Generalized compressible fluid flows and solutions of the Camassa-Holm variational model

Thomas Gallouët, Andrea Natale, François-Xavier Vialard

GENERALIZED COMPRESSIBLE FLUID FLOWS AND SOLUTIONS OF THE CAMASSA-HOLM VARIATIONAL MODEL

THOMAS GALLOUËT, ANDREA NATALE AND FRANÇOIS-XAVIER VIALARD

ABSTRACT. The Camassa-Holm equation on a domain $M \subset \mathbb{R}^d$, in one of its possible multi-dimensional generalizations, describes geodesics on the group of diffeomorphisms with respect to the $H(\text{div})$ metric. It has been recently reformulated as a geodesic equation for the L^2 metric on a subgroup of the diffeomorphism group of the cone over M. We use such an interpretation to construct an analogue of Brenier's generalized incompressible Euler flows for the Camassa-Holm equation. This involves describing the fluid motion using probability measures on the space of paths on the cone, so that particles are allowed to split and cross. Differently from Brenier's model, however, we are also able to account for compressibility by employing an explicit probabilistic representation of the Jacobian of the flow map. We formulate the boundary value problem associated to the Camassa-Holm equation using such generalized flows. We prove existence of solutions and that, for short times, smooth solutions of the Camassa-Holm equations are the unique solutions of our model. We propose a numerical scheme to construct generalized solutions on the cone and present some numerical results illustrating the relation between the generalized Camassa-Holm and incompressible Euler solutions.

1. Introduction

The Camassa-Holm (CH) equation is the geodesic equation for the H^1 metric on the group of diffeomorphims of the circle or the real line [9]. This can be derived as an approximation for ideal fluid flow with a free boundary in the shallow water regime. In this context, [22] showed that its natural generalization to a higher dimensional domain $M \subset \mathbb{R}^d$ consists in replacing the H^1 norm with the $H(\text{div})$ norm. In other words, the CH equation is the Euler-Lagrange equation for the Lagrangian

$$l(u) = a \int_M \|u\|^2 \,\mathrm{d}\rho_0 + b \int_M |\text{div}(u)|^2 \,\mathrm{d}\rho_0 \,, \tag{1.1}$$

where u is the Eulerian velocity field, $a, b > 0$ are constants and ρ_0 is the Lebesgue measure on M. This is a particular instance of a class of right-invariant Lagrangians on the diffeomorphism group of M considered in [21], which for $d = 3$ can be written as

$$l(u) = a \int_M \|u\|^2 \,\mathrm{d}\rho_0 + b \int_M |\text{div}(u)|^2 \,\mathrm{d}\rho_0 + c \int_M \|\text{curl}(u)\|^2 \,\mathrm{d}\rho_0 \,. \tag{1.2}$$

Such Lagrangians give rise to several important fluid dynamics models, including the EPDiff equation for the H^1 Sobolev norm of vector fields and the Euler-α model [17, 18], both of which have also been regarded as possible multi-dimensional versions of the CH equation, but also the Hunter-Saxton equation [20].

In one dimension, the CH equation is bi-Hamiltonian and completely integrable. It also possesses soliton solutions named peakons, i.e. non-smooth traveling wave solutions which interact and collide without changing their shapes. On the real line, these (weak) solutions have the following expression

$$u(x,t) = p(t) e^{-|x-q(t)|/\alpha} \,, \tag{1.3}$$

where $p(t)$ and $q(t)$ determine the height and speed of the wave, respectively, and $\alpha > 0$ is an independent constant determining its width [16]. Peakons always emerge from appropriate smooth initial data satisfying a certain decay property on the real line, yielding therefore a model for wave breaking [12, 27]. In other words, since the emergence of peakons corresponds to blow-up in an appropriate norm, strong solutions may have finite existence time. Furthermore, even weak solutions cannot be defined globally [27]; the collision of peakons, for instance, gives

Date: June 28, 2018.

an explicit example of finite time breakdown (blow up) of solutions. In this case, at the collision time, the Lagrangian map ceases to be injective and after this, weak solutions are not uniquely defined.

Recently, [14] put forward a novel interpretation of the CH equation which emphasizes its connection with the incompressible Euler equations. In order to describe this, we consider first the incompressible Euler model. In this case, the configuration space of the system is given by a subgroup of the diffeomorphism group which consits of all diffeomeorphisms preserving the Lebesgue measure, i.e. satisfying

$$(\varphi_t)_\# \rho_0 = \rho_0 \,. \tag{1.4}$$

In fact, this can be seen as an isotropy subgroup once we interpret the push-forward as an action of the diffeomorphism group on the space of densities on M. This point of view establishes a remarkable connection with optimal transport theory [7] (see also [29] for a description of the geometrical connection between the diffeomorphism group and the space of densities). Incompressible Euler flows are minimizers of the action

$$\int_0^T \int_M \frac{1}{2} \|\dot{\varphi}_t\|^2 \, \mathrm{d}\rho_0 \mathrm{d}t \,, \tag{1.5}$$

subject to (1.4) and with the additional constraint that $\varphi_0 = \mathrm{Id}$, the identity map on M, and $\varphi_T = h$, a given diffeomorphism on M, which prescribes the final position of each particle in M at the final time T.

Shnirelman proved that the infimum of this problem is not generally attained when $d \geq 3$ and that even when $d = 2$ there exist final configurations h which cannot be connected to the identity map with finite action [31]. This motivated Brenier to introduce a relaxation whose solutions are not diffeomorphims, but rather describe the flow in a probabilistic fashion. More precisely, Brenier defined generalized incompressible flows as probability measures $\boldsymbol{\mu}$ on $\Omega(M)$, the space of continuous curves on the domain $\mathrm{x} : t \in [0,T] \to x_t \in M$, satisfying

$$(e_t)_\# \boldsymbol{\mu} = \rho_0 \,, \tag{1.6}$$

where $e_t : \Omega(M) \to M$ is the evaluation map at time t defined by $e_t(\mathrm{x}) = x_t$. In this interpretation, the marginals $(e_0, e_t)_\# \boldsymbol{\mu}$ are probability measures on the product $M \times M$ and describe how particles move and spread their mass across the domain. Of course, classical deterministic solutions, i.e. curves of volume preserving diffeomorphisms $t \mapsto \varphi_t$, also fit in this definition and correspond to the case where the marginals $(e_0, e_t)_\# \boldsymbol{\mu}$ are concentrated on the graph of φ_t. Then, equation (1.6) is the equivalent of the incompressibility constraint in the generalized setting; in fact, when $\boldsymbol{\mu}$ is deterministic it coincides with (1.4). The minimization problem in terms of generalized flows consists in minimizing the action

$$\int_{\Omega(M)} \int_0^T \frac{1}{2} \|\dot{x}_t\|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\mu}(\mathrm{x}) \tag{1.7}$$

among generalized incompressible flows, with the constraint $(e_0, e_T)_\# \boldsymbol{\mu} = (\mathrm{Id}, h)_\# \rho_0$. Brenier proved that this model is consistent with classical solutions of the incompressible Euler equations [7]. In particular, smooth solutions correspond to the unique minimizers of the generalized problem if the pressure has bounded Hessian and for sufficiently small times. On the other hand, for any coupling there exists a unique pressure, defined as a distribution, associated to generalized solutions. This result was later improved by Ambrosio and Figalli [1] who showed that the pressure can be actually defined as a function and defined optimality conditions for generalized flows based on this result.

In analogy to the incompressible Euler case, the CH equation can be reformulated as a geodesic equation for the L^2 cone metric on a certain isotropy subgroup of the diffeomorphism group of $M \times \mathbb{R}_{>0}$. More precisely, CH flows are represented by time dependent maps in the form

$$(\varphi, \lambda) : (x, r) \in M \times \mathbb{R}_{>0} \to (\varphi(x), \lambda(x) r) \in M \times \mathbb{R}_{>0} \,, \tag{1.8}$$

where $\varphi : M \to M$ and $\lambda : M \to \mathbb{R}_{>0}$. This set of maps is a group under composition and is known as the automorphism group of $M \times \mathbb{R}_{>0}$. The isotropy subgroup is given by

$$\varphi_\# (\lambda^2 \rho_0) = \rho_0 \,. \tag{1.9}$$

Differently from (1.4), this condition does not enforce incompressibility but it relates φ and λ by requiring $\lambda = \sqrt{\mathrm{Jac}(\varphi)}$. Therefore, automorphisms satisfying (1.9) provide us with an alternative way to represent diffeomorphisms of M. Importantly, in this picture we cannot capture the blow up of solutions as induced by peakon collisions, as in this case the Jacobian would locally vanish. In addition, the metric space $M \times \mathbb{R}_{>0}$ equipped with the cone metric is not complete, which complicates the construction of generalized solutions following Brenier's approach. We are then led to work with the cone $\mathcal{C} = (M \times \mathbb{R}_{\geq 0})/(M \times \{0\})$, which allows us to represent solutions with vanishing Jacobian by paths on the cone reaching the apex.

In this paper, we construct a framework to solve the boundary value problem associated to the CH equation using generalized flows interpreted as probability measures $\boldsymbol{\mu}$ on the space $\Omega(\mathcal{C})$ of continuous paths on the cone $\mathrm{z} : t \in [0,T] \to z_t = [x_t, r_t] \in \mathcal{C}$. We will show that in this setting, the isotropy subgroup condition in (1.9) is then replaced by

$$(\pi_x)_\#(r^2 (e_t)_\# \boldsymbol{\mu}) = \rho_0 \,, \tag{1.10}$$

where π_x is the projection from the cone onto M and now $e_t : \Omega(\mathcal{C}) \to \mathcal{C}$ is the evaluation map for continuous paths on the cone. The generalized minimization problem for CH consists in minimizing the action

$$\int_{\Omega(\mathcal{C})} \int_0^T \|\dot{z}_t\|^2_{g_\mathcal{C}} \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \tag{1.11}$$

among generalized flows satisfying (1.10) and an appropriate coupling constraint. The issue of choosing the correct coupling constraint will occupy an important part in the paper. It will be evident that enforcing the coupling of points on the cone in the same way Brenier did for Euler is inappropriate for our case. The same holds also when enforcing the constraint on the base space M only. We will define a weaker form of coupling which allows us to prove existence of solutions but it is still compatible with the original model. In particular, we will prove three main results on this problem: existence of solutions of generalized problem; existence and uniqueness of the pressure as a distribution; and correspondence with smooth solutions of the CH equation.

The main difficulty in carrying out this program lies on the necessity to work on an unbounded cone domain and on the impossibility to "cut it" without limiting the class of functions that can be represented by the model. This issue is directly linked to the choice of the correct coupling constraint. In fact, we will introduce a sufficiently weak coupling constraint in order to be able to represent a sufficiently large class of generalized flows on the cone and consequently prove existence of solutions. In principle, this allows us to represent solutions that charge paths reaching the apex of the cone and therefore are characterized, in some sense, by a vanishing Jacobian locally in space. It is natural to ask whether these solutions are actually realized by appropriate couplings. In this paper, we will not answer this question but we will address its complementary side, that is, we will show that smooth solutions of the CH equation (that do not reach the apex) are the unique minimizers for our model for sufficiently short times and upon some regularity conditions on the pressure similar to the Euler case. Interestingly, the decoupling between the Lagrangian flow map and its Jacobian we use to define generalized solutions has also been used in [23] to construct global weak solutions of the CH equation. However, in their case, one continues solutions after the blowup by allowing the square root of the Jacobian to become negative, which does not occur in our construction.

It should be noted that the cone construction has been developed and used extensively in [24] in order to characterize the metric side of the Wasserstein-Fisher-Rao (WFR) distance (which is also called Hellinger-Kantorovich distance) on the space of positive Radon measures. In fact, as noted in [14] this has the same relation to the CH equation as the Wasserstein L^2 distance does to the incompressible Euler equations. In the geodesic problem associated the WFR distance the isotropy subgroup relation in (1.10) is used to prescribe the initial and final density. The resulting problem can then be expressed without recurring to the cone construction, yielding an optimal entropy-transport problem, a widespread form of unbalanced optimal transport based on the Kullback-Leibler divergence [11, 10, 24]. Unfortunately, we cannot easily relate such a formulation to our generalized CH problem and in fact this latter does not coincide with a multi-marginal entropy-transport problem. This means that we will develop our construction on the cone without looking at the possibility of reducing the problem using objects defined on the base space M only.

The framework we develop here for the CH equation opens several new directions in terms of the numerical treatment of these equations. This can be useful both to study the nature of generalized CH flows and as a basis to develop novel numerical tools to simulate classical solutions to this problem. In the case of the incompressible Euler equations, recent advances in numerical methods for optimal transport have inspired several methods which can also be reformulated for the CH problem. The introduction of the Sinkhorn's algorithm for the entropic regularization of optimal transport problems [13] has paved the way for the development of a number of efficient algorithms for several applications [5]. This methodology has been used in [6] to develop a numerical scheme to compute generalized incopressible Euler flows. In this paper, we construct a similar numerical scheme for the generalized CH problem, which however cannot represent the "blow up", i.e. solutions with vanishing Jacobian. It should be noted that different methodologies based on semi-discrete optimal transport [25] could provide a better description of generalized CH flows. Semi-discrete schemes for the incompressible Euler problem have been developed in [26] for the boundary value problem and in [15] for the initial value problem, and indicate a promising direction for the development of numerical schemes for the CH model as well.

Besides numerical applications, our approach to solve the variational CH model also suggests several new research directions. First of all, a natural question is whether our construction can provide any insight on the continuation of CH solutions after blow up, for instance in relation to the analytical approach in [23]. In addition, it is also natural to ask whether different models arising from the right-invariant Lagrangian (1.2) on the group of either compressible or incompressible diffeomorphisms can be treated in the same way as the CH model. A unified approach to treat this Lagrangian could shed light on the relation between several important fluid dynamics models and provide a deeper understanding of their solutions.

The rest of the paper is structured as follows. In section 2 we describe the notation and provide some background measure theoretical notions. In section 3 we describe the variational interpretation of the generalized CH equation as geodesic equation on the group of automorphisms of the cone. In section 4 we build on such an interpretation proposing a definition for compressible generalized flows which allows us to define solutions of the boundary value problem associated to the CH equation. In section 5 we prove that for any given final configuration of the flow defining the boundary conditions of the generalized CH problem there exists a unique pressure defined as a distribution; this mimics Brenier's analogue result for the incompressible Euler equations [8]. In section 6 we prove that smooth solutions of the CH equation are the unique minimizers of our generalized model for sufficiently short times. In section 7 we construct a numerical algorithm based on entropic regularization and Sinkhorn's algorithm to compute generalized CH flows and provide some numerical results. Conclusions and open questions are collected in section 8.

2. Notation and preliminaries

In this section, we describe the notation and some basic results used throughout the paper. Because of the similarities between our setting and the one of [24], we will adopt a similar notation for the cone construction and the measure theory objects we will employ.

2.1. Function spaces.

Given two metric spaces X and Y, we denote by $C^0(X;Y)$ the space of continuous functions $f : X \to Y$, and with $C^0(X)$ the space of real-valued continuous functions $f : X \to \mathbb{R}$. If X is compact $C^0(X)$ is a Banach space with respect to the sup norm $\|\cdot\|_{C^0}$. The set of Lipschitz continuous function on X is denoted by $C^{0,1}(X)$ and the associated norm is given by

$$\|f\|_{C^{0,1}} \coloneqq \sup_{x\in X} |f(x)| + \sup_{x,y\in X, x\neq y} \frac{|f(x)-f(y)|}{d_X(x,y)}\,, \tag{2.1}$$

where d_X denotes the distance function on X. If X is a manifold, we will denote by $\mathrm{Diff}(X)$ the group of smooth diffeomorphisms of X.

2.2. The cone and metric structures.

Let $M \subset \mathbb{R}^d$ be a compact domain. We will denote by g the Euclidean metric tensor on M, with $d_M : M \times M \to \mathbb{R}_{\geq 0}$ the Euclidean distance on M and with $\|\cdot\|_g$ the Euclidean norm. We denote by $\mathcal{C} \coloneqq (M \times \mathbb{R}_{\geq 0})/(M \times \{0\})$ the cone over M.

A point on the cone is an equivalence class $p = [x, r]$, with equivalence relation given by

$$(x_1, r_1) \sim (x_2, r_2) \Leftrightarrow (x_1, r_1) = (x_2, r_2) \text{ or } r_1 = r_2 = 0\,. \tag{2.2}$$

The distinguished point of the cone $[x, 0]$ is the apex of $\mathcal{C}$ and it is denoted by o. Every point on the cone different from the apex can be identified with a couple (x, r) where $x \in M$ and $r \in \mathbb{R}_{>0}$. Moreover, we fix a point $\bar{x} \in M$ and we introduce the projections $\pi_x : \mathcal{C} \to M$ and $\pi_r : \mathcal{C} \to \mathbb{R}_{\geq 0}$ defined by

$$\pi_x([x, r]) = \begin{cases} x & \text{if } r > 0\,, \\ \bar{x} & \text{if } r = 0\,, \end{cases} \qquad \pi_r([x, r]) = r\,. \tag{2.3}$$

We endow the cone with the metric tensor $g_{\mathcal{C}} = r^2 g + \mathrm{d}r^2$ defined on $M \times \mathbb{R}_{>0}$. We denote the associated norm by $\|\cdot\|_{g_{\mathcal{C}}}$. We use the superscripts g and $g_{\mathcal{C}}$ for differential operators, e.g., ∇^g, div^g and so on, to indicate that they are computed with respect to either one of these metrics. The distance on the cone $d_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_{\geq 0}$ is given by

$$d_{\mathcal{C}}([x_1, r_1], [x_2, r_2])^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos(d_M(x_1, x_2) \wedge \pi)\,. \tag{2.4}$$

The closed subset of the cone composed of points below a given radius $R > 0$ is denoted by $\mathcal{C}_R$, or more precisely

$$\mathcal{C}_R \coloneqq \{[x, r] \in \mathcal{C}\,;\, r \leq R\}\,. \tag{2.5}$$

Given an interval $I \subset \mathbb{R}$, we denote by $C^0(I; \mathcal{C})$ and $AC(I; \mathcal{C})$ the spaces of, respectively, continuous and absolutely continuous curves $\mathrm{z} : t \in I \to z_t \in \mathcal{C}$. We will generally use the notation

$$\mathrm{x} : t \in I \to x_t = \pi_x(z_t) \in M\,, \qquad \mathrm{r} : t \in I \to r_t = \pi_r(z_t) \in [0, +\infty)\,, \tag{2.6}$$

so that $\mathrm{z} = [\mathrm{x}, \mathrm{r}]$ and $z_t = [x_t, r_t]$. Note that if z is continuous (resp. absolutely continuous), then so is the path r but not x. However, x is continuous (resp. locally absolutely continuous) when restricted to the open set $\{t \in I; r_t > 0\}$. Then, if we define $\dot{\mathrm{z}} : t \in I \to \dot{z}_t \in \mathbb{R}^{d+1}$ by

$$\dot{z}_t = \begin{cases} (\dot{x}_t, \dot{r}_t) & \text{if } r_t > 0 \text{ and the derivatives exist,} \\ (0, 0) & \text{otherwise,} \end{cases} \tag{2.7}$$

we have that $\|\dot{z}_t\|_{g_{\mathcal{C}}}$ coincides for a.e. $t \in I$ with the metric derivative of z with respect to the distance $d_{\mathcal{C}}$ [24]. We denote by $AC^p(I; \mathcal{C})$ the space of absolutely continuous curves such that $\|\dot{\mathrm{z}}\|_{g_{\mathcal{C}}} \in L^p(I)$. Then, the following variational formula for the distance function holds

$$d_{\mathcal{C}}(p, q)^2 = \inf \left\{ \int_0^1 \|\dot{z}_t\|_{g_{\mathcal{C}}}^2 \,\mathrm{d}t\,;\, \mathrm{z} \in AC^2([0, 1]; \mathcal{C})\,,\, z_0 = p\,,\, z_1 = q \right\}. \tag{2.8}$$

We will extensively use the class of homogeneous functions on the cone defined as follows. A function $f : \mathcal{C}^n \to \mathbb{R}$ is p-homogeneous (in the radial direction) if for any constant $\lambda \in \mathbb{R}$ and for all n-tuples $([x_1, r_1], \ldots, [x_n, r_n]) \in \mathcal{C}^n$,

$$f([x_1, \lambda r_1], \ldots, [x_n, \lambda r_n]) = |\lambda|^p f([x_1, r_1], \ldots, [x_n, r_n])\,. \tag{2.9}$$

In particular, a p-homogeneous function $f : \mathcal{C} \to \mathbb{R}$ satisfies $f([x, \lambda r]) = |\lambda|^p f([x, r])$. Similarly, a functional $\sigma : C^0(I; \mathcal{C}) \to \mathbb{R}$ is p-homogeneous if for any constant $\lambda \in \mathbb{R}$ and for any path $\mathrm{z} \in C^0(I; \mathcal{C})$,

$$\sigma(t \mapsto [x_t, \lambda r_t]) = |\lambda|^p \sigma(\mathrm{z})\,, \tag{2.10}$$

where $\mathrm{z} : t \in I \to [x_t, r_t] \in \mathcal{C}$.

2.3. **Measure theoretic background.** Let X be a Polish space, i.e. a complete and separable metric space. We denote by $\mathcal{M}(X)$ the set of non-negative and finite Borel measures on X. The set of probability measures on X is denoted by $\mathcal{P}(X)$. Let Y be another Polish space and $F : X \to Y$ a Borel map. Given a measure $\mu \in \mathcal{M}(X)$ we denote by $F_\# \mu \in \mathcal{M}(Y)$ the push-forward measure defined by $(F_\# \mu)(A) \coloneqq \mu(F^{-1}(A))$ for any Borel set $A \subset Y$. Given a Borel set $B \subset X$ we let $\mu \llcorner B$ the restriction of μ to B defined by $\mu \llcorner B(C) \coloneqq \mu(B \cap C)$ for any Borel set $C \subseteq X$. Note that we will generally use bold symbols to denote measures on product spaces, e.g., $\boldsymbol{\mu} \in \mathcal{M}(X \times \ldots \times X)$.

We endow $\mathcal{P}(X)$ with the topology induced by narrow convergence, which is the convergence in duality with the space of real-valued continuous bounded functions $C_b^0(X)$. Then, $\mathcal{P}(X)$ can be identified with a subset of $[C_b^0(X)]^*$ with the weak-* topology (see Remark 5.1.2 in

[3]). Moreover, given a lower semi-continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, the functional $\mathcal{F} : \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{F}(\mu) \coloneqq \int_X f \, \mathrm{d}\mu \tag{2.11}$$

is also lower-semicontinuous (see Lemma 1.6 in [30]) .

As usual in this setting, we will use Prokhorov's theorem for a characterization of compact subsets of $\mathcal{P}(X)$ endowed with the narrow topology.

Theorem 2.1. *A set $\mathcal{K} \subset \mathcal{P}(X)$ is relatively sequentially compact in $\mathcal{P}(X)$ if and only if it is tight, i.e. for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that $\mu(X \setminus K_\epsilon) < \epsilon$ for any $\mu \in \mathcal{K}$.*

We also need a criterion to pass to the limit when computing integrals of unbounded functions: for this will use the concept of uniform integrability. Given a set $\mathcal{K} \subset \mathcal{P}(X)$, we say that a Borel function $f : X \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is uniformly integrable with respect to $\mathcal{K}$ if for any $\mu \in \mathcal{K}$ and any $\epsilon > 0$ there exists a $k > 0$ such that

$$\int_{f(x)>k} f(x) \, \mathrm{d}\mu(x) < \epsilon \, . \tag{2.12}$$

Lemma 2.2 (Lemma 5.1.17 in [3])**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}(X)$ narrowly convergent to $\mu \in \mathcal{P}(X)$ and let $f \in C^0(X)$. If f is uniformly integrable with respect to the set $\{\mu_n\}_{n\in\mathbb{N}}$ then*

$$\lim_{n\to+\infty} \int_X f \, \mathrm{d}\mu_n = \int_X f \, \mathrm{d}\mu \, . \tag{2.13}$$

For a fixed $T > 0$, we will denote by $\Omega(X) \coloneqq C^0([0, T]; X)$ the space of continuous paths on X. This is a Polish space so that we can use the tools introduced in this section also for probability measures $\boldsymbol{\mu} \in \mathcal{P}(\Omega(X))$. We call such probability measures *generalized flows* or also *dynamic plans*. When $X = \mathcal{C}$ we will often use Ω to denote $\Omega(\mathcal{C})$.

Since we will work with homogeneous functions on the cone, we also introduce the space of probability measures $\mathcal{P}_p(X)$, defined by

$$\mathcal{P}_p(X) \coloneqq \left\{ \mu \in \mathcal{P}(X) \, ; \, \int_X d_X(x, \bar{x})^p \, \mathrm{d}\mu(x) < +\infty \text{ for some } \bar{x} \in X \right\} . \tag{2.14}$$

Then, if $\boldsymbol{\mu} \in \mathcal{P}_p(\mathcal{C}^n)$, where $\mathcal{C}$ is the cone over the compact domain $M \subset \mathbb{R}^d$, it is easy to verify that any p-homogeneous function on $\mathcal{C}^n$ is $\boldsymbol{\mu}$-integrable.

Finally, we will denote by ρ_0 the Lebesgue measure on M normalized so that $\rho_0(M) = 1$.

3. The variational formulation on the cone

In this section we describe the geometric structure of the CH equation using the group of automorphisms of the cone. Such a formulation was introduced in [14] and it was used to interpret the CH equation as an incompressible Euler equations on the cone. In fact, in itself it is similar to that of the incompressible Euler equations originally considered by Arnold [4]. In this section we will only focus on smooth solutions, but we will later use the variational interpretation presented here to guide the construction of generalized solutions of the CH equation. We will keep the discussion formal at this stage and we will use some standard geometric tools and notation commonly adopted in similar contexts.

Consider a compact smooth domain $M \subset \mathbb{R}^d$. For any $\varphi \in \mathrm{Diff}(M)$ and $\lambda \in C^\infty(M; \mathbb{R}_{>0})$, we let $(\varphi, \lambda) : \mathcal{C} \to \mathcal{C}$ be the map defined by $(\varphi, \lambda)([x, r]) = [\varphi(x), \lambda(x)r]$. The automorphism group $\mathrm{Aut}(\mathcal{C})$ is the collection of such maps, i.e.

$$\mathrm{Aut}(\mathcal{C}) = \{(\varphi, \lambda) : \mathcal{C} \to \mathcal{C}; \, \varphi \in \mathrm{Diff}(M), \, \lambda \in C^\infty(M; \mathbb{R}_{>0})\} \, . \tag{3.1}$$

The group composition law is given by

$$(\varphi, \lambda) \cdot (\psi, \mu) = (\varphi \circ \psi, (\lambda \circ \psi)\mu) \, , \tag{3.2}$$

the identity element is $(\mathrm{Id}, 1)$, where Id is the identity map on M, and the inverse is given by

$$(\varphi, \lambda)^{-1} = (\varphi^{-1}, \lambda^{-1} \circ \varphi^{-1}) \, . \tag{3.3}$$

The tangent space of $\mathrm{Aut}(\mathcal{C})$ at (φ, λ) is denoted by $T_{(\varphi,\lambda)}\mathrm{Aut}(\mathcal{C})$. This is the set of tangent vectors

$$(\dot\varphi, \dot\lambda) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\varphi_t, \lambda_t)\,, \tag{3.4}$$

where $t \mapsto (\varphi_t, \lambda_t)$ is a curve on $\mathrm{Aut}(\mathcal{C})$ defined on an open interval around 0 and satisfying $(\varphi_0, \lambda_0) = (\varphi, \lambda)$. The tangent space $T_{(\varphi,\lambda)}\mathrm{Aut}(\mathcal{C})$ can be identified with the space of vector fields $C^\infty(M, \mathbb{R}^{d+1})$. The collection all the tangent spaces is the tangent bundle $T\mathrm{Aut}(\mathcal{C})$.

We endow $T\mathrm{Aut}(\mathcal{C})$ with the $L^2(M;\mathcal{C})$ metric inherited from $g_{\mathcal{C}}$. This is defined as follows: given $(\dot\varphi, \dot\lambda) \in T_{(\varphi,\lambda)}\mathrm{Aut}(\mathcal{C})$,

$$\|(\dot\varphi, \dot\lambda)\|^2_{L^2(M;\mathcal{C})} \coloneqq \int_M (\lambda^2 \|\dot\varphi\|_g^2 + \dot\lambda^2)\, \mathrm{d}\rho_0\,, \tag{3.5}$$

where $\|\cdot\|_g$ is the norm on M associated to g and ρ_0 is the Lebesgue measure on M.

In [14] the authors found that the CH equation on M coincides with the geodesic equation on the subgroup $\mathrm{Aut}_{\rho_0}(\mathcal{C}) \subset \mathrm{Aut}(\mathcal{C})$ defined as follows:

$$\mathrm{Aut}_{\rho_0}(\mathcal{C}) \coloneqq \{(\varphi, \lambda) \in \mathrm{Aut}(\mathcal{C})\,;\, \varphi_\#(\lambda^2\rho_0) = \rho_0\}\,. \tag{3.6}$$

In other words, the group $\mathrm{Aut}_{\rho_0}(\mathcal{C})$ can be regarded as the configuration space for the CH equation in the same way as the $\mathrm{Diff}_{\rho_0}(M)$ is the configuration space for the incompressible Euler equations, with

$$\mathrm{Diff}_{\rho_0}(M) \coloneqq \{\varphi \in \mathrm{Diff}(M)\,;\, \varphi_\#\rho_0 = \rho_0\}\,. \tag{3.7}$$

In order to see this, we first observe that the $L^2(M;\mathcal{C})$ metric is right invariant when restricted to $\mathrm{Aut}_{\rho_0}(\mathcal{C})$, meaning that it does not change when moving on this subgroup by right translations. In particular, for any $(\psi, \mu) \in \mathrm{Aut}_{\rho_0}(\mathcal{C})$, consider the right translation map $R_{(\psi,\mu)} : \mathrm{Aut}_{\rho_0}(\mathcal{C}) \to \mathrm{Aut}_{\rho_0}(\mathcal{C})$ defined by $R_{(\psi,\mu)}(\varphi, \lambda) = (\varphi, \lambda)\cdot(\psi, \mu)$. Its tangent map at (φ, λ) is given by

$$TR_{(\psi,\mu)}(\dot\varphi, \dot\lambda) = (\dot\varphi \circ \psi, (\dot\lambda \circ \psi)\mu). \tag{3.8}$$

Then,

$$\begin{aligned}
\|TR_{(\psi,\mu)}(\dot\varphi, \dot\lambda)\|^2_{L^2(M;\mathcal{C})} &= \int_M (\lambda^2 \circ \psi \mu^2 \|\dot\varphi\|_g^2 \circ \psi + \dot\lambda^2 \circ \psi\mu^2)\, \mathrm{d}\rho_0 \\
&= \int_M (\lambda^2 \|\dot\varphi\|_g^2 + \dot\lambda^2) \circ \psi\mu^2\, \mathrm{d}\rho_0 \\
&= \int_M (\lambda^2 \|\dot\varphi\|_g^2 + \dot\lambda^2)\, \mathrm{d}\psi_\#(\mu^2\rho_0) \\
&= \|(\dot\varphi, \dot\lambda)\|^2_{L^2(M;\mathcal{C})}\,.
\end{aligned} \tag{3.9}$$

Geodesics on $\mathrm{Aut}_{\rho_0}(\mathcal{C})$ correspond to stationary paths on $T\mathrm{Aut}_{\rho_0}(\mathcal{C})$ for the action functional

$$\int_0^T L((\varphi, \lambda), (\dot\varphi, \dot\lambda))\, \mathrm{d}t \tag{3.10}$$

for a given $T > 0$, where the Lagrangian $L((\varphi, \lambda), (\dot\varphi, \dot\lambda)) = \|(\dot\varphi, \dot\lambda)\|^2_{L^2(M;\mathcal{C})}$. The invariance of the metric implies that the geodesic equation can be expressed in terms of right trivialized (Eulerian) velocities only, or in other words in terms of the variables

$$(u, \alpha) = TR_{(\varphi,\lambda)^{-1}}(\dot\varphi, \dot\lambda) = (\dot\varphi \circ \varphi^{-1}, (\dot\lambda\lambda^{-1}) \circ \varphi^{-1})\,. \tag{3.11}$$

Now, the constraint $\varphi_\#(\lambda^2\rho) = \rho$ can be rewritten as $\lambda = \sqrt{\mathrm{Jac}(\varphi)}$. Moreover, we have that for any $f \in C^\infty(M)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_M f\, \mathrm{d}\varphi_\#(\lambda^2\rho_0) &= \int_M g(\nabla^g f \circ \varphi, \dot\varphi)\lambda^2\, \mathrm{d}\rho_0 + \int_M 2\lambda\dot\lambda f \circ \varphi\, \mathrm{d}\rho_0 \\
&= \int_M (g(\nabla^g f, u) + 2\alpha f) \circ \varphi\lambda^2\, \mathrm{d}\rho_0 \\
&= \int_M (-\mathrm{div}^g u + 2\alpha) f\, \mathrm{d}\rho_0\,.
\end{aligned} \tag{3.12}$$

Hence the constraint becomes $2\alpha = \mathrm{div}^g u$ in terms of Eulerian variables. Moreover, by right invariance,

$$L((\varphi,\lambda),(\dot\varphi,\dot\lambda)) = L((\mathrm{Id},1),(u,\alpha)) = \int_M \|u\|_g^2 + \frac{1}{4}(\mathrm{div}^g u)^2 \,\mathrm{d}\rho_0\,, \tag{3.13}$$

which is the Lagrangian for the CH equation. Note that the coefficient $1/4$ is directly related to the choice of $g_{\mathcal{C}}$ as cone metric. Using different coefficients in $g_{\mathcal{C}}$ we can obtain the general form of the Lagrangian in equation (1.1).

In order to compute the geodesic equation we consider the following augmented Lagrangian

$$L((\varphi,\lambda),(\dot\varphi,\dot\lambda)) = \int_M (\lambda^2\|\dot\varphi\|_g^2 + \dot\lambda^2)\,\mathrm{d}\rho_0 - \int_M P\,\mathrm{d}(\varphi_\#(\lambda^2\rho_0) - \rho_0)\,, \tag{3.14}$$

where $P : M \to \mathbb{R}$ is the Lagrange multiplier enforcing the constraint. Taking variations we obtain

$$\delta L = \int_M (2\lambda\delta\lambda\|\dot\varphi\|_g^2 + 2\lambda^2 g(\dot\varphi,\delta\dot\varphi) + 2\dot\lambda\delta\dot\lambda)\,\mathrm{d}\rho_0 - \int_M (g(\nabla^g P\circ\varphi,\delta\varphi)\lambda^2 + 2P\circ\varphi\,\lambda\delta\lambda)\,\mathrm{d}\rho_0\,. \tag{3.15}$$

Hence the Euler-Lagrange equations associated with L read as follows

$$\begin{cases} \lambda\ddot\varphi + 2\dot\lambda\dot\varphi + \frac{1}{2}\lambda\nabla^g P\circ\varphi = 0\,,\\ \ddot\lambda - \lambda\|\dot\varphi\|_g^2 + \lambda P\circ\varphi = 0\,, \end{cases} \tag{3.16}$$

which can be expressed in terms of (u,α) via right trivialization, yielding

$$\begin{cases} \dot u + \nabla_u^g u + 2u\alpha = -\frac{1}{2}\nabla^g P\,,\\ \dot\alpha + u\cdot\nabla\alpha + \alpha^2 - \|v\|_g^2 = -P\,. \end{cases} \tag{3.17}$$

Using the relation $\alpha = \mathrm{div}^g(u)/2$, finally gives us the CH equation for u.

Remark 3.1. *Note that in the literature for the CH equation the "pressure field" is sometimes defined in a different way so that, when M is one-dimensional, the first equation in (3.17) can be written as*

$$\partial_t u + u\partial_x u = -\partial_x p\,, \tag{3.18}$$

for an appropriate function p (see, e.g., [19]). Throughout the paper we will instead intend by pressure the Lagrange multiplier P considered above.

4. The generalized CH formulation

In view of the interpretation of the CH equation as geodesic flow on $\mathrm{Aut}_{\rho_0}(\mathcal{C})$, we now turn our attention to the following minimization problem:

Problem 4.1 (Deterministic CH flow problem)**.** *Given a diffemorphism $h \in \mathrm{Diff}(M)$, find a curve $t \in [0,T] \mapsto (\varphi_t,\lambda_t) \in \mathrm{Aut}_{\rho_0}(\mathcal{C})$ satisfying*

$$(\varphi_0,\lambda_0) = (\mathrm{Id},1)\,, \qquad (\varphi_T,\lambda_T) = (h,\sqrt{\mathrm{Jac}(h)})\,, \tag{4.1}$$

and minimizing the action in equation (3.10).

There is a remarkable analogy between this problem and the equivalent version for the incompressible Euler equations. This raises the question of whether we can define generalized solutions for this problem in the same way Brenier did for the Euler case. In this section we address this question by formulating the generalized CH flow problem, proving existence of solutions and discussing their nature. In the following the Lebesgue measure on the base space M is renormalized in such a way that $\rho_0(M) = 1$.

By generalized flow or dynamic plan we mean a probability measure on the space of continuous paths of the cone $\boldsymbol{\mu} \in \mathcal{P}(\Omega)$. This is a generalization for curves on the automorphism group since for any smooth curve $(\varphi,\lambda) : t \in [0,T] \to (\varphi_t,\lambda_t) \in \mathrm{Aut}_{\rho_0}(\mathcal{C})$, we can associate the generalized flow $\boldsymbol{\mu}$ defined by

$$\boldsymbol{\mu} = (\varphi,\lambda)_\#\rho_0\,. \tag{4.2}$$

More explicitly, for any Borel functional $\mathcal{F} : \Omega \to \mathbb{R}$,

$$\int_\Omega \mathcal{F}(\mathrm{z})\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_M \mathcal{F}([\varphi(x),\lambda(x)])\mathrm{d}\rho_0(x)\,, \tag{4.3}$$

where $[\varphi(x), \lambda(x)] : t \in [0,T] \to [\varphi_t(x), \lambda_t(x)] \in \mathcal{C}$.

The condition $(\varphi_t)_\# \lambda_t^2 \rho_0 = \rho_0$ is equivalent to requiring $\lambda : t \in [0,T] \to \lambda_t := \sqrt{\mathrm{Jac}(\varphi_t)} \in C^\infty(M; \mathbb{R}_{>0})$. We want to generalize this condition for arbitrary $\boldsymbol{\mu} \in \mathcal{P}(\Omega)$. Let $e_t : \Omega \to \mathcal{C}$ be the evaluation map at time $t \in [0,T]$. Then, if $\boldsymbol{\mu}$ is defined as in (4.2), we have

$$\mathfrak{h}_t^2(\boldsymbol{\mu}) := (\pi_x)_\# [r^2 (e_t)_\# \boldsymbol{\mu}] = \rho_0 \,. \tag{4.4}$$

In fact, for any $f \in C^0(M)$,

$$\begin{aligned} \int_M f \,\mathrm{d}\mathfrak{h}_t^2(\boldsymbol{\mu}) &= \int_\Omega f(x_t) r_t^2 \,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\ &= \int_\Omega f(x_t) r_t^2 \,\mathrm{d}(\varphi, \lambda)_\# \rho_0 \\ &= \int_M f \circ \varphi_t \lambda_t^2 \,\mathrm{d}\rho_0 \\ &= \int_M f \,\mathrm{d}(\varphi_t)_\# \lambda_t^2 \rho_0 \\ &= \int_M f \,\mathrm{d}\rho_0 \,, \end{aligned} \tag{4.5}$$

where for any path z and any time t, $x_t := \pi_x(z_t)$ and $r_t := \pi_r(z_t)$. By similar calculations, we also obtain

$$(e_0, e_T)_\# \boldsymbol{\mu} = \boldsymbol{\gamma} := [(\varphi_0, \lambda_0), (\varphi_T, \lambda_T)]_\# \rho_0 \,. \tag{4.6}$$

In other words, enforcing the boundary conditions in the generalized setting boils down to constraining a certain marginal of $\boldsymbol{\mu}$ to coincide with a given *coupling plan* $\boldsymbol{\gamma}$ on the cone, i.e. a probability measure in $\mathcal{P}(\mathcal{C} \times \mathcal{C})$.

Consider now the energy functional $\mathcal{A} : \Omega \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ defined by

$$\mathcal{A}(\mathrm{z}) := \begin{cases} \int_0^T \|\dot{z}_t\|_{gc}^2 \,\mathrm{d}t & \text{if } \mathrm{z} \in AC^2([0,T]; \mathcal{C}) \,, \\ +\infty & \text{otherwise} \,. \end{cases} \tag{4.7}$$

Setting $\mathcal{F}(\mathrm{z}) = \mathcal{A}(\mathrm{z})$ in (4.3) we obtain the action for the CH equation expressed in Lagrangian coordinates. This motivates the following definition for the generalized CH flow problem.

Problem 4.2 (Generalized CH flow problem). *Given a coupling plan on the cone $\boldsymbol{\gamma} \in \mathcal{P}_2(\mathcal{C}^2)$, find the dynamic plan $\boldsymbol{\mu} \in \mathcal{P}(\Omega)$ satisfying: the* homogeneous coupling constraint

$$\int_\Omega f(z_0, z_T) \,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_{\mathcal{C}^2} f \,\mathrm{d}\boldsymbol{\gamma} \,, \tag{4.8}$$

for all 2-homogeneous continuous functions $f : \mathcal{C}^2 \to \mathbb{R}$; the homogeneous marginal constraint

$$\int_\Omega \int_0^T f(t, x_t) r_t^2 \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_M \int_0^T f(t,x) \,\mathrm{d}t \,\mathrm{d}\rho_0(x) \quad \forall\, f \in C^0([0,T] \times M) \,; \tag{4.9}$$

and minimizing the action

$$\mathcal{A}(\boldsymbol{\mu}) := \int_\Omega \mathcal{A}(\mathrm{z}) \,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \,. \tag{4.10}$$

We remark three basic facts on this formulation:

- we substituted the constraint in (4.4) by its integral version in equation (4.9) as this form will be easier to manipulate in the following. However, the two formulations are equivalent when restricting to generalized flows with finite action (see lemma 4.3);
- we replaced the strong coupling constraint (4.6) by a weaker version, which is always implied by the former as long as $\boldsymbol{\gamma} \in \mathcal{P}_2(\mathcal{C}^2)$ and in particular when $\boldsymbol{\gamma}$ is deterministic, i.e. when it is induced by a diffeomorphism as in equation (4.6);
- we allow for general coupling plans in $\mathcal{P}_2(\mathcal{C}^2)$ so that the integral on the right-hand side of equation (4.8) is finite. However, we will mostly be interested in the case where the coupling is deterministic.

The first of the points above is made explicit in the following lemma, whose proof is postponed to the appendix.

Lemma 4.3. *For any generalized flow $\boldsymbol{\mu}$ with $\mathcal{A}(\boldsymbol{\mu}) < +\infty$ and satisfying the homogeneous coupling constraint in equation (4.8), the homogeneous marginal constraint in equation (4.9) is equivalent to the constraint*

$$\mathfrak{h}_t^2(\boldsymbol{\mu}) = \rho_0 \tag{4.11}$$

for all $t \in [0, T]$.

The main result of this section is contained in the following proposition, which states that generalized CH flows are well-defined as solutions of problem (4.2).

Proposition 4.4 (Existence of minimizers)**.** *Provided that there exists a dynamic plan $\boldsymbol{\mu}^*$ such that $\mathcal{A}(\boldsymbol{\mu}^*) < +\infty$, the minimum of the action in problem 4.2 is attained.*

Before providing the proof of proposition 4.4, we introduce a useful rescaling operation which will allow us to preserve the homogenous constraint when passing to the limit using sequences of narrowly convergent dynamic plans. Such an operation was introduced in [24] in order to deal with the analogous problem arising from the formulation of entropic optimal transport on the cone. Adapting the notation in [24] to our setting, we define for a functional $\theta : \Omega \to \mathbb{R}$,

$$\mathrm{prod}_\theta(\mathrm{z}) \coloneqq (t \in [0, T] \mapsto [x_t, r_t/\theta(\mathrm{z})]) \,. \tag{4.12}$$

Then, given a dynamic plan $\boldsymbol{\mu}$, if $\theta(\mathrm{z}) > 0$ for $\boldsymbol{\mu}$-almost any path z, we can define the dilation map

$$\mathrm{dil}_{\theta,2}(\boldsymbol{\mu}) \coloneqq \mathrm{prod}_{\theta\#}(\theta^2 \boldsymbol{\mu}) \,. \tag{4.13}$$

Since the constraint in equation (4.9) is 2-homogeneous in the radial coordinate r, it is invariant under the dilation map, meaning that if $\boldsymbol{\mu}$ satisfies (4.9), also $\mathrm{dil}_{\theta,2}(\boldsymbol{\mu})$ does. For the same reason, we also have

$$\mathcal{A}(\mathrm{dil}_{\theta,2}(\boldsymbol{\mu})) = \mathcal{A}(\boldsymbol{\mu}) \,. \tag{4.14}$$

The map $\mathrm{dil}_{\theta,2}$ performs a *rescaling* on the measure $\boldsymbol{\mu}$ in the sense specified by the following lemma.

Lemma 4.5. *Given a measure $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ and a 1-homogeneous functional $\sigma : \Omega \to \mathbb{R}$ such that $\sigma(\mathrm{z}) > 0$ for $\boldsymbol{\mu}$-almost every path z, suppose that*

$$C \coloneqq \left(\int_\Omega (\sigma(\mathrm{z}))^2 \, \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \right)^{1/2} < +\infty \,; \tag{4.15}$$

if $\tilde{\boldsymbol{\mu}} = \mathrm{dil}_{\sigma/C,2}(\boldsymbol{\mu})$ then $\tilde{\boldsymbol{\mu}}(\Omega) = 1$ and

$$\tilde{\boldsymbol{\mu}}(\{\mathrm{z} \in \Omega \,;\, \sigma(\mathrm{z}) = C\}) = 1 \,. \tag{4.16}$$

Proof. We prove this by direct calculation. Let $\theta \coloneqq \sigma/C$. By 1-homogeneity of σ, for $\boldsymbol{\mu}$-almost every path z

$$\sigma(\mathrm{prod}_\theta(\mathrm{z})) = \frac{\sigma(\mathrm{z})}{|\theta(\mathrm{z})|} = C \,. \tag{4.17}$$

Then,

$$\begin{aligned} \int_{\{\mathrm{z}\in\Omega\,;\,\sigma(\mathrm{z})=C\}} \mathrm{d}\tilde{\boldsymbol{\mu}}(\mathrm{z}) &= \int_{\{\mathrm{z}\in\Omega\,;\,\sigma(\mathrm{z})=C\}} \mathrm{d}\mathrm{prod}_{\theta\#}(\theta^2\boldsymbol{\mu})(\mathrm{z}) \\ &= \int_{\{\mathrm{z}\in\Omega\,;\,\sigma(\mathrm{prod}_\theta(\mathrm{z}))=C\}} \theta^2 \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\ &= \frac{1}{C^2} \int_\Omega (\sigma(\mathrm{z}))^2 \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = 1 \,. \end{aligned} \tag{4.18}$$

By similar calculations we also have $\tilde{\boldsymbol{\mu}}(\Omega) = 1$. □

Besides the rescaling operator and lemma 4.5, we will also need the following result which will allow us to construct suitable minimizers of the action in problem 4.2.

Lemma 4.6. *The set of measures with uniformly bounded action $\mathcal{A}(\boldsymbol{\mu}) \leq C$ and satisfying the homogeneous constraint in equation (4.9) is relatively sequentially compact for the narrow topology.*

Proof. Due to Therorem 2.1, it is sufficient to prove that sequences of admissible measures are tight. For a given path z with $\mathcal{A}(\mathrm{z}) \leq K$, for all $0 \leq s \leq t \leq T$,

$$d_{\mathcal{C}}(z_s, z_t) \leq \int_s^t \|\dot{z}_{t^*}\|_{g_{\mathcal{C}}} \, \mathrm{d}t^* \leq K^{1/2} |t-s|^{1/2} \,, \tag{4.19}$$

which implies that level sets of $\mathcal{A}(\mathrm{z})$ are equicontinuous. Consider now the set

$$\Omega_R \coloneqq \Omega(\mathcal{C}_R) = \{\mathrm{z} \in \Omega \,;\, \forall\, t \in [0,T] \,,\, r_t \leq R\} \,; \tag{4.20}$$

For any $K > 0$, the set $\{\mathrm{z} \in \Omega_R \,;\, \mathcal{A}(\mathrm{z}) \leq K\}$ is also equicontinuous; moreover, since paths in this set are bounded at any time, it is contained in a compact subset of Ω, by the Ascoli-Arzelà theorem.

In order to use such sets to prove tightness we need to be able to control the measure of $\Omega \setminus \Omega_R$. In particular, we now show that there exists a constant $C' > 0$ such that

$$\boldsymbol{\mu}(\Omega \setminus \Omega_R) \leq \frac{C'}{R^2} \,. \tag{4.21}$$

Let us fix a $t^* \in (0,T)$, $\epsilon > 0$ and an interval $I_\epsilon = (t^* - \epsilon/2, t^* + \epsilon/2) \subset [0,T]$. Moreover, consider the following set of paths

$$\{\mathrm{z} \in \Omega \,;\, \forall\, t \in I_\epsilon \,,\, r_t > R\} \,. \tag{4.22}$$

Then integrating the constraint in equation (4.9) over such a set with f being any continuous function such that $f(t,\cdot) = 1$ for $t \in I_\epsilon$, $f(t,\cdot) = 0$ for $t \in [0,T] \setminus I_{2\epsilon}$ and $0 \leq f \leq 1$, we obtain

$$\boldsymbol{\mu}(\{\mathrm{z} \in \Omega \,;\, \forall\, t \in I_\epsilon \,,\, r_t > R\}) \leq \frac{2}{R^2} \,. \tag{4.23}$$

Since the estimate is uniform in ϵ this means that

$$\boldsymbol{\mu}(\{\mathrm{z} \in \Omega \,;\, r_{t^*} > R\}) \leq \frac{2}{R^2} \,. \tag{4.24}$$

Now, consider the set $\mathcal{A}(\mathrm{z}) < Q$ because of equation (4.19) we have

$$|r_t - r_s| \leq Q^{1/2} |t-s|^{1/2} \,. \tag{4.25}$$

This implies that if Q is sufficiently small

$$\{\mathrm{z} \in \Omega \setminus \Omega_R \,; \mathcal{A}(\mathrm{z}) < Q\} \subseteq \{\mathrm{z} \in \Omega \,;\, r_{T/2} > R/2\} \,. \tag{4.26}$$

More precisely this holds for

$$Q^{1/2} \left| \frac{T}{2} \right|^{1/2} < \frac{R}{2} \,, \tag{4.27}$$

and hence for $Q < R^2/(2T)$. Therefore, if $Q < R^2/(2T)$,

$$\begin{aligned} \boldsymbol{\mu}(\Omega \setminus \Omega_R) &\leq \boldsymbol{\mu}((\Omega \setminus \Omega_R) \cap \{\mathrm{z} \,;\, \mathcal{A}(\mathrm{z}) < Q\}) + \boldsymbol{\mu}(\{\mathrm{z} \,;\, \mathcal{A}(\mathrm{z}) \geq Q\}) \\ &\leq \boldsymbol{\mu}(\{\mathrm{z} \in \Omega \,;\, r_{T/2} > R/2\}) + \frac{C}{Q} \\ &\leq \frac{8}{R^2} + \frac{C}{Q} \,. \end{aligned} \tag{4.28}$$

Taking $Q = R^2/(4T)$, we deduce that

$$\boldsymbol{\mu}(\Omega \setminus \Omega_R) \leq \frac{4(CT+2)}{R^2} \,, \tag{4.29}$$

which proves equation (4.21).

Recall that $\{\mathrm{z} \in \Omega_R \,;\, \mathcal{A}(\mathrm{z}) \leq K\}$ is contained in a compact set for any $K > 0$ and $R > 0$. For any $\epsilon > 0$, set $R = (8(CT+1)/\epsilon)^{1/2}$. For any admissible $\boldsymbol{\mu}$, we have

$$\begin{aligned} \boldsymbol{\mu}(\Omega \backslash \{\mathrm{z} \in \Omega_R \,;\, \mathcal{A}(\mathrm{z}) \leq 2C\epsilon^{-1}\}) &\leq \boldsymbol{\mu}(\Omega \backslash \{\mathrm{z} \,; \mathcal{A}(\mathrm{z}) \leq 2C\epsilon^{-1}\}) + \boldsymbol{\mu}(\Omega \backslash \Omega_R) \\ &\leq \frac{\epsilon}{2C} \int_\Omega \mathcal{A}(\mathrm{z}) \, \mathrm{d}\boldsymbol{\mu}(\mathrm{z}) + \frac{\epsilon}{2} \leq \epsilon \,, \end{aligned} \tag{4.30}$$

which proves tightness. □

We are now ready to prove existence of optimal solutions for the generalized CH problem.

Proof of proposition 4.4. The functional $\mathcal{A}(\mathrm{z})$ is lower semi-continuous; hence so is $\mathcal{A}(\boldsymbol{\mu})$. Consider a minimizing sequence $\boldsymbol{\mu}_n$ with $n \in \mathbb{N}$. By assumption we can take $\mathcal{A}(\boldsymbol{\mu}_n) \leq C$ for all $n \in \mathbb{N}$. Let $\mathrm{o} : t \in [0,T] \to o \in \mathcal{C}$ the path on the cone assigning to every time the apex of the cone o. Let $\boldsymbol{\mu}_n^{\mathrm{o}} := \boldsymbol{\mu}_n \llcorner \Omega^{\mathrm{o}} \in \mathcal{M}(\Omega)$ the restriction of $\boldsymbol{\mu}_n$ to $\Omega^{\mathrm{o}} := \Omega \setminus \{\mathrm{o}\}$. Such an operation preserves both the action and the constraints.

Let $\sigma : \Omega \to \mathbb{R}$ be the 1-homogeneous functional defined by

$$\sigma(\mathrm{z}) := \left(r_0^2 + r_T^2 + \int_0^T r_t^2 \,\mathrm{d}t \right)^{1/2} . \tag{4.31}$$

For any $\boldsymbol{\mu}_n^{\mathrm{o}}$ in the sequence, we obviously have that $\sigma(\mathrm{z}) > 0$ for $\boldsymbol{\mu}_n^{\mathrm{o}}$-almost every path. Moreover, since $\boldsymbol{\mu}_n^{\mathrm{o}}$ satisfies both the homogeneous marginal and coupling constraint, for all $n \in \mathbb{N}$,

$$\int_\Omega \sigma(\mathrm{z})^2 \,\mathrm{d}\boldsymbol{\mu}_n(\mathrm{z}) = T + 2 \,. \tag{4.32}$$

Hence we can apply lemma 4.5 and define a sequence $\tilde{\boldsymbol{\mu}}_n \in \mathcal{P}(\Omega)$ by $\tilde{\boldsymbol{\mu}}_n := \mathrm{dil}_{\sigma/T,2} \boldsymbol{\mu}_n^{\mathrm{o}}$. In particular, for all $n \in \mathbb{N}$, $\tilde{\boldsymbol{\mu}}_n$ is concentrated on the set of paths such that $\sigma(\mathrm{z}) = \sqrt{T+2}$, i.e.

$$\tilde{\boldsymbol{\mu}}_n \left(\left\{ \mathrm{z} \in \Omega \,;\, r_0^2 + r_T^2 + \int_0^T r_t^2 \,\mathrm{d}t = T + 2 \right\} \right) = 1 \,. \tag{4.33}$$

Moreover, $\tilde{\boldsymbol{\mu}}_n$ satisfies the homogenous constraint and the coupling constraint, since these are both 2-homogeneous in the radial direction, and for the same reason $\mathcal{A}(\tilde{\boldsymbol{\mu}}_n) = \mathcal{A}(\boldsymbol{\mu}_n) \leq C$. This is enough to apply lemma 4.6; thus, we can extract a subsequence $(\tilde{\boldsymbol{\mu}}_n)_n \rightharpoonup \tilde{\boldsymbol{\mu}}_\infty \in \mathcal{P}(\Omega)$.

We now show that for any $f \in C^0([0,T] \times M)$ the functional

$$\mathcal{F}(\mathrm{z}) := \int_0^T f(t, x_t) r_t^2 \,\mathrm{d}t \tag{4.34}$$

is uniformly integrable with respect to the sequence $(\tilde{\boldsymbol{\mu}}_n)_n$, that is, for any $\epsilon > 0$ there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$

$$\int_{\Omega, \mathcal{F}(\mathrm{z}) > K} \mathcal{F}(\mathrm{z}) \,\mathrm{d}(\tilde{\boldsymbol{\mu}}_n)_n(\mathrm{z}) < \epsilon \,. \tag{4.35}$$

It is sufficient to consider the case $\|f\|_{C^0} = 1$, because the case $\|f\|_{C^0} = 0$ is trivial and otherwise we can always rescale the functional by dividing it by $\|f\|_{C^0}$. Recall the definition of the functional σ in equation (4.31); we have

$$\int_{\Omega, \mathcal{F}(\mathrm{z}) > K} \mathcal{F}(\mathrm{z}) \,\mathrm{d}(\tilde{\boldsymbol{\mu}}_n)_n(\mathrm{z}) \leq \|f\|_{C^0} \int_{\Omega, \sigma(\mathrm{z})^2 > K} \sigma(\mathrm{z})^2 \mathrm{d}(\tilde{\boldsymbol{\mu}}_n)_n(\mathrm{z}) \,. \tag{4.36}$$

However, by equation (4.33) the right-hand side is zero if $K > T + 2$, which proves uniform integrability. Hence, using lemma 2.2, we deduce that $\tilde{\boldsymbol{\mu}}_\infty$ satisfies the homogeneous marginal constraint. Similarly, we can deduce that $\tilde{\boldsymbol{\mu}}_\infty$ also satisfies the homogeneous coupling constraint since $(e_0, e_T)_\#(\tilde{\boldsymbol{\mu}}_n)_n$ is concentrated on $\mathcal{C}_R^2$ with $R = \sqrt{T+2}$; hence it is an optimal solution of problem 4.2. □

Corollary 4.7. *Suppose that* $h \in \mathrm{Diff}(M)$ *is in the connected component containing* Id. *Then, if* $\boldsymbol{\gamma} = [(\mathrm{Id}, 1), (h, \sqrt{\mathrm{Jac}(h)})]_\# \rho_0$*, the minimum of the action in problem 4.2 is attained.*

In general, we cannot ensure that there exists a minimizer $\boldsymbol{\mu}$ of problem 4.2 satisfying the *strong coupling constraint*:

$$(e_0, e_T)_\# \boldsymbol{\mu} = \boldsymbol{\gamma} \,. \tag{4.37}$$

However, we have the following characterization for the existence of such minimizers when $\boldsymbol{\gamma}$ is deterministic:

Proposition 4.8 (Existence of minimizers satisfying the strong coupling constraint)**.** *Suppose that* $\boldsymbol{\gamma} = [(\mathrm{Id}, 1), (h, \sqrt{\mathrm{Jac}(h)})]_\# \rho_0$*. Then, problem 4.2 admits a minimizer satisfying the strong coupling constraint if and only if there exists a minimizer* $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ *(hence not necessarily a probability measure) such that*

$$\boldsymbol{\mu}(\{\mathrm{z} \in \Omega \,;\, r_0 = 0\}) = 0 \,. \tag{4.38}$$

Moreover, for any $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ satisfying the homogeneous coupling constraint, equation (4.38) is equivalent to

$$\boldsymbol{\mu}(\{\mathrm{z} \in \Omega\,;\, r_0 = r_T = 0\}) = 0\,. \tag{4.39}$$

Proof. For the first part of the proposition, we only need to prove that the condition (4.38) implies the existence of a minimizer satisfying the strong coupling constraint. To show this, assume that equation (4.38) holds and consider the 1-homogeneous functional $\tilde{\sigma}(\mathrm{z}) : \Omega \to \mathbb{R}$ defined by $\tilde{\sigma}(\mathrm{z}) = r_0$. Clearly $\tilde{\sigma}(\mathrm{z}) > 0$ for $\boldsymbol{\mu}$-almost every path z. Moreover, we have

$$\int_\Omega (\tilde{\sigma}(\mathrm{z}))^2 \,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_\Omega r_0^2 \,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = 1\,. \tag{4.40}$$

Hence, by lemma 4.5, the measure $\boldsymbol{\mu}^1 := \mathrm{dil}_{r_0,2}\boldsymbol{\mu} \in \mathcal{P}(\Omega)$ is concentrated on paths such that $r_0 = 1$. Then for any $g \in C^0(M^2)$ we can take $f = gr_0^2$ in equation (4.8) yielding

$$\int_\Omega g(x_0, x_T) \,\mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) = \int_M g(x, h(x)) \,\mathrm{d}\rho_0(x)\,. \tag{4.41}$$

Similarly, letting $\zeta := \sqrt{\mathrm{Jac}(h)}$,

$$\begin{aligned}\int_\Omega (r_T - \zeta(x_0))^2 \,\mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) &= \int_\Omega (r_T^2 + \zeta(x_0)^2 - 2\zeta(x_0) r_T) \,\mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) \\ &= \int_\Omega (r_T^2 + r_0^2\zeta(x_0)^2 - 2\zeta(x_0) r_0 r_T) \,\mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) \\ &= 2\int_M \zeta(x)^2 \mathrm{d}\rho_0(x) - 2\int_M \zeta(x)^2 \mathrm{d}\rho_0(x) = 0\,,\end{aligned} \tag{4.42}$$

which means that for $\boldsymbol{\mu}$-almost every path $r_T = \zeta(x_0)$. Then, for any continous bounded function $f : \mathcal{C}^2 \to \mathbb{R}$, we have

$$\begin{aligned}\int_\Omega f(z_0, z_T) \,\mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) &= \int_\Omega f([x_0, 1], [x_T, \zeta(x_0)]) \,\mathrm{d}\boldsymbol{\mu}^1(\mathrm{z}) \\ &= \int_M f([x, 1], [\varphi(x), \zeta(x)]) \,\mathrm{d}\rho_0(x)\,,\end{aligned} \tag{4.43}$$

which proves the first part of the proposition.

For the second part, let $\boldsymbol{\mu} \in \mathcal{M}(\Omega)$ be any dynamic plan satisfying the homogeneous coupling constraint. We decompose $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}}^0$ where

$$\tilde{\boldsymbol{\mu}} := \boldsymbol{\mu} \llcorner \{\mathrm{z} \in \Omega\,;\, r_0 \neq 0\}\,, \qquad \tilde{\boldsymbol{\mu}}^0 := \boldsymbol{\mu} \llcorner \{\mathrm{z} \in \Omega\,;\, r_0 = 0\}\,. \tag{4.44}$$

Now, if we define $\tilde{\boldsymbol{\mu}}^1 := \mathrm{dil}_{r_0,2}\tilde{\boldsymbol{\mu}}$, then $\tilde{\boldsymbol{\mu}}^0 + \tilde{\boldsymbol{\mu}}^1$ still satisfies the homogeneous coupling constraint and moreover, for any $\alpha \in [0, 2)$,

$$\begin{aligned}\int_\Omega r_T^\alpha \,\mathrm{d}\tilde{\boldsymbol{\mu}}^1(\mathrm{z}) &= \int_\Omega r_0^{2-\alpha} r_T^\alpha \,\mathrm{d}\tilde{\boldsymbol{\mu}}^1(\mathrm{z}) \\ &= \int_\Omega r_0^{2-\alpha} r_T^\alpha \,\mathrm{d}(\tilde{\boldsymbol{\mu}}^0 + \tilde{\boldsymbol{\mu}}^1)(\mathrm{z}) \\ &= \int_M \zeta^\alpha \,\mathrm{d}\rho_0\,.\end{aligned} \tag{4.45}$$

Taking the limit for $\alpha \to 2$, by the dominated convergence theorem,

$$\int_\Omega r_T^2 \,\mathrm{d}\tilde{\boldsymbol{\mu}}^1(\mathrm{z}) = \int_M \zeta^2 \,\mathrm{d}\rho_0 = 1\,. \tag{4.46}$$

In turn, this implies that

$$\int_\Omega r_T^2 \,\mathrm{d}\tilde{\boldsymbol{\mu}}^0(\mathrm{z}) = 0\,, \tag{4.47}$$

which means that $\tilde{\boldsymbol{\mu}}^0$-almost every path z has $r_T = 0$. □

The proofs of proposition 4.4 and 4.8 give us several insights on the nature of the generalized solutions of the CH variational problem. First of all, it is evident that such solutions can only be unique up to rescaling. In fact, since all constraints are homogenous and preserved by rescaling, given one minimizer one can generate others using the dilation map as in lemma 4.5. In addition, if the coupling is deterministic, even using rescaling, in principle one might not be able to find a minimizer satisfying the coupling constraint in the classical sense. By proposition 4.8, this happens if all minimizers charge paths which start and end at the apex of the cone. In this case the optimal solutions use the reservoir of mass in the apex to enforce the homogeneous marginal constraint on some time interval contained in $(0,T)$. Then, the procedure in the proof of proposition 4.8 cannot produce a minimizer satisfying the strong coupling constraint. However, it can still help us visualize such solutions. In fact, given any minimizer $\boldsymbol{\mu}$ we can consider the measure $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \llcorner \{\mathrm{z} \in \Omega\,;\, r_0 \neq 0\}$ and use this as in proposition 4.8 to generate a measure $\tilde{\boldsymbol{\mu}}^1 = \mathrm{dil}_{r_0,2}\tilde{\boldsymbol{\mu}}$ which satisfies the strong coupling constraint but not necessarily the homogeneous marginal constraint. The lack of mass in the homogeneous marginals can be seen as a generalization of the occurrence of non-injective Lagrangian flow maps in the solution, which in turn correspond to the breakdown of classical (weak) solutions.

5. Existence and uniqueness of the pressure

In the previous section, we proved existence of minimizers of the generalized CH problem. In general, given that all constraints are homogeneous, such minimizers are only defined up to rescaling. However, even using rescaling, it might not always be possible to find a minimizer that satisfies the strong coupling constraint. Here, we show that independently of this, the pressure field P in the CH equation (3.17) is uniquely defined as a distribution for any given deterministic coupling constraint. This reproduces a similar result proved by Brenier for the incompressible Euler case [8].

The idea is to extend the set of admissible generalized flows in order to define appropriate variations of the action. By analogy to the Euler case, we consider dynamic plans whose homogeneous marginals are not the Lebesgue measure ρ_0, but are sufficiently close to it. Given a dynamic plan $\boldsymbol{\nu} \in \mathcal{P}(\Omega)$ we denote by $\rho^{\boldsymbol{\nu}} : [0,T] \times M \to \mathbb{R}$ the function defined by

$$\rho^{\boldsymbol{\nu}}(t,\cdot) := \frac{\mathrm{d}\mathfrak{h}_t^2 \boldsymbol{\nu}}{\mathrm{d}\rho_0}\,, \tag{5.1}$$

for any $t \in [0,T]$. For an admissible generalized flow $\boldsymbol{\nu}$, $\rho^{\boldsymbol{\nu}} = 1$. Dynamic plans $\boldsymbol{\nu}$ with $\rho^{\boldsymbol{\nu}} \neq 1$ correspond to generalized automorphisms of the cone with a mismatch between the radial variable and the Jacobian of the flow map on the base space.

Definition 5.1 (Almost diffeomorphisms)**.** A generalized almost diffeomorphism is a probability measure $\boldsymbol{\nu} \in \mathcal{P}(\Omega)$ such that $\rho^{\boldsymbol{\nu}} \in C^{0,1}([0,T] \times M)$ and

$$\|\rho^{\boldsymbol{\nu}} - 1\|_{C^{0,1}([0,T]\times M)} \leq \frac{1}{2}\,. \tag{5.2}$$

For any $\rho \in C^{0,1}([0,T] \times M)$ with $\rho > 0$, let $\Phi^\rho : \Omega \to \Omega$ be the map defined by

$$\Phi^\rho(\mathrm{z}) := (t \in [0,T] \mapsto [x_t, r_t\sqrt{\rho(t,x_t)}] \in \mathcal{C})\,. \tag{5.3}$$

We use this map in the following proposition, which is the equivalent of proposition 2.1 in [8] and justifies our choice for the space of densities in definition 5.1.

Proposition 5.2. *Fix a* $\rho \in C^{0,1}([0,T] \times M)$ *such that*

$$\|\rho - 1\|_{C^{0,1}} \leq \frac{1}{2}\,, \qquad \rho(0,\cdot) = \rho(1,\cdot) = 1\,. \tag{5.4}$$

Then, given any dynamic plan $\boldsymbol{\mu} \in \mathcal{P}(\Omega)$ *with finite action* $\mathcal{A}(\boldsymbol{\mu}) < +\infty$*, satisfying the homogeneous constraint in equation* (4.9)*, i.e.* $\rho^{\boldsymbol{\mu}} = \rho_0$*, and the coupling constraint* (4.8)*, the dynamic plan* $\boldsymbol{\nu} := \Phi^\rho_\# \boldsymbol{\mu} \in \mathcal{P}(\Omega)$ *still satisfies the coupling constraint and we have* $\rho^{\boldsymbol{\nu}} = \rho$*; moreover,*

$$\mathcal{A}(\boldsymbol{\nu}) \leq \mathcal{A}(\boldsymbol{\mu}) + \|\rho - 1\|_{C^{0,1}}\mathcal{A}(\boldsymbol{\mu}) + |\rho - 1|^2_{C^{0,1}}(T + \mathcal{A}(\boldsymbol{\mu}))\,. \tag{5.5}$$

Proof. The fact that $\rho^{\boldsymbol{\nu}} = \rho$ and that $\boldsymbol{\nu}$ satisfies the coupling constraint follows from direct computation. As for equation (5.5), observe that

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{\nu}) &= \int_\Omega \int_0^T \mathcal{A}(\Phi^\rho(\mathrm{z}))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&= \int_\Omega \int_0^T \rho(t,x_t)\|\dot z_t\|^2_{g_\mathcal{C}} + r_t\dot r_t\partial_t(\rho(t,x_t)) + r_t^2(\partial_t\sqrt{\rho(t,x_t)})^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&\le \|\rho\|_{C^0}\mathcal{A}(\boldsymbol{\mu}) + \int_\Omega \int_0^T r_t\dot r_t\partial_t(\rho(t,x_t)) + r_t^2(\partial_t\sqrt{\rho(t,x_t)})^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z})\,.
\end{aligned}
\tag{5.6}
$$

Moreover,

$$
\begin{aligned}
\int_\Omega \int_0^T r_t\dot r_t\partial_t(\rho(t,x_t))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) &\le |\rho-1|_{C^{0,1}}\int_\Omega \int_0^T r_t\dot r_t(1+\|\dot x_t\|_g)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&\le \frac{1}{2}|\rho-1|_{C^{0,1}}\mathcal{A}(\boldsymbol{\mu})\,,
\end{aligned}
\tag{5.7}
$$

and similarly, since $\rho \ge 1/2$,

$$
\begin{aligned}
\int_\Omega \int_0^T r_t^2(\partial_t\sqrt{\rho(t,x_t)})^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) &\le \frac{1}{2}\int_\Omega \int_0^T r_t^2(\partial_t(\rho(t,x_t)))^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&\le \frac{1}{2}|\rho-1|^2_{C^{0,1}}\int_\Omega \int_0^T r_t^2(1+\|\dot x_t\|_g)^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&\le |\rho-1|^2_{C^{0,1}}(T+\mathcal{A}(\boldsymbol{\mu}))\,.
\end{aligned}
\tag{5.8}
$$

Reinserting these estimates into equation (5.6) we obtain (5.5). ☐

Consider now the following space

$$
B_0 \coloneqq \{\rho \in C^{0,1}([0,T]\times M)\,;\, \rho(0,\cdot) = \rho(1,\cdot) = 0\}\,,
\tag{5.9}
$$

which we regard as a Banach space with the $C^{0,1}$ norm. The following theorem shows that we can define the pressure as an element $p \in B_0^*$ and it is the analogue of Theorem 6.2 in [2].

Theorem 5.3. *Let $\boldsymbol{\mu}^*$ be a minimizer for the generalized CH problem such that $\mathcal{A}(\boldsymbol{\mu}^*) < +\infty$. Then there exists $p \in B_0^*$ such that*

$$
\langle p, \rho^{\boldsymbol{\nu}} - 1\rangle \le \mathcal{A}(\boldsymbol{\nu}) - \mathcal{A}(\boldsymbol{\mu}^*)\,,
\tag{5.10}
$$

for all generalized almost diffeomorphisms $\boldsymbol{\nu}$ satisfying the coupling constraint (4.8).

Proof. First of all, observe that for any generalized almost diffeomorphism $\boldsymbol{\nu}$ satisfying the coupling constraint,

$$
\rho^{\boldsymbol{\nu}}(0,\cdot) = \rho^{\boldsymbol{\nu}}(1,\cdot) = 1\,;
\tag{5.11}
$$

hence $\rho^{\boldsymbol{\nu}} - 1 \in B_0$ and the pairing in equation (5.10) is well defined. Now, consider the convex set $C \coloneqq \{\rho \in B_0; \|\rho\|_{C^{0,1}} \le \frac{1}{2}\}$ and the functional $\phi : B_0 \to \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$
\phi(\rho) \coloneqq \begin{cases} \inf\{\mathcal{A}(\boldsymbol{\nu})\,;\, \rho^{\boldsymbol{\nu}} = \rho + 1 \text{ and (4.8) holds}\} & \text{if } \rho \in C\,, \\ +\infty & \text{otherwise}\,. \end{cases}
\tag{5.12}
$$

We observe that $\phi(0) = \mathcal{A}(\boldsymbol{\mu}^*) < +\infty$ and so ϕ is a proper convex function. We prove that it is bounded in a neighbourhood of $\rho = 0$. By proposition 5.2, for any $\rho \in C$ there exists a $\boldsymbol{\nu} \in \mathcal{P}(\Omega)$ satisfying $\rho^{\boldsymbol{\nu}} = \rho + 1$ and the coupling constraint, such that

$$
\mathcal{A}(\boldsymbol{\nu}) \le \mathcal{A}(\boldsymbol{\mu}^*) + \|\rho\|_{C^{0,1}}\mathcal{A}(\boldsymbol{\mu}^*) + |\rho|^2_{C^{0,1}}(T + \mathcal{A}(\boldsymbol{\mu}^*))\,,
\tag{5.13}
$$

which implies

$$
\phi(\rho) \le \phi(1) + \|\rho\|_{C^{0,1}}\mathcal{A}(\boldsymbol{\mu}^*) + |\rho|^2_{C^{0,1}}(T + \mathcal{A}(\boldsymbol{\mu}^*))\,.
\tag{5.14}
$$

Therefore, ϕ is bounded in a neighbourhood of $\rho = 0$. As a consequence, by standard convex analysis arguments, ϕ is also locally Lipschitz on the same neighborhood and the subdifferential of ϕ at 0 is not empty, i.e. there exists $p \in B_0^*$ such that

$$
\langle p, \rho\rangle \le \phi(\rho) - \phi(0)\,.
\tag{5.15}
$$

By the definition of ϕ, this implies

$$\langle p, \rho \rangle \leq \mathcal{A}(\boldsymbol{\nu}) - \mathcal{A}(\boldsymbol{\mu}^*) \,, \tag{5.16}$$

for all generalized almost diffeomorphisms $\boldsymbol{\nu}$ satisfying $\rho^{\boldsymbol{\nu}} = \rho + 1$ and the coupling constraint in (4.8). □

Theorem 5.3 tells us that $\boldsymbol{\mu}^*$ is also a minimizer for the augmented action

$$\mathcal{A}^p(\boldsymbol{\nu}) \coloneqq \mathcal{A}(\boldsymbol{\nu}) - \langle p, \rho^{\boldsymbol{\nu}} - 1 \rangle \,, \tag{5.17}$$

defined on generalized almost diffeomorphisms. Then, for any $\rho \in B_0$, $\boldsymbol{\mu}^*_\epsilon \coloneqq \Phi^{1+\epsilon\rho}_{\#} \boldsymbol{\mu}^*$ is a generalized almost diffeomorphism if ϵ is sufficiently small. Moreover, we must have

$$\left. \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathcal{A}(\boldsymbol{\mu}^*_\epsilon) \right|_{\epsilon=0} = 0 \,. \tag{5.18}$$

By the same calculation as in the proof of proposition 5.2, this implies

$$\langle p, \rho \rangle = \int_\Omega \int_0^T \rho(t, x_t) \|\dot{z}_t\|^2_{gc} + \partial_t(\rho(t, x_t)) r_t \dot{r}_t \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\mu}^*(\mathrm{z}) \,, \tag{5.19}$$

for any $\rho \in B_0$, which defines p uniquely as a distribution. This also implies that the functional ϕ is actually differentiable at 0 since its subdifferential reduces to a single element.

The existence of a unique pressure for generalized CH flows is a natural extension of a similar surprising result discovered by Brenier for incompressible Euler. In fact, it can be regarded as the second instance of the appearence of a recurring behavior in the solutions of a variational fluid model. It should also be noted that in our case, we explicitly used the cone structure to construct appropriate variations of the Lagrangian which simplified the proof if compared to the incompressible Euler case.

6. Correspondence with deterministic solutions

In this section we study the correspondence between generalized and classical solutions of the CH equation. In particular, we show that for sufficiently short times classical solutions generate dynamic plans which are the unique minimizers of problem 4.2. There are two main complications that arise in this context. One is due to the singularity of the cone geometry and the other to the homogeneity of the coupling constraint. The first will imply an additional bound on the time T for which the correspondence holds. The second will intervene in the proof of uniqueness and it will be addressed by using the characterization of minimizers in proposition 4.8.

We start by proving a modified version of a result presented in [14] stating that smooth solutions of the CH equations are length minimizing for short times in an L^∞ neighborhood on $\mathrm{Aut}_{\rho_0}(\mathcal{C})$. Let (φ, λ) be a smooth solution of the system (3.16) on the interval $[0, T]$. Let P be the associated pressure and $\Psi_p(t, x, r) \coloneqq P(t, x) r^2$. Following [7] we introduce the following functional on Ω,

$$\mathcal{B}(\mathrm{z}) \coloneqq \begin{cases} \int_0^T \|\dot{z}_t\|^2_{gc} - \Psi_p(t, x_t, r_t) \, \mathrm{d}t & \text{if } \mathrm{z} \in AC^2([0,T]; \mathcal{C}) \,, \\ +\infty & \text{otherwise} \,. \end{cases} \tag{6.1}$$

Moreover, we consider the function $b : \mathcal{C}^2 \to \mathbb{R}$ defined by

$$b(p, q) \coloneqq \inf\{\mathcal{B}(\mathrm{z}) \,;\, z_0 = p \,,\, z_T = q\} \,. \tag{6.2}$$

Lemma 6.1. *Let $M \subset \mathbb{R}^d$ be a convex domain and let (φ, λ) be a smooth solution of (3.16) on $[0, T] \times M$, with P being the associated pressure and $\Psi_p(t, x, r) \coloneqq P(t, x) r^2$. For any fixed $x \in M$, let $\mathrm{z}^* = [\mathrm{x}^*, \mathrm{r}^*] \in \Omega$ be the curve defined by $\mathrm{x}^* : t \to x^*_t \coloneqq \varphi_t(x)$ and $\mathrm{r}^* : t \to r^*_t \coloneqq \lambda_t(x)$. Let $r_{min} \coloneqq \inf_{(t,x) \in [0,T] \times M} \lambda_t(x)$, $r_{max} \coloneqq \sup_{(t,x) \in [0,T] \times M} \lambda_t(x)$ and $\varrho \coloneqq 2 r_{max}/r_{min}$. There exists a constant $C_0 > 0$ such that, if*

- *for all $t \in [0, T]$ and for all $w \in T_{z^*_t}\mathcal{C}$,*

$$|\mathrm{Hess}^{gc} \, \Psi_p(w, w)| < \frac{C_0 \pi^2}{T^2} \|w\|^2_{gc} \,; \tag{6.3}$$

- *for all $t_0, t_1 \in [0, T]$,*

$$d_{\mathcal{C}}(z_{t_0}, z_{t_1}) \leq \frac{r_{min}}{4} \,; \tag{6.4}$$

- *the following inequality holds:*

$$\left[\varrho^2 + (\varrho+1)^2\right] \|P\|_{C^0} < \frac{3}{2T^2}\,; \tag{6.5}$$

then, $\mathcal{B}(\mathrm{z}^*) = b(z_0^*, z_T^*)$*; moreover, for any other* $\mathrm{z} \in AC^2([0,T];\mathcal{C})$ *such that* $z_0 = z_0^*$ *and* $z_T = z_T^*$*,* $\mathcal{B}(\mathrm{z}) = \mathcal{B}(\mathrm{z}^*)$ *if and only if* $\mathrm{z} = \mathrm{z}^*$*.*

Remark 6.2. *The assumption in (6.3) amounts to requiring that the operator norm of the matrix*

$$g_{\mathcal{C}}^{-1/2}(\mathrm{Hess}^{g_{\mathcal{C}}}\,\Psi_p)g_{\mathcal{C}}^{-1/2} = \begin{pmatrix} 2P + (\nabla^g)^2 P & \nabla^g P \\ (\nabla^g P)^T & 2P \end{pmatrix} \tag{6.6}$$

be bounded by $\pi\sqrt{C_0}/T$*. This is verified for sufficiently small* T *if, e.g.,* $P \in L^\infty([0,T];C^2(M))$*. Similarly, the assumption in (6.5) is verified for sufficiently small* T *if* $P \in C^0([0,T]\times M)$*, since for a given smooth solution* φ *with* $\varphi_0 = \mathrm{Id}$*,* $\varrho = 2r_{max}/r_{min} \to 2$ *as* $T \to 0$*.*

The proof of lemma 6.1 is postponed to the appendix. Lemma 6.1 is the equivalent of lemma 5.2 in [7, 28] on the cone. As in [7], we can use it to prove the optimality of the plan concentrated on the continuous solution. In order to do this, we preliminary prove the following additional lemma.

Lemma 6.3. *The function* $b : \mathcal{C}^2 \to \mathbb{R}$ *defined in equation (6.2) is 2-homogeneous; moreover it is continuous if* $\|P\|_{C^0} \le T^{-2}$*.*

Proof. Since $\mathcal{B}$ is 2-homogeneous, b is also 2-homogeneous. As for the second point, we start by proving a bound from below on the functional $\mathcal{B}$. Observe that for any $\mathrm{z} \in AC^2([0,T^*];\mathcal{C})$,

$$\begin{aligned}
\mathcal{B}(\mathrm{z}) &\ge \int_0^T \|\dot z_t\|^2_{g_{\mathcal{C}}} - r_t^2 \|P\|_{C^0}\,\mathrm{d}t \\
&\ge \int_0^T \|\dot z_t\|^2_{g_{\mathcal{C}}} - \left(\int_0^t \dot r_{t^*}\,\mathrm{d}t^* + r_0\right)^2 \|P\|_{C^0}\,\mathrm{d}t\,, \\
&\ge \int_0^T \|\dot z_t\|^2_{g_{\mathcal{C}}} - 2\left[\left(\int_0^t \dot r_{t^*}\,\mathrm{d}t^*\right)^2 + r_0^2\right] \|P\|_{C^0}\,\mathrm{d}t\,.
\end{aligned} \tag{6.7}$$

Using Jensen's inequality

$$\begin{aligned}
\mathcal{B}(\mathrm{z}) &\ge \int_0^T \|\dot z_t\|^2_{g_{\mathcal{C}}} - 2\left(t\int_0^t \dot r_{t^*}^2\,\mathrm{d}t^* + r_0^2\right) \|P\|_{C^0}\,\mathrm{d}t \\
&\ge \int_0^T \|\dot z_t\|^2_{g_{\mathcal{C}}} - 2\left(t\int_0^T \dot r_{t^*}^2\,\mathrm{d}t^* + r_0^2\right) \|P\|_{C^0}\,\mathrm{d}t \\
&\ge \int_0^T r_t^2\|\dot x_t\|_g^2 + \dot r_t^2(1 - T^2\|P\|_{C^0})\,\mathrm{d}t - 2r_0^2 T\|P\|_{C^0} \\
&\ge (1 - T^2\|P\|_{C^0})\mathcal{A}(\mathrm{z}) - 2r_0^2T\|P\|_{C^0} \\
&\ge -2r_0^2T\|P\|_{C^0}\,.
\end{aligned} \tag{6.8}$$

This implies that $b(p,q)$ is bounded from below and minimizing sequences of paths need to be bounded in the radial direction with a bound depending on p,q, T and $\|P\|_{C^0}$. We now prove continuity of b with respect to the second argument. Fix $p, q \in \mathcal{C}$ and $\epsilon > 0$, and let $q^\epsilon \in \mathcal{C}$ any point such that $d(q, q_\epsilon) \le \epsilon$. For any $T^* \in [0,T]$

$$b(p, q_\epsilon) \le b_{T^*}(p,q) + b_{T-T^*}(q, q_\epsilon)\,, \tag{6.9}$$

where the subscripts on the functional b indicate that we replace T by T^* or $T-T^*$ in the definition of $\mathcal{B}$. Moreover,

$$
\begin{aligned}
b_{T^*}(p,q) &= \inf_{\mathrm{z}\in AC^2([0,T^*];\mathcal{C})} \left\{ \int_0^{T^*} \|\dot z_t\|^2_{g_\mathcal{C}} - \Psi_p(t,x_t,r_t)\,\mathrm{d}t\,;\, z_0=p\,,\, z_{T^*}=q \right\} \\
&= \inf_{\mathrm{z}\in AC^2([0,T];\mathcal{C})} \left\{ \frac{T}{T^*}\mathcal{B}(\mathrm{z}) + \left(\frac{T}{T^*}-\frac{T^*}{T}\right)\int_0^T \Psi_p(t,x_t,r_t)\,\mathrm{d}t\,;\, z_0=p\,,\, z_T=q \right\} \\
&\leq \frac{T}{T^*}b(p,q) + \frac{T^2-T^{*2}}{T^*}C\|P\|_{C^0}\,,
\end{aligned}
\tag{6.10}
$$

where to pass from the first to the second line we used a change of variables to rewrite the integrals over the interval $[0,T]$ and collected terms using the definition of $\mathcal{B}$ in equation (6.1); and where $C>0$ is a constant depending on p,q, T and $\|P\|_{C^0}$. Similarly,

$$
b_{T-T^*}(q,q_\epsilon) \leq \frac{T}{T-T^*}\epsilon^2 + (T-T^*)(\pi_r(q)+\epsilon)^2\|P\|_{C^0}\,. \tag{6.11}
$$

Taking $T^*=T-\epsilon$ and combining these estimates we obtain, for sufficiently small ϵ,

$$
b(p,q_\epsilon) - b(p,q) \leq \epsilon\left(\frac{2b(p,q)}{T} + (4C+2(\pi_r(q))^2)\|P\|_{C^0} + T\right)\,. \tag{6.12}
$$

We get a similar estimate switching q and q^ϵ, which proves continuity in the second argument, and also for the first argument, which concludes the proof. □

Therefore for any admissible dynamic plan $\boldsymbol{\mu}$, because of the homogeneous constraint in equation (4.9), we have

$$
\int_\Omega b(z_0,z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_{\mathcal{C}^2} b(p,q)\,\mathrm{d}\boldsymbol{\gamma}(p,q)\,. \tag{6.13}
$$

Theorem 6.4. *Under the assumptions of lemma 6.1, the dynamic plan $\boldsymbol{\mu}^*=(\varphi,\lambda)_\#\rho_0$ is the unique optimal solution of problem 4.2 up to rescaling (in the sense of lemma 4.5) with $\boldsymbol{\gamma}=[(\varphi_0,\lambda_0),(\varphi_T,\lambda_T)]_\#\rho_0$.*

Proof. Let $\boldsymbol{\mu}$ be any dynamic plan with finite action, i.e. $\mathcal{A}(\boldsymbol{\mu})<\infty$, and satisfying the constraints in (4.8) and (4.9). Consider the functional

$$
\mathcal{P}(\mathrm{z}) = \int_0^T \Psi_p(t,x_t,r_t)\mathrm{d}t\,. \tag{6.14}
$$

Then,

$$
\begin{aligned}
\int_\Omega \mathcal{P}(\mathrm{z})\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) &= \int_\Omega\int_0^T \Psi_p(t,x_t,r_t)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&= \int_\Omega\int_0^T P(t,x_t)r_t^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \\
&= \int_0^T\int_M P\,\mathrm{d}\rho_0\mathrm{d}t\,.
\end{aligned}
\tag{6.15}
$$

Hence,

$$
\mathcal{B}(\boldsymbol{\mu}) = \mathcal{A}(\boldsymbol{\mu}) - \int_0^T\int_M P\,\mathrm{d}\rho_0\mathrm{d}t\,, \tag{6.16}
$$

and since equation (6.16) also holds replacing $\boldsymbol{\mu}$ with $\boldsymbol{\mu}^*$,

$$
\mathcal{B}(\boldsymbol{\mu}) - \mathcal{B}(\boldsymbol{\mu}^*) = \mathcal{A}(\boldsymbol{\mu}) - \mathcal{A}(\boldsymbol{\mu}^*)\,. \tag{6.17}
$$

Moreover, integrating the functional b defined in (6.2) with respect to $\boldsymbol{\mu}$ we obtain

$$
\int_\Omega b(z_0,z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_{\mathcal{C}^2} b(p,q)\,\mathrm{d}\boldsymbol{\gamma}(p,q)\,, \tag{6.18}
$$

and we get the same result integrating with respect to $\boldsymbol{\mu}^*$. In particular, by lemma 6.1,

$$
\int_\Omega b(z_0,z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \mathcal{B}(\boldsymbol{\mu}^*)\,. \tag{6.19}
$$

By definition of b in (6.2), for any path $\mathrm{z} \in \Omega$, $\mathcal{B}(\mathrm{z}) \geq b(z_0, z_T)$ and therefore

$$\mathcal{B}(\boldsymbol{\mu}) \geq \int_\Omega b(z_0, z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \mathcal{B}(\boldsymbol{\mu}^*)\,, \tag{6.20}$$

which implies the same inequality for $\mathcal{A}$ due to equation (6.17). This proves that $\boldsymbol{\mu}^*$ is an optimal solution.

In order to prove uniqueness, let $\boldsymbol{\mu}$ be a solution of problem 4.2. Then, equations (6.17) and (6.18) imply

$$\int_\Omega \mathcal{B}(\mathrm{z}) - b(z_0, z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \mathcal{B}(\boldsymbol{\mu}) - \mathcal{B}(\boldsymbol{\mu}^*) = \mathcal{A}(\boldsymbol{\mu}) - \mathcal{A}(\boldsymbol{\mu}^*) = 0\,. \tag{6.21}$$

Since for any $\mathrm{z} \in \Omega$ we have $\mathcal{B}(\mathrm{z}) \geq b(z_0, z_T)$, then for $\boldsymbol{\mu}$-almost every path z, $\mathcal{B}(\mathrm{z}) = b(z_0, z_T)$. Clearly, also for $\boldsymbol{\mu}^*$-almost every path z, $\mathcal{B}(\mathrm{z}) = b(z_0, z_T)$. Now, if $\boldsymbol{\mu}$ satisfies the strong coupling constraint, for $\boldsymbol{\mu}$-almost every path z and for $\boldsymbol{\mu}^*$-almost every path z^*, we have $\mathcal{B}(\mathrm{z}) = \mathcal{B}(\mathrm{z}^*)$ but also $z_0 = z_0^*$ and $z_T = z_T^*$. This implies $\mathrm{z} = \mathrm{z}^*$ by lemma 6.1. In other words, $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^*$ are concentrated on the same paths so they must coincide. On the other hand, if $\boldsymbol{\mu}$ does not satisfy the strong coupling constraint, we need to prove that

$$\boldsymbol{\mu}^{\mathrm{o}}(\{\mathrm{z} \in \Omega\,;\, r_0 = r_T = 0\}) = 0\,, \tag{6.22}$$

where $\boldsymbol{\mu}^{\mathrm{o}} \coloneqq \boldsymbol{\mu} \llcorner \Omega^{\mathrm{o}}$, $\Omega \coloneqq \Omega \setminus \{\mathrm{o}\}$ and $\mathrm{o} : t \in [0,T] \to o \in \mathcal{C}$. In fact, in this case by proposition 4.8 we know that $\boldsymbol{\mu}^{\mathrm{o}}$ can be rescaled to a minimizer satisfying the strong coupling constraint. In order to show that (6.22) holds, observe that for any $\mathrm{z} \in AC^2([0,T];\mathcal{C})$ such that $r_0 = 0$,

$$\begin{aligned}
\mathcal{B}(\mathrm{z}) &\geq \int_0^T \|\dot{z}_t\|_{g_\mathcal{C}}^2 - r_t^2 \|P\|_{C^0}\,\mathrm{d}t \\
&\geq \int_0^T \|\dot{z}_t\|_{g_\mathcal{C}}^2 - t\int_0^t \dot{r}_{t^*}^2\,\mathrm{d}t^*\,\|P\|_{C^0}\,\mathrm{d}t \\
&\geq \int_0^T r_t^2 \|\dot{x}_t\|_g^2 + \left(1 - \frac{T^2}{2}\|P\|_{C^0}\right)\dot{r}_t^2\,\mathrm{d}t \\
&\geq C\mathcal{A}(\mathrm{z})\,,
\end{aligned} \tag{6.23}$$

where $C \coloneqq (1 - T^2\|P\|_{C^0}/2) \in (0,1)$ by the assumption in (6.5). Recall that for $\boldsymbol{\mu}$-almost every path z we have $\mathcal{B}(\mathrm{z}) = b(z_0, z_T)$. Then, if we define $\tilde{\Omega} \coloneqq \{\mathrm{z} \in \Omega\,;\, r_0 = r_T = 0\}$, we have

$$C\int_{\tilde{\Omega}} \mathcal{A}(\mathrm{z})\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) \leq \int_{\tilde{\Omega}} \mathcal{B}(\mathrm{z})\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = \int_{\tilde{\Omega}} b(z_0, z_T)\,\mathrm{d}\boldsymbol{\mu}(\mathrm{z}) = b(o,o)\,\boldsymbol{\mu}(\tilde{\Omega}) = 0\,, \tag{6.24}$$

Since $\mathcal{A}(\mathrm{z}) \geq 0$, we find that for $\boldsymbol{\mu}$-almost every path z such that $z_0 = z_T = o$, we have $\mathrm{z} = \mathrm{o}$. This implies equation (6.22) and we are done. ☐

This last proof finally validates our formulation for the generalized CH problem 4.2. In particular, it clearly shows that the choice of a homogeneous coupling constraint is appropriate for the problem. In fact, it allowed us to prove well-posedness on an unbounded cone domain in section 4 and crucially, it also allowed us to produce a simple characterization for minimizers satisfying the strong coupling constraint (see proposition 4.8), which led here to the correspondence with deterministic solutions. It should also be noted that in this section we used the regularity of a given pressure field to infer that minimizers are deterministic. In fact, from section 5, we know that a pressure field always exists as a distribution. Improving such a result in terms of regularity could help to provide a better characterization of generalized flow solutions also in cases not covered in this section.

7. Discrete generalized solutions

There are two main obstacles in translating problem 4.2 to the discrete setting. On one hand, we need to make computations on an unbounded domain; on the other, we need to enforce a large number of constraints since the coupling in (4.8) is defined via the pairing with any 2-homogeneous function on the cone. However, if one is interested in simulating solutions that do not blow up (i.e. satisfying the criteria in proposition 4.8), it is appropriate to enforce the strong coupling constraint in (4.6) instead of (4.8). Hence, if we substitute $\mathcal{C}$ by $\mathcal{C}_R$ for a fixed $R > 1$ and use the strong coupling constraint in the generalized CH problem, we obtain a modified formulation that is able to reproduce a particular class of solutions, namely those with bounded Jacobian and that do not blow up. In this section we describe a numerical algorithm based on entropy regularization and Sinkhorn's algorithm that solves such a modified formulation. Our scheme is based on similar methods for the incompressible Euler equations developed in [28, 6, 5]. We also provide some numerical results illustrating the behavior of generalized CH flows.

7.1. **Discrete formulation.** We set $M = [0,1]^d$ and consider a uniform discretization with points $\{x_i\}_{i=1}^{N_x}$, and a discretization of the interval $(0,R]$ with points $\{r_i\}_{i=1}^{N_r}$ such that $r_j = 1$ for a fixed $j \in \{1,\dots,N_r\}$. These induce a discretization of the cone with points $\{z_i\}_{i=1}^{N}$ where $N = N_x N_r$. Similarly, we also consider a uniform discretization $\{t_i\}_{i=1}^{K}$ of $[0,T]$. Generalized flows are then replaced by a coupling arrays $\boldsymbol{\mu} \in (\mathbb{R}^N_{\geq 0})^K$. Note that we can incorporate the boundary condition $\lambda_0 = 1$ by reducing the dimension of $\boldsymbol{\mu}$. In particular, we now denote by π_x and π_r the canonical projections from $M \times (0,R]$ to M and $(0,R]$ respectively. We use the same notation to indicate the maps $\pi_x : \{1,\dots,N\} \to \{1,\dots,N_x\}$ and $\pi_r : \{1,\dots,N\} \to \{1,\dots,N_r\}$ mapping directly the discretization indices. Then, we set for any $\{j_1,\dots,j_K\} \in \{1,\dots,N\}^K$,

$$\boldsymbol{\mu}_{j_1,\dots,j_K} = \mathbb{1}_{\{\pi_r(z_{j_1})=1\}} \tilde{\boldsymbol{\mu}}_{\pi_x(j_1),j_2,\dots,j_K}\,, \tag{7.1}$$

where $\mathbb{1}$ is the indicator function and $\tilde{\boldsymbol{\mu}} \in \mathbb{R}^{N_x}_{\geq 0} \times (\mathbb{R}^N_{\geq 0})^{K-1}$. We denote by Π_0 the set of couplings satisfying (7.1). The marginal at a given time t_k is a discrete measure on $M \times (0,R]$. We denote this by $S_k(\boldsymbol{\mu}) \in \mathbb{R}^N_{\geq 0}$, and it is defined as follows:

$$[S_k(\boldsymbol{\mu})]_j = \sum_{j_1,\dots,j_{k-1},j_{k+1},\dots,j_K} \boldsymbol{\mu}_{j_1,\dots,j_{k-1},j,j_{k+1},\dots,j_K}\,. \tag{7.2}$$

We denote by $M_n : \mathbb{R}^N_{\geq 0} \to \mathbb{R}^{N_x}_{\geq 0}$ the nth moment taken in the radial direction, i.e.

$$M_n[A]_i = \sum_{j,\pi_x(j)=i} \pi_r(z_j)^n A_j\,. \tag{7.3}$$

Hence the constraint in (4.4) becomes

$$M_2[S_k(\boldsymbol{\mu})]_i = 1/N_x\,. \tag{7.4}$$

Moreover, we denote by Π the set of admissible coupling arrays,

$$\Pi = \{\boldsymbol{\mu} \in \Pi_0;\, \forall i,\, M_2[S_k(\boldsymbol{\mu})]_i = 1/N_x\}\,. \tag{7.5}$$

The constraint on the coupling between time 0 and T can be enforced weakly by including it directly in the cost, which is given by the following array

$$C_{j_1,\dots,j_K} = \frac{K-1}{T}\sum_{k=1}^{K-1} d_{\mathcal{C}}(z_{j_k}, z_{j_{k+1}})^2 + \alpha\, d_{\mathcal{C}}(z_{j_K}, (h(\pi_x(z_{j_1})), \sqrt{\mathrm{Jac}(h)}))^2\,, \tag{7.6}$$

where $\alpha > 0$ is a parameter. The regularized discrete problem is then,

$$\min_{\boldsymbol{\mu}\in\Pi} \langle C, \boldsymbol{\mu}\rangle - \epsilon E(\boldsymbol{\mu})\,, \tag{7.7}$$

where $\epsilon > 0$ is another parameter and $E(\boldsymbol{\mu})$ is the entropy of the coupling defined by

$$E(\boldsymbol{\mu}) = -\langle \boldsymbol{\mu}, \log(\boldsymbol{\mu}) - 1\rangle\,. \tag{7.8}$$

Problem (7.7) can be solved by means of alternating projections which consist in enforcing recursively the marginal constraints at the different time levels. In particular, we consider the following augmented functional

$$\min_{\boldsymbol{\mu}} \langle C, \boldsymbol{\mu}\rangle - \epsilon E(\boldsymbol{\mu}) - \sum_{i,k} p_i^k (M_2[S_k(\boldsymbol{\mu})]_i - 1/N_x)\,, \tag{7.9}$$

where $p^k \in \mathbb{R}^{N_x}$ for all $k \in \{1,\dots,K\}$. From (7.9) we obtain

$$\boldsymbol{\mu}_{j_1,\dots,j_K} = e^{-\frac{C_{j_1,\dots,j_K}}{\epsilon}} e^{\sum_k p^k_{\pi_x(j_k)} r^2_{\pi_r(j_k)}}\,. \tag{7.10}$$

Enforcing the constraint at time level n allows us to solve for p^n given the set $\{p^k\}_{k\neq n}$. This amounts to solving the following nonlinear equation for all $i \in \{1,\dots,N_x\}$,

$$\sum_j B_{i,j} e^{p_i^n r_j^2} r_j^2 = 1/N_x\,, \tag{7.11}$$

where

$$B = S_n \left[e^{-\frac{C_{j_1,\dots,j_K}}{\epsilon}} e^{\sum_{k,k\neq n} p^k_{\pi_x(j_k)} r^2_{\pi_r(j_k)}} \right] . \tag{7.12}$$

Due to the structure of the cost, we only need to store two arrays $D^0, D^1 \in \mathbb{R}^N \times \mathbb{R}^N$, given by

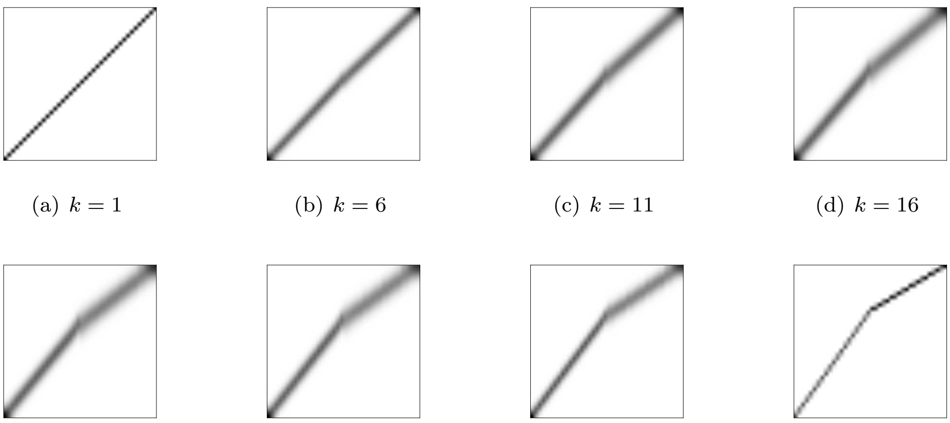
$$D^0_{i,j} = d_{\mathcal{C}}(z_i, z_j)^2 \,, \qquad D^1_{i,j} = d_{\mathcal{C}}(z_i, (h(\pi_x(z_j)), \sqrt{\mathrm{Jac}(h)}))^2 \,. \tag{7.13}$$

7.2. **Numerical results: from CH to Euler.** We now present some numerical results illustrating the behavior of generalized solutions of the CH problem and their relation to generalized incompressible Euler solutions. We consider two types of couplings to define the boundary conditions: a classical deterministic coupling, which we use to illustrate the emergence of discontinuities in the flow map, and a generalized coupling that obliges particles to cross each other so that the solution is not deterministic. For both cases, the domain will be the one-dimensional interval $M = [0,1]$ and $T = 1$.

A peakon-like solution. Consider the continuous map $h : [0,1] \to [0,1]$, defined by

$$h(x) = \begin{cases} 1.4\,x & \text{if } x \leq 0.5\,, \\ 0.6\,x + 0.4 & \text{if } x > 0.5\,. \end{cases} \tag{7.14}$$

We use this map to define the coupling on the cone as in equation (4.6). We compute the solution using the algorithm presented in the previous section with $N_x = 40$, $N_r = 41$, $0.55 \leq r \leq 1.45$, $K = 35$, $\alpha = 40$, $\epsilon = 5 \cdot 10^{-4}$. In figure 1 we show the evolution of the transport plan on the domain M given by $(e^M_{0,t_k})_\# \boldsymbol{\mu} \in \mathcal{P}(M^2)$, where $e^M_{0,t_k}(\mathrm{z}) := (x_0, x_{t_k})$, for selected times. In figure 2 we show the evolution of the marginals on the cone given by $(e_{t_k})_\# \boldsymbol{\mu} \in \mathcal{P}(\mathcal{C})$ for the same times. We remark that the dynamic plan is approximately deterministic since there is very little diffusion of the mass in the domain, which is at least partially due to the entropic regularization. In addition the discontinuity in the Jacobian of the coupling map propagates to the whole solution, which resembles a peakon with the discontinuity point corresponding to the peak of the peakon.



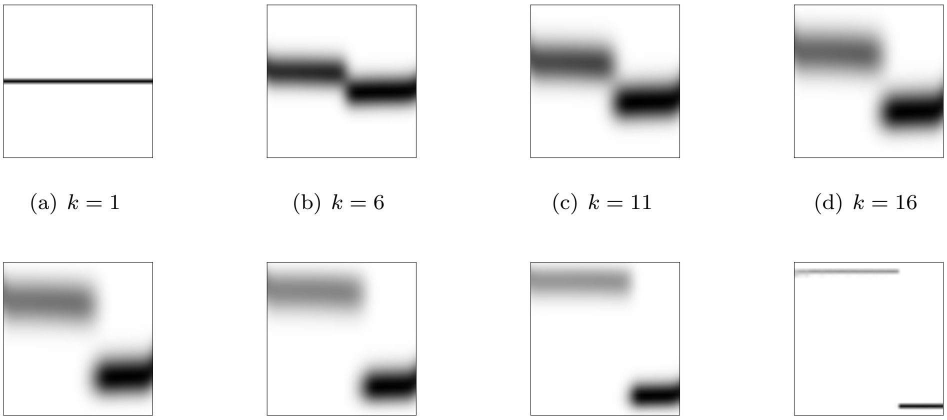
(a) $k = 1$ (b) $k = 6$ (c) $k = 11$ (d) $k = 16$

(e) $k = 20$ (f) $k = 25$ (g) $k = 30$ (h) $k = 35$

FIGURE 1. Transport couplings $(e^M_{0,t_k})_\# \boldsymbol{\mu}$ on $M \times M$ for the peakon-like solution associated to the boundary conditions specified by the map in equation (7.14).

A non-deterministic solution. The homogeneous marginal constraints allow us to consider very general couplings even defined by non-injective maps or maps that do not preserve the local orientation of the domain. Measure-preserving maps provide a special example since these were used by Brenier to define boundary conditions for generalized incompressible Euler flows. In fact if h is measure-preserving, i.e. $h_\# \rho_0 = \rho_0$, then we can use as coupling

$$\boldsymbol{\gamma} = [(\mathrm{Id}, 1), (h, 1)]_\# \rho_0 \,. \tag{7.15}$$



(a) $k = 1$ (b) $k = 6$ (c) $k = 11$ (d) $k = 16$

(e) $k = 20$ (f) $k = 25$ (g) $k = 30$ (h) $k = 35$

FIGURE 2. Fixed time marginals $(e_{t_k})_\# \boldsymbol{\mu}$ on the cone section $M \times [r_{min}, r_{max}]$ ($r_{min} = 0.55$, $r_{max} = 1.45$) for the peakon-like solution associated to the boundary conditions specified by the map in equation (7.14).

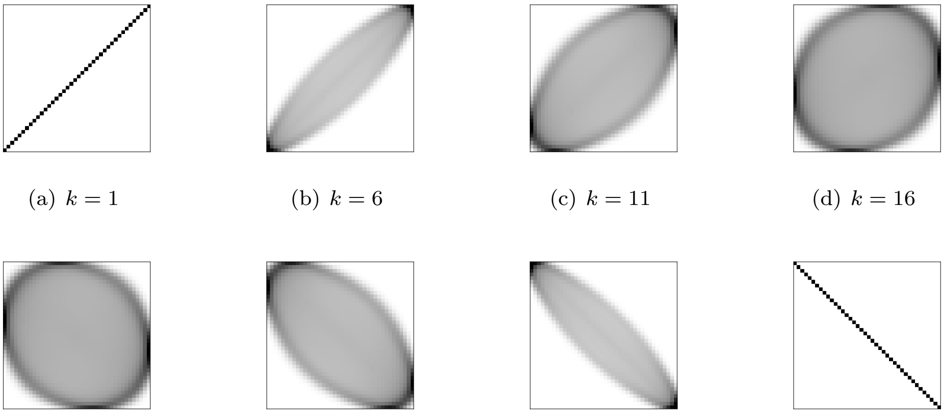
The existence of a generalized solution of the CH problem in this case is a direct consequence of the existence result proved by Brenier in [7]; this is because generalized Euler solution can be easily lifted to admissible solutions of our formulation concentrated on paths z with $r_t = 1$ for every time. Here, we take $h : [0, 1] \to [0, 1]$ to be the map

$$h(x) = 1 - x\,, \tag{7.16}$$

which can only be realized by a non-deterministic plan. We compute the discrete solution associated to such boundary conditions with $N_x = 40$, $N_r = 41$, $0.6 \leq r \leq 1.4$, $K = 35$, $\alpha = 40$, $\epsilon = 5 \cdot 10^{-4}$. As before, we show the evolution of the transport plan on the domain M given by $(e^M_{0,t_k})_\# \boldsymbol{\mu} \in \mathcal{P}(M^2)$ in figure 3. In figure 4 we show the evolution of the marginals on the cone given by $(e_{t_k})_\# \boldsymbol{\mu} \in \mathcal{P}(\mathcal{C})$. The transport plan evolution is remarkbly similar to that of the incompressible Euler equation for the same coupling (see, e.g., [6]). However, the two do not coincide as it is evident from the marginals on the cone in figure 4. In the case of incompressible Euler, these marginals are concentrated on $r = 1$ for every time, i.e. the transport plan remains measure-preserving during the evolution. This is clearly not the case for the generalized CH solution, for which also the Jacobian appears to be non-deterministic.

8. SUMMARY AND OUTLOOK

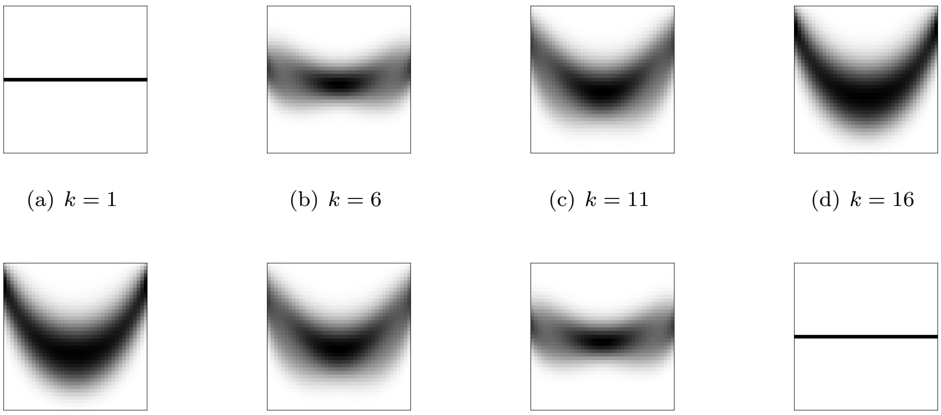
In this paper we studied the boundary value problem associated to the multi-dimensional CH equation, describing geodesics for the $H(\mathrm{div})$ metric on the diffeomorphim group. We constructed a framework to solve this problem based on Brenier's generalized formulation of the incompressible Euler equations. In such a framework, generalized solutions are probability measures on the space of continuous paths on an unbounded non-smooth manifold, the cone over the domain. This description effectively amounts to decoupling the evolution of the Lagrangian flow map to that of its Jacobian, and allowed us to represent arbitrary compressible flows. The crucial part of the formulation was in the choice of the constraints which are used to (re)enforce 1) the relation between the Jacobian and the flow map and 2) the boundary conditions of the problem, specifying the position of fluid particles at the final time. In our approach, both constraint are homogeneous, meaning that the boundary conditions are enforced in a weak sense. This led us to a formulation that allows particles to collapse at the same position in space (in a probabilistic sense), which is the phenomenon leading to the blow up of CH solutions, and prove existence of solutions in the generalized setting. Even if the boundary conditions are enforced weakly, the model still provides a relaxation of the CH variational problem; in fact, we proved that smooth solutions of the latter are the unique minimizers for sufficiently short times. Remarkably, we could also prove an equivalent version of Brenier's result on the pressure field



(a) $k = 1$ (b) $k = 6$ (c) $k = 11$ (d) $k = 16$

(e) $k = 20$ (f) $k = 25$ (g) $k = 30$ (h) $k = 35$

FIGURE 3. Transport couplings $(e^M_{0,t_k})_\#\boldsymbol{\mu}$ on $M \times M$ for the non-deterministic solution associated to the boundary conditions specified by the map in equation (7.16).



(a) $k = 1$ (b) $k = 6$ (c) $k = 11$ (d) $k = 16$

(e) $k = 20$ (f) $k = 25$ (g) $k = 30$ (h) $k = 35$

FIGURE 4. Fixed time marginals $(e_{t_k})_\#\boldsymbol{\mu}$ on the cone section $M \times [r_{min}, r_{max}]$ ($r_{min} = 0.6$, $r_{max} = 1.4$) for the non-deterministic associated to the boundary conditions specified by the map in equation (7.16).

in incompressible Euler. Specifically, we showed that for any deterministic boundary condition generalized solutions of CH are characterized by a unique pressure field intended as a distribution, independently of the occurrence of blow up. Finally, we constructed a numerical algorithm based on entropic regularization which is able to represent solutions that do not blow up and presented some numerical results illustrating the qualitative behavior of generalized CH solutions.

There are several natural questions that are left open and that will be addressed in future work:

- *Occurrence of blow up.* In our framework we did not rule out the possibility that generalized solutions charge paths reaching the apex of the cone, which correspond to the blow up scenario. It would be interesting to characterize the situations for which such a behavior occurs in terms of type of boundary conditions and their regularity, and the dimension and geometry of the base space M;
- *Regularity of the pressure.* Brenier's result on the existence and uniqueness of the pressure in incompressible Euler was subsequently improved by Ambrosio and Figalli [1] in

terms of regularity of the pressure field. It is natural to ask whether such result can be extended to the generalized CH problem. This question is related to the previous one, due to the fact that a sufficiently regular pressure field can prevent the occurrence of blow up as it can be deduced from the proofs in section 6;

- *Tight relaxation.* An even stronger result would entail determining in what hypothesis our formulation defines a tight relaxation of the $H(\mathrm{div})$ geodesic problem on the diffeomorphism group. For example, Brenier's relaxation of incompressible Euler is not tight in two dimensions but it is in three dimensions due to the work of Shnirelman [31]. It is unclear whether a similar result holds for the CH variational formulation we considered in this paper, and in general what is the correct relaxation for this problem.

Addressing these theoretical questions will also guide the development of numerical schemes which are better suited to the formulation considered in this paper than methods based on entropic regularization. A viable alternative in this context is given by semi-discrete methods, whose use for the generalized CH problem will also be studied in future work.

Acknowledgments

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n. PCOFUND-GA-2013-609102, through the PRESTIGE programme coordinated by Campus France. The authors would also like to acknowledge the support from the project MAGA ANR-16-CE40-0014 (2016-2020).

Appendix A. Proof of lemma 4.3

Proof. Here we prove that the homogeneous marginal constraint can be enforced at each time rather than in integral form as in equation (4.9).

First, we prove that the constraint in equation (4.9) implies the one in equation (4.11). In order to show this, for any fixed $t^* \in [0,T]$ and $f \in C^0(M)$, consider the following functionals

$$\mathcal{F}(\mathrm{z}) \coloneqq r_{t^*}^2 f(x_{t^*})\,, \qquad \mathcal{F}_n(\mathrm{z}) \coloneqq \int_0^T r_t^2 f(x_t)\delta_{n,t^*}(t)\,\mathrm{d}t\,, \tag{A.1}$$

where $\delta_{n,t^*} : [0,T] \to \mathbb{R}$, $n \in \mathbb{N}$, is a Dirac sequence of continuous functions converging to δ_{t^*}. Then for any $\mathrm{z} \in \Omega$, $\mathcal{F}_n(\mathrm{z}) \to \mathcal{F}(\mathrm{z})$ as $n \to +\infty$. Moreover, using Jensen's inequality,

$$\begin{aligned}\mathcal{F}_n(\mathrm{z}) &\leq \|f\|_{C^0} \int_0^T r_t^2 \delta_{n,t^*}\,\mathrm{d}t \\ &\leq 2\|f\|_{C^0}\left(r_0^2 + \int_0^T (r_t - r_0)^2\delta_{n,t^*}\,\mathrm{d}t\right) \\ &\leq 2\|f\|_{C^0}\left(r_0^2 + \int_0^T \dot{r}_t^2\,\mathrm{d}t \int_0^T t\,\delta_{n,t^*}\,\mathrm{d}t\right) \\ &\leq 2\|f\|_{C^0}\left(r_0^2 + T\mathcal{A}(\mathrm{z})\right)\,.\end{aligned} \tag{A.2}$$

The right-hand side is $\boldsymbol{\mu}$-integrable since $\mathcal{A}(\boldsymbol{\mu}) < +\infty$ and because of the coupling constraint. Hence, we get the result by the dominated convergence theorem.

Similarly, if $f \in C^0([0,T] \times M)$, we take

$$\mathcal{F}(\mathrm{z}) \coloneqq \int_0^T f(t,x_t) r_t^2\,\mathrm{d}t\,, \qquad \mathcal{F}_n(\mathrm{z}) \coloneqq \frac{T}{K}\sum_{k=0}^K f(t_k, x_{t_k}) r_{t_k}^2\,, \tag{A.3}$$

where $t_k \coloneqq kT/K$. Then for any $\mathrm{z} \in \Omega$, $\mathcal{F}_n(\mathrm{z}) \to \mathcal{F}(\mathrm{z})$ as $n \to +\infty$. Moreover,

$$\begin{aligned}\mathcal{F}_n(\mathrm{z}) &\leq 2\|f\|_{C^0}\left(r_0^2 + \frac{T}{K}\sum_{k=1}^K (r_{t_k} - r_0)^2\right) \\ &\leq 2\|f\|_{C^0}\left(r_0^2 + \frac{T}{K}\sum_{k=1}^K t_k \int_0^{t_k} \dot{r}_t^2\,\mathrm{d}t\right) \\ &\leq 2\|f\|_{C^0}\left(r_0^2 + T^2\mathcal{A}(\mathrm{z})\right)\,,\end{aligned} \tag{A.4}$$

and we can apply again the dominated convergence theorem to conclude the proof. □

Appendix B. Proof of lemma 6.1

Proof. Throughout this proof, all metric operations are performed with respect to the cone metric $g_{\mathcal{C}}$, so to simplify the notation we will simply use $\|\cdot\|$ for the norm and $\langle\cdot,\cdot\rangle$ for the inner product on $T\mathcal{C}$. Moreover, given a vector field u on the cone and a curve $t\mapsto p(t)\in\mathcal{C}$, $\nabla_t u(p(t)) \coloneqq \nabla_{\dot p(t)} u(p(t))$ is the covariant derivative of u at $p(t)$ with respect to the vector $\dot p(t)$.

Given a smooth solution (φ,λ) and a fixed $x\in M$, let $\mathrm{z}^* = [\mathrm{x}^*,\mathrm{r}^*]\in\Omega$ be the curve defined by $\mathrm{x}^* : t\to x_t^* \coloneqq \varphi_t(x)$ and $\mathrm{r}^* : t \to r_t^* \coloneqq \lambda_t(x)$. We want to show that for any curve $\mathrm{z}\in AC^2([0,T];\mathcal{C})$ such that $\mathrm{z}\neq \mathrm{z}^*$, $z_0 = z_0^*$ and $z_T = z_T^*$, we have $\mathcal{B}(\mathrm{z}) > \mathcal{B}(\mathrm{z}^*)$. We proceed in two steps: first we show that the inequality holds when z is smooth and when the geodesics between z_t^* and z_t are smooth for all $t\in[0,T]$; then we derive sufficient conditions for which the inequality holds also for curves z which are farther away from z^*.

Let $s\in[0,1]\mapsto c(t,s)\in\mathcal{C}$ be a family of geodesics parameterized by $t\in[0,T]$ such that $c(t,0) = z_t^*$ and $c(t,1) = z_t$. In order for such geodesics to be smooth we need to assume

$$d_M(x_t^*, x_t) < \pi\,, \qquad \forall\, t\in[0,T]\,. \tag{B.1}$$

Let $J(t,s)\coloneqq \partial_t c(t,s)$, which is a Jacobi field when restricted to any geodesic $c(t,\cdot)$ for any fixed $t\in[0,T]$. Moreover, $J(t,0) = \dot z_t^*$ and $J(t,1) = \dot z_t$. Hence we want to show that

$$\int_0^T \|J(t,0)\|^2 - \Psi_p(t,c(t,0))\,\mathrm{d}t \le \int_0^T \|J(t,1)\|^2 - \Psi_p(t,c(t,1))\,\mathrm{d}t\,. \tag{B.2}$$

Let $C\coloneqq \sup_{t\in[0,T]}\sup_{x\in M}\|\mathrm{Hess}\,\Psi_p\|$. The Taylor expansion of $\Psi_p(t,c(s,t))$ with respect to s at $s=0$ yields

$$\Psi_p(t,c(t,1)) - \Psi_p(t,c(t,0)) - \langle \nabla\Psi_p(t,c(t,0)), \partial_s c(t,0)\rangle \le \frac{C}{2}\int_0^1 \|\partial_s c(t,s)\|^2\,\mathrm{d}s\,. \tag{B.3}$$

Since $\partial_s c(t,s) = 0$ at $t=0$ and $t=T$, by the Poincaré inequality we also have

$$\int_0^T \|\partial_s c(t,s)\|^2\,\mathrm{d}t \le \frac{T^2}{\pi^2}\int_0^T |\partial_t\|\partial_s c(t,s)\|^2|\,\mathrm{d}t \le \frac{T^2}{\pi^2}\int_0^T \|\nabla_t\partial_s c(t,s)\|^2\,\mathrm{d}t\,. \tag{B.4}$$

Let $\dot J(t,s)\coloneqq \nabla_s\partial_t c(t,s)$ and exchanging the order of derivatives in the equation above we obtain

$$\int_0^T \|\partial_s c(t,s)\|^2\,\mathrm{d}t \le \frac{T^2}{\pi^2}\int_0^T \|\dot J(t,s)\|^2\,\mathrm{d}t\,. \tag{B.5}$$

Integrating over $[0,T]$ equation (B.3) and using equation (B.5) we get

$$\int_0^T \Psi_p(t,c(t,1)) - \Psi_p(t,c(t,0)) - \langle \nabla\Psi_p(t,c(t,0)), \partial_s c(t,0)\rangle\,\mathrm{d}t \le \frac{CT^2}{2\pi^2}\int_0^1 \|\dot J(t,s)\|^2\,\mathrm{d}s\,. \tag{B.6}$$

Consider the term involving the gradient of Ψ_p. Substituting $\nabla\Psi_p(t,c(t,0)) = -2\nabla_t \dot z_t^* = -2\nabla_t J(t,0)$, integrating by parts in t, and exchanging the order of derivatives for this term yields

$$\int_0^T \Psi_p(t,c(t,1)) - \Psi_p(t,c(t,0)) - 2\langle J(t,0), \dot J(t,0)\rangle\,\mathrm{d}t \le \frac{CT^2}{2\pi^2}\int_0^1 \|\dot J(t,s)\|^2_{g_{\mathcal{C}}}\,\mathrm{d}s\,. \tag{B.7}$$

Let $f(s)\coloneqq \int_0^T \|J(t,s)\|^2\,\mathrm{d}t$, then

$$f'(0) = \int_0^T 2\langle J(t,0), \dot J(t,0)\rangle\,\mathrm{d}t\,, \tag{B.8}$$

and

$$\begin{aligned} f(1) - f(0) - f'(0) &= \int_0^1 (1-s) f''(s)\,\mathrm{d}s \\ &= \int_0^1\int_0^T 2(1-s)(\|\dot J(t,s)\|^2 + \langle J(t,s), \nabla_s \dot J(t,s)\rangle)\,\mathrm{d}t\mathrm{d}s \\ &\ge \int_0^1\int_0^T 2(1-s)\|\dot J(t,s)\|^2\,\mathrm{d}t\mathrm{d}s\,, \end{aligned} \tag{B.9}$$

where the last inequality is due to the fact that for a Jacobi field $J(t,s)$,

$$\nabla_s \dot{J}(t,s) = -R(J(t,s), \partial_s c(t,s))\partial_s c(t,s)\,, \tag{B.10}$$

where R is the Riemann tensor, which for any tangent vectors X and Y at the same point on the cone over a flat manifold satisfies $\langle X, R(X,Y)Y\rangle \leq 0$. Moreover since the Jacobi fields are finite dimensional and $[0,T]\times M$ is compact, there exists a constant $C_0>0$ such that

$$f(1) - f(0) - f'(0) \geq \frac{C_0}{2}\int_0^1\int_0^T \|\dot{J}(t,s)\|^2\,\mathrm{d}t\mathrm{d}s\,. \tag{B.11}$$

Combining this with (B.7) and rearranging terms we obtain

$$\begin{aligned}\left(\frac{C_0}{2} - \frac{CT^2}{2\pi^2}\right)\int_0^1\int_0^T \|\dot{J}(t,s)\|^2\,\mathrm{d}t\mathrm{d}s + \int_0^T \|J(t,0)\|^2 - \Psi_p(t, c(t,0))\,\mathrm{d}t \\ \leq \int_0^T \|J(t,1)\|^2 - \Psi_p(t,c(t,1))\,\mathrm{d}t\,,\end{aligned} \tag{B.12}$$

which shows that z^* is minimizing among all paths $\mathrm{z}\in\Omega$ which satisfy (B.1).

Now, assume that for all $x\in M$, $d_{\mathcal{C}}(z_{t_0}, z_{t_1}) \leq \epsilon$, for all $t_0,t_1\in[0,T]$. Let

$$B_\delta \coloneqq \bigcap_{t\in[0,T]} \{q\in\mathcal{C}\,;\, d_{\mathcal{C}}(q, z_t^*) \leq \delta\}\,, \tag{B.13}$$

and take $\epsilon < \delta \coloneqq \frac{r_{min}}{2}$, where $r_{min} \coloneqq \inf_{(t,x)\in[0,T]\times M}\lambda_t(x)$. For any $q\in B_\delta$ and any $t\in[0,T]$ the geodesic path between q and z_t^* cannot pass through the apex, since otherwise the distance between the two points should be at least equal to r_{min}. In other words, we must have $d_M(q,z_t^*)<\pi$ and the path z^* is minimizing among all paths $\mathrm{z}\in\Omega$ contained in B_δ. Moreover, the geodesic path from z_0^* to z_T^* is also included in B_δ. Consider the following quantity

$$E(\delta,q,T^*) \coloneqq \inf_{p\in\partial B_\delta/\mathcal{C}(\partial M)}\left\{\inf_{\mathrm{z}\in AC^2([0,T^*];\mathcal{C})}\left\{\int_0^{T^*}\|\dot{z}_t\|^2 - \Psi_p(t,z_t)\,\mathrm{d}t\,;\, z_0 = q\in B_\delta\,,\, z_T = p\right\}\right\}, \tag{B.14}$$

which is the infimum action over the interval $[0,T^*]$ associated to a path starting at a point $q\in B_\delta$ and reaching its boundary ∂B_δ (but not points on ∂M) at time T^*. Given any path z such that $z_0=z_0^*$ and $z_T = z_T^*$ not contained in B_δ, we have

$$\mathcal{B}(\mathrm{z}) \geq \inf_{T_1+T_2\leq T}(E(\delta, z_0^*, T_1) + E(\delta, z_T^*, T_2))\,, \tag{B.15}$$

and we want to show that $\mathcal{B}(\mathrm{z}) > \mathcal{B}(\mathrm{z}^*)$. We have

$$\begin{aligned} E(\delta, z_0^*, T_1) &\geq \inf_p \inf_{\mathrm{z}} \int_0^{T_1}\|\dot{z}_t\|^2\,\mathrm{d}t - (r_{max}+\delta)^2 CT_1 \\ &\geq \frac{(\delta-\epsilon)^2}{T_1} - (r_{max}+\delta)^2 CT_1\,,\end{aligned} \tag{B.16}$$

where $C \coloneqq \sup_{(t,x)\in[0,T]\times M}|P(t,x)|$ and $r_{max} \coloneqq \sup_{(t,x)\in[0,T]\times M}\lambda_t(x)$. Hence, by equation (B.15),

$$\mathcal{B}(\mathrm{z}) \geq \frac{4(\delta-\epsilon)^2}{T} - (r_{max}+\delta)^2 CT\,. \tag{B.17}$$

On the other hand, we can deduce an upper bound for $\mathcal{B}(\mathrm{z}^*)$ using the geodesic path z^g between z_0^* and z_T^*, yielding

$$\mathcal{B}(\mathrm{z}) \leq \int \|\dot{z}_t^g\|^2\,\mathrm{d}t + r_{max}^2 CT \leq \frac{\epsilon^2}{T} + r_{max}^2 CT\,. \tag{B.18}$$

Therefore we find the following sufficient condition for optimality of the path z^*:

$$[r_{max}^2 + (r_{max}+\delta)^2]CT < \frac{4(\delta-\epsilon)^2}{T} - \frac{\epsilon^2}{T}\,. \tag{B.19}$$

The right-hand side is positive if $\epsilon < 2\delta/3$. Hence taking $\epsilon = \delta/2$ and substituting $\delta = \frac{r_{min}}{2}$,

$$\left[r_{max}^2 + \left(r_{max} + \frac{r_{min}}{2}\right)^2\right] CT < \frac{3r_{min}^2}{8T}\,, \tag{B.20}$$

which conclude s the proof. □

References

[1] Luigi Ambrosio and Alessio Figalli. On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations. *Calculus of Variations and Partial Differential Equations*, 31(4):497–509, 2008.

[2] Luigi Ambrosio and Alessio Figalli. Geodesics in the space of measure-preserving maps and plans. *Archive for rational mechanics and analysis*, 194(2):421–462, 2009.

[3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.

[4] Vladimir I Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applicationsa l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16(1):319–361, 1966.

[5] Jean-David Benamou, Guillaume Carlier, Marco Cuturi, Luca Nenna, and Gabriel Peyré. Iterative Bregman projections for regularized transportation problems. *SIAM Journal on Scientific Computing*, 37(2):A1111–A1138, 2015.

[6] Jean-David Benamou, Guillaume Carlier, and Luca Nenna. Generalized incompressible flows, multi-marginal transport and Sinkhorn algorithm. *arXiv preprint arXiv:1710.08234*, 2017.

[7] Yann Brenier. The least action principle and the related concept of generalized flows for incompressible perfect fluids. *Journal of the American Mathematical Society*, 2(2):225–255, 1989.

[8] Yann Brenier. The dual least action problem for an ideal, incompressible fluid. *Archive for rational mechanics and analysis*, 122(4):323–351, 1993.

[9] Roberto Camassa and Darryl D Holm. An integrable shallow water equation with peaked solitons. *Physical Review Letters*, 71(11):1661, 1993.

[10] Lénaic Chizat, Gabriel Peyré, Bernhard Schmitzer, and François-Xavier Vialard. An interpolating distance between optimal transport and Fisher–Rao metrics. *Foundations of Computational Mathematics*, 18(1):1–44, 2018.

[11] Lénaic Chizat, Gabriel Peyré, Bernhard Schmitzer, and François-Xavier Vialard. Unbalanced optimal transport: Dynamic and Kantorovich formulations. *Journal of Functional Analysis*, 274(11):3090 – 3123, 2018.

[12] Adrian Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. In *Annales de l'institut Fourier*, volume 50, pages 321–362. Chartres: L'Institut, 1950-, 2000.

[13] Marco Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. In *Advances in neural information processing systems*, pages 2292–2300, 2013.

[14] Thomas Gallouët and François-Xavier Vialard. The Camassa–Holm equation as an incompressible Euler equation: A geometric point of view. *Journal of Differential Equations*, 2017.

[15] Thomas O Gallouët and Quentin Mérigot. A Lagrangian scheme à la Brenier for the incompressible Euler equations. *Foundations of Computational Mathematics*, pages 1–31, 2017.

[16] Darryl D Holm and Jerrold E Marsden. Momentum maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation. In *The breadth of symplectic and Poisson geometry*, pages 203–235. Springer, 2005.

[17] Darryl D Holm, Jerrold E Marsden, and Tudor S Ratiu. Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Physical Review Letters*, 80(19):4173, 1998.

[18] Darryl D Holm, Jerrold E Marsden, and Tudor S Ratiu. The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Advances in Mathematics*, 137(1):1–81, 1998.

[19] Darryl D Holm and Martin F Staley. Wave structure and nonlinear balances in a family of evolutionary PDEs. *SIAM Journal on Applied Dynamical Systems*, 2(3):323–380, 2003.

[20] John K Hunter and Ralph Saxton. Dynamics of director fields. *SIAM Journal on Applied Mathematics*, 51(6):1498–1521, 1991.

[21] B Khesin, Jonatan Lenells, G Misiołek, and SC Preston. Geometry of diffeomorphism groups, complete integrability and geometric statistics. *Geometric and Functional Analysis*, 23(1):334–366, 2013.

[22] H-P Kruse, J Scheurle, and W Du. A two-dimensional version of the Camassa-Holm equation. In *Symmetry and Perturbation Theory: SPT 2001*, pages 120–127. World Scientific, 2001.

[23] Jae Min Lee. Global Lagrangian solutions of the Camassa-Holm equation. *arXiv preprint arXiv:1710.05484*, 2017.

[24] Matthias Liero, Alexander Mielke, and Giuseppe Savaré. Optimal entropy-transport problems and a new Hellinger–Kantorovich distance between positive measures. *Inventiones mathematicae*, 211(3):969–1117, 2018.

[25] Quentin Mérigot. A multiscale approach to optimal transport. In *Computer Graphics Forum*, volume 30, pages 1583–1592. Wiley Online Library, 2011.

[26] Quentin Mérigot and Jean-Marie Mirebeau. Minimal geodesics along volume-preserving maps, through semidiscrete optimal transport. *SIAM Journal on Numerical Analysis*, 54(6):3465–3492, 2016.

[27] Luc Molinet. On well-posedness results for Camassa-Holm equation on the line: a survey. *Journal of Nonlinear Mathematical Physics*, 11(4):521–533, 2004.

[28] Luca Nenna. *Numerical methods for multi-marginal optimal transportation*. PhD thesis, PSL Research University, 2016.

[29] Felix Otto. The geometry of dissipative evolution equations: the porous medium equation. 2001.

[30] Filippo Santambrogio. Optimal transport for applied mathematicians. *Birkäuser, NY*, pages 99–102, 2015.

[31] Alexander I Shnirelman. Generalized fluid flows, their approximation and applications. *Geometric & Functional Analysis GAFA*, 4(5):586–620, 1994.