundary conditions (2.15) and (2.16)

In this appendix we justify the boundary conditions (2.15) with (2.16) after the change of variables. From the boundary conditions (2.3) in the physical coordinate \((h, v)\), together with the definition of \(u_1\) and \(u_2\) given in (2.5), one has at \(x = L\)

\[
u_1(t, L) = B_2(h(t, L), Z(t), t) + \sqrt{\frac{g}{H_1}} h(t, L) =: F_1(h(t, L), Z(t), x, t),
\]

\[
u_2(t, L) = B_2(h(t, L), Z(t), t) - \sqrt{\frac{g}{H_1}} h(t, L) =: F_2(h(t, L), Z(t), x, t).
\]

From its definition, \(F_1\) is \(C^1\) and, from (2.4), and (1.21), there exists \(\nu_1 \in (0, \nu_1)\) such that, for any \(t \in [0, \infty)\), \(\partial_t F_0(0, Z(t), t) \neq 0\). Thus \(F_1\) is locally invertible with respect to its first variable, thus there exists \(\nu_2 \in (0, \nu_1)\) such that \(h(t, L) = F_1^{-1}(u_1(t, L), Z(t), t)\), where \(F_1^{-1}\) denotes the inverse with respect to the first variable. Besides, as \(F_1\) is of class \(C^2\) with respect to the two first variables, \(F_1^{-1}\) is also of class \(C^2\). Then, using (C.1)

\[
u_2(t, L) = F_2(F_1^{-1}(u_1(t, L), Z(t), t), Z(t), t) =: D_2(u_1(t, L), Z(t), t).
\]

and, using (2.4),

\[
\partial_t D_2(0, 0, t) = \partial_t F_2(0, 0, t) \partial_t (F_1^{-1})(0, 0, t)
\]

\[
= \partial_t F_2(0, 0, t) = \frac{\partial_t B_2(0, 0, t) - \sqrt{\frac{g}{H_1}}}{\partial_t B_2(0, 0, t) + \sqrt{\frac{g}{H_1}}}
\]

\[
= \frac{\lambda_1(L) - v_G(1 + k_p)}{\lambda_2(L) + v_G(1 + k_p)}.
\]
Now, as $\partial_2 F_1^{-1}(0,0,t) = -\partial_2 F_1(0,0,t)/\partial_1 F_1(0,0,t)$, using (2.4),

$$
\partial_2 D_2(0,0,t) = \partial_1 F_2(0,0,t) \partial_2 F_1^{-1}(0,0,t) + \partial_2 F_2(0,0,t) = -\partial_1 F_2(0,0,t) \partial_1 F_1(0,0,t) + \partial_2 F_2(0,0,t)
$$

$$
= \partial_2 B_2(0,0,t) \left( 1 - \frac{\partial_1 B_2(0,0,t)}{\partial_1 B_2(0,0,t) + \sqrt{g_2 / 2}} \right)
$$

which is no longer true when $(H_1 V_1)_x = 0$ which is no longer true when $(H_1, V_1)$ is not a steady-state. Let $f = (\lambda_2 \phi / \lambda_1)$, we have from (D.2)

$$
\partial_x f = \phi \frac{\lambda_1}{\lambda_2} (\lambda_1 \partial_x \lambda_2 - \lambda_2 \partial_x \lambda_1 + \lambda_2 \gamma_1 + \lambda_1 \delta_2)
$$

$$
= \phi \frac{\lambda_1}{\lambda_2} \left( (V_1 + \sqrt{gH_1})(-V_1) + \frac{\sqrt{gH_1}}{2H_1} H_{1x} + (-V_1 + \sqrt{gH_1})(V_1 + \frac{\sqrt{gH_1}}{2H_1} H_{1x}) \right)
$$

$$
+ (\sqrt{gH_1} - V_1) \left( \frac{3}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{3}{4} V_1 x + \frac{kV_1^2}{H_1^2} \right)
$$

$$
+ (V_1 + \sqrt{gH_1}) \left( -3 \sqrt{\frac{g}{H_1}} H_{1x} + \frac{3}{4} V_1 x + \frac{2kV_1^2}{H_1^2} \right)
$$

$$
= \frac{\phi}{\lambda_1} \left( \sqrt{gH_1} \left( -2V_1 + \frac{3}{2} V_1 x + \frac{2kV_1}{H_1} \right) - V_1 \left( \frac{3}{2} \sqrt{\frac{g}{H_1}} H_{1x} - \frac{kV_1^2}{H_1^2} \right) \right)
$$

$$
= \frac{\phi}{\lambda_1} \left( \frac{2kV_1}{H_1} \sqrt{gH_1} - \frac{kV_1^2}{H_1^2} \sqrt{\frac{H_1}{g}} V_1 + \frac{1}{2} \sqrt{\frac{g}{H_1}} \partial_x H_1 \right).
$$

And on the other hand:

$$
\frac{\phi \gamma_2}{\lambda_1} + \frac{\delta_1}{\lambda_2 \phi} f^2 = \frac{\phi}{\lambda_1} (\lambda_1 \gamma_2 + \lambda_2 \delta_1)
$$

$$
= \frac{\phi}{\lambda_1} \left( \frac{2kV_1}{H_1} \sqrt{gH_1} + \frac{kV_1^2}{H_1^2} \sqrt{\frac{H_1}{g}} V_1 V_1 + \frac{1}{2} \sqrt{\frac{g}{H_1}} \partial_x H_1 \right).
$$

Thus from (D.1) and (D.2)

$$
\partial_2 f = \left( \frac{\phi \gamma_2}{\lambda_1} + \frac{\delta_1}{\lambda_2 \phi} f^2 + \sqrt{\frac{g}{H_1}} \partial_x H_1 \right).
$$

D Proof of Lemma 2.2

In this appendix we prove Lemma 2.2. The proof is very similar to the proof given in [19] in the special case where $(H_1, V_1)$ is a steady state. However, it happens that the proof actually does not need the relation $(H_1 V_1)_x = 0$ which is no longer true when $(H_1, V_1)$ is not a steady-state. Let $f = (\lambda_2 \phi / \lambda_1)$, we have from (D.2)
And there exists $\delta_0$ such that, if $\|\partial_t H_1\|_{L^\infty((0, +\infty) \times (0, L)) \delta_0}$, 
\[
\frac{\phi}{\lambda_1^2} \left( \frac{2kV_1}{H_1} \sqrt{\frac{gH_1}{H_1}} + \frac{kV_1^2}{H_1} \sqrt{\frac{H_1}{g}} V_1 + \sqrt{\frac{g}{H_1} \partial_t H_1} \right) > 0, \quad \forall \ x \in [0, L], \ t \in [0, +\infty),
\]
(D.4)
and, from (D.1) and (D.3),
\[
\partial_x f = \left| \frac{\phi_2}{\lambda_1} + \frac{\delta_1}{\lambda_2} \phi \right|^2 + \sqrt{\frac{g}{H_1} \partial_t H_1},
\]
(D.5)
this ends the proof of Lemma 2.2.

References


