On canonical bases of a formal K-algebra

Abdallah Assi

THE CANONICAL FAN OF A FORMAL $\mathbb{K}$-ALGEBRA

ABDALLAH ASSI

Abstract. We associate with an algebra $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]] \subseteq \mathbb{K}[[x_1, \dots, x_n]]$ over a field $\mathbb{K}$ a fan called the canonical fan of $\mathbf{A}$. This generalizes the notion of the standard fan of an ideal.

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Introduction

Let $\mathbb{K}$ be a field and let $f_1, \dots, f_s$ be nonzero elements of the ring $\mathbf{F} = \mathbb{K}[[x_1, \dots, x_n]]$ of formal power series in $x_1, \dots, x_n$ over $\mathbb{K}$. Let $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]]$ be the $\mathbb{K}$-algebra generated by $f_1, \dots, f_s$. Set $U = \mathbb{R}_+^*$ and let $a \in U^n$. If $a = (a_1, \dots, a_n)$ then a defines a linear form on $\mathbb{R}^n$ which maps $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ to the inner product

$$< a, \alpha >= \sum_{i=1}^{n} a_i \alpha_i$$

of a with α. We denote abusively the linear form by a. Let $\underline{x} = (x_1, \dots, x_n)$ and let $f = \sum c_\alpha \underline{x}^\alpha$ be a nonzero element of $\mathbf{F}$. We set $\mathrm{Supp}(f) = \{\alpha | c_\alpha \neq 0\}$ and we call it the support of f. We set

$$\nu(f, a) = \min\{< a, \alpha > | \alpha \in \mathrm{Supp}(f)\}$$

and we call it the a-valuation of f). We set by convention $\nu(0, a) = +\infty$. Let

$$\mathrm{i}n(f, a) = \sum_{\alpha \in \mathrm{Supp}(f) | < a, \alpha >= \nu(f,a)} c_\alpha \underline{x}^\alpha.$$

We call $\mathrm{i}n(f, a)$ the a-initial form of f. Note that $\mathrm{i}n(f, a)$ is a polynomial. This notion can also be defined this way: we associate with f its Newton polyhedron defined to be $\Gamma_+(f) =$ the convex hull in $\mathbb{R}_+^n$ of $\bigcup_{\alpha \in \mathrm{supp}(f)} \alpha + \mathbb{R}_+^n$. The set of compact faces of $\Gamma_+(f)$ is finite. Let $\{\triangle_1, \dots, \triangle_t\}$ be this set. Given $i \in \{1, \dots, t\}$. We set $f_{\triangle_i} = \sum_{\alpha \in Supp(f) \cap \triangle_i} c_\alpha \underline{x}^\alpha$. Then $\{\mathrm{i}n(f, a) \mid a \in U^n\} = \{f_{\triangle_i} \mid 1 \leq i \leq t\}$.

Let the notations be as above, and let $\prec$ be a well ordering on $\mathbb{N}^n$. We set $\mathrm{e}xp(f, a) = \max_\prec \mathrm{Supp}\ (\mathrm{i}n(f, a))$ and $\mathrm{M}(f) = c_{\exp(f,a)} \underline{x}^{\exp(f,a)}$. We set $\mathrm{i}n(\mathbf{A}, a) = \mathbb{K}[[\mathrm{i}n(f, a) | f \in \mathbf{A} \setminus \{0\}]]$. We also set $\mathrm{M}(\mathbf{A}) = \mathbb{K}[[\mathrm{M}(f, a), f \in \mathbf{A} \setminus \{0\}]]$. The set $\mathrm{e}xp(\mathbf{A}, a) = \{\mathrm{e}xp(f, a) \mid f \in \mathbf{A} \setminus \{0\}\}$ is an affine subsemigroup of $\mathbb{N}^n$.

If $a \in \mathbb{R}^n$ then $\mathrm{i}n(f, a)$ may not be a polynomial, hence $\mathrm{e}xp(f)$ is not well defined. If $a = (a_1, \dots, a_n)$ with $a_{i_1} = \ldots = a_{i_l} = 0$ then we can avoid this difficulty in completing by the tangent cone order on $(x_{i_1}, \dots, x_{i_l})$. We shall however consider elements in U^n in order to avoid technical definitions and results.

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The aim of this paper is to study the stability of $in(\mathbf{A},a)$ and $\mathrm{M}(\mathbf{A},a)$ when a varies in U^n. Note that $exp(\mathbf{A},a)$ is not necessarily finitely generated (see example 12). It becomes so If the length of $\dfrac{\mathbf{F}}{\mathbf{A}}$ is finite. Under this condition, our main results are the following:

Theorem The set $\{in(\mathbf{A},a)|a\in U^n\}$ is a finite set. The same holds for the set $\{\mathrm{M}(\mathbf{A},a)|a\in U^n\}$.

Theorem Let S be a finitely generated affine semigroup. The set $E_S=\{a\in U^n \mid \exp(\mathbf{A},a)=S\}$ is a union of convex polyhedral cones and the set of E_S, S defines a fan of U^n.

These results generalize those of [8] for ideals in $\mathbb{K}[x_1,\dots,x_n]$ and [1], [2] for ideals in the ring of differential operators over $\mathbb{K}$.

The paper is organized as follows. In Section 1. we recall the notion of canonical basis of $\mathbf{A}$ with respect to $a\in U^n$, and we give an algorithm that computes an a-canonical basis starting with a set of generators of $\mathbf{A}$ (see [6] for the case $a=(-1,\dots,-1)$). In Section 2. we prove the finiteness theorem, and in Section 3. we prove the existence of a fan associated with $\mathbf{A}$.

1. Preliminary results

Let $\mathbf{A}=\mathbb{K}[[f_1,\dots,f_s]]$ be a subalgebra of $\mathbf{F}$ generated by $\{f_1,\dots,f_s\}\subseteq\mathbf{F}$ and let the notations be as above. In particular $\prec$ is a total well ordering on $\mathbb{N}^n$ compatible with sums. Let $a\in U^n$ and consider the total ordering on $\mathbb{N}^n$ defined by:

$$\alpha <_a \alpha' \quad\Leftrightarrow\quad \begin{cases} <a,\alpha><<a,\alpha'> \\ \text{or} \\ <a,\alpha>=<a,\alpha'> \text{ and } \alpha\prec\alpha' \end{cases}$$

The total ordering $<_a$ is compatible with sums in $\mathbb{N}^n$. We shall use sometimes the notations $\alpha\succ\beta$ for $\beta\prec\alpha$ and $\alpha>_a\beta$ for $\beta<_a\alpha$. We have the following:

Lemma 1. *There doesn't exist infinite sequences $(\alpha_k)_{k\geq 0}$ such that*

$$\alpha_0 >_a \alpha_1 >_a \dots >_a \alpha_k \dots$$

With $<a,\alpha_0>=<a,\alpha_k>$ for all $k\geq 1$.

Proof. This is a consequence of Dixon's Lemma, since such a sequence satisfies $\alpha_0\succ\alpha_1\succ\dots\succ\alpha_k\succ\dots$ □

Let $a\in U^n$. Then a defines a filtration on $\mathbf{F}:\mathbf{F}=\sum_{d\geq 0}\mathbf{F}_d$ where $\mathbf{F}_d$ is the $\mathbb{K}$-vector space generated by $\underline{x}^\alpha, <a,\alpha>=d$. Let $U_1={\mathbb{Q}^*}_+$. If $a\in U_1^n$ then, given two indices $d_1<d_2$, the set of indices d such that $d_1<d<d_2$ is clearly finite. In particular we get the following:

Lemma 2. *Let $\alpha,\beta\in U_1^n$ and assume that $\alpha>_a\beta$. There doesn't exist infinite sequences $(\alpha_k)_{k\geq 0}$ such that*

$$\alpha >_a \alpha_0 >_a \alpha_1 >_a \dots >_a \beta$$

Proof. The set of $<a,\gamma>, <a,\alpha>><a,\gamma>><a,\beta>$ is finite. Then the result is a consequence of Lemma 1 □

Definition 3. *Let $a\in U^n$ and let $f=\sum c_\alpha\underline{x}^\alpha$ be a nonzero element of $\mathbf{F}$. We say that f is a-homogeneous if $f\in\mathbf{F}_d$ for some d. This is equivalent to $\nu(\alpha,a)=\nu(f,a)$ for all $\alpha\in\mathrm{Supp}(f)$. Note that if $a\notin U_1^n$ then f is a-homogeneous if and only if f is a monomial.*

Definition 4. *Let $a\in U^n$ and let $\mathbf{H}$ be a subalgebra of $\mathbf{F}$. We say that $\mathbf{H}$ is a-homogeneous if it can be generated by a-homogeneous elements of $\mathbf{F}$.*

Every nonzero element $f = \sum c_\alpha \underline{x}^\alpha \in \mathbf{F}$ decomposes as $f = \sum_{k \geq d} f_k$, with $f_d = in(f,a)$ and for all $k > d$, if $f_k \neq 0$ then $f_k \in \mathbf{F}_k$.

Definition 5. *Let $a \in U^n$ and let $\{g_1, \ldots, g_r\} \subseteq \mathbf{A}$. We say that $\{g_1, \ldots, g_r\}$ is an a-canonical basis of $\mathbf{A}$ if $\mathrm{M}(\mathbf{A}, a) = \mathbb{K}[[\mathrm{M}(g_1, a), \ldots, \mathrm{M}(g_r, a)]]$. Clearly $\{g_1, \ldots, g_r\}$ is an a-canonical basis of $\mathbf{A}$ if and only if $exp(\mathbf{A}, a)$ is generated by $\{exp(g_1, a), \ldots, exp(g_r, a)\}$. In this case we write $exp(\mathbf{A}, a) = \langle exp(g_1, a), \ldots, exp(g_r, a)\rangle$*

An a-canonical basis $\{g_1, \ldots, g_r\}$ of $\mathbf{A}$ is said to be minimal if $\{\mathrm{M}(g_1, a), \ldots, \mathrm{M}(g_r, a)\}$ is a minimal set of generators of $\mathrm{M}(\mathbf{A}, a)$. It is said to be reduced if the following conditions are satisfied:

i) $\{g_1, \ldots, g_r\}$ is minimal.

ii) For all $1 \leq i \leq r, c_{\exp(g_i,a)} = 1$.

iii) For all $1 \leq i \leq r$, if $g_i - \mathrm{M}(g_i, a) \neq 0$ then $\underline{x}^\alpha \notin \mathbb{K}[[\mathrm{M}(g_1, a), \ldots, \mathrm{M}(g_r, a)]]$ for all $\alpha \in \mathrm{Supp}(g_i - \mathrm{M}(g_i, a))$.

Lemma 6. *If an a-reduced canonical basis exists, then it is unique.*

Proof. Let $F = \{g_1, \ldots, g_r\}$ and $G = \{g'_1, \ldots, g'_t\}$ be two a-reduced canonical bases of $\mathbf{A}$. Let $i = 1$. Since $\mathrm{M}(g_1, a) \in \mathbb{K}[[\mathrm{M}(g'_1, a), \ldots, \mathrm{M}(g'_t, a)]]$, then $\mathrm{M}(g_1, a) = \mathrm{M}(g'_1, a)^{l_1} \cdots \mathrm{M}(g'_t, a)^{l_t}$ for some $l_1, \ldots, l_t \in \mathbb{N}$. Every $\mathrm{M}(g'_i, a), i \in \{1, \ldots, t\}$ is in $\mathbb{K}[\mathrm{M}(g_1, a), \ldots, \mathrm{M}(g_r, a)]$. Then the equation above is possible only if $\mathrm{M}(g_1, a) = \mathrm{M}(g'_{k_1}, a)$ for some $k_1 \in \{1, \ldots, t\}$. This gives an injective map from $\{\mathrm{M}(g_1, a), \ldots, \mathrm{M}(g_r, a)\}$ to $\{\mathrm{M}(g'_1, a), \ldots, \mathrm{M}(g'_t, a)\}$. We construct in the same way an injective map from $\{\mathrm{M}(g'_1, a), \ldots, \mathrm{M}(g'_t, a)\}$ to $\{\mathrm{M}(g_1, a), \ldots, \mathrm{M}(g_r, a)\}$. Hence $r = t$ and both sets are equal. Suppose, without loss of generality that $\mathrm{M}(g_i, a) = \mathrm{M}(g'_i, a)$ for all $i \in \{1, \ldots, r\}$. If $g_i \neq g'_i$ then $\mathrm{M}(g_i - g'_i) \in \mathrm{M}(\mathbf{A}, a)$ because $g_i - g'_i \in \mathbf{A}$. This contradicts iii). $\square$

We now recall the division process in $\mathbf{A}$ (see [6] for the tangent cone order $a = (-1, \ldots, -1)$ and [4] for $n = 1$).

Theorem 7. *Let $a \in U_1^n$ and let $\{F_1, \ldots, F_s\} \subseteq \mathbb{K}[[\underline{x}]]$. Let F be a nonzero element of $\mathbb{K}[[\underline{x}]]$. There exist $H \in \mathbb{K}[[F_1, \ldots, F_s]]$ and $R \in \mathbf{F}$ such that the following conditions hold:*

(1) $F = H + R$

(2) If $R = \sum_\beta b_\beta \underline{x}^\beta$, then for all $\alpha \in \mathrm{Supp}(R), \underline{x}^\beta \notin \mathbb{K}[[\mathrm{M}(F_1, a), \ldots, \mathrm{M}(F_s, a)]]$.

(3) Set $H = \sum_\alpha c_\alpha F_1^{\alpha_1} \cdot \ldots \cdot F_s^{\alpha_s}$. If $H \neq 0$ then $\exp(F, a) = \max_{<_a}\{\exp(F_1^{\alpha_1} \cdot \ldots \cdot F_s^{\alpha_s}, a), c_\alpha \neq 0\}$.

Proof. We define the sequences $(F^k)_{k\geq 0}, (h^k)_{k \geq 0}, (r^k)_{k\geq 0}$ in $\mathbf{F}$ by $F^0 = F, h^0 = r^0 = 0$ and $\forall k \geq 0:$

(i) If $\mathrm{M}(F^k, a) \in \mathbb{K}[[\mathrm{M}(F_1, a), \ldots, \mathrm{M}(F_s, a)]]$, write $\mathrm{M}(F^k, a) = c_\alpha \mathrm{M}(F_1, a)^{\alpha_1} \cdots \mathrm{M}(F_s, a)^{\alpha_s}$. We set

$$F^{k+1} = F^k - c_\alpha F_1^{\alpha_1} \cdots F_s^{\alpha_s}, \quad h^{k+1} = h^k + c_\alpha F_1^{\alpha_1} \cdots F_s^{\alpha_s}, \quad r^{k+1} = r^k$$

(ii) If $\mathrm{M}(F^k, a) \notin \mathbb{K}[[\mathrm{M}(F_1, a), \ldots, \mathrm{M}(F_s, a)]]$, we set

$$F^{k+1} = F^k - \mathrm{M}(F^{k+1}, a), \quad h^{k+1} = h^k, \quad r^{k+1} = r^k + \mathrm{M}(F^k, a)$$

in such a way that for all $k \geq 0$, $exp(F^k, a) <_a exp(F^{k+1}, a)$ and $F = F^{k+1} + h^{k+1} + r^{k+1}$. If $F^l = 0$ for some $l \geq 1$ then we set $H = h^l$ and $R = r^l$. We then easily verify that H, R satisfy conditions (1) to (3). Suppose that $\{F^k | k \geq 0\}$ is an infinite set. Note that, by Lemma 1, given $k \geq 1$, if $F^k \neq 0$ then there exists $k_1 > k$ such that $\nu(F^k, a) < \nu(F^{k_1}, a)$. Hence, there exists a subsequence $(F^{j_l})_{l\geq 1}$ such that $\nu(F^{j_1}, a) < \nu(F^{j_2}, a) < \cdots$ In particular, if we set $G = \lim_{k\to+\infty} F^k, H = \lim_{k\to+\infty} h^k$, and $R = \lim_{k\to+\infty} r^k$, then $G = 0$, $F = H + R$, and H, R satisfy conditions (1) to (3). This completes the proof. $\square$

Definition 8. *We call the polynomial R of Theorem 7 the a-remainder of the division of F with respect to $\{F_1,\dots,F_s\}$ and we denote it by $R=R_a(F,\{F_1,\dots,F_s\})$.*

Suppose that $\{f_1,\dots,f_s\}$ is an a-canonical basis of $\mathbf{A}$. If $\mathrm{M}(f_i,a)\in\mathbb{K}[[(\mathrm{M}(f_j,a)|j\neq i)]]$ for some $1\leq i\leq s$, then obviously $\{f_j|j\neq i\}$ is also an a-canonical basis of $\mathbf{A}$, consequently we can get this way a minimal a-canonical basis of $\mathbf{A}$. Assume that $\{f_1,\dots,f_s\}$ is minimal and let $1\leq i\leq s$. If $a\in U_1^n$ then, dividing $f=f_i-\mathrm{M}(f_i,a)$ by $\{f_1,\dots,f_s\}$, and replacing f_i by $\mathrm{M}(f_i,a)+R_a(f_i-\mathrm{M}(f_i,a),\{f_1,\dots,f_r\})$, we obtain an a-reduced canonical basis of $\mathbf{A}$.

The next proposition gives a criterion for a finite set of $\mathbf{A}$ to be an a-canonical basis of $\mathbf{A}$.

Proposition 9. *Let $a\in U_1^n$. The set $\{f_1,\dots,f_s\}\subseteq\mathbf{A}$ is an a-canonical basis of $\mathbf{A}$ if and only if $R_a(f,\{f_1,\dots,$
$f_s\})=0$ for all $f\in\mathbf{A}$.*

Proof. Suppose that $\{f_1,\dots,f_s\}$ is an a-canonical basis of $\mathbf{A}$ and let $f\in\mathbf{A}$. Let $R=R_a(f,\{f_1,\dots,f_s\})$. If $R\neq 0$ then $\mathrm{M}(R,a)\notin\mathrm{M}(\mathbf{A},a)$. This is a contradiction because $R\in\mathbf{A}$. Conversely, suppose that $R_a(f,\{f_1,\dots,f_s\})=0$ for all $f\in\mathbf{A}$ and let $f\in\mathbf{A}$. If $\mathrm{M}(f,a)\notin\mathbb{K}[[\mathrm{M}(f_1,a),\dots,\mathrm{M}(f_s,a)]]$ then $\mathrm{M}(f,a)$ is a monomial of $R_a(f,\{f_1,\dots,f_s\})$, which is 0. This is a contradiction. □

The criterion given in Proposition 9 is not effective since we have to divide infinitely many elements of $\mathbf{F}$. In the following we shall see that it is enough to divide a finite number of elements.

Let $\phi:\mathbb{K}[X_1,\dots,X_s]\longmapsto\mathbb{K}[\mathrm{M}(f_1,a),\dots,\mathrm{M}(f_s,a)]$ be the morphism of rings defined by $\phi(X_i)=\mathrm{M}(f_i,a)$ for all $1\leq i\leq s$. We have the following

Lemma 10. *The ideal $Ker(\phi)$ is a binomial ideal, i.e., it can be generated by binomials.*

Proof. Suppose that f_i is monic for all $i\in\{1,\dots,s\}$ and write $\mathrm{M}(f_i,a)=x_1^{\theta_1^i}\dots x_n^{\theta_n^i}=\underline{x}^{\theta^i}$.

Let F be a polynomial of $\mathrm{K}er(\phi)$ and write $F=M_1+\ldots+M_p$ where M_i is a monomial for all $i\in\{1,\dots,p\}$. We shall prove by induction on p, that F is a finite sum of binomials, each of them is in $\mathrm{K}er(\phi)$. Write $M_i=b_iX_1^{\beta_1^i}\cdots X_s^{\beta_s^i}$. If $p=2$ then F is a binomial. Suppose that $p\geq 3$. We have $\phi(M_1)=b_1\mathrm{M}(f_1,a)^{\beta_1^1}\cdots\mathrm{M}(f_s,a)^{\beta_s^1}$, which is a monomial in $x_1,\dots,x_n$. Write $\phi(M_1)=b_1x_1^{\theta_1}\dots x_n^{\theta_n}$. Since $\phi(F)=0$ then $\phi(M_i)=b_ix_1^{\theta_1}\dots x_n^{\theta_n}$ for some $i\in\{1,\dots,p\}$. whence $\phi(M_1-\dfrac{b_1}{b_i}M_i)=0$. Write

$$F=M_1-\frac{b_1}{b_i}M_i+(b_i+\frac{b_1}{b_i})M_i+\sum_{j\neq 1,i}M_j$$

If $F_1=(b_i+\dfrac{b_1}{b_i})M_i+\sum_{j\neq 1,i}M_j$ then the cardinality of monomials of F_1 is at most $p-1$. By induction, F_1 is a sum of binomials, each of them is in $\mathrm{Ker}(\phi)$. Consequently the same holds for F. □

Let $\bar{S}_1,\dots,\bar{S}_m$ be a system of generators of $\mathrm{Ker}(\phi)$, and assume, by Lemma 10, that $\bar{S}_1,\dots,\bar{S}_m$ are binomials in $\mathbb{K}[X_1,\dots,X_s]$. Assume that $f_1,\dots,f_s$ are monic with respect to $<_a$. For all $1\leq i\leq m$, we can write $S_i(X_1,\dots,X_s)=X_1^{\alpha_1^i}\dots X_s^{\alpha_s^i}-X_1^{\beta_1^i}\dots X_s^{\beta_s^i}$. Let $S_i=\bar{S}_i(f_1,\dots,f_s)$. We have $S_i=f_1^{\alpha_1}\cdots f_s^{\alpha_s}-f_1^{\beta_1}\cdots f_s^{\beta_s}$, and $\mathrm{e}xp(S_i)>_a exp(f_1^{\alpha_1}\cdots f_s^{\alpha_s})=\mathrm{e}xp(f_1^{\beta_1}\cdots f_s^{\beta_s})$

Proposition 11. *Let $a\in U_1^n$. With the notations above, the following conditions are equivalent:*

(1) The set $\{f_1,\dots,f_s\}$ is an a-canonical basis of $\mathbf{A}$.
(2) For all $i\in\{1,\dots,m\},R_a(S_i,\{f_1,\dots,f_s\})=0$.

Proof. (1) implies (2) by Proposition 9.

(2) $\Longrightarrow$ (1): We shall prove that $R_a(f, \{f_1, \ldots, f_s\}) = 0$ for all $f \in \mathbf{A}$. Let f be a nonzero element of $\mathbf{A}$ and let $R = R_a(f, \{f_1, \ldots, f_s\})$. Then $R \in \mathbf{A}$. If $R \neq 0$ then $\mathrm{M}(R, a) \in \mathrm{M}(\mathbf{A}, a)$. Write

$$R = \sum_{\theta} c_\theta f_1^{\theta_1} \cdots f_s^{\theta_s}$$

and let $\alpha = inf_{\theta, c_\theta \neq 0}(exp(f_1^{\theta_1} \cdots f_s^{\theta_s}, a))$. Since $exp(R, a) \notin exp(\mathbf{A}) = \langle exp(f_1, a), \ldots, exp(f_s, a)\rangle$ then $exp(R, a) >_a \alpha$. Let $\{\theta^1, \ldots, \theta^l\}$ be such that $\alpha = \exp(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}, a)$ for all $i \in \{1, \ldots, l\}$ (such a set is clearly finite). If $\sum_{i=1}^l c_{\theta^i} \mathrm{M}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}, a) \neq 0$, then $exp(R, a) \in \langle exp(f_1, a), \ldots, exp(f_s, a)\rangle$, which is a contradiction. Hence, $\sum_{i=1}^l c_{\theta^i} \mathrm{M}(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}, a) = 0$, and consequently $\sum_{i=1}^l c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i}$ is an element of $\ker(\phi)$. In particular

$$\sum_{i=1}^{l} c_{\underline{\theta}^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} = \sum_{k=1}^{m} \lambda_k \bar{S}_k$$

with $\lambda_k \in \mathbb{K}[X_1, \ldots, X_s]$ for all $k \in \{1, \ldots, m\}$. Whence

$$\sum_{i=1}^{l} c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i} = \sum_{k=1}^{m} \lambda_k(f_1, \ldots, f_s) S_k.$$

From the hypothesis $\mathrm{R}_a(S_k, \{f_1, \ldots, f_s\}) = 0$ for all $k \in \{1, \ldots, m\}$. Hence there is an expression of S_k of the form $S_k = \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ with $exp(f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}, a) >_a exp(S_k, a)$. Replacing $\sum_{i=1}^l c_{\underline{\theta}^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$ by $\sum_{k=1}^m \lambda_k(f_1, \ldots, f_s) \sum_{\underline{\beta}^k} c_{\underline{\beta}^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ in the expression of R, we can rewrite R as

$$R = \sum_{\underline{\theta'}} c_{\underline{\theta'}} f_1^{\theta'_1} \cdots f_s^{\theta'_s}$$

with $\alpha_1 = inf_{\theta', c_{\theta'} \neq 0} exp(f_1^{\theta'_1} \cdots f_s^{\theta'_s}, a)) >_a \alpha$. Then we restart with this representation. We construct this way an infinite sequence $\exp(R, a) >_a \ldots >_a \alpha_1 >_a \alpha$, which contradicts Lemma 2. $\square$

The characterization given in Proposition 11 suggests an algorithm that construct, starting with a set of generators of $\mathbf{A}$, an a-canonical basis of $\mathbf{A}$. However, such a canonical basis can be infinite as it is shown in the following example:

Example 12. *(see* [9]*) Let* $\mathbf{A} = \mathbb{K}[[x + y, xy, xy^2]]$ *and let* $a = (2, 1)$*. Then* $\mathrm{M}(x + y, a) = x, \mathrm{M}(xy, a) = xy$*, and* $\mathrm{M}(xy^2, a) = xy^2$*. The kernel of the map:*

$$\phi : \mathbb{K}[X, Y, Z] \longmapsto \mathbb{K}[x, y], \phi(X) = x, \phi(Y) = xy, \phi(Z) = xy^2$$

is generated by $\bar{S}_1 = XZ - Y^2$*. Hence* $S = (x + y)xy^2 - x^2y^2 = -xy^3 = R_a(-xy^3, \{x + y, xy, xy^2\})$*. Then* xy^3 *is a new element of the a-canonical basis of* $\mathbf{A}$*. If we restart with the representation* $\mathbf{A} = \mathbb{K}[[x + y, xy, xy^2, xy^3]]$*, then a new element,* xy^4*, will be added to the a-canonical basis of* $\mathbf{A}$*. In fact,* xy^n *is an element of the minimal reduced a-canonical basis of* $\mathbf{A}$ *for all* $n \geq 1$*. In particular the a-canonical basis of* $\mathbf{A}$ *is infinite.*

In the following we shall assume that the length $l(\dfrac{\mathbf{F}}{\mathbf{A}})$ is finite. This guarantees the finiteness of a canonical basis. Under this hypothesis, using the results above, we get the following algorithm:

Algorithm. Let $\mathbf{A} = \mathbb{K}[[f_1, \ldots, f_s]]$ and let $a \in U_1^n$. Let $\{\bar{S}_1, \ldots, \bar{S}_m\}$ be a set of generators of the map ϕ of Proposition 10 and let $S_i = \bar{S}_i(f_1, \ldots, f_s)$ for all $i \in \{1, \ldots, m\}$.

(1) If $R_a(S_i, \{f_1, \ldots, f_s\}) = 0$ for all $i \in \{1, \ldots, m\}$ then $\{f_1, \ldots, f_s\}$ is an a-canonical basis of $\mathbf{A}$.
(2) If $R = R_a(S_i, \{f_1, \ldots, f_s\}) \neq 0$ for some $i \in \{1, \ldots, m\}$ then we set $R = f_{s+1}$ and we restart with $\{f_1, \ldots, f_r, f_{r+1}\}$. Note that in this case, we have $\langle exp(f_1, a), \ldots, exp(f_s, a)\rangle \subset \langle exp(f_1, a), \ldots, exp(f_s, a), exp(f_{s+1}, a)\rangle \subseteq exp(\mathbf{A}, a)$. By hypothesis, $l(\frac{\mathbf{F}}{\mathbf{A}}) < +\infty$, hence, after a finite number of operations, we get an a-canonical basis of $\mathbf{A}$.

2. A finiteness Theorem

Let the notations be as in Section 1. In particular $\mathbf{A} = \mathbb{K}[[f_1, \ldots, f_s]]$ with $\{f_1, \ldots, f_s\} \subseteq \mathbf{F}$. We shall assume that $l(\frac{\mathbf{F}}{\mathbf{A}}) < +\infty$. The aim of this section is to prove that the set of $in(A, a), a \in U^n$ is finite. We first recall this result when $\mathbf{A} = \mathbb{K}[f]$ then we prove some preliminary results which will also be used later in the paper.

Lemma 13. *Let f be a nonzero element of $\mathbb{K}[[x_1, \ldots, x_n]]$. The set $\{\mathrm{M}(f, a), a \in U^n\}$ (resp. $\{in(f, a), a \in U^n\}$) is finite.*

Proof. Write $f = \sum_\alpha c_\alpha \underline{x}^\alpha$ and let $E = \cup_{\alpha \in Supp(f)} \alpha + \mathbb{N}^n$. Then $E + \mathbb{N}^n \subseteq E$, and consequently there exists a finite set of E, say $\{\alpha_1, \ldots, \alpha_r\}$, such that $E = \cup_{i=1}^r \alpha_i + \mathbb{N}^n$. By definition $\alpha_i \in Supp(f)$ for all $i \in \{1, \ldots, r\}$. If we choose the α_i's such that $\alpha_i \notin \cup_{j \neq i} \alpha_j + \mathbb{N}^n$, then $\{exp(A, a), a \in U^n\} = \{\alpha_1, \ldots, \alpha_r\}$. This proves our assertion. □

Lemma 14. *Let $a \neq b$ be two elements of U^n. Let $\{g_1, \ldots, g_r\}$ be an a-reduced canonical basis of $\mathbf{A}$. If $\mathrm{M}(\mathbf{A}, a) = \mathrm{M}(\mathbf{A}, b)$ then $\{g_1, \ldots, g_r\}$ is also a b-reduced canonical basis of $\mathbf{A}$.*

Proof. Let $i \in \{1, \ldots, r\}$ and write $g_i = \mathrm{M}(g_i, a) + \sum c_\beta \underline{x}^\beta$ where for all β, if $c_\beta \neq 0$, then $\underline{x}^\beta \notin \mathrm{M}(\mathbf{A}, a)$. Since $\mathrm{M}(\mathbf{A}, a) = \mathrm{M}(\mathbf{A}, b)$ then for all β, if $c_\beta \neq 0$, then $\underline{x}^\beta \notin \mathrm{M}(\mathbf{A}, a)$. This implies that $\mathrm{M}(g_i, a) = \mathrm{M}(g_i, b)$. Hence $\{g_1, \ldots, g_r\}$ is a b-canonical basis of $\mathbf{A}$, and the same argument shows that this basis is also reduced. □

Lemma 15. *Let $a \neq b$ be two elements of U^n. If $\mathrm{M}(\mathbf{A}, a) \neq \mathrm{M}(\mathbf{A}, b)$, then $\mathrm{M}(\mathbf{A}, a) \not\subseteq \mathrm{M}(\mathbf{A}, b)$.*

Proof. Assume that $\mathrm{M}(\mathbf{A}, a) \subseteq \mathrm{M}(\mathbf{A}, b)$ and let $\{g_1, \ldots, g_r\}$ be a b-reduced canonical basis of $\mathbf{A}$. By hypothesis, there is $1 \leq i \leq r$ such that $\mathrm{M}(g_i, b) \in \mathrm{M}(\mathbf{A}, b) \setminus \mathrm{M}(\mathbf{A}, a)$. Write $g_i = \mathrm{M}(g_i, b) + \sum c_\beta \underline{x}^\beta$. For all β, if $c_\beta \neq 0$, then $\underline{x}^\beta \notin \mathrm{M}(\mathbf{A}, b)$, hence $\underline{x}^\beta \notin \mathrm{M}(\mathbf{A}, a)$, which implies that $\mathrm{M}(g_i, a) \notin \mathrm{M}(\mathbf{A}, a)$ ($\mathrm{M}(\mathbf{A}, a) \subseteq \mathrm{M}(\mathbf{A}, b)$). This is a contradiction because $g_i \in \mathbf{A}$. □

Corollary 16. *If $a \in U^n \setminus U_1^n$ then there exists $b \in U_1^n$ such that $\mathrm{M}(\mathbf{A}, a) = \mathrm{M}(\mathbf{A}, b)$.*

Proof. Let $\{g_1, \ldots, g_r\}$ be an a-reduced canonical basis of $\mathbf{A}$. By hypothesis, there exists $\epsilon > 0$ such that for all $i \in \{1, \ldots, r\}$ and for all $b \in B(a, \epsilon)$, $\mathrm{M}(g_i, a) = \mathrm{M}(g_i, b)$ (where $B(a, \epsilon)$ denotes the ball of ray ϵ centered at a). Take $b \in U^n \cap B(a, \epsilon)$. We have $\mathrm{M}(\mathbf{A}, a) \subseteq \mathrm{M}(\mathbf{A}, b)$. By Lemma 15, $\mathrm{M}(\mathbf{A}, a) = \mathrm{M}(\mathbf{A}, b)$ □

Corollary 17. *Let $a \neq b$ be two elements of U^n. If $in(\mathbf{A}, a) \neq in(\mathbf{A}, b)$, then $in(\mathbf{A}, a) \not\subseteq in(\mathbf{A}, b)$*

Proof. We shall prove that if $in(\mathbf{A}, a) \subseteq in(\mathbf{A}, b)$ then $in(\mathbf{A}, a) = in(\mathbf{A}, b)$. Let to this end $\{g_1, \ldots, g_r\}$ be an a-reduced canonical basis of $\mathbf{A}$ and let $i \in \{1, \ldots, r\}$. Write $in(g_i, a) = M_1 + \ldots + M_t$ where M_j is b-homogeneous for all $i \in \{1, \ldots, t\}$. By hypothesis $in(g_i, a) \in in(\mathbf{A}, b)$, hence $M_j \in in(\mathbf{A}, b)$ for all $j \in \{1, \ldots, t\}$. Suppose that $\mathrm{M}(g_i, a)$ is a monomial of M_1. We have $M_1 \in in(\mathbf{A}, b)$. But M_1 is also a-homogeneous. It follows that $\mathrm{M}(M_1, b) = \mathrm{M}(M_1, a) = \mathrm{M}(g_i, a)$, in particular $\mathrm{M}(g_i,a) \in \mathrm{M}(\mathbf{A},b)$. This proves that $\mathrm{M}(\mathbf{A},a) \subseteq \mathrm{M}(\mathbf{A},b)$. By Lemma 15 we have $\mathrm{M}(\mathbf{A},a) = \mathrm{M}(\mathbf{A},b)$ and $\{g_1,\ldots,g_r\}$ is also a b-reduced canonical basis of $\mathbf{A}$. Finally $\mathrm{in}(\mathbf{A},a)$ is generated by $\{in(g_1,a),\ldots,in(g_s,a)\}$ (resp. $\mathrm{in}(\mathbf{A},b)$ is generated by $\{in(g_1,b),\ldots,in(g_r,b)\}$). Now the argument above shows that for all $i \in \{1,\ldots,r\}$, $\mathrm{in}(g_i,a) = \mathrm{in}(g_i,b)$. This proves the equality. □

Remark 18. *1. Let $a \neq b$ be two elements of U^n. The proof of Corollary 17 implies that if $in(\mathbf{A},a) = \mathrm{in}(\mathbf{A},b)$ and if $\{g_1,\ldots,g_r\}$ is an a-reduced canonical basis of $\mathbf{A}$ then $\{g_1,\ldots,g_r\}$ is also a b-reduced canonical basis of $\mathbf{A}$.*

2. Corollary 17 implies the following: if $a \in U^n \setminus U_1^n$ then there exists $b \in U_1^n$ such that $\mathrm{in}\mathbf{A},a) = \mathrm{M}(\mathbf{A},a) = \mathrm{M}(\mathbf{A},b) = \mathrm{in}(\mathbf{A},b)$..

We can now state and prove the following finiteness theorem:

Theorem 19. *Let $A = \mathbb{K}[[f_1,\ldots,f_s]]$ and let the notations be as above. The set $\mathrm{M}(A) = \{\mathrm{M}(A;a) | a \in U^n\}$ is finite. In particular the set $I(A) = \{\mathrm{in}(\mathbf{A},a) | a \in U^n\}$ is finite.*

Proof. We need only to prove that $\mathrm{M}(A)$ is a finite set. Assume that M is infinite. By Lemma 13 there is an infinite set $U_1 = \{a_1,a_2,\ldots\}$ in U^n such that for all $1 \leq k \leq s$ and for all $a \in U_1$, $\mathrm{M}(f_k,a) = m_k$ where m_k is a nonzero monomial of f_k. Let $J_1 = \mathbb{K}[[m_1,\ldots,m_s]]$: $J_1 \subseteq \mathrm{M}(\mathbf{A},a)$ for all $a \in U_1$. Obviously $J_1 \neq \mathrm{M}(\mathbf{A},a_1)$ (otherwise, $\mathrm{M}(\mathbf{A},a_1) \subset \mathrm{M}(\mathbf{A},a_2)$, for example. This contradicts Lemme 15). We claim that there is $f_{s+1} \in \mathbf{A}$ such that for all $\beta \in \mathrm{Supp}(f_{s+1}), \underline{x}^\beta \notin J_1$. Let to this end $m \in \mathrm{M}(\mathbf{A},a_1) \setminus J_1$ and let $f \in \mathbf{A}$ such that $\mathrm{M}(f,a_1) = m$. Obviously $m \notin J_1$. We set $f_{s+1} = R_{a_1}(f,\{f_1,\ldots,f_s\})$. By lemma 13, there is monomial m_{s+1} of f_{s+1} and an infinite subset $U_2 \subseteq U_1$ such that for all $a \in U_2, \mathrm{M}(f_{s+1},a) = m_{s+1}$.

Let $J_2 = \mathbb{K}[[m_1,\ldots,m_s,m_{s+1}]]$: $J_1 \subset J_2$. The same process applied to $\{f_1,\ldots,f_{s+1}\}, J_2$ and U_2 will construct $m_{s+2} \notin J_2$, $f_{s+2} \in \mathbf{A}$, and an infinite subset $U_3 \subseteq U_2$ such that for all $a \in U_3, \mathrm{M}(f_{s+2},a) = m_{s+2}$. We get this way an infinite increasing sequence $J_1 \subset J_2 \subset J_3 \subset \ldots$ and for all i, there is $a_i \in U^n$ such that $J_i \subseteq \mathrm{M}(\mathbf{A},a_i)$. This is a contradiction because $l(\dfrac{\mathbf{F}}{\mathbf{A}})$ is finite. □

Definition 20. *The set $\{g_1,\ldots,g_r\}$ of $\mathbf{A}$ which is an a- canonical basis of $\mathbf{A}$ for all $a \in U^n$, is called the universal canonical basis of $\mathbf{A}$.*

3. The Newton fan

Let $\mathbf{A} = \mathbb{K}[[f_1,\ldots,f_s]]$ and let the notations be as in Section 2. In this section we aim to study the stabitily of $exp(\mathbf{A},a)$ and $in(\mathbf{A},a)$ when a very in U^n. Let S be a finitely generated affine semigroup of $\mathbb{N}^n$. Let

$$E_S = \{a \in U^n \mid exp(\mathbf{A},a) = S\}$$

We have the following:

Theorem 21. *There exists a partition $\mathcal{P}$ of U^n into convex rational polyherdal cones such that for all $\sigma \in \mathcal{P}$, $exp(\mathbf{A},a)$ and $in(\mathbf{A},a)$ do not depend on $a \in \sigma$.*

In order to prove Theorem 21 we start by fixing some notations. Let S be a finitely generated affine semigroup of $\mathbb{N}^n$ and let $a \in E_S$. Let $\{g_1,\ldots,g_r\}$ be the a-reduced canonical basis of $\mathbf{A}$. By Lemma 15, Lemma 17, and Remark 18, $\{g_1,\ldots,g_r\}$ is also the b-reduced canonical basis of $\mathbf{A}$ for all $b \in E_S$. Denote by $\sim$ the equivalence relation on U^n defined from $\{g_1,\ldots,g_r\}$ by

$$a \sim b \Longleftrightarrow \mathrm{in}(g_i,a) = \mathrm{in}(g_i,b) \quad \text{for all} \quad i \in \{1,\ldots,r\},$$

Proposition 22. *$\sim$ defines on U^n a finite partition into convex rational polyhedral cones and E_S is a union of a part of these cones.*

Proof. Let $c, d \in U^n$ such that $c \sim d$ and let e in the segment $[c, d]$, also let $\theta \in [0, 1]$ such that $e = \theta c + (1 - \theta))d$. Then $a \in U^n$ and $\text{in}(g_i, e) = \text{in}(g_i, c) = \text{in}(g_i, d)$ by an immediate verification. Moreover, $c \sim t \cdot c$ for all $c \in U^n$ and $t > 0$. Therefore the equivalence classes are convex rational polyhedral cones (the rationality results from Corollary 16 and Remark 18). On the other hand, if $c \sim d$ and $c \in E_S$, then $d \in E_S$. This proves that E_S is a union of classes for $\sim$, the number of classes being finite by Theorem 19 □

Proof of Theorem 21 We define $\mathcal{P}$ in the following way: for each S we consider the restriction $\mathcal{P}_S$ on E_S of the above partition, and then $\mathcal{P}$ is the finite union of the $\mathcal{P}_S$'s. On each cone of the partition, $in(\mathbf{A}, a)$ and $exp(\mathbf{A}, a)$ are fixed. Conversely assume that $in(\mathbf{A}, a)$ is fixed and let b such that $in(\mathbf{A}, a) = in(\mathbf{A}, b)$. By Corollary 17 and Remark 18, an a-reduced canonical basis $\{g_1, \ldots, g_r\}$ of $\mathbf{A}$ is also a b-reduced canonical basis of $\mathbf{A}$. Moreover, $in(g_i, a) = in(g_i, b)$ and $exp(\mathbf{A}, a) = exp(\mathbf{A}, b)$. This ends the proof of the theorem except for the convexity of E_S proved below.

Lemma 23. *E_S is a convex set: if $a \neq b \in E_S$ then $[a, b] \subseteq E_S$*

Proof. Let $a, b \in E_S$ and let $\lambda \in]0, 1[$. Let $\{g_1, \ldots, g_r\}$ be an a (and then b) reduced canonical basis of $\mathbf{A}$. Let $i \in \{1, \ldots, r\}$ and set $M = \text{in}(g_i, a)$. Write $M = M_1 + \ldots + M_t$ where M_k is b-homogeneous for all $k \in \{1, \ldots, t\}$ and $\nu(M_1, b) > \nu(M_k, b)$ for all $k \in \{2, \ldots, t\}$. We have $\nu(g_i, a) = \nu(M_1, a) = \nu(M_k, a)$ and $\nu(M_1, b) > \nu(M_k, b)$ for all $k \in \{2, \ldots, t\}$. This implies that $\nu(g_i, \theta a + (1 - \theta)b) = \nu(M_1, \theta a + (1 - \theta)b) > \nu(M_k, \theta a + (1 - \theta)b))$ for all $k \in \{2, \ldots, t\}$, hence $exp(g_i, \theta a + (1 - \theta)b) = exp(g_i, a) = \exp(g_i, b)$. In particular $exp(\mathbf{A}, a) \subseteq exp(\mathbf{A}, \theta a + (1 - \theta)b)$. By Lemma 15 we get the equality. This proves that $exp(\mathbf{A}, \theta a + (1 - \theta)b) = S$. □

In the following we shall give some precisions about the partition above. Obviously if S_1 and S_2 are two distinct finitely generated affine semigroups and if $a \in E_{S_1}$ and $b \in E_{S_2}$ then $in(\mathbf{A}, a) \neq in(\mathbf{A}, b)$ and $exp(\mathbf{A}, a) \neq exp(\mathbf{A}, b)$, hence, by Lemma 15 and Corollary 17, neither $in(\mathbf{A}, a) \subseteq in(\mathbf{A}, b)$ nor $in(\mathbf{A}, b) \subseteq in(\mathbf{A}, a)$, and the same conclusion holds for $exp(\mathbf{A}, a)$ and $exp(\mathbf{A}, b)$.

Next we shall characterize open cones of the partition with maximal dimension. Let f be a nonzero element of $\mathbb{K}[[x_1, \ldots, x_n]]$ and let $a \in U^n$. We say that $in(f, a)$ is multihomogeneous if $in(f, a)$ is b-homogeneous for all $b \in U^n$. This is equivalent to saying that $in(f, a)$ is a monomial.

Definition 24. *Let $a \in U^n$. We say that $\text{in}(\mathbf{A}, a)$ is a multihomogeneous algebra if it is generated by multihomogeneous elements. Note that in this case, If $g \in \text{in}(\mathbf{A}, a)$ then every monomial of g is also in $\text{in}(\mathbf{A}, a)$.*

Lemma 25. *Let $a \in U^n$ and let $\{g_1, \ldots, g_r\}$ be an a-reduced canonical basis of $\mathbf{A}$. Then $\text{in}(\mathbf{A}, a)$ is a multihomogeneous algebra if and only if $\text{in}(g_i, a)$ is a monomial for all $i \in \{1, \ldots, r\}$.*

Proof. We only need to prove the if part. Let $i \in \{1, \ldots, r\}$ and write $in(g_i, a) = M_1 + \ldots + M_t$ with $exp(g_i, a) = exp(M_1, a)$. Assume that $t > 1$. Since $in(\mathbf{A}, a)$ is multihomogeneous then $M_i \in in(\mathbf{A}, a)$ for all $i \in \{2, \ldots, t\}$. But $\{g_1, \ldots, g_r\}$ is reduced. This is a contradiction. Hence $t = 1$ and $in(g_i, a)$ is a monomial. □

Proposition 26. *The set of $a \in U^n$ for which $\text{in}(\mathbf{A}, a)$ is multihomogeneous defines the open cones of dimension n of $\mathcal{P}$.*

Proof. Let $a \in U^n$ and let $\{g_1, \ldots, g_r\}$ be an a-reduced canonical basis of $\mathbf{A}$. For all $i \in \{1, \ldots, r\}$, $in(g_i, a)$ is a monomial, hence there exists $\epsilon > 0$ such that $in(g_i, b) = in(g_i, a)$ for all $b \in B(a, \epsilon)$ (where $B(a, \epsilon)$ is the ball centered at a of ray ϵ). This proves, by Lemma 15 and Corollary 17, that $\{g_1, \ldots, g_r\}$ is also a b-reduced canonical basis of $\mathbf{A}$ for all $b \in B(a, \epsilon)$. Conversely, if a is in an open cone of E_S for some S, then for all b in a neighbourhood, an a-reduced canonical basis $\{g_1, \ldots, g_r\}$ of $\mathbf{A}$ is also a b-reduced canonical basis of $\mathbf{A}$. This implies that $\mathrm{in}(g_i, a)$ is a monomial. This proves our assertion. $\square$

Definition 27. *$\mathcal{P}$ is called the standard fan of $\mathbf{A}$.*

Remark 28. *(1) Although we do not have a proof for the existence of a fan for subalgebras $\mathbf{A}$ with $l(\dfrac{\mathbf{F}}{\mathbf{A}})$ not necessarily finite, we think that this fan does exist. This is true of course if $\exp(\mathbf{A}, a)$ is finitely generated for all $a \in U^n$, but it is not easy to verify this condition a priori.*

(2) (see [4]) Suppose that $n = 1$, i.e. $\mathbf{A} = \mathbb{K}[[f_(t), \ldots, f_s(t)]] \subseteq \mathbb{K}[[t]]$. Then $X_1 = f_1(t), \ldots, X_s = X_s(t)$ represents the expansion of a curve in $\mathbb{K}^s$ near the origin. In tis case, $\mathrm{in}(\mathbf{A}, a)$ does not depend on $a \in U^n$ and if the parametrization is primitive then $l(\dfrac{\mathbb{K}[[t]]}{\mathbf{A}}) < +\infty$. If $s = 2$ then $\mathrm{in}(\mathbf{A}, a)$ is a free numerical semigroup and the arithmetic of this semigroup contains a lot of information about the singularity of the curve at the origin.

(3) (see also [5]) If $\mathbf{A} = \mathbb{K}[f_1, \ldots, f_s]$ is a subalgebra of $\mathbf{P} = \mathbb{K}[x_1, \ldots, x_n]$ then Theorem 19 and Theorem 21 remain valid when we vary $a \in \mathbb{R}^n_+$, under the assumption that $l(\dfrac{\mathbf{P}}{\mathbf{A}}) < +\infty$ (note that in this case, $\mathrm{in}(f, a)$ is a polynomial for all $a \in \mathbb{R}^n_+$ and for all $f \in \mathbf{P}$).

Example 29. *Let $f(X_1, \ldots, X_n, Y) \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ and suppose that f has a parametrization of the form $X_1 = t_1^{e_1}, \ldots, X_n = t_n^{e_n}, Y = Y(t_1, \ldots, t_n) \in \mathbb{K}[[t_1, \ldots, t_n]]$ (for instance, this is true if f is a quasi-ordinary polynomial, i.e. the discriminant of f is of the form $X_1^{N_1} \cdots X_s^{N_s}(c + \phi(X_1, \ldots, X_s))$ with $c \in \mathbb{K}^*$ and $\phi(0, \ldots, 0) = 0$). Then $\dfrac{\mathbb{K}[[X_1, \ldots, X_n]][Y]}{f} \simeq \mathbf{A} = \mathbb{K}[[t_1^{e_1}, \ldots, t_n^{e_n}, Y(t_1, \ldots, t_n)]]$. In this case, $(e_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, e_n)$ belong to $\exp(\mathbf{A}, a)$ for all $a \in U^n$. Moreover, $\exp(\mathbf{A}, a)$ is is a free finitely generated affine semigroup in the sense of [3]. In this case, $l(\dfrac{\mathbf{F}}{\mathbf{A}})$ need not to be finite, but results of Theorem 19 and Theorem 21 are valid.*

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Université d'Angers, Mathématiques, 49045 Angers ceded 01, France

E-mail address: assi@math.univ-angers.fr