A 4-choosable graph that is not $(8:2)$-choosable

Zdeněk Dvořák, Xiaolan Hu, Jean-Sébastien Sereni

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Zdeněk Dvořák† Xiaolan Hu‡ Jean-Sébastien Sereni§

Abstract

In 1980, Erdős, Rubin and Taylor asked whether for all positive integers a, b, and m, every $(a : b)$-choosable graph is also $(am : bm)$-choosable. We provide a negative answer by exhibiting a 4-choosable graph that is not $(8 : 2)$-choosable.

Coloring the vertices of a graph with sets of colors (that is, each vertex is assigned a fixed-size subset of the colors such that adjacent vertices are assigned disjoint sets) is a fundamental notion, which in particular captures fractional colorings. The fractional chromatic number of a graph G can indeed be defined to be the infimum (which actually is a minimum) of the ratios a/b such that, if every vertex of G is replaced by a clique of order b and every edge of G is replaced by a complete bipartite graph between the relevant cliques, then the chromatic number of the obtained graph is at most a.

In their seminal work on list coloring, Erdős, Rubin and Taylor [2] raised several intriguing questions about the list version of set coloring. Before stating them, let us review the relevant definitions.

Set coloring. A function that assigns a set to each vertex of a graph is a *set coloring* if the sets assigned to adjacent vertices are disjoint. For positive integers a and $b \leq a$, an $(a : b)$*-coloring* of a graph G is a set coloring with range $\binom{\{1,\ldots,a\}}{b}$, *i.e.*, a set coloring that to each vertex of G assigns a b-element subset of $\{1, \ldots, a\}$. The concept of $(a : b)$-coloring is a generalization of the conventional vertex coloring. In fact, an $(a : 1)$-coloring is exactly an ordinary proper a-coloring.

A *list assignment* for a graph G is a function L that to each vertex v of G assigns a set $L(v)$ of colors. A set coloring φ of G is an L*-set coloring* if $\varphi(v) \subseteq L(v)$ for every $v \in V(G)$. For a positive integer b, we say that φ is an $(L : b)$*-coloring* of G if φ is an L-set coloring and $|\varphi(v)| = b$ for every $v \in V(G)$. If such an $(L : b)$-coloring exists, then G is $(L : b)$*-colorable.* For an integer $a \geq b$, we say that G is $(a : b)$*-choosable* if G is $(L : b)$-colorable for every list assignment L

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such that $|L(v)| = a$ for each $v \in V(G)$. We abbreviate $(L:1)$-coloring, $(L:1)$-colorable, and $(a:1)$-choosable to L-coloring, L-colorable, and a-choosable, respectively.

Questions and results. It is straightforward to see that if a graph is $(a:b)$-colorable, it is also $(am:bm)$-colorable for every positive integer m: we can simply replace every color in an $(a:b)$-coloring by m new colors. However, this argument fails in the list coloring setting, leading Erdős, Rubin and Taylor [2] to ask whether every $(a:b)$-choosable graph is also $(am:bm)$-choosable whenever $m \geq 1$. A positive answer to this question is sometimes referred to as "the $(am:bm)$-conjecture". Using the characterization of 2-choosable graphs found in *loc. cit.*, Tuza and Voigt [4] provided a positive answer when $a = 2$ and $b = 1$. In the other direction, Gutner and Tarsi [3] demonstrated that if k and m are positive integers and k is odd, then every $(2mk:mk)$-choosable graph is also $2m$-choosable.

Formulated differently, the question is to know whether every $(a:b)$-choosable graph is also $(c:d)$-choosable whenever $c/d = a/b$ and $c \geq a$. This formulation raises the same question when $c/d > a/b$, which was also asked by Erdős, Rubin and Taylor [2]. About ten years ago, Gutner and Tarsi [3] answered this last question negatively, by studying the kth choice number of a graph for large values of k. More precisely, the *kth choice number* of a graph G is $\mathrm{ch}_{:k}(G)$, the least integer a for which G is $(a:k)$-choosable. Their result reads as follows.

Theorem 1 (Gutner & Tarsi, 2009)**.** *Let G be a graph. For every positive real ϵ, there exists an integer k_0 such that $\mathrm{ch}_{:k}(G) \leq k(\chi(G)+\epsilon)$ for every $k \geq k_0$.*

As a direct corollary, one deduces that for all integers $m \geq 3$ and $\ell > m$, there exists a graph that is $(a:b)$-choosable and not $(\ell:1)$-choosable with $\frac{a}{b} = m$. (To see this, one can for example apply Theorem 1 with $\varepsilon = 1$ to the disjoint union of a clique of order $m-1$ and a complete balanced bipartite graph with choice number $\ell+1$.)

Another related result that should be mentioned here was obtained by Alon, Tuza and Voigt [1]. They proved that for every graph G,

$$\inf\left\{\frac{a}{b}\,\middle|\, G \text{ is } (a:b)\text{-choosable}\right\} = \inf\left\{\frac{a}{b}\,\middle|\, G \text{ is } (a:b)\text{-colorable}\right\}.$$

In other words, the fractional choice number of a graph equals its fractional chromatic number.

The purpose of our work is to provide a negative answer to Erdős, Rubin and Taylor's question when $a = 4$ and $b = 1$.

Theorem 2. *There exists a graph G that is* 4*-choosable, but not* $(8:2)$*-choosable.*

We build such a graph by incrementally combining pieces with certain properties. Each piece is defined, and its relevant properties established, in the forthcoming lemmas.

Gadgets and lemmas. A *gadget* is a pair (G, L_0), where G is a graph and L_0 is an assignment of lists of even size. A *half-size list assignment* for the gadget is a list assignment L for G such that $|L(v)| = |L_0(v)|/2$ for every $v \in V(G)$. Let us start the construction by a key observation on list colorings of 5-cycles.

Lemma 3. *Consider the gadget* (C, L_0)*, where* $C = v_1v_2v_3v_4v_5$ *is a* 5*-cycle,* $L_0(v_1) = \{1, 2, 5, 6\}$*,* $L_0(v_2) = \{1, 4, 5, 6\}$*,* $L_0(v_3) = L_0(v_4) = \{3, 4, 5, 6\}$ *and* $L_0(v_5) = \{2, 4, 5, 6\}$*. Then* C *is* L*-colorable for every half-list assignment* L *such that* $|L(v_1) \cap L(v_3)| \leq 1$*, but* C *is not* $(L_0 : 2)$*-colorable.*

Proof. The first statement is well known, but let us give the easy proof for completeness: since $|L(v_1) \cap L(v_3)| \leq 1$, we have $|L(v_1) \cup L(v_3)| \geq 3$, and thus $L(v_1)$ or $L(v_3)$ contains a color c_6 not belonging to $L(v_2)$. By symmetry, we can assume that $c_6 \in L(v_1)$. We color v_1 by c_6 and then for $i = 5, 4, 3, 2$ in order, we color v_i by a color $c_i \in L(v_i) \setminus \{c_{i+1}\}$. The resulting coloring is proper—we have $c_2 \neq c_6$, since $c_6 \notin L(v_2)$.

Suppose now that C has an $(L_0 : 2)$-coloring, and for $c \in \{1, \ldots, 6\}$ let V_c be the set of vertices of C on which the color c is used. Since two colors are used on each vertex of C, we have $\sum_{c=1}^{6} |V_c| = 10$. On the other hand, V_c is an independent set of a 5-cycle, and thus $|V_c| \leq 2$ for every color c. Furthermore, color 1 only appears in the lists of v_1 and v_2, which are adjacent in C. It follows that $|V_1| \leq 1$. The situation is similar for color 2, which appears only in the lists of v_1 and v_5, and also for color 3, which only appears in the lists of v_3 and v_4. Consequently, $\sum_{c=1}^{6} |V_c| \leq 3 \cdot 2 + 3 \cdot 1 = 9$, which is a contradiction. □

Corollary 4. *Consider the gadget* (G_1, L_1)*, where* G_1 *consists of a* 5*-cycle* $C = v_1v_2v_3v_4v_5$ *and a path* v_1xyv_3*, with* $L_1(v_1) = L_1(v_3) = \{1, \ldots, 6\}$*,* $L_1(v_2) = \{1, 4, 5, 6\}$*,* $L_1(v_4) = \{3, 4, 5, 6\}$*,* $L_1(v_5) = \{2, 4, 5, 6\}$*,* $L_1(x) = \{1, 2, 3, 4\}$ *and* $L_1(y) = \{1, 2\}$*. Then* G_1 *is* L*-colorable for every half-list assignment* L *such that* $L(v_1) = L(v_3)$*, but* G_1 *is not* $(L_1 : 2)$*-colorable.*

Proof. Let L be a half-list assignment for G_1. First L-color y and x by colors $c_y \in L(y)$ and $c_x \in L(x) \setminus \{c_y\}$, respectively. Since $c_x \neq c_y$ and $L(v_1) = L(v_3)$, there exist sets $L'(v_1) \subseteq L(v_1) \setminus \{c_x\}$ and $L'(v_3) \subseteq L(v_3) \setminus \{c_y\}$ of size two such that $L'(v_1) \neq L'(v_3)$. Let $L'(v_i) = L(v_i)$ for $i \in \{2, 4, 5\}$. Lemma 3 implies that C is L'-colorable, which yields an L-coloring of G.

In an $(L_1 : 2)$-coloring, the vertex y would have to be assigned $\{1, 2\}$ and x would have to be assigned $\{3, 4\}$, and thus the sets of available colors for v_1 and for v_3 would have to be $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$, respectively. However, no such $(L_1 : 2)$ coloring of C exists according to Lemma 3. □

Next we construct auxiliary gadgets, which will be combined with the gadget from Corollary 4 to deal with the case where $L(v_1) \neq L(v_3)$. Let G be a graph, let S be a subset of vertices of G and L a list assignment for G. An *L-coloring* of S is a coloring of the subgraph of G induced by S. Moreover, if S' is a subset of vertices of G that contains S and φ' is an L-coloring of S', then φ' *extends* φ if $\varphi'|S = \varphi$. Let (G, L_0) be a gadget, let v_1 and v_3 be distinct vertices of G, and let S be a set of vertices of G not containing v_1 and v_3. The gadget is (v_1, v_3, S)*-relaxed* if every half-list assignment L satisfies at least one of the following conditions.

(i) There exists an L-coloring ψ_0 of $\{v_1, v_3\}$ such that every L-coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L-coloring of G.

(ii) $L(v_1) = L(v_3)$ and there exists and L-coloring ψ_0 of S such that every L-coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L-coloring of G.

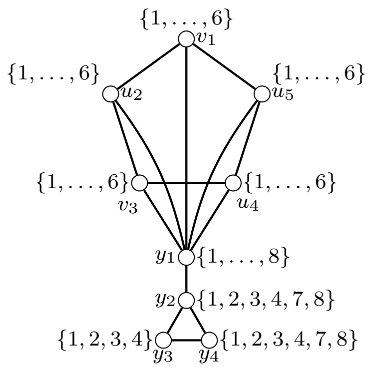

Figure 1: The gadget (G_2, L_2) of Lemma 5

Lemma 5. *Consider the gadget (G_2, L_2) presented in Figure 1, where G_2 consists of a 5-cycle $C_2 = v_1u_2v_3u_4u_5$, a vertex y_1 adjacent to all vertices of C_2, a triangle $y_2y_3y_4$, and an edge y_1y_2, with $L_2(v) = \{1, \dots, 6\}$ for every $v \in V(C_2)$, $L_2(y_1) = \{1, \dots, 8\}$, $L_2(y_2) = L_2(y_4) = \{1, 2, 3, 4, 7, 8\}$, and $L_2(y_3) = \{1, 2, 3, 4\}$. The gadget is $(v_1, v_3, \{y_4\})$-relaxed, and $\varphi(y_4) \cap \{7, 8\} \neq \varnothing$ for every $(L_2 : 2)$-coloring φ of G_2.*

Proof. Let L be a half-list assignment for G_2. If not all vertices of C_2 have the same list, then choose a color $c \in L(y_1) \setminus L(y_2)$, and observe there exists an L-coloring of $G_2[V(C_2) \cup \{y_1\}]$ such that y_1 has color c. Let ψ_0 be the restriction of this coloring to $\{v_1, v_3\}$. Clearly, every L-coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L-coloring of G, and thus (i) holds.

If all the vertices of C_2 have the same list (and hence in particular $L(v_1) = L(v_3)$), then let c be a color in $L(y_1) \setminus L(v_1)$. Observe that there exists an L-coloring of $G_2[\{y_1, y_2, y_3, y_4\}]$ such that y_1 has color c. Let ψ_0 be the restriction of this coloring to y_4. Again, every L-coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L-coloring of G, and thus (ii) holds.

It remains to show that if φ is an $(L_2 : 2)$-coloring of G_2 then $\varphi(y_4) \cap \{7, 8\} \neq \varnothing$. Suppose, on the contrary, that $\varphi(y_4) \cap \{7, 8\} = \varnothing$. It follows that $\varphi(y_4) \cup \varphi(y_3) = \{1, 2, 3, 4\}$, and hence $\varphi(y_2) = \{7, 8\}$. As a result, $\varphi(y_1) \subseteq \{1, \dots, 6\}$ and, by symmetry, we can assume that $\varphi(y_1) = \{5, 6\}$. This implies that $\varphi(v) \subseteq \{1, 2, 3, 4\}$ for each $v \in V(C_2)$. In particular, $\varphi|V(C_2)$ is a $(4 : 2)$-coloring of C_2, which is a contradiction since the 5-cycle C_2 has fractional chromatic number 5/2. □

Lemma 6. *Consider the gadget (G_3, L_3), obtained from the gadget (G_2, L_2) of Lemma 5 as follows (see Figure 2 for an illustration of G_3). The graph G_3 consists of G_2 and for $i \in \{1, 2\}$, the vertices $z_{i,1}, \dots, z_{i,7}$; the edges $y_4z_{i,1}$ and $y_4z_{i,2}$; the edge $z_{i,j}z_{i,k}$ for every j and every k such that $1 \leq j < k \leq 4$ and $(j, k) \neq (1, 2)$; the edges of the triangle $z_{i,5}z_{i,6}z_{i,7}$ and the edge $z_{i,4}z_{i,5}$. Let $L_3(v) = L_2(v)$ for $v \in V(G_2)$, and for $i \in \{1, 2\}$ let $L_3(z_{i,1}) = \{1, 2, 3, 6 + i\}$, $L_3(z_{i,2}) = \{4, 5, 6, 6 + i\}$, $L_3(z_{i,3}) = \{1, \dots, 6\}$, $L_3(z_{i,4}) = \{1, \dots, 8\}$, $L_3(z_{i,5}) = L_3(z_{i,7}) = \{1, 2, 3, 4, 7, 8\}$ and $L_3(z_{i,6}) = \{1, 2, 3, 4\}$. The gadget (G_3, L_3) is*

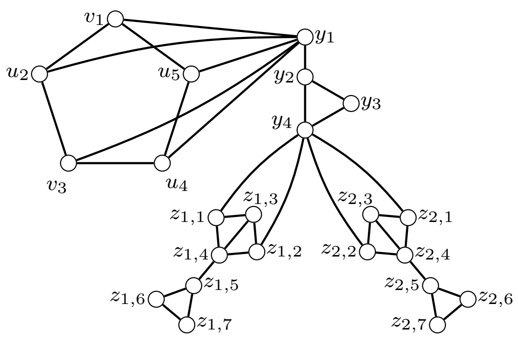

Figure 2: The graph G_3 from Lemma 6

$(v_1, v_3, \{z_{1,7}, z_{2,7}\})$-relaxed, and $\varphi(z_{1,7}) = \{7, 8\}$ or $\varphi(z_{2,7}) = \{7, 8\}$ for every $(L_3 : 2)$-coloring φ of G_3.

Proof. Let L be a half-list assignment for G_3. The gadget $(G_2, L_3|V(G_2))$ is $(v_1, v_3, \{y_4\})$-relaxed by lemma 5. Suppose first that (i) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ_0 be the corresponding L-coloring of $\{v_1, v_3\}$. For $i \in \{1, 2\}$, if $L(z_{i,1}) \cap L(z_{i,2}) \neq \varnothing$, then let c_i be a color in $L(z_{i,1}) \cap L(z_{i,2})$. Otherwise, $|L(z_{i,1}) \cup L(z_{i,2})| = 4 > |L(z_{i,3})|$, and thus we can choose a color $c_i \in (L(z_{i,1}) \cup L(z_{i,2})) \setminus L(z_{i,3})$. Let c be a color in $L(y_4) \setminus \{c_1, c_2\}$. By (i) for G_2, we know that ψ_0 extends to an L-coloring ψ of G_2 such that $\psi(y_4) = c$. If $L(z_{i,1}) \cap L(z_{i,2}) \neq \varnothing$, then color both $z_{i,1}$ and $z_{i,2}$ by c_i, otherwise color one of them by c_i and the other one by an arbitrary color from its list that is different from c. There are at least two colors in $L(z_{i,4})$ distinct from the colors of $z_{i,1}$ and $z_{i,2}$, choose such a color c'_i so that $L(z_{i,5}) \setminus \{c'_i\} \neq L(z_{i,6})$. Color $z_{i,4}$ by c'_i and extend the coloring to $z_{i,3}$, which is possible by the choice of c_i. Observe that any L-coloring of $z_{i,7}$ extends to an L-coloring of the triangle $z_{i,5}z_{i,6}z_{i,7}$ where the color of $z_{i,5}$ is not c'_i. We conclude that (G_3, L_3) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ'_0 be the corresponding L-coloring of y_4. For $i \in \{1, 2\}$, greedily extend ψ'_0 to an L-coloring of $z_{i,1}, \ldots, z_{i,7}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{z_{1,7}, z_{2,7}\}$. Observe that (G_3, L_3) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_3 : 2)$-coloring of G_3. Lemma 5 implies that $\varphi(y_4) \cap \{7, 8\} \neq \varnothing$. By symmetry, we can assume that $7 \in \varphi(y_4)$. It follows that $\varphi(z_{1,1}) \subset \{1, 2, 3\}$ and $\varphi(z_{1,2}) \subset \{4, 5, 6\}$, and thus $\varphi(z_{1,1}) \cup \varphi(z_{1,2}) \cup \varphi(z_{1,3}) = \{1, \ldots, 6\}$. Consequently, $\varphi(z_{1,4}) = \{7, 8\}$, and $\varphi(z_{1,5})$ is a subset of $\{1, 2, 3, 4\}$. This yields that $\varphi(z_{1,5}) \cup \varphi(z_{1,6}) = \{1, 2, 3, 4\}$, and therefore $\varphi(z_{1,7}) = \{7, 8\}$. □

Lemma 7. *Consider the gadget (G_4, L_4) obtained from the gadget (G_3, L_3) of Lemma 6 as follows (see Figure 3 for an illustration of G_4). The graph G_4 consists of G_3; the three triangles $w_1w_2w_3$ and $w_{i,1}w_{i,2}w_{i,3}$ for $i \in \{1, 2\}$; and the edges $z_{1,7}w_1$, $z_{2,7}w_1$, $w_3w_{1,1}$ and $w_3w_{2,1}$. Let $L_4(v) = L_3(v)$ for $v \in V(G_3)$, and for $i \in \{1, 2\}$, let $L_4(w_1) = L_4(w_3) = L_4(w_{i,1}) = L_4(w_{i,3}) = \{1, 2, 3, 4, 7, 8\}$*

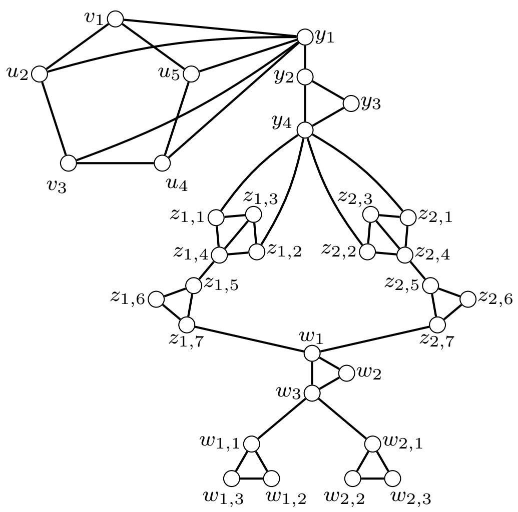

Figure 3: The graph G_4 from Lemma 7.

and $L_4(w_2) = L_4(w_{i,2}) = \{1, 2, 3, 4\}$*. The gadget is* $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$*-relaxed, and* $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7, 8\}$ *for every* $(L_4 : 2)$*-coloring* φ *of* G_4*.*

Proof. Let L be a half-list assignment for G_4. The gadget $(G_3, L_4|V(G_3))$ is $(v_1, v_3, \{z_{1,7}, z_{2,7}\})$-relaxed by Lemma 5. Suppose first that (i) holds for the restriction of L to G_3 (with $S = \{z_{1,7}, z_{2,7}\}$), and let ψ_0 be the corresponding L-coloring of $\{v_1, v_3\}$. Choose a color $c_1 \in L(z_{1,7})$ so that $L(w_1) \setminus \{c_1\} \neq L(w_2)$. If $c_1 \in L(w_1)$, then choose $c_2 \in L(z_{2,7}) \setminus (L(w_1) \setminus \{c_1\})$, otherwise choose $c_2 \in L(z_{2,7})$ so that $L(w_1) \setminus \{c_2\} \neq L(w_2)$. Choose a color $c_3 \in L(w_3)$ so that $L(w_{i,1}) \setminus \{c_3\} \neq L(w_{i,2})$ for $i \in \{1, 2\}$. By (i) for G_3, there exists an L-coloring of G_3 extending ψ_0 and assigning c_i to $z_{i,7}$ for each $i \in \{1, 2\}$. Color w_3 by c_3 and observe the L-coloring can be extended to w_1 and w_2 thanks to the choice of c_1 and c_2. Moreover, the choice of c_3 ensures that for each $i \in \{1, 2\}$, we can color $w_{i,3}$ with any color in $L(w_{i,3})$ and further extend the coloring to $w_{i,1}$ and $w_{i,2}$. We conclude that (G_4, L_4) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{z_{1,7}, z_{2,7}\}$), and let ψ_0' be the corresponding L-coloring of $\{z_{1,7}, z_{2,7}\}$. Greedily extend ψ_0' to an L-coloring of w_1, w_2, w_3, and $w_{i,1}$, $w_{i,2}$, $w_{i,3}$ for $i \in \{1, 2\}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{w_{1,3}, w_{2,3}\}$. Observe that (G_4, L_4) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_4 : 2)$-coloring of G_4. By Lemma 6 and by symmetry, we can assume that $\varphi(z_{1,7}) = \{7, 8\}$. Consequently, $\varphi(w_1) \subset \{1, 2, 3, 4\}$, and thus $\varphi(w_1) \cup \varphi(w_2) = \{1, 2, 3, 4\}$, which yields that $\varphi(w_3) = \{7, 8\}$. We conclude analogously that $\varphi(w_{1,3}) = \{7, 8\} = \varphi(w_{2,3})$. □

We can now combine (G_1, L_1) with (G_4, L_4) to obtain a gadget (G_5, L_5) that is L-colorable from each half-list assignment L, but not $(L_5 : 2)$-colorable.

Lemma 8. *Consider the gadget* (G_5, L_5) *obtained from the gadgets* (G_1, L_1) *of Corollary 4 and* (G_4, L_4) *of Lemma 7 as follows (see Figure 4 for an illustration*

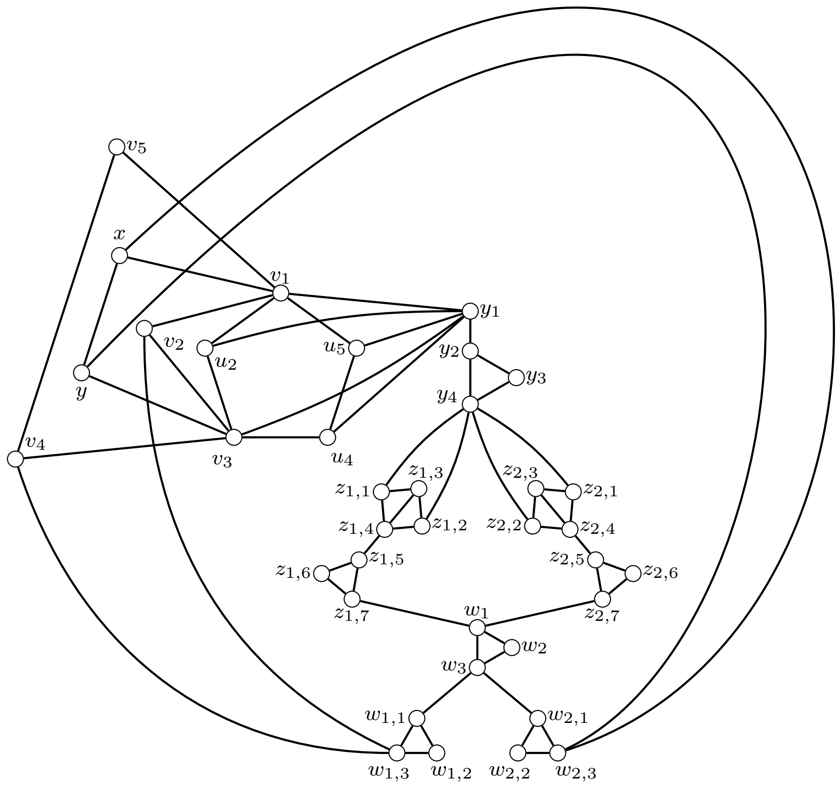

Figure 4: The graph G_5 from Lemma 8.

of G_5). The graph G_5 is obtained from the union of the graphs G_1 and G_4 (intersecting in $\{v_1, v_3\}$) by adding the edges $w_{1,3}v_2$, $w_{1,3}v_4$, $w_{2,3}x$ and $w_{2,3}y$. Let $L_5(v) = L_4(v)$ for $v \in V(G_4)$, $L_5(v) = L_1(v)$ for $v \in V(G_1) \setminus \{v_2, v_4, x, y\}$, and $L_5(v) = L_1(v) \cup \{7, 8\}$ for $v \in \{v_2, v_4, x, y\}$. Then G_5 is L-colorable for every half-list assignment L, but not $(L_5 : 2)$-colorable.

Proof. Let L be a half-list assignment for G_5. The gadget $(G_4, L_5|V(G_4))$ is $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$-relaxed by Lemma 7. Suppose first that (i) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L-coloring of $\{v_1, v_3\}$. Greedily extend ψ_0 to an L-coloring ψ of G_1. Choose $c_1 \in L(w_{1,3}) \setminus \{\psi(v_2), \psi(v_4)\}$ and $c_2 \in L(w_{2,3}) \setminus \{\psi(x), \psi(y)\}$. By (i) for G_4, there exists an L-coloring of G_4 that extends ψ_0 and assigns to $w_{i,3}$ the color c_i for each $i \in \{1, 2\}$. This yields, together with ψ, an L-coloring of G_5.

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L-coloring of $\{w_{1,3}, w_{2,3}\}$. Note that $L(v_1) = L(v_3)$ in this case. Corollary 4 implies that G_1 has an L-coloring ψ such that $\psi(v_2) \in L(v_2) \setminus \{\psi_0(w_{1,3})\}$, $\psi(v_4) \in L(v_4) \setminus \{\psi_0(w_{1,3})\}$, $\psi(x) \in L(x) \setminus \{\psi_0(w_{2,3})\}$, and $\psi(y) \in L(y) \setminus \{\psi_0(w_{2,3})\}$. By (ii), the restriction of $\psi \cup \psi_0$ to $\{v_1, v_3, w_{1,3}, w_{2,3}\}$ extends to an L-coloring of G_4, which together with ψ gives an L-coloring of G_5.

It remains to show that G_5 is not $(L_5 : 2)$-colorable. If φ were an $(L_5 : 2)$-coloring of G_5, then by Lemma 7 we would have $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7, 8\}$, and thus the restriction of φ to G_1 would be an $(L_1 : 2)$-coloring, thereby contradicting Corollary 4. □

The final graph. We are now in a position to prove Theorem 2 by simply using a standard construction to ensure uniform lengths of lists.

Proof of Theorem 2. Let G be a graph and L' an assignment of lists of size 8 obtained as follows. Let K be a clique with vertices $r_1, \ldots, r_4$, and let $L'(r_1) = \cdots = L'(r_4) = \{9, \ldots, 16\}$. For every $(L' : 2)$-coloring ψ of K, let G_ψ be a copy of the graph G_5 from the gadget (G_5, L_5) of Lemma 8, and for each vertex $v \in V(G_\psi)$ such that $|L_5(v)| = 2k$ with $k \in \{2, 3\}$, we add the edges $vr_1, \ldots, vr_{4-k}$ and let $L'(v) = L_5(v) \cup \bigcup_{i=1}^{4-k} \psi(r_i)$. If G had an $(L' : 2)$-coloring φ, then letting ψ be the restriction of φ to K, the restriction of φ to G_ψ would be an $(L_5 : 2)$-coloring of G_5, thereby contradicting Lemma 8.

Consider now a list assignment L for G such that $|L(v)| = 4$ for every $v \in V(G)$. Let φ be any L-coloring of K. For each $(L' : 2)$-coloring ψ of K, let L_ψ be the list assignment for G_ψ obtained by, for each $v \in V(G_\psi)$, removing the colors of neighbors in K according to φ, and possibly removing further colors to ensure that $|L_\psi(v)| = |L_5(v)|/2$. By Lemma 8, the graph G_ψ has an L_ψ-coloring. The union of these colorings and φ yields an L-coloring of G. □

Concluding remarks. It follows from Theorem 2 that for each integer $a \geq 4$, there exists a graph G_a that is a-choosable but not $(2a : 2)$-choosable—if we have such a graph G_a, taking the disjoint union of $\binom{2(a+1)}{2}$ copies of G_a and adding a vertex adjacent to all other vertices yields G_{a+1}, by an argument analogous to the list uniformization procedure used for the proof of Theorem 2. It is natural to ask whether there exists a graph that is 3-choosable but not $(6 : 2)$-choosable. We believe this to be the case; in particular, Corollary 4 only requires lists of size at most 6. However, it does not seem easy to construct a gadget that satisfies the properties stated in Lemma 5 without using a vertex with a list of size 8.

References

[1] N. Alon, Z. Tuza, and M. Voigt, *Choosability and fractional chromatic numbers*, Discrete Mathematics, 165–166 (1997), pp. 31 – 38. Graphs and Combinatorics.

[2] P. Erdős, A. L. Rubin, and H. Taylor, *Choosability in graphs*, Congr. Numer., 26 (1980), pp. 125–157.

[3] S. Gutner and M. Tarsi, *Some results on $(a\colon b)$-choosability*, Discrete Math., 309 (2009), pp. 2260–2270.

[4] Z. Tuza and M. Voigt, *Every 2-choosable graph is $(2m, m)$-choosable*, J. Graph Theory, 22 (1996), pp. 245–252.