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► To cite this version:

Clement Aldebert, David Nerini, Mathias Gauduchon, J.C. Poggiale. Does structural sensitivity alter complexity–stability relationships?. *Ecological Complexity*, 2016, 28, pp.104 - 112. 10.1016/j.ecocom.2016.07.004 . hal-01827263

HAL Id: hal-01827263

<https://hal.science/hal-01827263>

Submitted on 11 Dec 2018

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1 Does structural sensitivity alter complexity-stability 2 relationships?

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12 Keywords

13 food webs ; dynamical systems ; structural sensitivity ; complexity-stability relationships ; functional
14 response

15 Highlights

- 16 • We study food webs of varying trophic complexity (number of species, connectance)
- 17 • Food web dynamics is modeled using either Holling's or Ivlev's functional response
- 18 • Community persistence is mainly driven by trophic complexity
- 19 • Occurrence of fluctuating dynamics is mainly driven by the functional response
- 20 • Conclusions are robust to some changes in parameter values and model assumptions

21 Abstract

22 Structural sensitivity, namely the sensitivity of a model dynamics to slight changes in its mathematical
23 formulation, has already been studied in some models with a small number of state variables. The
24 aim of this study is to investigate the impact of structural sensitivity in a food web model. Especially,
25 the importance of structural sensitivity is compared to that of trophic complexity (number of species,
26 connectance), which is known to strongly influence food web dynamics. Food web structures are built
27 using the niche model. Then food web dynamics are modeled using several type II functional responses
28 parameterized to fit the same predation fluxes. Food web persistence was found to be mostly determined
29 by trophic complexity. At the opposite, even if food web connectance promotes equilibrium dynamics,
30 their occurrence is mainly driven by the choice of the functional response. These conclusions are robust to
31 changes in some parameter values, the fitting method and some model assumptions. In a one-prey/one-
32 predator system, it was shown that the possibility that multiple stable states coexist can be highly
33 structural sensitive. Quantifying this type of uncertainty at the scale of ecosystem models will be both a
34 natural extension to this work and a challenging issue.

1 Introduction

Predictions made by mathematical models can be sensitive to model formulation (Anderson *et al.*, 2010; Fulton *et al.*, 2003b; Fussmann & Blasius, 2005, among others). However, this sensitivity has rarely been tested in theoretical and operational ecosystem models (Arhonditsis & Brett, 2004; Fulton *et al.*, 2003a). In ecological models with multiple interacting populations, phenomena observed at the community scale are usually represented by simplifying smaller scale processes. For instance, collective and individual behaviours as well as physiological processes involved in predation are collapsed into one function, the functional response (Gentleman *et al.*, 2003; Jeschke *et al.*, 2002). Numerous mathematical formulations of a given biological phenomenon are relevant in the sense that: (i) their properties and assumptions about underlying processes are consistent with the knowledge of the system to model, (ii) they equivalently fit empirical data (Cordoleani *et al.*, 2011; Mullin *et al.*, 1975). Moreover, some of these functions may have the same mathematical properties (pointwise properties, monotonicity, convexity, ...). However, the choice of a particular function among relevant ones can affect the dynamics predicted by the same model. Differences occur in predicted steady-state values, equilibrium vs. oscillating dynamics and in the system response to external disturbances (Aldebert *et al.*, 2016). Uncertainty due to this choice of a function is coined as “structural sensitivity” (*sensu* Cordoleani *et al.*, 2011).

Structural sensitivity has been theoretically studied in simple models with a few state variables, mainly predator-prey and food chain models (Adamson & Morozov, 2012, 2014; Fussmann & Blasius, 2005; Gross *et al.*, 2004; Myerscough *et al.*, 1996). The aim of this study is to extend these results to more complex models such as food webs. Previous results on predator-prey models may suggest that food web models are sensitive to the choice of type-II functional response. We propose to compare food web dynamics under changes in both functional response formulation and trophic complexity (number of trophic species and trophic links).

Trophic complexity is known to affect food web dynamics and stability. Complexity-stability relationships have been conceptually studied by MacArthur (1955) and then more formally by May (1972, 1973). May’s work has led to a long-standing debate which is still open after decades of field and theoretical researches (Loreau, 2010; May, 1999; McCann, 2000). Food webs exhibit a huge number of different structures. A relevant analysis of their common properties requires to reproduce their diversity. Numerous food webs with empirically consistent structural properties and a desired trophic complexity can be

64 built by simple random models (Cattin *et al.*, 2004; Williams & Martinez, 2000, among others). These
65 models have been used to statistically investigate complexity-stability relationships in food web models
66 based on different ecological phenomena (Brose *et al.*, 2006; Heckmann *et al.*, 2012; Kartascheff *et al.*,
67 2009, 2010; Plitzko *et al.*, 2012; Stouffer & Bascompte, 2010, 2011; Uchida & Drossel, 2007; Williams,
68 2008; Williams & Martinez, 2004).

69 The questions we address in this paper are: (i) are dynamics predicted by a food web model more im-
70 pacted by structural sensitivity or by trophic complexity? (ii) Does structural sensitivity alter complexity-
71 stability relationships?

72 Next section presents the studied food web model. It is an extension of a predator-prey model in
73 which structural sensitivity has already been explored (Aldebert *et al.*, 2016). Structural sensitivity in
74 this model is compared to the impact of trophic complexity in section 3.1. Observed results are then
75 explained from the knowledge of predator-prey models (section 3.2) and their robustness to changes
76 in the method used to fit functional responses is tested (section 3.3). Then, the relative importance
77 of trophic complexity, functional response formulation and parameter values is estimated (section 3.4).
78 Next, robustness to changes in model assumptions is assessed, and complexity-stability relationships are
79 compared to empirical findings (section 3.5). Paper ends with a more general discussion about structural
80 sensitivity and modelling of biological systems (section 3.6).

81 2 Models

82 2.1 Food web structure

83 Food webs are composed by S species (*sensu* trophic species) and one resource. Species are linked by L
84 trophic interactions, so that food web connectance is $C = L/S^2$ (directed connectance, Martinez, 1991).
85 The niche model (Williams & Martinez, 2000) is used to randomly build numerous food webs with the
86 desired number of species and connectance. The niche model generates quickly numerous food webs
87 with patterns that are consistent with empirical data (Allesina *et al.*, 2008; Cattin *et al.*, 2004; Williams
88 & Martinez, 2000). It is based on the principle of ecological niche (Hutchinson, 1957). A species i is
89 characterized by a niche value n_i uniformly drawn in the interval $[0, 1]$, the niche axis.

90 The niche model is described in section 1 of Supporting Online Material (SOM). Food webs are made
91 of distinct species, that are either a primary producer or a predator. Attribution of trophic links allows for

cannibalism and trophic loops. We added a rejection step after food webs construction to avoid unrealistic patterns. We only studied food webs with a realized connectance that deviated at most by 0.01 of the expected one, that are connected (no disconnected parts), and in which all predators feed (as a prey or through a food chain) upon at least one primary producer.

2.2 Food web dynamics

Food web dynamics is modeled using a dynamical system of S differential equations. It is a bio-energetic model extended for a multi-species system (Plitzko *et al.*, 2012; Yodzis & Innes, 1992, among others). This deterministic model is continuous in time with unstructured populations. Each species i is described by its biomass B_i , with dynamics given by the ordinary differential equation:

$$\frac{dB_i}{dt} = \lambda q_i^\phi B_i + \lambda \sum_{j \in R_i} G_{i,j}^\phi B_j - \sum_{j \in C_i} G_{j,i}^\phi B_j - \alpha_i B_i - \beta_i B_i^2 \quad i = 1, \dots, S. \quad (1)$$

Right terms of model (1) handle respectively a gain in biomass by primary production, sum of gains by predation, sum of losses by predation, linear mortality and respiration, density-dependent mortality (intra-specific competition, diseases). Species i possesses a set of prey (predator) species denoted as R_i (C_i). By definition, primary producers have $R_i = \emptyset$ and top-predators have $C_i = \emptyset$. The parameter λ is the assimilation efficiency. For the sake of simplicity, λ is assumed to be the same for all species. Parameter α_i is the linear mortality rate and parameter β_i is the per-capita intra-specific competition rate of species i . The letter ϕ indicates the specific formulation used for the Holling-type II functional response $G_{i,j}^\phi$. For simplicity, all species are assumed to have the same formulation. This one is either Holling's disc equation (1959; 1965) denoted as $G_{i,j}^H$ or Ivlev's functional response (1955) denoted as $G_{i,j}^I$ (later called Holling's FR and Ivlev's FR):

$$G_{i,j}^H = \frac{a_i^H f_{i,j} B_j}{1 + h_i^H a_i^H T_i}, \quad G_{i,j}^I = \frac{1}{h_i^I} (1 - \exp(-h_i^I a_i^I T_i)) \frac{f_{i,j} B_j}{T_i} \quad \text{with } T_i = \sum_{j \in R_i} f_{i,j} B_j.$$

Both functional responses are extended for a predator with multiple prey species by assuming that it does not switch between preys (Gentleman *et al.*, 2003). For Holling's FR, parameters a_i^H and h_i^H are respectively the attack rate and the handling time of the predator. For Ivlev's FR, parameter $1/h_i^I$ is the maximal digestion rate and $a_i^I h_i^I$ is the satiation coefficient of the predator. The total amount of prey

available for species i is the weighted sum of its prey species biomass T_i . The weighting parameter $f_{i,j}$ is constant and it can be considered as the foraging effort or the feeding preference of predator i for its prey species j (obviously, $f_{i,j} = 0$ and $G_{i,j}^\phi = 0$ if $j \notin R_i$). This means that the total functional response of a predator $G_i^{\phi,tot}(T_i) = \sum_{j \in R_i} G_{i,j}^\phi$ is a function of T_i (figure 1) and that $G_{i,j}^\phi = G_i^{\phi,tot} f_{i,j} B_j / T_i$. Both functional responses also fulfills properties:

$$G_i^{\phi,tot} \in \mathcal{C}^2, G_i^{\phi,tot}(0) = 0, G_i^{\phi,tot}(T_i) \geq 0, G_i^{\phi,tot'}(T_i) > 0, G_i^{\phi,tot''}(T_i) < 0, \lim_{T_i \rightarrow +\infty} G_i^{\phi,tot}(T_i) < +\infty,$$

with $\mathcal{C}^2$ being the class of twice continuously differentiable functions. Other properties means that $G_i^{\phi,tot}$ is null in absence of prey, increases with prey biomass, is concave and saturates at high prey biomass.

Functional response's parameters have the same mathematical meaning in both formulations:

$$G_i^{\phi,tot'}(0) = a_i^\phi, \lim_{T_i \rightarrow +\infty} G_i^{\phi,tot}(T_i) = \frac{1}{h_i^\phi}.$$

Thus, a_{pred}^ϕ gives the slope of the functional response at the origin, and $1/h_{pred}^\phi$ gives the asymptotic value of the functional response when it saturates at high prey biomass.

The term of primary productivity q_i^ϕ has the same equation as the functional response $G_{i,j}^\phi$ with a constant pool of resources B_{res} :

$$q_i^H = \begin{cases} \frac{a_i^H B_{res}}{1 + h_i^H a_i^H B_{res}} & \text{if } i \in P_I \\ 0 & \text{otherwise} \end{cases}, \quad q_i^I = \begin{cases} \frac{1}{h_i^I} (1 - \exp(-h_i^I a_i^I B_{res})) & \text{if } i \in P_I \\ 0 & \text{otherwise.} \end{cases}$$

This term vanishes for predators, with P_I being the set of primary producers. Setting a constant pool of resource implies two assumptions. First, the time scale of the resource dynamics is such that it is always at quasi-equilibrium in comparison to the time scale of population dynamics. Second, the equilibrium value is independent of the food web dynamics, it only depends on the environment, which is supposed to be constant.

Let $M_i = 10^{x n_i}$ be the body mass of species i , with a scale parameter x . We assume that some parameter values scale allometrically across species (Brown *et al.*, 2004):

Figure 1. Functional responses used in the model. a: total functional response of species i (total amount of preys eaten by predator and by time unit) calculated as a function of its total prey biomass available, using Holling's disc equation (solid) or Ivlev's functional response (dashed). The latter is parameterized in order to minimize the weighted Euclidean distance between the two formulations. For the sake of visibility, only a part of the fitting range is shown. b: weighting function used to calculate the distance between formulations. See section 2.2 for formulations and section 2.3 for parameterization.

$$a_i^\phi = a^\phi M_i^{-1/4}, h_i^\phi = h^\phi M_i^{1/4}, \alpha_i = \alpha M_i^{-1/4}, \beta_i = \beta M_i^{-1/4}.$$

As species abundance is quantified in terms of biomass, the scaling exponent $-1/4$ corresponds to an increase of individual metabolic rate with body mass power $3/4$, divided by individual body mass. The $3/4$ exponent is controversial. Empirical data and theoretical studies indicate values between $2/3$ and 1 (Kooijman, 2010). Nevertheless, the exact value of this exponent has a limited impact on species extinctions in similar food web models (Kartascheff *et al.*, 2010).

Note that if two species have the same trophic interactions and parameter values, model (1) is sensitive to the aggregation of these identical species as no direct inter-specific competition is considered. It has no impact on our numerical results as identical (or infinitely close) species are infinitely rare in food webs built by the niche model. This problem in model consistency can be solved by considering an inter-specific competition term, with a competition strength that decreases with a measure of distance between species.

Table 1. Parameter values used in the food web model. Parameter values from Heckmann *et al.* (2012) were estimated from empirical data sets (up to > 700 organisms from unicellular eukaryotes to plants and mammals, which span 20 orders of magnitude in body mass, Brose *et al.*, 2006; Brown *et al.*, 2004) using allometric scaling or set to values similar to other studies for comparison (like Kartascheff *et al.*, 2009, 2010).

biological meaning	parameter	value	source	unit
mortality rate	α	0.3	(Heckmann <i>et al.</i> , 2012)	time ⁻¹
per-capita competition rate	β	0.5	(Heckmann <i>et al.</i> , 2012)	biomass ⁻¹ time ⁻¹
assimilation efficiency	λ	0.65	(Heckmann <i>et al.</i> , 2012)	-
resource biomass	B_{res}	500	(Heckmann <i>et al.</i> , 2012)	biomass
magnitude of body mass range	x	8	(Plitzko <i>et al.</i> , 2012)	-
Holling’s disc equation:				
attack rate	a^H	6	(Heckmann <i>et al.</i> , 2012)	biomass ⁻¹ time ⁻¹
handling time	h^H	0.35	(Heckmann <i>et al.</i> , 2012)	time
Ivlev’s functional response:				
maximal consumption rate	$1/h^I$	1/0.36	see text	time ⁻¹
satiation coefficient	$a^I h^I$	3.17×0.36	see text	biomass ⁻¹

2.3 Parameterization and numerical study

Parameter values are displayed in table 1. The range of trophic complexity (20 to 60 species, connectance of 0.10 to 0.30) is set to include values used in previous studies (Gross *et al.*, 2009; Kartascheff *et al.*, 2009, 2010; Plitzko *et al.*, 2012, among others) for comparison. The number of simulations is limited by assuming that a species i has the same feeding preference $f_{i,j} = 1/|R_i|$ for all its prey species $j \in R_i$ (with $|R_i|$ being the number of prey species of species i). This choice implies that T_i is the average biomass of prey species $\sum_{j \in R_i} B_j / |R_i|$ (weak generalist model, *sensu* Williams, 2008).

Parameters in Ivlev’s FR are chosen to minimize the weighted Euclidean distance between both functional responses:

$$S_{A,\omega}(a^I, h^I) = d_{A,\omega}^2(G_i^{H,tot}, G_i^{I,tot}) = \int_A \omega(T_i) \left(G_i^{H,tot}(T_i) - G_i^{I,tot}(T_i) \right)^2 dT_i, \quad (2)$$

with a weighting function $\omega(T_i)$ on a fitting range A of T_i values. Total prey biomass ranges from 0 to constant resource biomass for primary producers, so we set $A = [0, B_{res}]$. For a fixed amount of total prey biomass $\sum_{j \in R_i} B_j$, the resulting T_i depends on $|R_i|$. To balance this effect *a priori*, the distance is

156 weighted by (figure 1):

$$\omega_{\mathcal{P}}(T_i) = \frac{1}{S_{max}T_{max}} \sum_{n=1}^N n \text{ with } N = \min(\lfloor T_{max}/T_i \rfloor, S_{max}) \text{ and } T_{max} = \max(A), \quad (3)$$

157 which is the frequency distribution of T_i values for $\sum_{j \in R_i} B_j \in A$ and for a predator's number of prey
 158 species which ranges between one and $S_{max} = 60$. Minimizing the cost function $S_{[0, B_{res}], \omega_{\mathcal{P}}}$ using the
 159 simplex method by Nelder & Mead (1965) give values $a^I = 3.17$ and $h^I = 0.36$. The sensitivity of our
 160 results to choices in A and $\omega(T_i)$ is discussed in section 3.3.

161 Let us recall that, when an equilibrium is reached, the right-hand side of equation (1) vanishes for all
 162 species, with $B_1^*, B_2^*, \dots, B_S^*$ being the species biomass at an equilibrium (and T_i^* the total prey biomass
 163 available of species i at equilibrium). The jacobian matrix J^ϕ of system (1) is presented in appendix S1.

164 The algorithm presented in section 2 of SOM (with a discussion about its limits) is used to auto-
 165 matically determine the kind of asymptotic dynamics reached by a food web. The proposed algorithm
 166 classifies asymptotic dynamics in three types: dynamics in which at least one species goes extinct, equi-
 167 librium in which all species coexist, fluctuating dynamics (e.g. limit cycle, torus, strange attractor) in
 168 which all species coexist. The proportion of food webs in which all species coexist (at equilibrium or
 169 with fluctuating dynamics) is a measure of food web persistence. The proportion of persistent food webs
 170 (i.e. food webs in which all species coexist) which exhibit a fluctuating dynamics is a measure of food
 171 web variability. Food webs construction and simulation were achieved by C++ programs using GNU
 172 Scientific Library for C/C++ (Galassi *et al.*, 2013). Post-simulation analysis and figures were performed
 173 using R language (R Core Team, 2013).

174 3 Results and Discussion

175 3.1 Structural sensitivity vs. trophic complexity

176 Both functional responses predict similar patterns of food web persistence as a function of trophic com-
 177 plexity (figure 2a,c). Within this range, Ivlev's FR predicts less extinctions than Holling's FR does (from
 178 -11% to $+1\%$ food webs with at least one species extinction). With both functional responses, the
 179 proportion of non-persistent food webs increases with trophic complexity. This increase is higher with
 180 the number of species ($\times 1.4$ with Holling's FR and $\times 1.5$ with Ivlev's FR) than with connectance ($\times 1.2$

181 and $\times 1.3$). Functional responses are parameterized to predict quantitatively close predation fluxes, which
 182 explains their similar predictions of food web persistence.

183 Conversely, food web variability is mainly driven by functional response formulation within the tested
 184 range of trophic complexity (figure 2b,d). Ivlev's FR predicts 2.3 times more equilibrium dynamics (from
 185 98 % to 99 % of persistent food webs reach an equilibrium) than Holling's FR does (from 25 % to
 186 80 %). The proportion of equilibrium dynamics increases more with connectance (on average $\times 3.2$ with
 187 Holling's FR and $+1$ % with Ivlev's FR) than with the number of species ($\times 1.1$ with Holling's FR, no
 188 trend with Ivlev's FR). Even if the effect of trophic complexity can change with other measures of food
 189 web persistence and variability, the latter is still higher with Ivlev's FR (section 5 in SOM).

190 **3.2 Understanding structural sensitivity in food webs**

191 Model (1) applied to a one-predator/one-prey system owns an equilibrium which is stable with Ivlev's FR
 192 and unstable with Holling's FR in 26 % to 49 % of the parameter space explored in Aldebert *et al.* (2016).
 193 With Holling's FR, dynamics converge on a stable limit cycle. Equilibrium stability is different because
 194 the slope of Holling's FR at equilibrium is lower. The slope of the functional response at equilibrium in
 195 food webs can be described by its elasticity:

$$\gamma_i := g_i^{\phi'}(1) \in [0, 1] \text{ with } t_i := \frac{T_i}{T_i^*} \text{ and } g_i^{\phi}(t_i) := \frac{G_i^{\phi, tot}(t_i T_i^*)}{G_i^{\phi, tot}(T_i^*)},$$

196 where the normalized functional response equals 1 at equilibrium ($g_i^{\phi}(1) = 1$). A high elasticity γ_i
 197 stabilizes equilibria in predator-prey (Aldebert *et al.*, 2016; Yeakel *et al.*, 2011) and food web models
 198 (section 6 of SOM).

199 The elasticity depends on prey biomass at equilibrium, which has a similar distribution between
 200 functional responses (figure 3). However, Ivlev's FR has a higher elasticity than Holling's FR for low
 201 prey biomass values (figure 4b). As a consequence, the realized γ_i distribution in food web simulations
 202 is 1.6 times higher with Ivlev's FR ($\bar{\gamma} = 0.60$) than with Holling's FR ($\bar{\gamma} = 0.37$, figure 4a). With the
 203 latter, one can expect that stable equilibria are closer to a bifurcation threshold (with respect to γ_i).
 204 Thus, more food webs are likely to be on the unstable side of this bifurcation with Holling's FR. This
 205 can explain why this function leads more frequently to fluctuating dynamics. This reasoning extends the
 206 previous mechanism found in predator-prey models (Fussmann & Blasius, 2005) to the scale of complex

Holling's disc equation

Ivlev's functional response

Figure 2. Food web dynamics as a function of trophic complexity predicted by Holling's (a, b) and Ivlev's functional responses (c, d). a, c: proportion of food webs with at least one species extinction. b, d: proportion of persistent food webs with a stable positive equilibrium. The number of food webs studied for each pair of parameters is enough to obtain 10 000 food webs with a stable positive equilibrium.

Figure 3. Estimated densities of species biomass (a) and total prey biomass available for predator species (b) in food webs reaching a positive equilibrium with Holling’s (plain) or Ivlev’s (dashed) functional response. Computed from 160 000 food webs of varying complexity (4 levels of connectance and 4 numbers of species, 10 000 food webs by pair of values). Density estimates are realized using non-parametric kernel methods (Simonoff, 1996).

3.3 Structural sensitivity and the method used to fit the functional responses

As the slope of the functional response drives food web variability, one can think about different ways to parameterize Ivlev’s FR. The relevance of these different approaches from a biological point of view is discussed in section 3.6. Here, we focus on robustness to changes in parameter values estimated for Ivlev’s FR. Different parameter sets are estimated by minimizing the weighted distance (2): (i) over different ranges of T_i values, and (ii) using either the weighting function $\omega_{\mathcal{P}}$ (3) (figure 1) or a uniform weighting function $\omega_{\mathcal{U}}(T_i)$. One last parameter set is estimated using the empirical distribution of total prey biomass observed in persistent food webs at equilibrium with Holling’s FR (figure 3b).

Parameter sets that give a better fit to Holling’s FR at low prey biomass (closer a^ϕ value) predict a closer value of food web variability between functional responses (table 2 and figure 5). However, they also predict more distant primary productivity ($\approx 1/h^\phi$ value) and predicted value of food web persistence. Furthermore, equilibrium dynamics are still more frequent with Ivlev’s FR within the tested range of

Figure 4. Elasticity of the functional response: estimated density in food webs reaching a positive equilibrium (a) and as a function of total prey biomass available (b) with Holling’s (plain) or Ivlev’s (dashed) functional response. Computed for predator species from 160 000 food webs of varying complexity (4 levels of connectance and 4 numbers of species, 10 000 food webs by pair of values). Density estimates are realized using non-parametric kernel methods (Simonoff, 1996).

220 trophic complexity (section 3 in SOM). The parameter set that leads to the closer food web variability
 221 ($\omega_{\mathcal{P}}$ within $[0, 3.5]$) does not lead to the best fit of functional responses within the range of prey biomass
 222 observed (column $S_{\omega_{\mathcal{U}}, [0, 3.5]}$ in table 2), as it is optimized for even smaller biomass values. In other
 223 words, this parameter set optimizes only the slope at the origin (a^{ϕ}), as we know that the functional
 224 response slope drives food web variability. Only one parameter set is usually considered in previous
 225 studies on structural sensitivity in predator-prey models (Aldebert *et al.*, 2016; Cordoleani *et al.*, 2011;
 226 Fussmann & Blasius, 2005; Myerscough *et al.*, 1996). We checked here with different parameter sets that
 227 functional response formulation influences food web variability, even if this influence is lower with some
 228 fitting methods.

229 3.4 Structural sensitivity vs. parameter sensitivity

230 Even with the simplifying assumption that parameter values scale allometrically, a full parameter sensi-
 231 tivity analysis requires a high computational effort. Indeed, to be reasonably exhaustive it is necessary
 232 to study thousands of food webs with different numbers of species and connectance levels for each pa-

Table 2. Impact of the method used to fit the functional responses. The columns indicate respectively: the weighting function used, the range of T_i values within which the weighted distance between functions is minimized, the pair of parameters values obtained (a^ϕ and h^ϕ) for Ivlev’s functional response, the corresponding distance (computed with the same weighting function for comparison) between functions at low ($[0, 3.5]$) and high ($[3.5, 500]$) total prey biomass available, the predicted dynamics (proportion of food webs with extinction(s), proportion of persistent food webs reaching an equilibrium) in 30-species food webs with a connectance of 0.15. The number of food webs studied for each pair of parameters is enough to obtain 1 000 food webs with a stable positive equilibrium. The bold line corresponds to the parameters values used in the rest of the study. In addition, one parameter set for Ivlev’s functional response has been estimated using the empirical distribution of T_i observed in persistent food webs at equilibrium with Holling’s disc equation (figure 3b). Dynamics predicted by Holling’s disc equation are indicated for comparison.

weighting function	$T_i \in A$	a^ϕ	h^ϕ	$S_{\omega_{\mathcal{U}},[0,3.5]}$	$S_{\omega_{\mathcal{U}},[3.5,500]}$	extinction(s)	equilibrium
$\omega_{\mathcal{U}}$	[0, 500]	2.70	0.35	0.049	< 0.001	0.79	1.00
	[0, 20]	3.12	0.37	0.026	0.021	0.83	0.99
	[0, 15]	3.22	0.37	0.025	0.021	0.84	0.98
	[0, 10]	3.39	0.38	0.017	0.046	0.85	0.96
	[0, 3.5]	4.03	0.41	0.004	0.165	0.89	0.89
$\omega_{\mathcal{P}}$	[0, 500]	3.17	0.36	0.036	0.005	0.83	0.98
	[0, 20]	4.67	0.41	0.007	0.165	0.91	0.72
	[0, 15]	4.82	0.42	0.007	0.215	0.92	0.68
	[0, 10]	4.98	0.43	0.010	0.269	0.92	0.60
	[0, 3.5]	5.31	0.45	0.023	0.387	0.93	0.51
T_i distribution from fig. 3b		4.50	0.44	0.015	0.327	0.91	0.80
Holling’s disc equation		6	0.35			0.86	0.43

parameter set. However, we investigate the robustness of our results to the functional response parameter values. We vary the parameter values used for Holling’s FR (a^H and h^H) within a range $[-20 \%, +20 \%]$ (discretized by steps of 10 %) and re-estimate the parameters for Ivlev’s FR by fitting the two functions. The relative variation considered is comparable to uncertainties in parameter estimation from empirical data (Cordoleani *et al.*, 2011). Food webs with the same numbers of species and connectance levels as in figure 2 are studied for each parameter set and each functional response for a total amount of $\approx 6.10^7$ food webs studied.

Within the tested range of trophic complexity, food web persistence is strongly correlated (0.76) to the number of species, whereas food web variability is strongly correlated (0.82) to functional response formulation (3). Both persistence and variability are also correlated (0.37 and 0.36) to connectance. In comparison, food web dynamics are weakly correlated to parameter values. A change in parameter values has a weaker impact on model dynamics than a change in model formulation, even if both changes

Figure 5. Impact of the method used to fit the functional responses. a: Holling’s disc equation (plain), Ivlev’s functional response with the parameter set used in the study (bold dashed) and with parameter sets obtained with different fitting methods (thin dashed, see table 2). b and c: food web dynamics (30 species, connectance of 0.15) predicted for these parameter sets. Gray labels and arrows summarize the main trend of results: a better fit at low prey biomass (a) leads to a closer predicted food web variability (c), but also to more distant functions at other prey biomass values (a) and more distant predicted food web persistence (b).

correspond to a quantitatively similar variation measured between the functional responses (Adamson & Morozov, 2012, 2014; Cordoleani *et al.*, 2011; Wood & Thomas, 1999). Indeed, for a quantitatively similar variation, a change of formulation may affect the function shape and lead to a higher change in functional response slope near equilibrium, i.e. in equilibrium stability.

3.5 Model assumptions, complexity-stability relationships and biological systems

Different combinations of model assumptions are made by recent studies on food webs, often for simplifying reasons (Brose *et al.*, 2006; Gross *et al.*, 2009; Heckmann *et al.*, 2012; Kartascheff *et al.*, 2010; Plitzko *et al.*, 2012; Stouffer & Bascompte, 2010, 2011; Williams, 2008). The impact of these assumptions is detailed in section 4 of SOM and summarized in table 4. In all cases, equilibrium dynamics are more frequent with Ivlev’s FR than with Holling’s FR. In addition, both functional responses show the same qualitative effect of each assumption.

Table 3. Correlations between food web dynamics, trophic complexity, the functional response used and its parameterization. We simulated food webs of varying complexity (4 numbers of species times 4 connectance levels) with different functional response formulation (Holling’s or Ivlev’s FRs) and parameterization (5 values of a^ϕ times 5 values of h^ϕ), leading to $n = 800$ combinations. For each one, food web dynamics is summarized by the proportion of food webs with extinction(s) and the proportion of persistent food webs reaching a stable equilibrium. A positive correlation with the functional response corresponds to an increase with Ivlev’s FR. NS means that the corresponding p-value is higher than 0.01 (Student’s test on the regression coefficient). For each combination, the number of food webs studied is enough to obtain 1 000 food webs with a stable positive equilibrium. The total number of food webs simulated is $\approx 6.10^7$.

	number of species	connectance	a^ϕ	h^ϕ	functional response
extinction(s)	0.76	0.37	0.12	0.07	-0.17
equilibrium	NS	0.36	-0.22	0.07	0.82

Table 4. Model assumptions and complexity-stability relationships. Three changes in model assumptions are tested: studying food webs with a fixed number of five primary producer species (PP), setting the body mass of primary producers to 1 and deleting cannibalistic links. For each changes, results (proportion of food webs with extinction(s) and proportion of persistent food webs that reach an equilibrium) are summarized for Holling’s and Ivlev’s functional responses. Trophic complexity (number of species S and connectance C) is varied within the same range as in figure 2. The obtained range of results and the mean impact of trophic complexity are shown. The “-” indicates that the impact of the number of species is not discussed because the number of persistent food webs studied has been decreased for computational reasons.

assumption	functional response	extinction(s)			equilibrium		
		range	S	C	range	S	C
number of PP	Holling	[0.48, 1.00]	$\times 1.6$	$\times 0.8$	[0.15, 0.78]	-	$\times 4.3$
	Ivlev	[0.47, 1.00]	$\times 1.7$	$\times 0.9$	[0.92, 0.99]	-	$\times 1.03$
body mass of PP	Holling	[0.48, 0.99]	$\times 1.6$	$\times 1.2$	[0.38, 0.86]	$\times 1.02$	$\times 2.2$
	Ivlev	[0.43, 1.00]	$\times 1.7$	$\times 1.3$	1.00	=	=
body mass of PP + no cannibalism	Holling	[0.60, 1.00]	$\times 1.4$	$\times 1.2$	[0.35, 0.81]	$\times 1.05$	$\times 2.2$
	Ivlev	[0.51, 1.00]	$\times 1.5$	$\times 1.3$	1.00	=	=

257 These changes in model assumptions show that food web persistence decreases with connectance due
 258 to a decrease of the proportion of primary producer species in food webs made by the niche model (SOM,
 259 figure 2). However, persistence is increased by the connectance *per se*, i.e. the number of pathways
 260 where energy can flow. Furthermore, connectance increases the occurrence of equilibrium dynamics and
 261 improves species survival by keeping their dynamics far from the extinction threshold. More equilibrium
 262 dynamics occur at high connectance because it increases the number of weak trophic links. Weak links
 263 stabilize equilibria by dampening predator-prey oscillations (McCann *et al.*, 1998).

264 The impact of the number of species on system dynamics can be related to general questions on the
 265 effect of biodiversity on ecosystem functioning (May, 1999; McCann, 2000). Jiang & Pu (2009) have made
 266 a meta-analysis of empirical studies on the effect of biodiversity. Both observational and experimental
 267 studies on multi-trophic communities indicate that the temporal variability of state variables (biomasses,
 268 processes, etc.) decreases at the population level (15 studies), and even more at the community level (25
 269 studies). These trends are predicted by the model, which predicts more equilibrium dynamics (figure 2
 270 and table 4) and a lower temporal variability of population biomass (section 5 in SOM) in larger persistent
 271 food webs. In addition, larger food webs have a lower variability of their predicted total biomass (data
 272 not shown). Note that in the model we used, there is no feedback of population dynamics on the trophic
 273 structure, which is set *a priori*. Thus, the link between diversity and ecosystem persistence can be better
 274 explored using evolutionary models that build persistent networks of interaction based on population
 275 dynamics (Drossel *et al.*, 2001; Loeuille & Loreau, 2005; Rossberg *et al.*, 2006).

276 3.6 Structural sensitivity and modelling biological systems

277 We have shown that quantitatively close mathematical functions can lead to qualitatively different food
 278 web dynamics. However, the amount of differences in dynamics depends on the way the functions were
 279 parameterized. In general, parameterization is realized by either fitting the functions to data on the
 280 process itself (Anderson *et al.*, 2010; Cordoleani *et al.*, 2011; Fussmann *et al.*, 2000; Poggiale *et al.*, 2010),
 281 or fitting the full model predictions (other parameters can be optimized at the same time) to data on the
 282 temporal evolution of the system (Canale *et al.*, 1973; Kooi & Kooijman, 1994).

283 Here, we used the first method in different ways. The most classical ones, namely a fit with a uniform
 284 weight over the range of biomass observed or a fit using the empirical biomass distribution, lead to
 285 significant differences in model dynamics between functional responses. These differences become lower

only if the fit is done so that functions are close in term of property that drives model dynamics (here the functional response slope at the origin). So, model dynamics have to be known *a priori* to be able to use this approach, which is in fact more similar to the second method of fit.

The second method focuses on the dynamics predicted by the model. Different functions can be parameterized in order to predict close dynamics. However, these dynamics are specific to the set of environmental conditions (i.e. other parameter values) used. If these conditions are changed (e.g. resource availability, mortality due to external factors), the different functions are likely to predict different dynamics as they are outside the range of optimization. The underlying idea is similar to the method of generalized modelling (Gross & Feudel, 2006) and the approach proposed by Adamson & Morozov (2012). These approaches do not specify model formulation and describe only the local properties of the model (like the slope of a function) near an equilibrium. So, results on equilibrium stability are independent of a specific formulation. However, these approaches provide no information on the system dynamics far from this equilibrium, like the existence of alternative stable states. In conclusion, if different functions are parameterized to predict similar dynamics in a given situation, this is likely to work only in the vicinity of this given situation (e.g. a given equilibrium). Choosing to do so or to use the first method depends on the purpose of the study.

Stable equilibria and limit cycles can coexist in the food web model applied to a one-predator/one-prey system, due to density-dependent mortality (Aldebert *et al.*, 2016). Density-dependent mortality allows the coexistence of different asymptotic states that depend on functional response formulation. As a consequence, functional response formulation drives model predictions in situations where external disturbances and recovery policies are applied (resilience, hysteresis phenomena). Such situations have not been investigated in food webs in this study, as we did one dynamical simulation per food web. Multiple simulations with different initial conditions are required for each food web to try to detect different asymptotic dynamics. Indeed, the existence of such dynamics cannot be fully determined using bifurcation analysis in systems with tens of state variables like food webs. However, quantifying uncertainties in food web model predictions in situations with external disturbances would be an important step toward more accurate model predictions.

4 Conclusion

We investigated the sensitivity of a food web model to the choice of functional response formulation (Holling’s or Ivlev’s FRs). We found a little effect of functional response formulation on food web persistence, whereas food web variability is significantly lower with Ivlev’s functional response. Functional response slope at equilibrium explains this lower variability. In addition, food web variability is more driven by functional response formulation than by its parameter values and the tested range of trophic complexity (number of species, connectance). However, complexity-stability relationships are not qualitatively affected by functional response formulation. These conclusions are robust both to different combinations of model assumptions (cannibalism, primary production) and to different fitting methods to parameterize functional responses.

Because of intrinsic data variability and because model formulation is always a simplified representation of complex biological processes, the choice of functional response may remain uncertain for many species. The results demonstrate that this uncertainty in the formulation of a food web model can lead to uncertainties in the type of asymptotic dynamics it predicts. In addition, uncertainties in predicted system resilience in case of external disturbances are known to arise in some predator-prey models, and so are likely to occur in food web models. The quantification of these potential uncertainties in food web resilience may be a challenging way of research toward more accurate model predictions.

Acknowledgments

The authors thank four anonymous reviewers for their very constructive comments on earlier versions of the manuscript. We also thank Bastien Bellemin-Noel for language proof-reading. The authors acknowledge the staff of the “Cluster de calcul intensif HPC” Platform (OSU Institut Pythéas, Aix-Marseille Université, INSU-CNRS) for providing computing facilities. The PhD scholarship of C.A. was funded by the French Ministry for Education and Research.

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452 Appendix S1: the jacobian matrix

$$\begin{aligned}
 J_{i,i}^{\phi} &= \lambda \left[q_i^{\phi} + G_i^{\phi,tot} + B_i \frac{\partial G_i^{\phi,tot}}{\partial B_i} \right] - G_{i,i}^{\phi} - \sum_{k \in C_i} B_k \frac{\partial G_{k,i}^{\phi}}{\partial B_i} - \alpha_i - 2\beta_i B_i \quad i = 1, \dots, S \\
 J_{i,j}^{\phi} &= \lambda B_i \frac{\partial G_i^{\phi,tot}}{\partial B_j} - G_{j,i}^{\phi} + \sum_{k \in C_i} \frac{\partial G_{k,i}^{\phi}}{\partial B_j} \quad i, j = 1, \dots, S, \quad j \neq i
 \end{aligned}$$