Holomorphic functions with universal boundary behaviour

S. Charpentier

HOLOMORPHIC FUNCTIONS WITH UNIVERSAL BOUNDARY BEHAVIOUR

S. CHARPENTIER

ABSTRACT. We are interested in functions analytic in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ with a wild behaviour near the boundary $\mathbb{T}$ of $\mathbb{D}$. For instance, the main result implies the existence of a residual subset of $H(\mathbb{D})$ whose every element f satisfies the property that, given any compact subset K of $\mathbb{T}$, different from $\mathbb{T}$, given any continuous functions φ on K, and any compact set L of $\mathbb{D}$, there exists an increasing sequence $(r_n)_n \subset [0,1)$ converging to 1, such that $f(r_n(\zeta - z) + z)$ converges to $\varphi(\zeta)$, uniformly for $(\zeta, z) \in K \times L$, as n goes to ∞. Among other things radial growth of such functions and connections with universal Taylor series are investigated. Functions in the disc algebra whose derivatives disjointly satisfy a somewhat similar universal property, are also exhibited. In some sense, the results are sharp.

1. Introduction

Let us denote by $\mathbb{B}_N$ the unit ball of $\mathbb{C}_N$ and $H(\mathbb{B}_N)$ the space of functions holomorphic in $\mathbb{B}_N$, endowed with the locally uniform convergence topology. For a *typical radial weight* v on $\mathbb{B}_N$ (*i.e.* a continuous positive function on $\mathbb{B}_N$ such that $v(z) = v(|z|)$ for all $z \in \mathbb{B}_N$, and $v(z) \to 0$ as $|z| \to 1$), we define the *growth space* $H_v^\infty(\mathbb{B}_N)$ as

$$H_v^\infty := \left\{ f \in H(\mathbb{B}_N),\ \|f\|_v := \sup_{z \in \mathbb{B}_N} |f(z)|\, v(z) < \infty \right\}. \tag{1.1}$$

Let $H_v^0(\mathbb{B}_N)$ denote the closure of the polynomials in $H_v^\infty(\mathbb{B}_N)$.

In [3], Bayart proved that, for any typical radial weight v on $\mathbb{B}_N$, generically every element f of $H_v^0(\mathbb{B}_N)$ is universal in the sense that, given any measurable function φ on $\partial\mathbb{B}_N$, there exists a sequence $(r_n)_{n\geq 0} \subset [0,1)$, converging to 1 such that for any $z \in \mathbb{D}$, $f(r_n(\zeta - z) + z)$ converges to $\varphi(\zeta)$ as $n \to \infty$, for almost every ζ in the boundary $\partial\mathbb{B}_N$ of $\mathbb{B}_N$. Here a property is said to be generic if it is satisfied by every element of a residual subset of the ambient space (namely a subset containing a dense G_δ-subset, that is a dense countable intersection of open sets). This result improved classical results previously obtained by, *e.g.*, Bagemihl and Seidel [1] and Kahane and Katznelson [16] in the unit disc $\mathbb{D}$, Iordan [15] and Dupain [12] in $\mathbb{B}_N$, $N \geq 1$.

A natural question is whether some functions holomorphic in $\mathbb{B}_N$ may enjoy this kind of universal property at *every* point of the boundary, and along every path in $\mathbb{B}_N$ with an endpoint in $\partial\mathbb{B}_N$. The answer is no for $N > 1$, as Globevnik and Stout proved in [14, Section III] that for any function holomorphic in $\mathbb{B}_N$, $N > 1$, there exists $p \in \partial\mathbb{B}_N$ and a path γ with p as one endpoint such that $f \circ \gamma$ is constant. However it is a rather classical fact that there exists a function analytic in $\mathbb{D} := \mathbb{B}_1$ which is unbounded along any path to any point in $\mathbb{T} := \partial\mathbb{D}$. Therefore, we can ask the following: does there exist a function f in $H(\mathbb{D})$ such that, given any (continuous) path $\gamma : [0,1] \to \overline{D}$ with $\gamma([0,1)) \subset \mathbb{D}$ and $\gamma(1) \in \mathbb{T}$, $f(\gamma([0,1)) = \mathbb{C}$? We positively answer this question by proving the following stronger result, itself consequence of the first part of our main result, Theorem 2.4. More precisely,

2010 *Mathematics Subject Classification.* 30K05, 40A05, 41A10, 47A16.
Key words and phrases. universality.

Theorem 1. *There exists a dense G_δ-subset $\mathcal{V}_u(\mathbb{D})$ of $H(\mathbb{D})$ every element f of which satisfies the following universal property: given any compact subset K of $\mathbb{T}$, different from $\mathbb{T}$, any continuous functions φ on K and any compact subset L of $\mathbb{D}$, there exists a sequence $(r_n)_{n\geq 0} \subset [0,1)$ converging to 1, such that*

$$\sup_{\zeta\in K}\sup_{z\in L} |f(r_n(\zeta - z) + z) - \varphi(\zeta)| \to 0, \textit{ as } n \to \infty.$$

Back to Bayart's result, we can now wonder whether some *universal* functions in $\mathcal{V}_u(\mathbb{D})$ may grow as slow as possible to ∞ near the boundary. It turns out that no, as we will see that any such function must grow rather fast to the boundary.

At this point we shall draw attention to some possible analogy between universal functions in $\mathcal{V}_u(\mathbb{D})$ and the well-studied notion of universal Taylor series. In 1996, Nestoridis [21] exhibited a dense G_δ-subset $\mathcal{U}(\mathbb{D})$ of $H(\mathbb{D})$ consisting in functions $f = \sum_{k\geq 0} a_k z^k$ with the property that given any compact subset $K \subset \mathbb{C} \setminus \mathbb{D}$, with connected complement, and any function h continuous on K and holomorphic in the interior of K, there exists an increasing sequence $(\lambda_n)_n \subset \mathbb{N}$, such that

$$\sup_{z\in K}\left|\sum_{k=0}^{\lambda_n} a_k z^k - h(z)\right| \to 0, \text{as } n \to \infty.$$

Universal Taylor series have been extensively studied during the last two decades. It appears that the growth to the boundary of $\mathbb{D}$ of functions in $\mathcal{V}_u(\mathbb{D})$ seems to ressemble to that of universal Taylor series. Nevertheless, while functions in $\mathcal{V}_u(\mathbb{D})$ are trivially Abel summable at no point of $\mathbb{T}$, there are functions in $\mathcal{U}(\mathbb{D})$ which are Abel summable at some points of $\mathbb{T}$ [10]. We will see in Paragraph 2.4 another major difference between those two kinds of universal functions, making the sets $\mathcal{V}_u(\mathbb{D})$ and $\mathcal{U}(\mathbb{D})$ uncomparable in terms of inclusion.

Back again to Bayart's result, it is tempting to ask if functions in the disc algebra $A(\mathbb{D})$ (*i.e.* the functions in $H(\mathbb{D})$ which extends continuously to the boundary) are far from having wild universal boundary behaviour. Obviously no functions from $A(\mathbb{D})$ can be universal in the sense of Bayart. However we will see that, in some sense, they are as close as possible to functions with *a.e.* universal boundary behaviour. It will be illustrated by the second part of Theorem 2.4. Subsequently, we will also get the following, where m stands for the Lebesgue measure on the boundary of $\mathbb{D}$.

Theorem 2. *There exists a residual subset $\mathcal{V}_m^s(\mathbb{D})$ of $A(\mathbb{D})$ whose every element satisfies the following property: Given any countable family $(\varphi_l)_{l\in\mathbb{N}^*}$ of measurable functions on $\mathbb{T}$, there exists an increasing sequence $(r_n)_n \subset [0,1)$ converging to 1 and a subset $E \subset \mathbb{T}$ with $m(E) = 1$, such that for any $1 \leq l < \infty$, any $\zeta \in E$ and any $z \in \mathbb{D}$,*

$$f^{(l)}(r_n(\zeta - z) + z) \to \varphi_l(\zeta) \textit{ as } n \to \infty.$$

Here $f^{(l)}$ denotes the derivative of order l of f. An important feature in this result is that the sequence $(r_n)_n$ does not depend on $l \in \mathbb{N}^*$. By the way, this implies that *a.e.* convergence cannot be replaced with uniform convergence as in Theorem 1.

The proof of Theorem 2.4 use polynomial approximation based on Mergelyan's theorem. We shall mention that a similar - but weaker - version of Theorem 2 was proven in [9] using only basic tools of linear algebra. By the Riemann mapping Theorem and its refinement, the statement of Theorem 2.4 may be adapted to bounded simply connected domain with regular enough boundary. For general bounded simply connected domains, it would be of interest to search for a possible statement.

Section 2 is devoted to the statement of Theorem 2.4, its consequences, to some discussions related to the sharpness of the results and possible connections with universal Taylor series. The proof of Theorem 2.4 is detailed in Section 3.

Notations. For a compact set K of $\mathbb{C}$, we will denote by $\mathcal{C}(K)$ the space of continuous functions on K endowed with the supremum norm, that we will sometimes denote by $\|\cdot\|_K$.

2. The results

2.1. **Definitions and statements of the main theorem.** Before stating our results let us introduce some terminology. Let $I \subset \mathbb{N}$ and $(\lambda^l)_{l\in I}$ be a countable family of sequences in $\mathbb{C}^{\mathbb{N}}$. For any $l \in I$ we will denote by λ_i^l, $i \in \mathbb{N}$, the i-th term of the l-th sequence λ^l.

Definition 2.1. *We say that $(\lambda^l)_{l\in I}$ is* admissible *if the two following conditions hold:*

(1) For any $l \in I$, λ_i^l is non-zero for any i large enough;
(2) For any $l \in I$, $\limsup_i |\lambda_i^l|^{1/i} \leq 1$.

If I is reduced to a single element, we will simply say that the only sequence $(\lambda_i)_{i\in\mathbb{N}}$ contained in I is admissible whenever it satisfies (1) and (2) in the previous definition.

Given a single sequence $\lambda := (\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{C}^{\mathbb{N}}$, we define the linear map T_λ which takes a formal power series $\sum_i a_i z^i$ to $\sum_i \lambda_i a_i z^i$. If λ is admissible (here I contains only one element), then the power series $\sum_i \lambda_i z^i$ has radius of convergence greater than 1, and $T_\lambda(f)$ is the Hadamard product of f and $\sum_i \lambda_i z^i$. In particular, $T_\lambda(f)$ is continuous from $H(\mathbb{D})$ into itself, and of course from X to $H(\mathbb{D})$ for any X continuously embedded into $H(\mathbb{D})$.

In the sequel, our sequences $(\lambda^l)_{l\in I}$ may be required to satisfy some additional condition.

Definition 2.2. *We say that $(\lambda^l)_{l\in I}$ is* well-scaled *if there exists an increasing sequence of positive integers $(\nu_i)_{i\in\mathbb{N}}$ such that, for any $l, l' \in I$, $l < l'$ implies*

$$\frac{|\lambda_{\nu_i}^l|}{|\lambda_{\nu_i}^{l'}|} \to 0 \text{ or } \infty \text{ as } i \to \infty.$$

We will exhibit analytic functions in the disc enjoying two kinds of extreme behaviour at the boundary.

Definition 2.3. *Let ρ be a subset of $]0,1[$ with 1 as a limit point and let $\Lambda := (\lambda^l)_{l\in I}$, $I \subset \mathbb{N}$, be an admissible countable family of sequences in $\mathbb{C}^{\mathbb{N}}$. We denote by $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ and $\mathcal{V}_m(\mathbb{D}, \Lambda, \rho)$ the subsets of $H(\mathbb{D})$ consisting in those f satisfying, respectively, Properties* **(P1)** *and* **(P2)** *below:*

(P1) *Given any compact subset K of $\mathbb{T}$, different from $\mathbb{T}$, any continuous function φ on K, any compact subset L of $\mathbb{D}$, and any $l \in I$, there exists an increasing sequence $(r_n)_n \subset \rho$ converging to 1, such that*

$$\sup_{\zeta\in K} \sup_{z\in L} |T_{\lambda^l}(f)\,(r_n(\zeta - z) + z) - \varphi(\zeta)| \to 0 \text{ as } n \to \infty.$$

(P2) *Given any countable family $(\varphi_l)_{l\in I}$ of measurable functions on $\mathbb{T}$, there exist an increasing sequence $(r_n)_n \subset \rho$ converging to 1 and a subset $E \subset \partial\mathbb{D}$, with $m(E) = 1$, such that for any $l \in I$, any $z \in \mathbb{D}$ and any $\zeta \in E$,*

$$T_{\lambda^l}(f)\,(r_n(\zeta - z) + z) \to \varphi_l(\zeta) \text{ as } n \to \infty.$$

The main theorem asserts that the sets $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ and $\mathcal{V}_m(\mathbb{D}, \Lambda, \rho)$ are rather *large* in $H(\mathbb{D})$. Also, being regular at the boundary does not even prevent from belonging to $\mathcal{V}_m(\mathbb{D}, \rho, \Lambda)$.

Theorem 2.4. *Let ρ be a subset of $]0,1[$ with 1 as a limit point and let $\Lambda := (\lambda^l)_{l\in I}$, $I \subset \mathbb{N}$, be an admissible countable family of sequences in $\mathbb{C}^{\mathbb{N}}$.*

(1) The set $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ is non-empty if and only if

$$\limsup_{i\to\infty} |\lambda_i^l|^{1/i} = 1,\ l \in I. \tag{2.1}$$

If the latter conditiong holds, then $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ is a dense G_δ-subset of $H(\mathbb{D})$.

(2) *If Λ is well-scaled, then*
   (a) *The set $\mathcal{V}_m(\mathbb{D},\Lambda,\rho)$ is residual in $H(\mathbb{D})$.*
   (b) *The set $A(\mathbb{D})\cap\mathcal{V}_m(\mathbb{D},\Lambda,\rho)$ is residual in $A(\mathbb{D})$.*

We postpone the proof of this theorem to Section 3. The remaining of this section is devoted to comments, corollaries and discussion about the *optimality* of Theorem 2.4.

2.2. **Some direct consequences.** First of all, it is easy to check that any functions satisfying **(P1)** also have pointwise and, by Lusin's theorem, *a.e.* universal behaviour at the boundary of $\mathbb{D}$, in the following sense:
(i) Given any continuous function φ on $\mathbb{T}$, any compact subset L of $\mathbb{D}$ and any $l\in I$, there exists an increasing sequence $(r_n)_n\subset\rho$ converging to 1, such that for any $\zeta\in\mathbb{T}$,

$$\sup_{z\in L}|T_{\lambda^l}(f)\,(r_n(\zeta-z)+z)-\varphi(\zeta)|\to 0 \text{ as } n\to\infty.$$

(ii) Given any measurable function φ on $\mathbb{T}$ and any $l\in I$, there exist an increasing sequence $(r_n)_n\subset\rho$ converging to 1 such that for any $z\in\mathbb{D}$ and *a.e.* $\zeta\in\mathbb{T}$,

$$T_{\lambda^l}(f)\,(r_n(\zeta-z)+z)\to\varphi(\zeta) \text{ as } n\to\infty.$$

Note that functions satisfying **(P2)** satisfy a stronger property than (ii), since the sequence $(r_n)_n$ in **(P2)** is independent of l. This is the main reason why we assume the sequence Λ to be well-scaled in Theorem 2.4 (2). As we will see, in general there is no function in $H(\mathbb{D})$ satisfying **(P1)** with a sequence $(r_n)_n$ independent of $l\in I$, whenever I has at least two elements (see Proposition 2.8 below).

It is not difficult to construct a function f holomorphic in $\mathbb{D}$ such that given any (continuous) path $\gamma:[0,1]\to\overline{\mathbb{D}}$ with $\gamma([0,1[)\subset\mathbb{D}$ and $\gamma(1)\in\mathbb{D}$, $f(\gamma(t))$ does not have a limit as $t\to 1$. We may observe that every function in $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ satisfies a stronger property.

Proposition 2.5. *Any function in $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ has the property that given any (continuous) path $\gamma:[0,1]\to\overline{\mathbb{D}}$ with $\gamma([0,1[)\subset\mathbb{D}$ and $\gamma(1)\in\mathbb{T}$, $T_\lambda f(\gamma([0,1]))$ is dense in $\mathbb{C}$ for any $\lambda\in\Lambda$.*

It is worth mentioning that the latter does not extend to the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$, $N\geq 2$. Indeed, as proven in [14, Section III], for any function holomorphic in $\mathbb{B}_N$, there exists $p\in\partial\mathbb{B}_N$ and a path γ with p as one endpoint such that $f\circ\gamma$ is constant. Yet it would be of interest to seek for a relevant weaker version of Theorem 2.4 (1) in $\mathbb{B}_N$, $N\geq 2$.

Given two power series $\sum_i a_iz^i$ and $\sum_i b_iz^i$ we denote by $f*g$ the Hadamard product of f by g, that is

$$f*g(z)=\sum_i a_ib_iz^i,\ z\in\mathbb{D}.$$

We directly deduce from Theorem 2.4 the following corollary.

Corollary 2.6. *Let ρ be a subset of $]0,1[$ with 1 as a limit point and let $\mathcal{G}:=(g_l)_{l\in I}$, $I\subset\mathbb{N}$, be countably many functions in $H(\mathbb{D})$. There exists a G_δ-subset $\mathcal{V}_u(\mathbb{D},\mathcal{G},\rho)$ whose every element f satisfies the following property: Given any proper compact subset K of $\mathbb{T}$, any continuous functions φ on K, any compact subset L of $\mathbb{D}$, and any $l\in I$, there exists an increasing sequence $(r_n)_n\subset\rho$ converging to 1, such that*

$$\sup_{\zeta\in K}\sup_{z\in L}|f*g_l(r_n(\zeta-z)+z)-\varphi|\to 0 \textit{ as } n\to\infty.$$

We can also specify the sequences λ^l and deduce from Theorem 2.4 little bit more precise versions of Theorems 1 and 2 of the introduction. We denote by $f^{(l)}$ the derivative of order

l of f if $l \in \mathbb{N}$, and its antiderivative of order l if $l \in \mathbb{Z} \setminus \mathbb{N}$. For us, the antiderivative of f will be the primitive of f in $\mathbb{D}$ which vanishes at 0. More generally, for $l \in \mathbb{Z}$,

$$f^{(l)}(z) = \int_0^z f^{(l+1)}(w)dw.$$

Corollary 2.7. *Let ρ be a subset of $]0,1[$ with 1 as a limit point. There exist a G_δ-subset $\mathcal{V}_u^s(\mathbb{D},\rho)$ of $H(\mathbb{D})$, and a residual subset $\mathcal{V}_m^s(\mathbb{D},\rho)$ of $H(\mathbb{D})$, such that $\mathcal{V}_m^s(\mathbb{D},\rho) \cap A(\mathbb{D})$ is residual in $A(\mathbb{D})$ and :*
*(1) Any f in $\mathcal{V}_u^s(\mathbb{D},\rho)$ satisfies the following property **(Q1)**: Given any compact subset K of $\mathbb{T}$, $K \neq \mathbb{T}$, any continuous functions φ on K, any compact subset L of $\mathbb{D}$, and any $l \in \mathbb{Z}$, there exists an increasing sequence $(r_n)_n \subset \rho$ converging to 1 such that*

$$\sup_{\zeta \in K} \sup_{z \in L} \left| f^{(l)}(r_n(\zeta - z) + z) - \varphi(\zeta) \right| \to 0 \text{ as } n \to \infty.$$

*(2) Any f in $\mathcal{V}_m^s(\mathbb{D},\rho)$ satisfies the following property **(Q2)**: Given any countable family $(\varphi_l)_{l \in \mathbb{N}^*}$ of measurable functions on $\partial\mathbb{D}$, there exists an increasing sequence $(r_n)_n \subset \rho$ converging to 1 and a subset $E \subset \mathbb{T}$ with $m(E) = 1$, such that for any $1 \leq l < \infty$, any $\zeta \in E$ and any $z \in \mathbb{D}$,*

$$f^{(l)}(r_n(\zeta - z) + z) \to \varphi_l(\zeta) \text{ as } n \to \infty.$$

Proof. We denote by $\mathcal{V}_u^s(\mathbb{D},\rho)$ (resp. $\mathcal{V}_m^s(\mathbb{D},\rho)$) the subset of $H(\mathbb{D})$ consisting in these functions which satisfy Property **(Q1)** (resp. **(Q2)**). Then we define $\lambda_i^0 = 1$, $i \in \mathbb{N}$. For $1 \leq l < \infty$, we set

$$\lambda_i^l = \begin{cases} 0 & \text{if } 0 \leq i \leq l-1 \\ i(i-1)\dots(i-l+1) & \text{if } i \geq l, \end{cases}$$

and for $l < 0$, we set $\lambda_i^l = [(i+1)(i+2)\cdots(i+|l|)]^{-1}$.

Observe that if $f = \sum_i a_i z^i$, then for $l \in \mathbb{Z}$, $f^{(l)}(z) = z^{-l} T_{\lambda^l} f(z)$, $z \in \mathbb{D} \setminus \{0\}$. Moreover, $\Lambda_1 := (\lambda^l)_{l \in \mathbb{Z}}$ and $\Lambda_2 := (\lambda^l)_{l \in \mathbb{N}^*}$ are admissible and $\limsup_i |\lambda_i^l|^{1/i} \geq 1$, $l \in \mathbb{Z}$. Also, Λ_2 is well-scaled. So $\mathcal{V}_u(\mathbb{D},\Lambda_1,\rho)$ (respectively $\mathcal{V}_m(\mathbb{D},\Lambda_2,\rho) \cap A(\mathbb{D})$) is a dense G_δ-subset of $H(\mathbb{D})$ (resp. a residual subset of $A(\mathbb{D})$), for any $\rho \subset]0,1[$ with 1 as a limit point.

By Theorem 2.4, in order to prove (2), we only have to check that $\mathcal{V}_m^s(\mathbb{D},\rho) \supset \mathcal{V}_m(\mathbb{D},\Lambda_2,\rho)$. We fix $(\varphi_l)_{l \in \mathbb{N}}$ a sequence of measurable functions on $\mathbb{T}$ and denote by $\tilde{\varphi}_{l \in \mathbb{N}}$ the functions given by $\tilde{\varphi}_l(\zeta) = \zeta^l \varphi_l(\zeta)$, $\zeta \in \mathbb{T}$. Let us fix any $f = \sum_i a_i z^i \in \mathcal{V}_m(\mathbb{D},\Lambda,\rho)$. There exist $E \subset \mathbb{T}$ with $m(E) = 1$ and an increasing sequence $(r_n)_n \subset \rho$ converging to 1, such that for any $1 \leq l < \infty$, any $\zeta \in E$ and any $z \in \mathbb{D}$,

$$\left| \sum_{i \geq l} \lambda_i^l a_i (r_n(\zeta - z) + z)^i - \tilde{\varphi}_l(\zeta) \right| \to 0 \text{ as } n \to 0.$$

Therefore, multiplying by $|r_n(\zeta - z) + z|^{-l}$ (note that $(r_n(\zeta - z) + z)^{-l} \to \zeta^{-l}$, $n \to 0$), we get

$$\left| \sum_{i \geq l} \lambda_i^l a_i (r_n(\zeta - z) + z)^{i-l} - \varphi_l(\zeta) \right| \to 0 \text{ as } n \to 0.$$

We conclude by recalling that $\sum_{i \geq l} \lambda_i^l a_i (r_n(\zeta - z) + z)^{i-l} = f^{(l)}(r_n(\zeta - z) + z)$.

For (1), the equality $\mathcal{V}_u^s(\mathbb{D},\rho) = \mathcal{V}_u(\mathbb{D},\Lambda_1,\rho)$ is proven exactly as above, upon approximating $\zeta^{-l}\varphi(\zeta)$ and multiplying by $|r_n(\zeta - z) + z|^l$ for the inclusion $\mathcal{V}_u^s(\mathbb{D},\rho) \subset \mathcal{V}_u(\mathbb{D},\Lambda_1,\rho)$. □

We conclude this paragraph by drawing attention to a simple instance of set of the type $\mathcal{V}_u(\mathbb{D}\Lambda,\rho)$, which may be of interest in itself. It is the class corresponding to Λ reduced to the single sequence $(1,1,\dots,)$ and $\rho = [0,1]$; we simply denote it by $\mathcal{V}_u(\mathbb{D})$. Namely $\mathcal{V}_u(\mathbb{D})$

is the subset of $H(\mathbb{D})$ consisting in those functions f which satisfies the following: For any compact subset K of $\mathbb{T}$ different from $\mathbb{T}$, any continuous function φ on K, and any compact subset L of $\mathbb{D}$, there exists an increasing sequence $(r_n)_n$ converging to 1, such that

$$\sup_{\zeta\in K}\sup_{z\in L}|f\left(r_n(\zeta-z)+z\right)-\varphi(\zeta)|\to 0,\ \text{ as } n\to\infty.$$

Theorem 2.4 and Proposition 2.17 (see below) ensure that $\mathcal{V}_u(\mathbb{D})$ is a spaceable and densely lineable G_δ-dense subset of $H(\mathbb{D})$. This set will also appear in Paragraph 2.4.

2.3. **Optimality of Theorem 2.4.** We may first wonder whether there exist functions in $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ satisfying **(P1)** along some sequence $(r_n)_n$ independent of $l\in I$, that is like in Property **(P2)**. Actually the set $\mathcal{V}_u^s(\mathbb{D},\rho)$ introduced in Corollary 2.7 contains no such element. More generally, we have the following:

Proposition 2.8. *Let ρ be a subset of $]0,1[$ with 1 as a limit point. Let us denote by $\widetilde{\mathcal{V}_u^{1,2}}(\mathbb{D},\rho)$ the subset of $H(\mathbb{D})$ consisting in functions satisfying the following: Given any compact subset K of $\mathbb{T}$ different from $\mathbb{T}$ and any continuous functions φ_1,φ_2 on K, there exists an increasing sequence $(r_n)_n\subset\rho$ converging to 1 such that*

$$\sup_{\zeta\in K}|f(r_n\zeta)-\varphi_1(\zeta)|\to 0 \text{ as } n\to\infty \quad\text{and}\quad \sup_{\zeta\in K}|f'(r_n\zeta)-\varphi_2(\zeta)|\to 0 \text{ as } n\to\infty.$$

The set $\widetilde{\mathcal{V}_u^{1,2}}(\mathbb{D},\rho)$ is empty.

Proof. It is almost obvious. Assume by contradiction that some f belongs to $\widetilde{\mathcal{V}_u^{1,2}}(\mathbb{D},\rho)$. Then for $\varepsilon>0$ and any constant $A>0$ large enough, there exists $n\in\mathbb{N}$ such that

$$\sup_{\vartheta\in[0,\pi]}\left|f(r_ne^{i\vartheta})\right|<\varepsilon \quad\text{and}\quad \sup_{\vartheta\in[0,\pi]}\left|f'(r_ne^{i\vartheta})-A\right|<\varepsilon/\pi. \tag{2.2}$$

The second inequality implies $|\int_0^\pi f'(r_ne^{i\vartheta})d\vartheta|>A-\varepsilon$. Since $f\in H(\mathbb{D})$, it is a primitive of f' along the path $\vartheta\mapsto r_ne^{i\vartheta}$, and the previous inequality, together with the first one of (2.2), imposes $2\varepsilon>A-\varepsilon$, which is impossible, up to choose A large enough. □

The previous proof shows more generally that if $f\in H(\mathbb{D})$ and $f'(r_n\zeta)$ tends to ∞ as $n\to\infty$, uniformly for ζ in some compact subset K of $\mathbb{T}$, then $f(r_n\zeta)$ cannot be bounded uniformly for ζ in K. In particular, Theorem 2.4 cannot be improved by asking the sequence $(r_n)_n$ to be independent of l in (1), or by replacing $a.e$ convergence in (2) by uniform convergence on any compact subset of $\mathbb{T}$ different from $\mathbb{T}$.

Yet since the intersection of two residual subsets is still residual, $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)\cap\mathcal{V}_m(\mathbb{D},\Lambda,\rho)$ is residual in $H(\mathbb{D})$ whenever Λ is an admissible well-scaled family of sequences.

Further we can still ask the following: *Are there some admissible Λ for which*

$$\mathcal{V}_u(\mathbb{D},\Lambda,\rho)\cap A(\mathbb{D})\neq\emptyset?$$

The following proposition is the first step toward a negative answer to this question. It shows that, like for universal Taylor series, the Taylor coefficients of functions in $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ must grow rather fast to infinity.

Proposition 2.9. *Let ρ be a subset of $]0,1[$ with 1 as a limit point, let $(\lambda_n)_{n\in\mathbb{N}}$ be an admissible sequence and let $(\gamma_n)_{n\in\mathbb{N}^*}$ be a decreasing sequence such that $\sum_n\gamma_n/n<\infty$. If $\sum_n a_nz^n\in\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$, then*

$$\limsup_n\frac{|a_n|}{e^{n\gamma_n}}=+\infty.$$

Proof. Let $(\gamma_n)_{n\in\mathbb{N}^*}$ be as in the statement. We first prove that $\sum_n(|a_n|/e^{n\gamma_n}) = +\infty$. For some $N_0 \geq 1$ and any $M > N \geq N_0$ large enough, it is exhibited in [18, Lemma 2.1] a positive 2π-periodic function $f_{N,M}$ with support in $|\vartheta| \leq 1/2 \pmod{2\pi}$, such that for any $N \leq m \leq M$,

$$|\hat{f}_{N,M}(-m)| \leq \frac{e^{N_0}}{e^{m\gamma_m}}.$$

Now let $N \geq 0$ be fixed. Since $\sum_n a_n z^n \in \mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$, for some $l \in I$, there exists a sequence $(r_n)_{n\in\mathbb{N}} \subset \rho$ such that for any $n \in \mathbb{N}$,

$$\|\sum_i \lambda_i^l a_i r_n^i \zeta^i - (1 + \sum_{i=0}^{N-1} \lambda_i^l a_i \zeta^i)\|_K < \frac{1}{4},$$

where $K := \{e^{i\vartheta};\ |\vartheta| \leq 1/2\}$. By uniform continuity, there exists $n_0 \in \mathbb{N}$ such that $\|\sum_{i=0}^{N-1} \lambda_i^l a_i \zeta^i - \sum_{i=0}^{N-1} \lambda_i^l a_i r_{n_0}^i \zeta^i\|_K < \frac{1}{4}$. Thus we deduce from the triangle inequality that

$$\mathrm{Re}\left(\sum_{i\geq N} \lambda_i^l a_i r_{n_0}^i \zeta^i\right) > \frac{1}{2},\ \zeta \in K.$$

It also follows from the compactness of K and the fact that Λ is admissible that there exists $M_0 \geq N$ such that for any $M \geq M_0$,

$$\mathrm{Re}\left(\sum_{i=N}^{M} \lambda_i^l a_i r_{n_0}^i \zeta^i\right) > \frac{1}{2},\ \zeta \in K.$$

Therefore, multiplying the previous by the positive quantity $\int_{-1/2}^{1/2} \hat{f}_{N,M}(\vartheta)d\vartheta$, we get for infinitely many $M > N$,

$$\begin{aligned}
\pi = \frac{1}{2}\int_{-1/2}^{1/2} f(\vartheta)d\vartheta &\leq \mathrm{Re}\left(\sum_{k=N}^{M} \lambda_k^l a_k r_{n_0}^k \int_{-1/2}^{1/2} \hat{f}_{N,M}(\vartheta)e^{ik\vartheta}d\vartheta\right) \\
&\leq 2\pi \sum_{k=N}^{M} |\lambda_k^l| r_{n_0}^k |a_k| |\hat{f}_{N,M}(-k)| \\
&\leq 2\pi \sum_{k=N}^{M} \frac{|\lambda_k^l| r_{n_0}^k |a_k| e^{N_0}}{e^{k\gamma_k}}.
\end{aligned}$$

Thus $\sum_{k\in\mathbb{N}} |\lambda_k^l| r_{n_0}^k |a_k| e^{-k\gamma_k} = +\infty$ and, according to the proof of [18, Theorem 2.2], this gives

$$\limsup_k \frac{|\lambda_k^l| r_{n_0}^k |a_k|}{e^{k\gamma_k}} = +\infty.$$

Finally, using that $\limsup_k |\lambda_k^l|^{1/k} \leq 1$, we obtain $\limsup_k |a_k|/e^{k\gamma_k} = +\infty$. □

Trivially, $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ does not share any common element with the Nevanlinna class. By [17, Theorem 3], Proposition 2.9 implies that, in contrast with universal functions considered by Bayart in [3], any function in $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ must have a fast radial growth to the boundary. To be more precise, let us introduce some terminology. For a continuous increasing function $\phi : [1, +\infty) \to [1, +\infty)$, let v_ϕ denote the radial weight defined by

$$v_\phi(z) = \exp\left(\phi\left(\frac{1}{1-|z|}\right)\right). \tag{2.3}$$

In [17, Section 2], the author calls *non universal* a function in $H(\mathbb{D})$ satisfying the following definition.

Definition 2.10. *A function f in $H(\mathbb{D})$ is said to be of* non-universal type *if $f \in H_{v_\phi}(\mathbb{D})$ for some ϕ such that*

$$\int_1^\infty \log \phi(t) \frac{dt}{t^2} < +\infty. \tag{2.4}$$

For example, functions smaller or comparable to $\phi(t) = e^{t/\log^\beta(t+1)}$ at $+\infty$, where $\beta > 1$, satisfy (2.4). The terminology *non-universal* comes from [17, Theorem 3] which asserts that if f is of non-universal type, then f is not a universal Taylor series. In fact, the proof of this result consists in showing that a Taylor series whose Taylor coefficients satisfy the conclusion of Proposition 2.9 cannot be of non-universal type. Thus, by Proposition 2.9, we deduce:

Corollary 2.11. *Let ρ be a subset of $]0,1[$ with 1 as a limit point and Λ an admissible countable family of complex sequences. If f is of non-universal type, then f does not belong to $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$.*

We end this paragraph by explaining that, like for universal Taylor series, the rate of growth imposed by Corollary 2.11 is not far from being optimal. The next proposition is the boundary universality version of [17, Theorem 8].

Proposition 2.12. *Let ρ be a subset of $]0,1[$ with 1 as a limit point and Λ an admissible countable family of complex sequences satisfying* (2.1)*. There exist a positive real number M and a universal function $f \in \mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ such that*

$$|f(z)| \leq C \exp\left(\exp\left(\frac{M}{1-|z|}\log\log\frac{4}{1-|z|}\right)\right)$$

for every $z \in \mathbb{D}$, where C is a constant.

Let $\phi_0(t) = \exp(Mt\log\log 4t)$ and let $H^0_{v_{\phi_0}}(\mathbb{D})$ denote the closure of the polynomials in $H_{v_{\phi_0}}(\mathbb{D})$, where v_{ϕ_0} is given by (2.3). The key-ingredient is the following highly non-trivial approximation lemma.

Lemma 2.13 (Lemma 2 of [17])**.** *Let K be a proper compact subset of $\mathbb{T}$ and let $\varepsilon > 0$. Then there exists a polynomial P such that*

$$\sup_{z\in K}|P(z)| < \varepsilon \quad \textit{and} \quad \sup_{z\in\mathbb{D}}\left|\frac{P}{v_{\phi_0}}(z)\right| < \varepsilon.$$

The remaining of the proof, that we omit, consists in applying the Baire Category Theorem in $H^0_{v_{\phi_0}}(\mathbb{D})$ and follows the same lines as in the proof of Theorem 2.4 (1), see Paragraph 3.1 for the details.

2.4. **Boundary universal functions *versus* universal Taylor series.** We recall that $\mathcal{U}(\mathbb{D})$ denotes the set of universal Taylor series (see the Introduction for the definition). By the Baire category theorem $\mathcal{U}(\mathbb{D}) \cap \mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ is a dense G_δ-subset of $H(\mathbb{D})$, whenever Λ is admissible and satisfies (2.1). As recalled in the introduction, there exist functions in $\mathcal{U}(\mathbb{D})$ which are Abel summable at some points of $\mathbb{T}$. Trivially this cannot hold for any function in $\mathcal{V}_u(\mathbb{D})$, so that

$$\mathcal{U}(\mathbb{D}) \not\subset \mathcal{V}_u(\mathbb{D}).$$

Nevertheless, the previous paragraph makes it appear that elements from $\mathcal{U}(\mathbb{D})$ and $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ share common properties. So we can now wonder whether there exist some Λ and ρ for which $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ is contained in $\mathcal{U}(\mathbb{D})$. We will see that this not the case. Our argument will be based on the crucial fact that any universal Taylor series possess Ostrowski-gaps [13]. We recall the definition of Ostrowski-gaps.

Definition 2.14. *Let $\sum_{i\geq 0} a_i z^i$ be a power series with radius of convergence $R \in (0, +\infty)$. We say that it has Ostrowski-gaps (p_m, q_m) if (p_m) and (q_m) are sequences of natural numbers such that*

(1) $p_1 < q_1 \leq p_2 < q_2 \leq \cdots \leq p_m < q_m \leq \ldots$ and $\lim\limits_{m\to+\infty} \dfrac{q_m}{p_m} = +\infty$,

(2) for $I = \bigcup_{m\geq 1}\{p_m + 1, \ldots, q_m\}$, we have $\lim_{i\in I} |a_i|^{1/i} = 0$.

Let us now show how to produce a function in $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$ without Ostrowski-gaps, Λ being admissible. We start with $\sum_{k\geq 0} a_k z^k \in \mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$. Next, for $A > 1$ we set $b_k = A^{-k} a_k/|a_k|$ (with the convention $a_k/|a_k| = 0$ if $a_k = 0$) and observe that $\sum_{k\geq 0} \lambda_k b_k z^k$ clearly belongs to $A(\mathbb{D})$, for any λ in Λ admissible. Thus, as it is easily checked, the function $f = \sum_{k\geq 0}(a_k + b_k) z^k$ is an element of $\mathcal{V}_u(\mathbb{D}, \Lambda, \rho)$. Moreover it does not possess Ostrowski-gaps, for $|a_k + b_k|^{1/k} \geq 1/A$, $k > 0$.

All in all, we have checked the following.

Proposition 2.15. *For any $\rho \subset]0, 1[$ with 1 as a limit point and any Λ admissible countable family of complex sequences, we have*

$$\mathcal{U}(\mathbb{D}) \not\subset \mathcal{V}_u(\mathbb{D}) \quad \text{and} \quad \mathcal{V}_u(\mathbb{D}, \Lambda, \rho) \not\subset \mathcal{U}(\mathbb{D}).$$

Actually a argument similar to that presented before the previous proposition was used in [8, Section 4] to produce Taylor series without Ostrowski-gaps which are universal in the following sense. We denote by $C(R)$ the set $\{z \in \mathbb{C};\ 1 \leq |z| \leq R\}$.

Definition 2.16. *We define $\mathcal{U}(\mathbb{D}, R)$ as the subset of $H(\mathbb{D})$ consisting in those Taylor series $\sum_{k\geq 0} a_k z^k$ with the property that, given any compact subset $K \subset C(R)$ with connected complement, and any function h continuous on K, analytic in the interior of K, there exists an increasing sequence $(\lambda_n)_n \subset \mathbb{N}$ such that*

$$\sup_{z\in K} \left| \sum_{k=0}^{\lambda_n} a_k z^k - h(z) \right| \to 0, \text{ as } n \to \infty.$$

We may ask the following:

Question. *Do there exist Λ and ρ such that*

$$\emptyset \neq \mathcal{V}_u(\mathbb{D}, \Lambda, \rho) \subset \mathcal{U}(\mathbb{D}, 1)?$$

2.5. **A point of view in Operator Theory.** We shall say that Theorem 2.4 has an operator theoretic flavour. Let $(T_n)_n$ be a sequence of bounded linear operators between topological vector spaces X and Y. We recall that $(T_n)_n$ is said to be *universal* if there exists $x \in X$ such that the set $\{T_n x;\ n \geq 0\}$ is dense in Y. Let now $T_{n,m} : X \to Y$, $n, m \in \mathbb{N}$ be bounded linear operators. The family $\{(T_{n,m})_n;\ m \in \mathbb{N}\}$ is said to be *disjoint universal* if there exists $x \in X$ such that the set

$$\{(T_{n,0}x, T_{n,1}x, \ldots)\,; n \geq 0\}$$

is dense in $Y \times Y \times \ldots$, endowed with the product topology induced by that of Y. The recent notion of disjoint universality was introduced in [5, 6] and recently studied in [7, 11, 20, 22].

Let now $(r_n)_n$ be a sequence in $[0, 1]$ converging to 1 and for $(\lambda, z) \in \mathbb{C}^{\mathbb{N}} \times \mathbb{D}$ with λ admissible, define the continuous linear map

$$T_n(\lambda, z) : \left\{ \begin{array}{lcl} H(\mathbb{D}) & \to & \mathcal{C}(\mathbb{T}) \text{ (resp. } \mathcal{M}(\mathbb{T})) \\ f = \sum_i a_i w^i & \mapsto & \sum_i \lambda_i a_i \left(r_n(\zeta - z) + z\right)^k, \end{array} \right.$$

where $\mathcal{C}(\mathbb{T})$ stands for the space of continuous functions on $\mathbb{T}$, endowed with the supremum norm, and where $\mathcal{M}(\mathbb{T})$ denotes the space of measurable functions on $\mathbb{T}$, endowed with the metrizable topology of convergence in (Lebesgue) measure. Observe that $f \in H(\mathbb{D})$ satisfies **(P2)** or (ii) (at the beginning of Paragraph 2.2) if and only if f has the same universal boundary behaviour with *a.e.* convergence replaced by the convergence in Lebesgue measure. Thus if $\Lambda := (\lambda^l)_{l\in I}$ is countable and f belongs to $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ (resp. $\mathcal{V}_m(\mathbb{D},\Lambda,\rho)$) then, for $z \in \mathbb{D}$ fixed, f is a common universal vector for the family of sequences of operators $T_n(\lambda^l, z)$, indexed by $l \in \mathbb{N}$ (resp. a disjoint universal vector for $\{(T_n(\lambda^l, z))_n,\ l \in \mathbb{N}\}$). In the study of universal sequences of operators we are sometimes interested in exhibiting structures of some subsets of universal objects. For instance, one says that the set of universal objects is densely lineable if it contains a subspace which is dense in the ambient space, and one says it is spaceable whenever it contains an infinite dimensional closed subspace. Under the assumptions that there exists large sets of universal objects, there exist criteria to prove dense lineability and spaceability. Without going into details, the following can be proven upon applying existing criteria or modifying very standard proofs contained in [2, 4, 7, 19].

Proposition 2.17. *Let ρ be a subset of $]0,1[$ with 1 as a limit point and let $\Lambda := (\lambda^l)_{l\in I}$, $I \subset \mathbb{N}$, be an admissible countable family of sequences in $\mathbb{C}^{\mathbb{N}}$.*

- *(1) If $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$ is non-empty, then it is densely lineable and spaceable in $H(\mathbb{D})$;*
- *(2) If Λ is well-scaled, then*
  - *(a) The set $\mathcal{V}_m(\mathbb{D},\Lambda,\rho)$ is densely lineable and spaceable in $H(\mathbb{D})$.*
  - *(b) The set $A(\mathbb{D}) \cap \mathcal{V}_m(\mathbb{D},\Lambda,\rho)$ is densely lineable and spaceable in $A(\mathbb{D})$.*

3. Proof of Theorem 2.4

3.1. **Proof of Part (1).** The "only if" part is obvious. Indeed, since Λ is admissible, we need only check that if $\limsup_i |\lambda_i^l|^{1/i} < 1$ for some $l \in I$, then $\mathcal{V}_u(\mathbb{D},\Lambda,\rho) = \emptyset$. Now, if $\limsup_i |\lambda_i^l|^{1/i} < 1$ and $f \in H(\mathbb{D})$, then the radius of convergence of $T_{\lambda^l} f$ is greater than 1, so that $T_{\lambda^l} f$ is continuous on the closure of $\mathbb{D}$ and then cannot belong to $\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$.

The "if" part requires a bit more efforts. Let ρ and $\Lambda = (\lambda^l)_{l\in I}$ be as in the statement of the theorem and let $(P^j)_{j\in\mathbb{N}}$ be the family of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We denote by $D(0,t)$, $t \geq 0$, the *closed* disc centered at 0 with radius t. Let $\vartheta_0 \in \pi(\mathbb{R} \setminus \mathbb{Q})$ and for $m, n \in \mathbb{N}$, let us denote by $C_{m,n}$ the set

$$C_{m,n} = \mathbb{T} \setminus \{e^{i\vartheta};\ |\vartheta - m\vartheta_0| < 1/n\}. \tag{3.1}$$

For $l \in I$, $j, k, m, n \in \mathbb{N}$ and $0 < r < 1$, we introduce the set $U(j, k, l, m, n, r, s)$ defined by

$$U(j,k,l,m,n,r,s) := \left\{ f \in H(\mathbb{D});\ \sup_{\zeta\in C_{m,n}} \sup_{z\in D(0,\frac{s}{s+1})} |T_{\lambda^l} f(r(\zeta - z) + z) - P^j(\zeta)| < \frac{1}{k} \right\}.$$

We claim that

$$\mathcal{V}_u(\mathbb{D},\Lambda,\rho) = \bigcap_{l\in I} \bigcap_{j,k,m,n,s} \bigcup_{r\in\rho} U(j,k,l,m,n,r,s).$$

Indeed, let f belong to the right-hand side of the previous equation. We fix $l \in I$, $\varepsilon > 0$, a compact set K in $\mathbb{T}$, different from $\mathbb{T}$, a continuous functions h on K, and a compact subset L of $\mathbb{D}$. First we choose s large enough so that $L \subset D(0, s/(s+1))$. By Mergelyan's theorem, there exists $j \in \mathbb{N}$ such that $\|h(\zeta) - P^j(\zeta)\|_K < \varepsilon/2$. Since K is different from $\mathbb{T}$ and $\vartheta_0 \in \pi(\mathbb{R} \setminus \mathbb{Q})$, there exists $m, n \in \mathbb{N}$ such that $K \subset C_{m,n}$. We finally set $k \in \mathbb{N}$ such that $1/k < \varepsilon/2$. By assumption, there exists $r \in \rho$ such that

$$\sup_{\zeta\in C_{m,n}} \sup_{z\in D(0,\frac{s}{s+1})} |T_{\lambda^l} f(r(\zeta - z) + z) - P^j(\zeta)| < \varepsilon/2,$$

which immediately gives

$$\sup_{\zeta\in K}\sup_{z\in L}|T_{\lambda^l}f(r(\zeta-z)+z)-h\|_K<\varepsilon,$$

i.e. $f\in\mathcal{V}_u(\mathbb{D},\Lambda,\rho)$. The other inclusion is straightforward.

By the Baire Category Theorem, and since I is countable, we are then reduced to prove the following.

Proposition 3.1. *For any $l\in I$, any $j,k,m,n,s\in\mathbb{N}$, the set $\cup_{r\in\rho}U(j,k,l,m,n,r,s)$ is open and dense in $H(\mathbb{D})$.*

We need the following known generalization of Cauchy formula. As it is very short, we include its proof.

Lemma 3.2. *Let $f=\sum_i a_iz^i$ and $g=\sum_i b_iz^i$ be two functions in $H(\mathbb{D})$. Then, for any $0<r<1$ and any $0\leq|z|<r$,*

$$f\star g(z)=\sum_i a_ib_iz^i=\frac{1}{2i\pi}\int_{\partial D(0,r)}f(w)g\left(\frac{z}{w}\right)\frac{dw}{w}.$$

Proof. Let $0<r<1$ and $0\leq|z|<r$. Since the convergence of $\sum_i a_iw^i$ and $\sum_i f(w)b_i\left(\frac{z^i}{w^{i+1}}\right)$ is uniform with respect to $|w|=r$, we have

$$\begin{aligned}\int_{\partial D(0,r)}f(w)g\left(\frac{z}{w}\right)\frac{dw}{w} &= i\sum_n b_nz^n\sum_k a_k\int_0^{2\pi}r^{k-n}e^{i(k-n)\vartheta}d\vartheta\\ &= 2i\pi\sum_n a_nb_nz^n,\end{aligned}$$

hence the result. □

Proof of Proposition 3.1. We use the notation $C_{m,n}$, $m,n\in\mathbb{N}$, as introduced in (3.1), and we recall that $D(0,t)$ stands for the *closed* disc centered at 0 with radius t. For $0<r,t<1$, we denote by $C_{m,n}(r,t)$ the set

$$C_{m,n}(r,t)=\{r(\zeta-z)+z;\,\zeta\in C_{m,n},\,z\in D(0,t)\}.$$

An easy computation shows that $C_{m,n}(r,t)\subset A_{m,n}(r,t)$ where

$$A_{m,n}(r,t)=\left\{z=|z|e^{i\vartheta}\in\mathbb{D};\,(1+t)r-t\leq|z|\leq(1-t)r+t,\,e^{i\vartheta}\in C_{m,n}\right\}.$$

$A_{m,n}(r,t)$ is a compact susbet of $\mathbb{D}$, from what we deduce that each set $U(j,k,l,m,n,r,s)$ is open in $H(\mathbb{D})$, using the continuity of T_{λ^l} from $H(\mathbb{D})$ into itself (we recall that Λ is admissible). For the density, we first prove that each set

$$\bigcup_{r\in\rho}U(j,k,l,m,n,r,s),$$

intersects any neighborhood of 0. Let us fix $\varepsilon>0$, and $0<R<R'<1$. By uniform continuity of P^j on compact sets, and since 1 is a limit point of ρ, there exists $r\in\rho$ with $(1+t)r-t>R'$, such that

$$\sup_{\zeta\in C_{m,n}}\sup_{z\in D(0,\frac{s}{s+1})}\left|P^j\left(r(\zeta-z)+z\right)-P^j(\zeta)\right|<\frac{1}{2k}. \tag{3.2}$$

Let $\eta>1$ such that $\eta R/R'<1$. Since Λ is admissible, and since $\liminf_i|\lambda_i^l|^{1/i}\geq1$ by assumption, there exists $i_0\in I$ such that for any $i\geq i_0$, $\lambda_i^l\neq0$ and $|\lambda_i^l|^{1/i}\geq\eta R/R'$. By the choice of r, the compact set $D(0,R')\cup A_{m,n}(r,s/(s+1))$ has connected complement, and

[2, Lemma 5] - a refinement of Mergelyan's theorem - ensures that there exists a polynomial $\tilde{P} := \sum_i a_i z^i$, with valuation greater than i_0[1], such that

$$\|\tilde{P}\|_{D(0,R')} < \frac{\varepsilon(\eta-1)}{\eta} \quad \text{and} \quad \sup_{w\in A_{m,n}(r,\frac{s}{s+1})} |\tilde{P}(w) - P^j(w)| < \frac{1}{2k}. \tag{3.3}$$

Since the valuation of $\tilde{P}$ is at least i_0, we can define the polynomial P as

$$P(z) := T_{1/\lambda^l}\tilde{P} = \sum_i \frac{a_i}{\lambda_i^l} z^i.$$

By (3.2), the right part of (3.3), and the fact that $C_{m,n}(r, s/(s+1)) \subset A_{m,n}(r, s/(s+1))$, we have

$$\sup_{\zeta\in C_{m,n}} \sup_{z\in D(0,\frac{s}{s+1})} |T_{\lambda^l}P\left(r(\zeta - z) + z\right) - P^j(\zeta)| < 1/k.$$

Let us now prove that $\|P\|_{D(0,R)} < \varepsilon$. We denote by d the degree of P and define the polynomial Q by

$$Q(z) = \sum_{i=i_0}^d \frac{z^i}{\lambda_i^l},$$

so that $P = \tilde{P} \star Q$. By Lemma 3.2, we get for $|z| \leq R$,

$$\begin{aligned}
|P(z)| &= \frac{1}{2\pi}\left|\int_{\partial D(0,R')} \tilde{P}(w) Q\left(\frac{z}{w}\right)\frac{dw}{w}\right| \\
&\leq \frac{1}{2\pi}\|\tilde{P}\|_{D(0,R')}\left|\int_0^{2\pi}\sum_{k=i_0}^d \frac{1}{\lambda_k^l}\frac{z^k}{(R')^k e^{ik\vartheta}}d\vartheta\right| \\
&\leq \frac{\varepsilon(\eta-1)}{\eta}\sum_{k=i_0}^d\left(\frac{1}{|\lambda_k^l|^{1/k}}\frac{R}{R'}\right)^k \\
&\leq \varepsilon,
\end{aligned}$$

where we have used (3.3) and the choice of i_0. Thus P belongs to $\bigcup_{r\in\rho} U(j,k,l,m,n,r)$, and the latter set intersects any neighborhood of 0.

To finish, let us now fix $\varepsilon >$, $0 < R < 1$ and $g \in H(\mathbb{D})$. By Runge's theorem, there exists a polynomial Q such that $\|Q - g\|_{D(0,R)} < \varepsilon/2$. The previous then ensures the existence of a polynomial P_0 such that $\|P_0\|_{D(0,R)} < \varepsilon/2$ and

$$\sup_{\zeta\in C_{m,n}} \sup_{z\in D(0,\frac{s}{s+1})} |T_{\lambda^l}P_0\left(r(\zeta - z) + z\right) - (P^j(\zeta) - T_{\lambda^l}Q(\zeta))| < \frac{1}{k}.$$

It follows that $P := P_0 + Q$ belongs to $\bigcup_{r\in\rho} U(j,k,l,m,n,r)$ and satisfies $\|P - g\|_{D(0,R)} < \varepsilon$, as desired. □

3.2. **Proof of Part (2).** As the proofs of (2) (a) is very similar to that of (2) (b), but simpler, we only deal with (2) (b). The key-ingredient is the following approximation lemma.

Lemma 3.3. *Let $(\lambda_n^1)_n$ and $(\lambda_n^2)_n$ be two sequences of complex numbers such that*

$$\frac{\lambda_n^1}{\lambda_n^2} \to 0 \text{ as } n \to \infty.$$

We assume that λ_k^1 and λ_k^2 are non-zero for any k large enough. Then for any continuous function φ on $\partial\mathbb{D}$, any $N \in \mathbb{N}$, and any $\varepsilon > 0$, there exists a compact subset E of $\partial\mathbb{D}$ and a polynomial $P = \sum_k \alpha_k z^k$ such that

[1] For a polynomial $P(z) = \sum_i a_i z^i$, the valuation of P is defined as the smallest integer i such that $a_i \neq 0$.

(1) $m(E) > 1 - \varepsilon$;
(2) $val(P) \geq N$;
(3) $\sum_k |\lambda_k^1 \alpha_k| < \varepsilon$;
(4) $|\sum_k \lambda_k^2 \alpha_k \zeta^k - \varphi(\zeta)| < \varepsilon$ *for every* $\zeta \in E$.

Proof of Lemma 3.3. We fix φ and ε as in the statement. Set

$$M = 1 + \max_{\zeta \in \partial\mathbb{D}} |\varphi(\zeta)|.$$

Let also G be a compact arc in $\partial\mathbb{D}$ with $m(G) > 1 - \varepsilon/2$. Using Mergelyan's theorem (more precisely [2, Lemma 5]), we define a polynomial Q such that $Q(0) = 0$ and $|Q(\zeta) - 1| < \varepsilon/M$ for any $\zeta \in G$. Let also F be a compact arc in $\partial\mathbb{D}$ with connected complement, such that $m(F) > 1 - \varepsilon/2$. By Mergelyan's theorem, there exists a polynomial R such that

$$|R(\zeta) - \varphi(\zeta)| < \varepsilon,\ \zeta \in F.$$

Let us denote by d and e the degrees of R and Q respectively, and write

$$R(z) = \sum_{k=0}^{d} a_k z^k, \quad Q(z) = \sum_{k=1}^{e} b_k z^k \quad \text{and} \quad R(z)Q(z^\mu) = \sum_{k=\mu}^{e\mu+d} c_k z^k,$$

where $\mu \in \mathbb{N}$. Let us denote by γ and η the maximum of the moduli of the coefficients of R and Q, respectively. Since $\lambda_k^1/\lambda_k^2 \to 0$ as $k \to \infty$, we can choose $\mu \geq N$ large enough in order to have, for any $k \geq \mu$:

(a) $\lambda_k^i \neq 0$, $1 \leq i \leq 2$;
(b) $c_k = a_i b_j$ if $k = i + j\mu$, $0 \leq i \leq d$, $1 \leq j \leq e$, and $c_k = 0$ else;
(c) $|\lambda_k^1/\lambda_k^2| < (de\gamma\eta)^{-1}\varepsilon$;

Note that (b) is possible because $Q(0) = 0$. Then we define $\alpha_k = c_k/\lambda_k^2$ and observe that, since at most de coefficients c_k are non-zero, the polynomial

$$P(z) := \sum_{k=\mu}^{e\mu+d} \alpha_k z^k,$$

satisfies Property (3). (2) is obviously satisfied. Moreover $\sum_k \lambda_k^2 \alpha_k z^k = R(z)Q(z^\mu)$. Now, let us consider the arc $I := \{z \in \partial\mathbb{D};\ z^\mu \in G\}$. Since $z \mapsto z^\mu$ preserves the Lebesgue measure on $\partial\mathbb{D}$, if we set $E = I \cap F$, then we have $m(E) > 1 - \varepsilon$, and for every $\zeta \in E$,

$$|\sum_k \lambda_k^2 \alpha_k \zeta^k - \varphi(\zeta)| \leq |R(\zeta) - \varphi(\zeta)||Q(\zeta^\mu)| + |Q(\zeta^\mu)\varphi(\zeta) - \varphi(\zeta)| < 3\varepsilon.$$

Since ε is arbitrary, it shows that P also satisfies Property (4) for E satisfying Property (1). □

We come back to the proof of Theorem 2.4 (2). Without loss of generality, we assume that $I = \mathbb{N}^*$. Up to re-order the family $(\lambda^l,\ 1 \leq l < N)$ and up to some minor changes in the forthcoming proof, we may and shall assume that for any k and any $1 \leq l < l' < N$, $1 \leq |\lambda_k^l| \leq |\lambda_k^{l'}|$.

Let Λ and ρ be as in the statement of the theorem, and fix an increasing sequence $(r_n)_n \subset \rho$, converging to 1. Let $(f_j)_j$ be a dense sequence in $\mathcal{C}(\mathbb{T})$ and $(L_k)_k$ an exhaustion of compact subsets of $\mathbb{D}$. Let also $(\phi(n))_n$ be an enumeration of the countable subset of $\mathbb{N}^{\mathbb{N}}$ consisting in all sequences with finitely many non-zero coordinates. We denote by $\phi(n)(l)$ the l-th coordinate of $\phi(n)$. Let us also define $\mathcal{E}_k$ as the set of those compact subsets E in $\partial\mathbb{D}$ such that $m(E) \geq 1 - 1/2^{k+1}$. Then for $j, k, n, s \geq 1$, let us consider the set

$$U(j,k,n,s) = \bigcup_{m\geq n} \bigcup_{E\in\mathcal{E}_k} \left\{ f \in A(\mathbb{D});\ \sup_{1\leq l\leq k} \sup_{z\in L_s} \sup_{\zeta\in E} |T_{\lambda^l} f(r_m(\zeta - z) + z) - f_{\phi(j)(l)}(\zeta)| < \frac{1}{2^k} \right\}.$$

The first step of the proof consists in showing that

$$\mathcal{V}_m(\mathbb{D}, \Lambda, \rho) \supset \bigcap_{j,k,n,s} U(j,k,n,s).$$

Let us fix $f \in \cap_{j,k,n,s} U(j,k,n,s)$ and a sequence $(\varphi_l)_l$ of measurable functions on $\partial\mathbb{D}$. We first build by induction on k an increasing sequence $(v_k)_k \subset \mathbb{N}$ and a sequence E_k of subsets of $\partial\mathbb{D}$ with $m(E_k) \geq 1 - 1/2^k$, such that

$$\sup_{1\leq l\leq k} \sup_{\zeta\in E_k} \sup_{z\in L_k} |T_{\lambda^l} f(r_{v_k}(\zeta - z) + z) - \varphi_l(\zeta)| < \frac{1}{2^k}. \tag{3.4}$$

We set $v_0 = 0$ and $E_0 = \emptyset$, and assume that the construction has been made until step $k-1$. By Lusin's theorem, there exists $G_{k,l} \subset \partial\mathbb{D}$ with $m(G_{k,l}) \geq 1 - \frac{1}{k2^{k+1}}$, $1 \leq l \leq k$, and $j_k \in \mathbb{N}$ such that

$$\sup_{\zeta\in G_{k,l}} \left| f_{\phi(j_k)(l)}(\zeta) - \varphi_l(\zeta) \right| < \frac{1}{2^{k+1}},\ 1 \leq l \leq k.$$

We set $G_k = \cap_{1\leq l\leq k} G_{k,l}$ and observe that $m(G_k) \geq 1 - 1/2^{k+1}$. Now we use that $f \in U(j_k, k+1, r_{v_{k-1}}+1, k)$ to get the existence of $v_k > v_{k-1}$ and $F_k \in \mathcal{E}_k$, with $m(F_k) \geq 1 - 1/2^{k+1}$, such that

$$\sup_{1\leq l\leq k} \sup_{\zeta\in F_k} \sup_{z\in L_k} |T_{\lambda^l} f(r_{v_k}(\zeta - z) + z) - f_{\phi(j_k)(l)}(\zeta)| < \frac{1}{2^{k+1}}.$$

We finish the construction by setting $E_k = F_k \cap G_k$.

It remains to prove that there exists $E \subset \partial\mathbb{D}$ with $m(E) = 1$ such that for any $l \in \mathbb{N}^*$, any $\zeta \in E$ and any $z \in \mathbb{D}$,

$$|T_{\lambda^l} f(r_{v_k}(\zeta - z) + z) - \varphi_l(\zeta)| \to 0 \text{ as } k \to \infty.$$

For $M \in \mathbb{N}$, we set $H_M := \cap_{k\geq M} E_k$ and $E := \cup_{M\geq 0} H_M$. By construction, $m(E) = 1$. Let us now fix $l \in \mathbb{N}^*$, $\zeta \in E$ and $z_0 \in \mathbb{D}$. There exists $M_0 \in \mathbb{N}$ such that, for any $k \geq M_0$, $k \geq l$, $\zeta \in E_k$ and $z_0 \in L_k$. So, by Inequality (3.4), the inequality

$$|T_{\lambda^l} f(r_{v_k}(\zeta - z) + z) - \varphi_l(\zeta)| < \frac{1}{2^k}$$

holds for any $k \geq M_0$. Letting $k \to \infty$, we get the desired conclusion.

In order to apply the Baire Category Theorem and get that $\mathcal{V}_m(\mathbb{D}, \Lambda, \rho)$ is a dense G_δ-subset of $A(\mathbb{D})$, it suffices to show that $U(j,k,n,s)$ is open and dense in $A(\mathbb{D})$ for any $j, k, n, s \in\geq 1$. The fact that it is open follows directly from the continuity of the map T_{λ^l} from $A(\mathbb{D})$ to $H(\mathbb{D})$ (we recall that Λ is admissible).

Let us now prove that $U(j,k,n,s)$ is dense in $A(\mathbb{D})$. As usual, it enough to prove that $U(j,k,n,s)$ intersects any neighborhood of 0. Let $\varepsilon > 0$ be fixed. We build by induction $k+1$ subsets $E_1, \dots, E_k$ of $\partial\mathbb{D}$ and $k+1$ polynomials $P_0, \dots, P_k$. We set $E_0 = \partial\mathbb{D}$, $P_0 = 0$. Then we choose the sets E_l and polynomials $P_l = \sum_i \alpha_i^l z^i$, $1 \leq l \leq k$, by applying k times Lemma 3.3, in order to have

(1) For $l = 1$ and any $\zeta \in E_1$,
  (i) $m(E_1) > 1 - \frac{1}{k2^k}$;
  (ii) $\sum_i |\alpha_i^1| < \varepsilon/k$;
  (iii) $|\sum_i \lambda_i^1 \alpha_i^1 \zeta^i - f_{\phi(j)(1)}(\zeta)| < \frac{1}{k2^k}$;
(2) and for any $2 \leq l \leq k$ and any $\zeta \in E_l$,

(a) $m(E_l) > 1 - \frac{1}{k2^k}$;
(b) $\mathrm{val}(P_l) > \deg(P_{l-1})$;
(c) $\sum_i |\lambda_i^{l-1}\alpha_i^l| < \varepsilon/k$;
(d) $|\sum_i \lambda_i^l \alpha_i^l \zeta^i - (f_{\phi(j)(l)}(\zeta) - (\sum_{s=1}^{l-1}\sum_i \lambda_i^l \alpha_i^s \zeta^i))| < \frac{1}{k2^k}$.

Then we set $E = \cap_l E_l$ and $R = \sum_l P_l$. We observe that $m(E) > 1 - 1/2^k$, *i.e.* $E \in \mathcal{E}_k$, and that $\|R\|_{\mathbb{D}} < \varepsilon$ (by (ii) and (c)). Moreover, let us fix $1 \le l \le k$. We have, for any $\zeta \in E$,

$$\begin{aligned}
|T_{\lambda^l} R(\zeta) - f_{\phi(j)(l)}(\zeta)| &= |\sum_{s=1}^{l-1}\sum_i \lambda_i^l \alpha_i^s \zeta^i + \sum_i \lambda_i^l \alpha_i^l \zeta^i + \sum_{s=l+1}^{k}\sum_i \lambda_i^l \alpha_i^s \zeta^i - f_{\phi(j)(l)}(\zeta)| \\
&< \frac{1}{k2^k} + \sum_{s=l+1}^{k}\sum_i |\lambda_i^l \alpha_i^s| \quad \text{(by (iii) or (d))} \\
&< \frac{1}{k2^k} + \frac{(k-1)}{k2^k} \quad \text{(by (c))} \\
&= \frac{1}{2^k}.
\end{aligned}$$

Now, by uniform continuity of R on the compact set $\{r(\zeta - z) + z);\ \zeta \in E,\ z \in L_s\}$, we get the existence of some $m \ge n$, independent of $1 \le l \le k$, such that for any $z \in L_s$ and any $\zeta \in E$,

$$|T_{\lambda^l} R(r_m(\zeta - z) + z) - f_{\phi(j)(l)}(\zeta)| < \frac{1}{2^k}.$$

Thus $U(j,k,n,s)$ meets any neighborhood of 0 in $A(\mathbb{D})$ and the proof is complete.

ACKNOWLEDGEMENT

The origin of this note goes back to discussions with Vassili Nestoridis, during a Research in Pairs at CIRM Luminy in May 2018. I would like to thank the CIRM for the very good conditions of work. The author is partly supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).

REFERENCES

1. F. Bagemihl and W. Seidel, *Some boundary properties of analytic functions*, Mathematische Zeitschrift **61** (1954), no. 1, 186–199 (en).
2. F. Bayart, *Linearity of sets of strange functions*, Mich. Math. J. **53** (2005), no. 2, 291–303 (en).
3. ______, *Universal Radial Limits Of Holomorphic Functions*, Glasgow Mathematical Journal **47** (2005), no. 2, 261–267 (en).
4. F. Bayart, K.-G. Grosse-Erdmann, V. Nestoridis, and C. Papadimitropoulos, *Abstract theory of universal series and applications*, Proc. London Math. Soc. **96** (2008), 417–463.
5. L. Bernal-González, *Disjoint hypercyclic operators*, Studia Mathematica **182** (2007), no. 2, 113–131 (en).
6. J. Bès and A. Peris, *Disjointness in hypercyclicity*, Journal of Mathematical Analysis and Applications **336** (2007), no. 1, 297–315 (en).
7. S. Charpentier, *Countably universal series in the complex plane*, submitted (2017).
8. S. Charpentier and A. Mouze, *Universal Taylor series and summability*, Rev. Mat. Complut. **28** (2015), no. 1, 153–167.
9. S. Charpentier and V. Nestoridis, *On the boundary behaviour of derivatives of functions in the disc algebra*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 7, 732–736. MR 3811745
10. G. Costakis, *On the Radial Behavior of Universal Taylor Series*, Monatshefte für Mathematik **145** (2005), no. 1, 11–17 (en).
11. G. Costakis and N. Tsirivas, *Doubly universal Taylor series*, J. Approx. Theory **180** (2014), 21–31.
12. Y. Dupain, *Extension à la dimension n d'un théorème de Ortel et Schneider*, Mathematische Zeitschrift **206** (1991), no. 1, 71–80 (fr).
13. W. Gehlen, W. Luh, and J. Müller, *On the existence of O-universal functions*, Complex Variables Theory Appl. **41** (2000), no. 1, 81–90.

14. J. Globevnik and E. L. Stout, *Holomorphic functions with highly noncontinuable boundary behavior*, Journal d'Analyse Mathématique **41** (1982), no. 1, 211–216 (en).
15. A. Iordan, *On the radial cluster sets for holomorphic functions in the unit ball of* $\mathbb{C}^n$, Mathematische Zeitschrift **202** (1989), no. 2, 199–206 (en).
16. J.-P. Kahane and Y. Katznelson, *Sur le comportement radial des fonctions analytiques*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A718–A719. MR 0277721
17. A. Melas, *On the growth of universal functions*, Journal d'Analyse Mathématique **82** (2000), no. 1, 1–20 (en).
18. A. Melas, V. Nestoridis, and I. Papadoperakis, *Growth of coefficients of universal Taylor series and comparison of two classes of functions*, Journal d'Analyse Mathématique **73** (1997), no. 1, 187–202 (en).
19. Q. Menet, *Hypercyclic subspaces and weighted shifts*, Advances in Mathematics **255** (2014), 305–337. MR 3167484
20. A. Mouze, *On doubly universal functions*, J. Approx. Theory **226** (2018), 1–13.
21. V. Nestoridis, *Universal Taylor series*, Ann. Inst. Fourier (Grenoble) **46** (1996), 1293–1306.
22. V. Vlachou, *Disjoint Universality for Families of Taylor-type Operators*, J. Math. Anal. Appl **448** (2017), no. 2, 1318–1330.

Stéphane Charpentier, Institut de Mathématiques de Marseille, UMR 7353, Aix-Marseille Université, Technopôle Château-Gombert, 39 rue F. Joliot Curie, 13453 Marseille Cedex 13, France

E-mail address: `stephane.charpentier.1@univ-amu.fr`