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Abstract. In this paper, we study the ruin problem with investment in a general framework where the business part X is a Lévy process and the return on investment R is a semimartingale. Under some conditions, we obtain upper and lower bounds on the finite and infinite time ruin probabilities as well as the logarithmic asymptotic for them. When R is a Lévy process, we retrieve some well-known results. Finally, we obtain conditions on the exponential functionals of R for ruin with probability one, and we express these conditions using the semimartingale characteristics of R in the case of Lévy processs.

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1. Introduction and Main Results

The estimation of the probability of ruin of insurance companies is a fundamental problem for market actors. Classically, a Poisson process with drift was used to model the value of an insurance company and in that case, under some assumptions on the parameters of the process, the probability of ruin decreases at least as an exponential function of the initial capital, see e.g. [1]. Over time, the compound Poisson process has been replaced by more complex models. In a first generalisation, the value of the company is modeled by a Lévy process and then the ruin probability behaves essentially like the tail of the Lévy measure and, in the light-tailed case, this means that this probability decreases at least as an exponential function (see [1], [18], [20], and [36]). To generalise even further, it can be assumed that insurance companies invest their capital in a financial market. The main question is then: how does the probability of ruin changes with this additional source of risk?

In this general setting, the value of an insurance company with initial capital $y > 0$, denoted by $Y = (Y_t)_{t \geq 0}$, is given as the solution of the following linear stochastic differential equation

$$Y_t = y + X_t + \int_0^t Y_{s-} dR_s, \text{ for all } t \geq 0, \tag{1}$$

where $X = (X_t)_{t \geq 0}$ and $R = (R_t)_{t \geq 0}$ are two independent one dimensional stochastic processes defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and chosen so that (1) makes sense. In risk theory, the process X represents the profit and loss of the business activity and R represents the return of the investment. The main problem then concerns the study of the stopping time defined by

$$\tau(y) = \inf\{t \geq 0 | Y_t < 0\}$$

with $\inf\{\emptyset\} = +\infty$ and the evaluation of the ruin probability before time $T > 0$, namely $\mathbf{P}(\tau(y) \leq T)$, and the ultimate ruin probability $\mathbf{P}(\tau(y) < +\infty)$. The ruin problem in this general setting was first studied in [26].

Before describing our set-up and our results, we give a brief review of the relevant litterature. The special case when $R_t = rt$, with $r > 0$, for all $t \geq 0$ (non-risky investment) is well-studied and we refer to [30] and references therein for the main results. In brief, in that case and under some additional conditions, the ruin probability decreases even faster than an exponential since the capital of the insurance company is constantly increasing.

The case of risky investment is also well-studied. In that case, it is assumed in general that X and R are independent Lévy processes. The first results in this setting appear in [17] (and later in [38]) where it was shown that under some conditions there exists $C > 0$ and $y_0 \geq 0$ such that for all $y \geq y_0$ and for some $b > 0$

$$\mathbf{P}(\tau(y) < +\infty) \geq C y^{-b}.$$

Qualitatively, this means that the ruin probability cannot decrease faster as a power function, i.e. the degrowth is much slower than in the no-investment case. Later, under some conditions on the Lévy triplets of X and R, it was shown in [29] that for some $\beta > 0$ and $\epsilon > 0$, there exists $C > 0$ such that, as $y \to \infty$,

$$y^{\beta}\,\mathbf{P}(\tau(y) < +\infty) = C + o(y^{-\epsilon}).$$

Recently, in [15], it is proven, under different assumptions on the Lévy triplets and when X has no negative jumps, that there exists $C > 0$ such that for the above $\beta > 0$

$$\lim_{y\to\infty} y^{\beta}\,\mathbf{P}(\tau(y) < +\infty) = C.$$

Results concerning bounds on $\mathbf{P}(\tau(y) < +\infty)$ are given in [17] where it is shown that, for all $\epsilon > 0$, there exists $C > 0$ such that for all $y \geq 0$ and the same $\beta > 0$

$$\mathbf{P}(\tau(y) < +\infty) \leq C y^{-\beta+\epsilon}.$$

In less general settings similar results are available. The case when X is a compound Poisson process with drift and exponential jumps and R is a Brownian motion with drift is studied in [12] (negative jumps only) and in [16] (positive jumps only). In [31] the model with negative jumps is generalized to the case where the drift of X is a bounded stochastic process.

Finally, some exact results for the ultimate ruin probability are available in specific models (see e.g. [30], [38]) and conditions for ruin with probability one are given, for different levels of generality, in [12], [15], [16], [17], [28] and [31].

From a practical point of view, insurance companies are interested in the estimation of the ruin probability $\mathbf{P}(\tau(y) \leq T)$ before time T more than in the ultimate ruin probability $\mathbf{P}(\tau(y) < +\infty)$ or its asymptotic when $y \to +\infty$. For this reason, the goal of this paper is to obtain the inequalities for $\mathbf{P}(\tau(y) \leq T)$. Moreover, in contrast to the insurance business activity, the return on the investment can not be modelled, in general, by a homogeneous process like a Lévy process. Indeed, the market conditions can change over time or switch between different states. This explains why we assume, in this paper, that R is a semimartingale.

Thus, in the following we suppose that the processes $X = (X_t)_{t\geq 0}$ and $R = (R_t)_{t\geq 0}$ are independent one-dimensional processes both starting from zero, and such that X is a Lévy process and R is a semimartingale. We suppose additionally that the jumps of R denoted $\Delta R_t = R_t - R_{t-}$ are strictly bigger than -1, for all $t > 0$.

We denote the generating triplet of the Lévy process X by (a_X, σ_X^2, ν_X) where $a_X \in \mathbb{R}$, $\sigma_X \geq 0$ and ν_X is a Lévy measure. We recall that the generating triplet characterizes the law of X via the characteristic function ϕ_X of X_t (see e.g. p.37 in [35]):

$$\phi_X(\lambda) = \exp\left(t\left(i\lambda a_X - \frac{\sigma_X^2\lambda^2}{2} + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x|\leq 1\}})\, \nu_X(dx)\right)\right)$$

where the Lévy measure ν_X satisfies

$$\int_{\mathbb{R}} \min(x^2, 1)\, \nu_X(dx) < \infty.$$

As well-known, the process X can then be written in the form:

$$\begin{aligned} X_t = a_X t + \sigma_X W_t + \int_0^t \int_{|x|\leq 1} x(\mu_X(ds, dx) - \nu_X(dx)ds) \\ + \int_0^t \int_{|x|>1} x\mu_X(ds, dx), \end{aligned} \tag{2}$$

where μ_X is the measure of jumps of X and W is standard Brownian Motion.

We recall that a semimartingale $R = (R_t)_{t\geq 0}$ can be also defined by its semimartingale decomposition, namely

$$\begin{aligned} R_t = B_t + R_t^c + \int_0^t \int_{|x|\leq 1} x(\mu_R(ds, dx) - \nu_R(ds, dx)) \\ + \int_0^t \int_{|x|>1} x\mu_R(ds, dx), \end{aligned} \tag{3}$$

where $B = (B_t)_{t\geq 0}$ is a drift part, $R^c = (R_t^c)_{t\geq 0}$ is the continuous martingale part of R, μ_R is the measure of jumps of R and ν_R is its compensator (see e.g. Chapter 2 of [14] for more information about these notions).

It is possible to check that in this case $[X, R]_t = 0$, for all $t \geq 0$, and that the equation (1) has a unique strong solution (see e.g. Theorem 11.3 in [27]): for $t > 0$

$$Y_t = \mathcal{E}(R)_t \left(y + \int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \tag{4}$$

where $\mathcal{E}(R)$ is Doléans-Dade's exponential,

$$\mathcal{E}(R)_t = \exp\left(R_t - \frac{1}{2}\langle R^c\rangle_t\right) \prod_{0<s\leq t} (1 + \Delta R_s)e^{-\Delta R_s}$$

(for more details about Doléans-Dade's exponential see e.g. Ch.1, §4f, p. 58 in [14]). Then the time of ruin is simply

$$\tau(y) = \inf\left\{ t \geq 0 \,\middle|\, \int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}} < -y \right\} \tag{5}$$

because $\mathcal{E}(R)_t > 0$, for all $t \geq 0$, and this last fact follows from the assumption that $\Delta R_t > -1$, for all $t \geq 0$.

In this paper, we show that the behaviour of $\tau(y)$ for finite horizon $T > 0$ depends strongly on the behaviour of the exponential functionals at T, i.e. on the behaviour of

$$I_T = \int_0^T e^{-\hat{R}_s} ds \text{ and } J_T(\alpha) = \int_0^T e^{-\alpha \hat{R}_s} ds$$

where $\alpha > 0$ and $\hat{R}_t = \ln \mathcal{E}(R)_t$, for all $t \geq 0$, and for infinite horizon on the behaviour of

$$I_\infty = \int_0^\infty e^{-\hat{R}_s} ds \text{ and } J_\infty(\alpha) = \int_0^\infty e^{-\alpha \hat{R}_s} ds.$$

For convenience we denote $J_T = J_T(2)$ and $J_\infty = J_\infty(2)$. More precisely, defining

$$\beta_T = \sup\left\{ \beta \geq 0 : \mathbf{E}(J_T^{\beta/2}) < \infty, \mathbf{E}(J_T(\beta)) < \infty \right\},$$

we prove the following theorem.

Theorem 1. *Let $T > 0$. Assume that $\beta_T > 0$ and that, for some $0 < \alpha < \beta_T$, we have*

$$\int_{|x|>1} |x|^\alpha \nu_X(dx) < \infty. \tag{6}$$

Then, for all $y > 0$,

$$\mathbf{P}(\tau(y) \leq T) \leq \frac{C_1 \mathbf{E}(I_T^\alpha) + C_2 \mathbf{E}(J_T^{\alpha/2}) + C_3 \mathbf{E}(J_T(\alpha))}{y^\alpha}, \tag{7}$$

where the expectations on the right hand side are finite and $C_1 \geq 0$, $C_2 \geq 0$, and $C_3 \geq 0$ are constants that depend only on α in an explicit way. Moreover, if (6) *holds for all $0 < \alpha < \beta_T$, then*

$$\limsup_{y \to \infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} \leq -\beta_T. \tag{8}$$

Remark 1. Theorem 1 is, up to our knowledge, the first result on the upper bound, when R is not deterministic, for the ruin probability before a finite time even in the case when R is a Lévy process. This theorem links the ruin probability with the tails of the Lévy measure of X and the exponential functionals of the process R which are well-studied objects (see Proposition 2 and the examples there for the values of β_T). When $\beta_T = +\infty$, Theorem 1 shows that under the mentioned conditions the ruin probability decreases faster than any power function when $y \to +\infty$. When $\beta_T < +\infty$, Theorem 1 implies that the ruin probability decreases at least as a power function as $y \to +\infty$.

In Theorem 2, under some simple conditions on the Lévy triplet of X, we give a lower bound for the ruin probability.

Theorem 2. *Let $T > 0$. Assume that for $\gamma_T \geq 1$ we have $\mathbf{E}(I_T^{\gamma_T}) = +\infty$. Additionally, assume that*

$$\int_{|x|>1} |x|\nu_X(dx) < +\infty \tag{9}$$

and that

$$a_X + \int_{|x|>1} x\nu_X(dx) < 0 \text{ or } \sigma_X > 0. \tag{10}$$

Then, for all $\delta > 0$, there exists a positive numerical sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $+\infty$ such that, for all $C > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathbf{P}(\tau(y_n) \leq T) \geq \frac{C}{y_n^{\gamma_T} \ln(y_n)^{1+\delta}}.$$

Moreover,

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} \geq -\gamma_T.$$

We remark that under the conditions of Theorems 1 and 2 with $\gamma_T \geq \beta_T$ we obtain immediately, under rather general conditions, the logarithmic asymptotic for the ruin probability:

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) \leq T)\right)}{\ln(y)} = -\beta_T.$$

From Theorems 1 and 2, we can also obtain similar results for the ultimate ruin probability. Define

$$\beta_\infty = \sup\left\{\beta \geq 0 : \mathbf{E}(I_\infty^\beta) < \infty, \mathbf{E}(J_\infty^{\beta/2}) < \infty, \mathbf{E}(J_\infty(\beta)) < \infty\right\}.$$

Then, since $(I_t)_{t\geq 0}$, $(J_t)_{t\geq 0}$ and $(J_t(\alpha))_{t\geq 0}$ are increasing, we obtain, letting $T \to \infty$ and using the monotone convergence theorem with the upper bound of Theorem 1, the following corollary.

Corollary 1. *Assume that* $\beta_\infty > 0$ *and that the condition (6) holds for some* $0 < \alpha < \beta_\infty$*, then*

$$\mathbf{P}(\tau(y) < \infty) \leq \frac{C_1\mathbf{E}(I_\infty^\alpha) + C_2\mathbf{E}(J_\infty^{\alpha/2}) + C_3\mathbf{E}(J_\infty(\alpha))}{y^\alpha},$$

where $C_1 \geq 0$*,* $C_2 \geq 0$*, and* $C_3 \geq 0$ *are constants that depend only on* α *in an explicit way. Moreover, if* (6) *holds for all* $0 < \alpha < \beta_\infty$*, then*

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) < \infty)\right)}{\ln(y)} \leq -\beta_\infty.$$

Remark 2. In the case when $\hat{R}$ is Lévy process and its Laplace transform has a strictly positive root β_0 (cf. Proposition 2 and the examples there), $\beta_\infty = \beta_0$. In particular, when

$$\hat{R}_t = a_{\hat{R}}t + \sigma_{\hat{R}}W_t$$

with $\hat{\sigma_R} > 0$, we get that $\beta_\infty = \beta_0 = 2a_R/\sigma_R^2 - 1$ and the results of Corollary 1 coincide with the conclusions of [12] and [15].

Concerning the lower bound, we get the following.

Corollary 2. *Assume that* $\mathbf{E}(I_\infty) < +\infty$ *and* $\mathbf{E}(J_\infty) < +\infty$ *and that there exists* $\gamma_\infty > 1$ *such that* $\mathbf{E}(I_\infty^{\gamma_\infty}) = +\infty$ *and* $\mathbf{E}(J_\infty^{\gamma_\infty/2}) = +\infty$*. Assume that* X *verifies* (9) *and* (10)*.Then,*

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y) < \infty)\right)}{\ln(y)} \geq -\gamma_\infty.$$

Again, under the assumptions of the Corollaries 1 and 2 with $\gamma_\infty \geq \beta_\infty$ we can obtain the logarithmic asymptotic for the ultimate ruin probability.

To complete our study of the ruin problem in this setting, we give sufficient conditions for ruin with probability one when X has positive jumps bounded by $a > 0$ and verifies one of the following conditions:

$$a_X < 0 \text{ or } \sigma_X > 0 \text{ or } \nu_X([-a, a]) > 0. \tag{11}$$

Theorem 3. *Assume that* X *verifies the conditions above. In addition assume that* $(\mathbf{P} - a.s.)$*,* $I_\infty = +\infty$*,* $J_\infty = +\infty$ *and that there exists a limit*

$$\lim_{t\to\infty} \frac{I_t}{\sqrt{J_t}} = L$$

with $0 < L < \infty$*. Then, for all* $y > 0$*,*

$$\mathbf{P}(\tau(y) < \infty) = 1.$$

In the case of Lévy processes we express the conditions on the exponential functionals in terms of their characteristics to get the following result which is similar to the known results in e.g. [16] and [28].

Corollary 3. *Assume that X verifies the conditions of Theorem 3. Suppose that R is a Lévy process with characteristic triplet (a_R, σ_R^2, ν_R) satisfying*

$$\int_{-1}^{\infty} |\ln(1+x)| \mathbf{1}_{\{|\ln(1+x)|>1\}} \nu_R(dx) < \infty \tag{12}$$

and

$$a_R - \frac{\sigma_R^2}{2} + \int_{-1}^{\infty} (\ln(1+x) - x\mathbf{1}_{\{|\ln(1+x)|\leq 1\}}) \nu_R(dx) < 0.$$

Then, for all $y > 0$,

$$\mathbf{P}(\tau(y) < \infty) = 1.$$

The rest of the paper is structured as follows. In Section 2, we point to the known results about exponential functionals of semimartingales, we give a simple way to obtain β_T and β_∞ in the case when R is a Lévy process (see Propositions 1 and 2) and apply it to some examples. In Section 3, we derive in Proposition 3 some identities en law, then we recall in Proposition 4 the Novikov-Bichteler-Jacod inequalities for discontinuous martingales, and finally we prove Theorem 1. In Section 4, we establish some lemmas and then we prove Theorem 2. In Section 5 we prove Theorem 3 and Corollary 3.

2. Exponential functionals of semimartingales

Exponential functionals of semimartingales (especially of Lévy processes) are very well-studied. The question of the existence of the moments of I_∞ and the formula in the case when R is a subordinator were considered in [6], [9] and [33]. In the case when R is a Lévy process, the question of the existence of the density of the law of I_∞, PDE equations for the density and the asymptotics for the law were investigated in [2], [3], [5], [10], [11], [13], [19], [24], [25] and [32]. In the more general case of processes with independent increments, conditions for the existence of the moments and reccurent equations for the moments were studied in [33] and [34]. The existence of the density of such functionals and the corresponding PDE equations were considered in [37]. Here, we give two simple results concerning the finiteness of β_T and β_∞ when R is a Lévy process and apply them to the computation of β_T and β_∞ in some examples.

First of all, we give some basic facts about the exponential transform $\hat{R} = (\hat{R}_t)_{t\geq 0}$ of R, i.e. the process defined by

$$\mathcal{E}(R)_t = \exp(\hat{R}_t).$$

Since

$$\mathcal{E}(\hat{R}_t) = \exp\left(R_t - \frac{1}{2}\langle R^c\rangle_t + \sum_{0<s\leq t}(\ln(1+\Delta R_s) - \Delta R_s)\right)$$

we get that

$$\hat{R}_t = R_t - \frac{1}{2}\langle R^c\rangle_t + \sum_{0<s\leq t}(\ln(1+\Delta R_s) - \Delta R_s).$$

When R is a semimartingale, the process $\hat{R}$ is also a semimartingale and the jumps of $\hat{R}$ are given by

$$\Delta\hat{R}_t = \ln(1+\Delta R_t), \text{ for all } t \geq 0.$$

Similarly, when R is a Lévy process, the process $\hat{R}$ is also a Lévy process.

Proposition 1. *Suppose that R is a Lévy process. For $\alpha > 0$ and $T > 0$ the following conditions are equivalent:*

(i) $\mathbf{E}(J_T(\alpha)) < \infty$,

(ii) $\int_{|x|>1} e^{-\alpha x}\nu_{\hat{R}}(dx) < \infty$,

(iii) $\int_{-1}^{\infty} \mathbf{1}_{\{|\ln(1+x)|>1\}}(1+x)^{-\alpha}\nu_R(dx) < \infty$.

Proof. By Fubini's theorem, we obtain

$$\mathbf{E}(J_T(\alpha)) = \mathbf{E}\left(\int_0^T e^{-\alpha\hat{R}_t}dt\right) = \int_0^T \mathbf{E}(e^{-\alpha\hat{R}_t})dt.$$

So, $\mathbf{E}(J_T(\alpha)) < \infty$ is equivalent to $\mathbf{E}(e^{-\alpha\hat{R}_t}) < \infty$, for all $t \geq 0$, which, by Theorem 25.3, p.159 in [35], is equivalent to

$$\int_{|x|>1} e^{-\alpha x}\nu_{\hat{R}}(dx) < \infty.$$

Then, note that

$$
\begin{aligned}
\int_{|x|>1} e^{-\alpha x}\nu_{\hat{R}}(dx) &= \int_0^1 \int_{|x|>1} e^{-\alpha x}\nu_{\hat{R}}(dx)ds \\
&= \mathbf{E}\left(\sum_{0<s\leq 1} e^{-\alpha\Delta \hat{R}_s}\mathbf{1}_{\{|\Delta\hat{R}_s|>1\}}\right) \\
&= \mathbf{E}\left(\sum_{0<s\leq 1} (1+\Delta R_s)^{-\alpha}\mathbf{1}_{\{|\ln(1+\Delta R_s)|>1\}}\right) \\
&= \int_{-1}^{\infty} \mathbf{1}_{\{|\ln(1+x)|>1\}}(1+x)^{-\alpha}\nu_R(dx).
\end{aligned}
$$

□

Proposition 1 allows us to compute β_T in some standard models of mathematical finance.

Example 1. Suppose that $\hat{R}$ is given by $\hat{R}_t = a_{\hat{R}}t + \sigma_{\hat{R}}W_t + \sum_{n=0}^{N_t} Y_n$, where $a_{\hat{R}} \in \mathbb{R}$, $\sigma_{\hat{R}} \geq 0$, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $\gamma > 0$, and $(Y_n)_{n\in\mathbb{N}}$ is a sequence of iid random variables. Suppose, in addition, that all processes involved are independent. If for $(Y_n)_{n\in\mathbb{N}}$ we take any sequence of iid random variables with $\mathbf{E}(e^{-\alpha Y_1}) < \infty$, for all $\alpha > 0$, then $\beta_T = +\infty$. If for $(Y_n)_{n\in\mathbb{N}}$ we take a sequence of iid random variables with $\mathbf{E}(e^{-\alpha Y_1}) < \infty$, when $\alpha < \alpha_0$, for some $\alpha_0 > 0$, and $\mathbf{E}(e^{-\alpha_0 Y_1}) = +\infty$, then $\beta_T = \alpha_0$.

Example 2. Suppose that $\hat{R}$ is a Lévy process with triplet $(a_{\hat{R}}, \sigma^2_{\hat{R}}, \nu_{\hat{R}})$, where $a_{\hat{R}} \in \mathbb{R}$, $\sigma_{\hat{R}} \geq 0$ and $\nu_{\hat{R}}$ is the measure on $\mathbb{R}$ given by

$$
\nu_{\hat{R}}(dx) = \left(C_1|x|^{-(1+\alpha_1)}e^{-\lambda_1|x|}\mathbf{1}_{\{x<0\}} + C_2x^{-(1+\alpha_2)}e^{-\lambda_2 x}\mathbf{1}_{\{x>0\}}\right)dx,
$$

where $C_1, C_2 > 0$, $\lambda_1, \lambda_2 > 0$ and $0 < \alpha_1, \alpha_2 < 2$. This specification includes as special cases the Kou, CGMY and variance-gamma models (see e.g. Section 4.5 p.119 in [8]). We will show that $\beta_T = \lambda_1$. Note that, using Proposition 1 and the change of variables $y = -x$, we see that $\mathbf{E}(J_T(\alpha)) < \infty$, for $\alpha > 0$, is equivalent to

$$
C_1\int_1^{\infty} y^{-(1+\alpha_1)}e^{-(\lambda_1-\alpha)y}dy + C_2\int_1^{\infty} x^{-(1+\alpha_2)}e^{-(\alpha+\lambda_2)x}dx < \infty.
$$

But, the first integral converges if $\alpha \leq \lambda_1$ and diverges if $\alpha > \lambda_1$ and second integral always converges. Now, if $\alpha \geq 2$, it is easy to show that $\mathbf{E}(J_T(\alpha)) < \infty$ implies $\mathbf{E}(J_T^{\alpha/2}) < \infty$ (see Lemma 1 below). Thus, if $\lambda_1 \geq 2$, we have $\beta_T = \lambda_1$.

Proposition 2. *Suppose that the Lévy process $\hat{R}$ admits a Laplace transform, for all $t \geq 0$, i.e. for $\alpha > 0$*

$$\mathbf{E}(\exp(-\alpha \hat{R}_t)) = \exp(t\psi_{\hat{R}}(\alpha))$$

and that its Laplace exponent $\psi_{\hat{R}}$ has a strictly positive root β_0. Then the following conditions are equivalent:

(i) $\mathbf{E}(I_\infty^\alpha) < \infty$,

(ii) $\mathbf{E}(J_\infty^{\alpha/2}) < \infty$,

(iii) $\mathbf{E}(J_\infty(\alpha)) < \infty$,

(iv) $\alpha < \beta_0$.

Therefore, $\beta_\infty = \beta_0$.

Proof. Note that, for any $\alpha > 0$ and $k > 0$,

$$\begin{aligned}\exp(t\psi_{\hat{R}}(\alpha)) = \mathbf{E}(\exp(-\alpha \hat{R}_t)) &= \mathbf{E}\left(\exp\left(-\frac{\alpha}{k} k\hat{R}_t\right)\right)\\ &= \exp\left(t\psi_{k\hat{R}}\left(\frac{\alpha}{k}\right)\right).\end{aligned}$$

Therefore, $\psi_{\hat{R}}(\alpha) = \psi_{k\hat{R}}\left(\frac{\alpha}{k}\right)$, for all $\alpha > 0$ and $k > 0$. Then, Lemma 3 in [32] yields the desired result. □

Remark 3. Note that the root of the Laplace exponent was already identified as the relevant quantity for the tails of $\mathbf{P}(\tau(y) < \infty)$ in [29].

Using Proposition 2 we can compute β_∞ in two important examples.

Example 3. Suppose that $R_t = a_R t + \sigma_R W_t$, for all $t \geq 0$, where $a_R \in \mathbb{R}$, $\sigma_R > 0$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion, then $\hat{R}_t = \left(a_R - \frac{\sigma_R^2}{2}\right) t + \sigma_R W_t$, for all $t \geq 0$. Thus, we obtain $\psi_{\hat{R}}(\alpha) = -\left(a_R - \frac{1}{2}\sigma_R^2\right)\alpha + \frac{\sigma_R^2}{2}\alpha^2$ and, by Proposition 2, $\beta_\infty = \frac{2a_R}{\sigma_R^2} - 1$. We remark that this coincides with the results in e.g. [12] and [16].

Example 4. Suppose that $\hat{R}_t = a_{\hat{R}} t + \sigma_{\hat{R}} W_t + \sum_{n=0}^{N_t} Y_n$, where $a_{\hat{R}} \in \mathbb{R}$, $\sigma_{\hat{R}} \geq 0$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $\gamma > 0$, and $(Y_n)_{n\in\mathbb{N}}$ is a sequence of iid random variables with $\mathbf{E}(e^{-\alpha Y_1}) < \infty$, for all $\alpha > 0$. Suppose, in addition, that all processes involved are independent. It is easy to see that, for all $\alpha > 0$,

$$\psi_{\hat{R}}(\alpha) = -a_{\hat{R}}\alpha + \frac{\sigma_{\hat{R}}^2}{2}\alpha^2 + \gamma\left(\mathbf{E}(e^{-\alpha Y_1}) - 1\right).$$

Now, it is possible to show (see e.g. [36]) that the equation $\psi_{\hat{R}}(\alpha) = 0$ has an unique non-zero solution if, and only if, $\hat{R}$ is not a subordinator and $\psi'(0+) < 0$ which, under some additional conditions to invert the differentiation and expectation operators, is equivalent to $a_{\hat{R}} > \gamma \mathbf{E}(Y_1)$. In that case, β_∞ is the unique non-zero real solution of this equation.

3. Upper bound for the ruin probability

In this section, we prove Theorem 1. We start with some preliminary results: Lemma 1 and Propositions 3 and 4.

Lemma 1. *For all $T > 0$, we have the following.*

(a) *If $0 < \alpha < 2$, then $\mathbf{E}(J_T^{\alpha/2}) < \infty$ implies $\mathbf{E}(I_T^\alpha) < \infty$ and $\mathbf{E}(J_T(\alpha)) < \infty$.*

(b) *If $\alpha \geq 2$, $\mathbf{E}(J_T(\alpha)) < \infty$ implies $\mathbf{E}(I_T^\alpha) < \infty$ and $\mathbf{E}(J_T^{\alpha/2}) < \infty$.*

Proof. First note that by the Cauchy-Schwarz inequality we obtain, for all $T > 0$,

$$I_T = \int_0^T \mathcal{E}(R)_s^{-1} ds \leq \sqrt{T} \left(\int_0^T \mathcal{E}(R)_s^{-2} ds \right)^{1/2} = \sqrt{T}\sqrt{J_T}.$$

So, $\mathbf{E}(I_T^\alpha) \leq T^{\alpha/2} \mathbf{E}(J_T^{\alpha/2})$, for all $\alpha > 0$.

Now, if $0 < \alpha < 2$, we have $\frac{2}{\alpha} > 1$ and by Hölder's inequality

$$J_T(\alpha) = \int_0^T \mathcal{E}(R)_s^{-\alpha} ds \leq T^{(2-\alpha)/2} \left(\int_0^T \mathcal{E}(R)_s^{-2} ds \right)^{\alpha/2} = T^{(2-\alpha)/2} J_T^{\alpha/2}.$$

These inequalities yield (a).

Now, if $\alpha \geq 2$, we have either $\alpha = 2$ which yields the desired result or $\alpha > 2$. In that case, we have $\frac{\alpha}{2} > 1$ and, by Hölder's inequality, we obtain

$$\begin{aligned} J_T &= \int_0^T \mathcal{E}(R)_s^{-2} ds \leq T^{(\alpha-2)/\alpha} \left(\int_0^T \mathcal{E}(R)_s^{-\alpha} ds \right)^{2/\alpha} \\ &= T^{(\alpha-2)/\alpha} J_T(\alpha)^{2/\alpha}. \end{aligned}$$

So, $\mathbf{E}(J_T^{\alpha/2}) \leq T^{(\alpha-2)/2} \mathbf{E}(J_T(\alpha))$, which yields (b). $\square$

Denote by $M^d = (M^d_t)_{t\geq 0}$ the local martingale defined as:

$$M^d_t = \int_0^t \int_{|x|\leq 1} \frac{x}{\mathcal{E}(R)_{s-}}(\mu_X(ds,dx) - \nu_X(dx)ds)$$

and by $U = (U_t)_{t\geq 0}$ the process given by

$$U_t = \int_0^t \int_{|x|>1} \frac{x}{\mathcal{E}(R)_{s-}}\mu_X(ds,dx).$$

If $\int_{|x|>1}|x|\nu_X(dx) < +\infty$, we can also define the local martingale $N^d = (N^d_t)_{t\geq 0}$ as

$$N^d_t = \int_0^t \int_{\mathbb{R}} \frac{x}{\mathcal{E}(R)_{s-}}(\mu_X(ds,dx) - \nu_X(dx)ds).$$

Proposition 3. *We have the following identity in law:*

$$\left(\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)_{t\geq 0} \stackrel{\mathcal{L}}{=} \left(a_X I_t + \sigma_X W_{J_t} + M^d_t + U_t\right)_{t\geq 0}.$$

Moreover, if $\int_{|x|>1}|x|\nu_X(dx) < +\infty$*, then,*

$$\left(\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)_{t\geq 0} \stackrel{\mathcal{L}}{=} \left(\delta_X I_t + \sigma_X W_{J_t} + N^d_t\right)_{t\geq 0},$$

where $\delta_X = a_X + \int_{|x|>1} x\nu_X(dx)$*.*

Proof. We show first that

$$\mathcal{L}\left(\left(\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)_{t\geq 0} \,|\, \mathcal{E}(R)_s = q_s, s\geq 0\right) = \mathcal{L}\left(\left(\int_0^t \frac{dX_s}{q_{s-}}\right)_{t\geq 0}\right)$$

To prove this equality in law we consider the representation of the stochastic integrals by Riemann sums (see [14], Proposition I.4.44, p. 51). We recall that for any increasing sequence of stopping times $\tau = (T_n)_{n\in\mathbb{N}}$ with $T_0 = 0$ such that $\sup_n T_n = \infty$ and $T_n < T_{n+1}$ on the set $\{T_n < \infty\}$, the Riemann approximation of the stochastic integral $\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}$ will be

$$\tau\left(\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{E}(R)_{T_n-}}\left(X_{T_{n+1}\wedge t} - X_{T_n\wedge t}\right)$$

The sequence $\tau_n = (T(n,m))_{m\in\mathbb{N}}$ of adapted subdivisions is called Riemann sequence if $\sup_{m\in\mathbb{N}}(T(n,m+1)\wedge t - T(n,m)\wedge t) \to 0$ as $n\to\infty$

for all $t > 0$. For our purposes we will take a deterministic Riemann sequence. Then, Proposition I.4.44, p.51 of [14] says that for all $t > 0$

$$\tau_n\left(\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \xrightarrow{\mathbf{P}} \int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}} \tag{13}$$

and

$$\tau_n\left(\int_0^t \frac{dX_s}{q_{s-}}\right) \xrightarrow{\mathbf{P}} \int_0^t \frac{dX_s}{q_{s-}} \tag{14}$$

where $\xrightarrow{\mathbf{P}}$ denotes the convergence in probability. According to the Kolmogorov theorem, the law of the process is entirely defined by its finite-dimensional distributions. Let us take for $k \geq 0$ a subdivision $t_0 = 0 < t_1 < t_2 \cdots < t_k$ and a continuous bounded function $F: \mathbb{R}^k \to \mathbb{R}$, to prove by standard arguments that

$$\mathbf{E}\left[F\left(\tau_n\left(\int_0^{t_1} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right), \cdots \tau_n\left(\int_0^{t_k} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right)\right) \mid \mathcal{E}(R)_s = q_s, s \geq 0\right]$$
$$= \mathbf{E}\left[F\left(\tau_n\left(\int_0^{t_1} \frac{dX_s}{q_{s-}}\right), \cdots \tau_n\left(\int_0^{t_k} \frac{dX_s}{q_{s-}}\right)\right)\right]$$

Taking into account (13) and (14), we pass to the limit as $n \to \infty$ and we obtain

$$\mathbf{E}\left[F\left(\int_0^{t_1} \frac{dX_s}{\mathcal{E}(R)_{s-}}, \cdots \int_0^{t_k} \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \mid \mathcal{E}(R)_s = q_s, s \geq 0\right]$$
$$= \mathbf{E}\left[F\left(\int_0^{t_1} \frac{dX_s}{q_{s-}}, \cdots \int_0^{t_k} \frac{dX_s}{q_{s-}}\right)\right]$$

and this proves the claim.

Using the decomposition (2) we get that

$$\begin{aligned}\int_0^t \frac{dX_s}{q_{s-}} = a_X \int_0^t \frac{ds}{q_s} + \sigma_X \int_0^t \frac{dW_s}{q_{s-}} & \\ + \int_0^t \int_{|x|\leq 1} \frac{x}{q_{s-}}(\mu_X(ds,dx) - \nu_X(ds,dx)) & \\ + \int_0^t \int_{|x|>1} \frac{x}{q_{s-}}\, \mu_X(ds,dx). &\end{aligned}$$

We denote the last two terms in the r.h.s. of the equality above by $M_t^d(q)$ and $U_t(q)$ respectively. Recall that since X is Lévy process the four processes appearing in the right-hand side of the above equality are independent. We use the well-known identity in law

$$\left(\int_0^t \frac{dW_s}{q_{s-}}\right)_{t\geq 0} \overset{\mathcal{L}}{=} \left(W_{\int_0^t \frac{ds}{q_s^2}}\right)_{t\geq 0}$$

to write

$$\left(a_X \int_0^t \frac{ds}{q_s},\ \sigma_X \int_0^t \frac{dW_s}{q_{s-}}, M_t^d(q), U_t(q)\right)_{t\geq 0}$$
$$\overset{\mathcal{L}}{=} \left(a_X \int_0^t \frac{ds}{q_s}, \sigma_X W_{\int_0^t \frac{ds}{q_s^2}}, M_t^d(q), U_t(q)\right)_{t\geq 0}.$$

Then, we take the sum of these processes and we integrate w.r.t. the law of $\mathcal{E}(R)$. This yields the first result.

The proof of the second part is the same except we take the following decomposition of X:

$$X_t = \delta_X t + \sigma_X W_t + \int_0^t \int_{\mathbb{R}} x(\mu_X(ds,dx) - \nu_X(dx)ds).$$

□

The last ingredient in the proof of Theorem 1 are the Novikov-Bichteler-Jacod maximal inequalities for compensated integrals with respect to random measures (see [4], [23] and also [22]) which we will state below after introducing some notations. Let $f : (\omega, t, x) \mapsto f(\omega, t, x)$ be a left-continuous and measurable random function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$. Specializing the notations of [23] to our case, we say that $f \in F_2$ if, for almost all $\omega \in \Omega$,

$$\int_0^t \int_{\mathbb{R}} f^2(\omega, s, x)\nu_X(dx)ds < \infty.$$

If $f \in F_2$, we can define the compensated integral by

$$C_f(t) = \int_0^t \int_{\mathbb{R}} f(\omega, s, x)\,(\mu_X(ds,dx) - \nu_X(dx)ds)$$

for all $t \geq 0$. For these compensated integrals, we then have the following inequalities.

Proposition 4 (c.f. Theorem 1 in [23]). *Let f be a left-continuous measurable random function with $f \in F_2$. Let $C_f = (C_f(t))_{t\geq 0}$ be the compensated integral of f as defined above.*

(a) *For all* $0 \le \alpha \le 2$,

$$\mathbf{E}\left(\sup_{0\le t\le T}|C_f(t)|^\alpha\right) \le K_1\mathbf{E}\left[\left(\int_0^T\int_{\mathbb{R}} f^2\nu_X(dx)ds\right)^{\alpha/2}\right].$$

(b) *For all* $\alpha \ge 2$,

$$\mathbf{E}\left(\sup_{0\le t\le T}|C_f(t)|^\alpha\right) \le K_2\mathbf{E}\left[\left(\int_0^T\int_{\mathbb{R}} |f|^2\nu_X(dx)ds\right)^{\alpha/2}\right] + K_3\mathbf{E}\left(\int_0^T\int_{\mathbb{R}} |f|^\alpha\nu_X(dx)ds\right)$$

where $K_1 \ge 0$, $K_2 \ge 0$, *and* $K_3 \ge 0$ *are constants depending only on* α *in an explicit way.*

Proof of Theorem 1. Note that

$$\sup_{0\le t\le T} -(a_X I_t + \sigma_X W_{J_t} + M_t^d + U_t) \le |a_X| I_T + \sup_{0\le t\le T}\sigma_X|W_{J_t}| + \sup_{0\le t\le T}|M_t^d| + \sup_{0\le t\le T}|U_t|,$$

and that for positive random variable Z_1, Z_2, Z_3, Z_4 we have

$$\{Z_1+Z_2+Z_3+Z_4>y\} \subseteq \left\{Z_1>\frac{y}{4}\right\}\cup\left\{Z_2>\frac{y}{4}\right\}\cup\left\{Z_3>\frac{y}{4}\right\}\cup\left\{Z_4>\frac{y}{4}\right\}.$$

Therefore, using Proposition 3, we obtain

$$\begin{aligned}\mathbf{P}(\tau(y)\le T) =& \mathbf{P}\left(\sup_{0\le t\le T} -(a_X I_t + \sigma_X W_{J_t} + M_t^d + U_t) > y\right)\\ \le& \mathbf{P}\left(|a_X|I_T > \frac{y}{4}\right) + \mathbf{P}\left(\sup_{0\le t\le T}\sigma_X|W_{J_t}| > \frac{y}{4}\right)\\ &+ \mathbf{P}\left(\sup_{0\le t\le T}|M_t^d| > \frac{y}{4}\right) + \mathbf{P}\left(\sup_{0\le t\le T}|U_t| > \frac{y}{4}\right).\end{aligned}$$

For the first term, using Markov's inequality, we obtain

$$\mathbf{P}\left(|a_X|I_T > \frac{y}{4}\right) \le \frac{4^\alpha|a_X|^\alpha}{y^\alpha}\mathbf{E}(I_T^\alpha).$$

For the second term, since $(J_t)_{0\le t\le T}$ is increasing we can change the time in the supremum and condition on $(\mathcal{E}(R)_t)_{0\le t\le T}$ to obtain

$$\begin{aligned}\mathbf{P}\left(\sup_{0\le t\le T}\sigma_X|W_{J_t}|>\frac{y}{4}\right)&=\mathbf{P}\left(\sup_{0\le t\le J_T}\sigma_X|W_t|>\frac{y}{4}\right)\\&=\mathbf{E}\left[\mathbf{P}\left(\sup_{0\le t\le J_T}\sigma_X|W_t|>\frac{y}{4}\,\middle|\,(\mathcal{E}(R)_t)_{0\le t\le T}\right)\right]\end{aligned}$$

Since W and R are independent, we obtain, using the reflection principle, the fact that $W_{\int_0^T q_t^{-2}dt}\overset{\mathcal{L}}{=}\left(\int_0^T q_t^{-2}dt\right)^{1/2}W_1$ and Markov's inequality, that

$$\begin{aligned}&\mathbf{P}\left(\sup_{0\le t\le J_T}\sigma_X|W_t|>\frac{y}{4}\,\middle|\,\mathcal{E}(R)_t=q_t,0\le t\le T\right)\\&=2\mathbf{P}\left(\left(\int_0^T q_t^{-2}dt\right)^{1/2}\sigma_X|W_1|>\frac{y}{4}\right)\\&\le 2\frac{4^\alpha\sigma_X^\alpha}{y^\alpha}\left(\int_0^T q_t^{-2}dt\right)^{\alpha/2}\mathbf{E}(|W_1|^\alpha).\end{aligned}$$

Then, since $\mathbf{E}(|W_1|^\alpha)=\frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)$, we obtain

$$\mathbf{P}\left(\sup_{0\le t\le T}\sigma_X|W_{J_t}|>\frac{y}{4}\right)\le\frac{2^{(5\alpha+2)/2}\Gamma\left(\frac{\alpha+1}{2}\right)\sigma_X^\alpha}{\sqrt{\pi}y^\alpha}\mathbf{E}(J_T^{\alpha/2}).$$

Note that the inequalities for the first two terms work for all $\alpha>0$.

Suppose now that $0<\alpha\le 1$. We see that $\mathcal{E}(R)_{t-}^{-1}(\omega)x\mathbf{1}_{\{|x|\le 1\}}\in F_2$. Therefore, using Markov's inequality and part (a) of Proposition 4, we obtain

$$\begin{aligned}&\mathbf{P}\left(\sup_{0\le t\le T}|M_t^d|>\frac{y}{4}\right)\le\frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}|M_t^d|^\alpha\right)\\&\le K_1\frac{4^\alpha}{y^\alpha}\mathbf{E}\left[\left(\int_0^T\int_{\mathbb{R}}\frac{x^2}{\mathcal{E}(R)_{s-}^2}\mathbf{1}_{\{|x|\le 1\}}\nu_X(dx)ds\right)^{\alpha/2}\right]\\&=K_1\frac{4^\alpha}{y^\alpha}\left(\int_{\mathbb{R}}x^2\mathbf{1}_{\{|x|\le 1\}}\nu_X(dx)\right)^{\alpha/2}\mathbf{E}(J_T^{\alpha/2}).\end{aligned}$$

For the last term, note that since $0<\alpha\le 1$, we have $\left(\sum_{i=1}^N x_i\right)^\alpha \le \sum_{i=1}^N x_i^\alpha$, for $x_i\ge 0$ and $N\in\mathbb{N}^*$ and, for each $t\ge 0$,

$$\begin{aligned}|U_t|^\alpha &\le \left(\sum_{0<s\le t}\mathcal{E}(R)_{s-}^{-1}|\Delta X_s|\mathbf{1}_{\{|\Delta X_s|>1\}}\right)^\alpha\\ &\le \sum_{0<s\le t}\mathcal{E}(R)_{s-}^{-\alpha}|\Delta X_s|^\alpha\mathbf{1}_{\{|\Delta X_s|>1\}}\\ &= \int_0^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx).\end{aligned}$$

Therefore, using Markov's inequality and the compensation formula (see e.g. Theorem II.1.8 p.66-67 in [14]), we obtain

$$\begin{aligned}\mathbf{P}\left(\sup_{0\le t\le T}|U_t|>\frac{y}{4}\right) &\le \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}|U_t|^\alpha\right)\\ &\le \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}\int_0^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)\\ &= \frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\int_0^T\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\nu_X(dx)ds\right)\\ &= \frac{4^\alpha}{y^\alpha}\left(\int_{\mathbb{R}}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\nu_X(dx)\right)\mathbf{E}(J_T(\alpha)).\end{aligned}$$

This finishes the proof when $0<\alpha\le 1$.

Suppose now that $1<\alpha\le 2$. The bound for $\mathbf{P}\left(\sup_{0\le t\le T}|M_t^d|>\frac{y}{4}\right)$ can be obtained in the same way as in the previous case. Applying Hölder's inequality we obtain

$$\begin{aligned}|U_t|^\alpha &\le \left(\int_0^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-1/\alpha}\mathcal{E}(R)_{s-}^{1/\alpha-1}|x|\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)^\alpha\\ &\le \left(\int_0^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-1}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)\\ &\quad\times\left(\int_0^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-1}\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)^{\alpha-1}\\ &\le \left(\int_0^t\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-1}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\mu_X(ds,dx)\right)^\alpha.\end{aligned}$$

Then, using Markov's inequality and the compensation formula, we obtain

$$\mathbf{P}\left(\sup_{0\le t\le T}|U_t|>\frac{y}{4}\right)\le\frac{4^\alpha}{y^\alpha}\mathbf{E}\left(\sup_{0\le t\le T}|U_t|^\alpha\right)$$
$$=\left(\int_{\mathbb{R}}|x|^\alpha\mathbf{1}_{\{|x|>1\}}\nu_X(dx)\right)^\alpha\mathbf{E}(I_T^\alpha).$$

This finishes the proof in the case $1<\alpha\le 2$.

Finally, suppose that $\alpha\ge 2$. The estimation for $\mathbf{P}\left(\sup_{0\le t\le T}|U_t|>\frac{y}{4}\right)$ still works in this case. Moreover, since $\mathcal{E}(R)_{t-}^{-1}(\omega)x\mathbf{1}_{\{|x|\le 1\}}\in F_2$, we obtain, applying part (b) of Proposition 4 that

$$\begin{aligned}\mathbf{P}\left(\sup_{0\le t\le T}|M_t^d|>\frac{y}{4}\right)\le& K_2\mathbf{E}\left[\left(\int_0^T\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-2}x^2\mathbf{1}_{\{|x|\le 1\}}\nu_X(dx)ds\right)^{\alpha/2}\right]\\&+K_3\mathbf{E}\left(\int_0^T\int_{\mathbb{R}}\mathcal{E}(R)_{s-}^{-\alpha}|x|^\alpha\mathbf{1}_{\{|x|\le 1\}}\nu_X(dx)ds\right)\\=&K_2\left(\int_{\mathbb{R}}x^2\mathbf{1}_{\{|x|\le 1\}}\nu_X(dx)\right)^{\alpha/2}\mathbf{E}(J_T^{\alpha/2})\\&+K_3\left(\int_{\mathbb{R}}|x|^\alpha\mathbf{1}_{\{|x|\le 1\}}\nu_X(dx)\right)\mathbf{E}(J_T(\alpha)).\end{aligned}$$

Note that the right-hand side is finite since $|x|^\alpha\mathbf{1}_{\{|x|\le 1\}}\le|x|^2\mathbf{1}_{\{|x|\le 1\}}$ when $\alpha\ge 2$. This finishes the proof of (7). Then we take ln from both sides of (7), we divide the inequality by $\ln(y)$ and we take $\lim_{y\to+\infty}$, and, then $\lim_{\alpha\to\beta_T}$ to get (8). □

4. Lower bound for the ruin probability

In this section, we prove Theorem 2 and, therefore, show that the upper bound obtained in Theorem 1 is asymptotically optimal for a large class of Lévy processes X. We start with some preliminary results. Denote $x^{+,p}=(\max(x,0))^p$, for all $x\in\mathbb{R}$ and $p>0$.

Lemma 2. *Suppose that a random variable $Z\ge 0$ ($\mathbf{P}-a.s.$) satisfies $\mathbf{E}(Z^p)=+\infty$, for some $p>0$. Then, for all $\delta>0$, there exists a positive numerical sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $+\infty$ such that, for all $C>0$, there exists $n_0\in\mathbb{N}$ such that for all $n\ge n_0$,*

$$\mathbf{P}(Z\ge y_n)\ge\frac{C}{y_n^p\ln(y_n)^{1+\delta}}.$$

Proof. If $Z \geq 0$ ($\mathbf{P} - a.s.$) is a random variable and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of class C^1 with positive derivative, then, using Fubini's theorem, we obtain

$$g(0) + \int_0^\infty g'(u)\mathbf{P}(Z \geq u)du = g(0) + \mathbf{E}\left(\int_0^Z g'(u)du\right) = \mathbf{E}(g(Z)).$$

Applying this to the function $g(z) = z^p$ with $p > 0$ we obtain, for all $y > 1$,

$$\int_y^\infty u^{p-1}\mathbf{P}(Z \geq u)du = \infty.$$

Moreover, for all $\delta > 0$,

$$\sup_{u \geq y}[u^p \ln(u)^{1+\delta}\mathbf{P}(Z \geq u)] \int_y^\infty \frac{du}{u\ln(u)^{1+\delta}} \geq \int_y^\infty u^{p-1}\mathbf{P}(Z \geq u)du.$$

So, since $\int_y^\infty \frac{du}{u\ln(u)^{1+\delta}} < \infty$, we obtain, for all $y > 1$,

$$\sup_{u \geq y}[u^p \ln(u)^{1+\delta}\mathbf{P}(Z \geq u)] = \infty.$$

Therefore, there exists a numerical sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $+\infty$ such that,

$$\lim_{n\to\infty} y_n^p \ln(y_n)^{1+\delta}\mathbf{P}(Z \geq y_n) = +\infty.$$

□

Lemma 3. *Assume that X and Y are independent random variables with $\mathbf{E}(Y) = 0$. Assume that $p \geq 1$. Then, $\mathbf{E}[X^{+,p}] \leq \mathbf{E}[(X+Y)^{+,p}]$.*

Proof. For each $x \in \mathbb{R}$, we define the function $h_x : y \mapsto (x+y)^{+,p}$ on $\mathbb{R}$. Since $p \geq 1$, h_x is a convex function and we obtain, using Jensen's inequality, that for each $x \in \mathbb{R}$,

$$\mathbf{E}[(x+Y)^{+,p}] = \mathbf{E}[h_x(Y)] \geq h_x(\mathbf{E}(Y)) = h_x(0) = x^{+,p}.$$

We obtain the desired result by integrating w.r.t. the law of X. □

Lemma 4. *Let $T > 0$. Assume that $a < 0$ or $\sigma > 0$ and that there exists $\gamma > 0$ such that $\mathbf{E}(I_T^\gamma) = \infty$. Then, $\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] = \infty$.*

Proof. Suppose first that $a < 0$ and $\sigma = 0$. Then,

$$\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] = |a|^\gamma \mathbf{E}(I_T^\gamma) = \infty.$$

Next, suppose that $a \leq 0$ and $\sigma > 0$. In that case, using the identities in law $W \overset{\mathcal{L}}{=} -W$ and $W_{J_T} \overset{\mathcal{L}}{=} \sqrt{J_T}W_1$, the Cauchy-Schwarz inequality and the independence between W_1 and J_T, we obtain

$$\begin{aligned}\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] &\geq \mathbf{E}[(\sigma\sqrt{J_T}W_1)^{+,\gamma}] = \sigma^\gamma \mathbf{E}(W_1^{+,\gamma})\mathbf{E}(J_T^{\gamma/2}) \\ &\geq \sigma^\gamma \mathbf{E}(W_1^{+,\gamma})T^{-\gamma/2}\mathbf{E}(I_T^\gamma) = \infty.\end{aligned}$$

Finally, if $a > 0$ and $\sigma > 0$, using the fact that $W \overset{\mathcal{L}}{=} -W$, that $W_{J_T} \overset{\mathcal{L}}{=} \sqrt{J_T}W_1$ and choosing $C > 1$, we obtain that

$$\begin{aligned}\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] &= \mathbf{E}[(-|a|I_T + \sigma\sqrt{J_T}W_1)^{+,\gamma}] \\ &\geq \mathbf{E}[(-|a|I_T + \sigma\sqrt{J_T}W_1)^{+,\gamma}\mathbf{1}_{\{\sigma\sqrt{J_T}W_1 \geq C|a|I_T\}}] \\ &\geq \mathbf{E}[((C-1)|a|I_T)^\gamma \mathbf{1}_{\{\sigma\sqrt{J_T}W_1 \geq C|a|I_T\}}].\end{aligned}$$

Since $\frac{I_T}{\sqrt{J_T}} \leq \sqrt{T}$, by Cauchy-Schwarz's inequality, we obtain using the independence between W_1 and I_T

$$\begin{aligned}\mathbf{E}[(-aI_T - \sigma W_{J_T})^{+,\gamma}] &\geq \mathbf{E}\left[((C-1)|a|I_T)^\gamma \mathbf{1}_{\left\{W_1 \geq \frac{C|a|\sqrt{T}}{\sigma}\right\}}\right] \\ &= \mathbf{P}\left(W_1 \geq \frac{C|a|\sqrt{T}}{\sigma}\right)(C-1)^\gamma |a|^\gamma \mathbf{E}(I_T^\gamma) = \infty.\end{aligned}$$

□

Proof of Theorem 2. The assumptions imply $\int_{|x|>1}|x|\nu_X(dx) < +\infty$ and so, by Proposition 3, we obtain

$$\mathbf{P}\left(\sup_{0\leq t\leq T}\left(-\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \geq y\right) \geq \mathbf{P}((-\delta_X I_T - \sigma_X W_{J_T} - N_T^d)^+ \geq y),$$

where δ_X and $N^d = (N_t^d)_{t\in[0,T]}$ are defined as in Proposition 3.

Then, by independence, we get

$$\begin{aligned}&\mathbf{E}[(-\delta_X I_T - \sigma_X W_{J_T} - N_T^d)^{+,\gamma_T}] \\ &\quad = \int_D \mathbf{E}[(-\delta_X I_T(q) - \sigma_X W_{J_T(q)} - N_T^d(q))^{+,\gamma_T}]\mathbf{P}_{\mathcal{E}(R)}(dq),\end{aligned}$$

where D is the Skorokhod space of càdlàg functions on $[0,T]$, the measure $\mathbf{P}_{\mathcal{E}(R)}$ is the law of $(\mathcal{E}(R)_t)_{t\in[0,T]}$, $I_T(q) = \int_0^T \frac{ds}{q_s}$, $J_T(q) = \int_0^T \frac{ds}{q_s^2}$ and

$$\begin{aligned}N_T^d(q) = &\int_0^T \int_{|x|\leq 1} \frac{x}{q_{s-}}(\mu_X(ds,dx) - \nu_X(dx)ds) \\ &+ \int_0^T \int_{|x|>1} \frac{x}{q_{s-}}(\mu_X(ds,dx) - \nu_X(dx)ds).\end{aligned}$$

Denote by $N'_T(q)$ and $N''_T(q)$ the two terms on the r.h.s. of the equation above. Fixing $q \in D$, we now prove that $\mathbf{E}(N'_T(q)) = 0$ and $\mathbf{E}(N''_T(q)) = 0$. First, note that by Theorem 1 p.176 in [21] and Theorem II.1.8 p.66-67 in [14], we find that

$$\begin{aligned}\mathbf{E}([N'_{\cdot}(q), N'_{\cdot}(q)]_T) &= \mathbf{E}\left(\int_0^T \int_{|x|\leq 1} \frac{x^2}{q_{s-}^2} \mu_X(ds, dx)\right) \\ &= \mathbf{E}\left(\int_0^T \int_{|x|\leq 1} \frac{x^2}{q_s^2} \nu_X(dx)ds\right) \\ &= \left(\int_0^T \frac{ds}{q_s^2}\right)\left(\int_{|x|\leq 1} x^2 \nu_X(dx)\right).\end{aligned}$$

Then, since q a strictly positive càdlàg function on a compact interval, the integral $\int_0^T \frac{ds}{q_s^2} < +\infty$ and since $\int_{|x|\leq 1} x^2 \nu_X(dx) < +\infty$ by definition of the Lévy measure, we have $\mathbf{E}([N'_{\cdot}(q), N'_{\cdot}(q)]_T) < +\infty$. This shows that $N'(q)$ is a (square integrable) martingale and so $\mathbf{E}(N'_T(q)) = 0$. For the second term, similarly we have

$$\int_0^T \int_{|x|>1} \frac{|x|}{q_s} \nu_X(dx)ds = \left(\int_0^T \frac{ds}{q_s}\right)\left(\int_{|x|>1} |x| \nu_X(dx)\right) < +\infty.$$

Therefore, by Proposition II.1.28 p.72 in [14] and Theorem II.1.8 p.66-67 in [14], we have

$$\begin{aligned}\mathbf{E}(N''_T(q)) = \mathbf{E}&\left(\int_0^T \int_{|x|>1} \frac{x}{q_{s-}} \mu_X(ds, dx)\right) \\ &- \mathbf{E}\left(\int_0^T \int_{|x|>1} \frac{x}{q_{s-}} \nu_X(dx)ds\right) = 0.\end{aligned}$$

Now, since the random variables $-\delta_X I_T(q) - \sigma_X W_{J_T(q)}$ and $-N_T^d(q)$ are independent and $\mathbf{E}(N_T^d(q)) = 0$, for all $q \in D$, we can apply Lemma 3 to obtain

$$\mathbf{E}[(-\delta_X I_T - \sigma_X W_{J_T} - N_T^d)^{+,\gamma_T}] \geq \mathbf{E}[(-\delta_X I_T - \sigma_X W_{J_T})^{+,\gamma_T}].$$

Then, using Lemma 2 and Lemma 4 with $a = \delta_X$, $\sigma = \sigma_X$, the variable $Z = (-\delta_X I_T - \sigma_X W_{J_T})^+$ and $p = \gamma_T$, we can conclude that for all $\delta > 0$, there exists a strictly positive sequence $(y_n)_{n\in\mathbb{N}}$ increasing to $+\infty$ such that, for all $C > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\mathbf{P}(\tau(y_n) \leq T) \geq \frac{C}{y_n^{\gamma_T} \ln(y_n)^{1+\delta}}.$$

The above implies that

$$\limsup_{y\to\infty} \frac{\ln\left(\mathbf{P}(\tau(y)\le T)\right)}{\ln(y)} \ge -\gamma_T + \lim_{n\to\infty} \frac{\ln(C) - \ln(\ln(y_n)^{1+\delta})}{\ln(y_n)} = -\gamma_T$$

and it finishes the proof. □

Proof of Corollary 2. First of all we show that the process N^d appearing in the proof of Theorem 2 is uniformly integrable. We take first

$$N'_t = \int_0^t \int_{|x|<1} \frac{x}{\mathcal{E}(R)_{s-}} (\mu_X(ds,dx) - \nu_X(ds,dx))$$

Since

$$\sup_{t\ge 0} \mathbf{E}[(N'_t)^2] = \mathbf{E}\left(\int_0^t \int_{|x|<1} \frac{x^2}{\mathcal{E}^2(R)_{s-}} \nu_X(ds,dx)\right)$$

$$= \mathbf{E}(J_\infty) \int_{\mathbb{R}} x^2 \mathbf{1}_{\{|x|<1\}} \nu_X(dx) < +\infty,$$

the process N' is uniformly integrable. Now, let

$$N''_t = \int_0^t \int_{|x|>1} \frac{x}{\mathcal{E}(R)_{s-}} \mu_X(ds,dx) - \int_0^t \int_{|x|>1} \frac{x}{\mathcal{E}(R)_{s-}} \nu_X(ds,dx)$$

By compensating theorem

$$\mathbf{E}\left(\int_0^{+\infty} \int_{|x|>1} \frac{|x|}{\mathcal{E}(R)_{s-}} \mu_X(ds,dx)\right) = \mathbf{E}\left(\int_0^{+\infty} \int_{|x|>1} \frac{|x|}{\mathcal{E}(R)_{s-}} \nu_X(ds,dx)\right)$$

$$= \mathbf{E}(I_\infty) \int_{|x|>1} |x|\, \nu_X(dx) < +\infty$$

and this shows that N'' has a finite ($\mathbf{P}$-a.s.) limit as $t \to +\infty$. Hence, N^d is uniformly integrable and $\mathbf{E}(N^d_\infty) = 0$. From Proposition 3 we get that

$$\int_0^{+\infty} \frac{dX_s}{\mathcal{E}(R)_{s-}} \stackrel{\mathcal{L}}{=} \delta_X I_\infty + \sigma_X W_{J_\infty} + N^d_\infty$$

Imitating the proof of Lemma 4 we conclude that

$$\mathbf{E}[(-\delta_X I_\infty - \sigma_X W_{J_\infty})^{+,\gamma_\infty}] = +\infty.$$

Finally, from Lemma 2 with $Z = (-\delta_X I_\infty - \sigma_X W_{J_\infty})^+$ and $p = \gamma_\infty$ we obtain the claimed result. □

5. Conditions for Ruin with Probability 1

In this section we prove Theorem 3 and Corollary 3.

Proof of Theorem 3. We have, for all $y > 0$,

$$\mathbf{P}(\tau(y) < \infty) = \mathbf{P}\left(\sup_{t\geq 0}\left(-\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \geq y\right)$$

$$\geq \mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) \geq y\right) \geq \mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) = +\infty\right).$$

But, the independence of X and R implies

$$\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}}\right) = +\infty\right) =$$

$$\int_D \mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{q_{s-}}\right) = +\infty\right) \mathbf{P}_{\mathcal{E}(R)}(dq)$$

where D is Skorokhod space of continuous from the right functions with left-hand limit on $\mathbb{R}_+$. We remark that the event

$$\left\{\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{q_{s-}}\right) = +\infty\right\}$$

is a tail event for an additive process $\left(-\int_0^t \frac{dX_s}{q_{s-}}\right)_{t\geq 0}$ and this event has either probability 0 or 1. We will now show that

$$\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{q_{s-}}\right) = +\infty\right) = 1$$

on the set

$$Q = \left\{q \in D : I_\infty(q) = +\infty, J_\infty(q) = +\infty, \lim_{t\to\infty} \frac{I_t(q)}{\sqrt{J_t(q)}} = L(q),\ 0 < L(q) < \infty\right\},$$

of probability 1. Here we denote as previously $I_t(q) = \int_0^t q_s^{-1} ds$ and $J_t(q) = \int_0^t q_s^{-2} ds$.

Imitating the proof of Proposition 4 for the truncation function $\mathbf{1}_{\{|x|\leq a\}}$, we have

$$\begin{aligned}
&\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{q_{s-}}\right)=+\infty\right)\\
&=\mathbf{P}\left(\limsup_{t\to\infty}\Big(-a_X I_t(q)-\sigma_X W_{J_t(q)}-M_t^{a,d}(q)-U_t^a(q)\Big)=+\infty\right)\\
&\geq \mathbf{P}\left(\limsup_{t\to\infty}\Big(-a_X I_t(q)-\sigma_X W_{J_t(q)}-M_t^{a,d}(q)\Big)=+\infty\right),
\end{aligned}$$

where

$$M_t^{a,d}(q)=\int_0^t\int_{\mathbb{R}} x\mathbf{1}_{\{|x|\leq a\}}(\mu_X(ds,dx)-\nu_X(dx)ds)$$

and

$$U_t^a(q)=\int_0^t\int_{\mathbb{R}} x\mathbf{1}_{\{|x|>a\}}\mu_X(ds,dx).$$

The last inequality follows from the assumption $\nu_X(]a,+\infty[)=0$ which implies that $U_t^a\leq 0$ for all $t\geq 0$.

Next, $H_t(q)=-\sigma_X W_{J_t(q)}-M_t^{a,d}(q)$ is a locally square-integrable martingale and using the independence of the terms in the Lévy-Itô decomposition of X, we can obtain its variance :

$$\mathbf{E}(H_t(q)^2)=\left(\sigma_X^2+\int_{\mathbb{R}} x^2\mathbf{1}_{\{|x|\leq a\}}\nu_X(dx)\right)J_t(q)=\sigma_\xi^2 J_t(q)$$

with $\sigma_\xi^2=\sigma_X^2+\int_{\mathbb{R}} x^2\mathbf{1}_{\{|x|\leq a\}}\nu_X(dx)$.
Now if $\sigma_\xi=0$, then by assumption we would have $a_X<0$ and $M_t^{a,d}=0$, for all $t\geq 0$, and thus

$$\mathbf{P}\left(\limsup_{t\to\infty}\left(-\int_0^t \frac{dX_s}{q_{s-}}\right)=+\infty\right)\geq \mathbf{P}\left(\limsup_{t\to\infty}\left(-a_X I_t(q)\right)=+\infty\right)=1$$

since $I_\infty(q)=+\infty$ on the set Q.

If $\sigma_\xi>0$, we take an increasing sequence $(t_n)_{n\in\mathbb{N}}$ starting from zero and tending to $+\infty$ and, then, for all $n\in\mathbb{N}^*$ and $0\leq k\leq n$ we set

$$\tilde{H}_{n,k}=\frac{H_{t_k}(q)-H_{t_{k-1}}(q)}{\sigma_\xi\sqrt{J_{t_n}(q)}}.$$

Then, we see that $(\tilde{H}_{n,k})_{n,k\in\mathbb{N}^*}$ is a martingale difference sequence (for the obvious filtration) which satisfies the conditions of the central limit theorem for such sequences (see e.g. Theorem 8 p.442 in [21]). Thus,

$$\frac{H_{t_n}(q)}{\sigma_\xi \sqrt{J_{t_n}(q)}} = \sum_{k=1}^{n} \tilde{H}_{n,k} \xrightarrow{d} \xi$$

as $n \to \infty$, where $\xi \sim \mathcal{N}(0,1)$. Thus,

$$-\frac{1}{\sqrt{J_{t_n}(q)}} \int_0^{t_n} \frac{dX_s}{q_{s-}} \xrightarrow{d} -a_X L(q) + \sigma_\xi \xi$$

as $n \to \infty$. Then, by the Skorokhod representation theorem, we can find a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$, a random variable $\tilde{\xi}$ and a process $\tilde{X}$ which are equal in law to the random variable ξ and the process X respectively such that

$$\lim_{n\to\infty} -\frac{1}{\sqrt{J_{t_n}(q)}} \int_0^{t_n} \frac{d\tilde{X}_s}{q_{s-}} = -a_X L(q) + \sigma_\xi \tilde{\xi} \;\; (\tilde{\mathbf{P}} - a.s.).$$

Thus, on the set $\{-a_X L(q) + \sigma_\xi \tilde{\xi} > 0\}$ of positive probability, we have

$$\limsup_{t\to\infty} \left(-\int_0^t \frac{d\tilde{X}_s}{q_{s-}} \right) \geq \lim_{n\to\infty} \left(-\int_0^{t_n} \frac{d\tilde{X}_s}{q_{s-}} \right) = +\infty,$$

and so also

$$\mathbf{P}\left(\limsup_{t\to\infty} \left(-\int_0^t \frac{dX_s}{q_{s-}} \right) = +\infty \right) > 0.$$

So, this last probability is equal to one for all $q \in Q$. But, since Q is itself an event of probability 1, we finally obtain

$$\mathbf{P}\left(\limsup_{t\to\infty} \left(-\int_0^t \frac{dX_s}{\mathcal{E}(R)_{s-}} \right) = +\infty \right) = 1.$$

□

Finally, we prove the result on the ruin with probability one for the case of Lévy processes.

Proof of Corollary 3. Note that the assumption (12) implies that the expectation $\mathbf{E}(|\hat{R}_1|) < \infty$ and, by the law of large numbers for Lévy processes, we get that

$$\lim_{t\to\infty} \frac{\hat{R}_t}{t} = \mathbf{E}(\hat{R}_1) = a_R - \frac{\sigma_R^2}{2} + \int_{-1}^{\infty} (\ln(1+x) - x\mathbf{1}_{\{|\ln(1+x)|\leq 1\}})\nu_R(dx).$$

But, the fact that $\lim_{t\to\infty}\frac{\hat{R}_t}{t} < 0$ is equivalent to $I_\infty = J_\infty = +\infty$ (**P**-a.s.) by Theorem 1 in [6]. So it is enough to check that the limit

$$\lim_{t\to\infty}\frac{I_t}{\sqrt{J_t}} = L$$

exists with $0 < L < \infty$ (**P**-a.s.).

We obtain, using de l'Hospital's rule and the time-reversion property of $\hat{R}$ that

$$\lim_{t\to\infty}\frac{I_t}{\sqrt{J_t}} = \lim_{t\to\infty} 2e^{\hat{R}_t}\sqrt{J_t} \stackrel{\mathcal{L}}{=} 2\left(\int_0^\infty e^{2\hat{R}_s}ds\right)^{1/2}$$

The last integral is finite (**P**-a.s.) again by Theorem 1 in [6]. □

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