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ABSTRACT. The paper is concerned with the finite-time stabilization of a hybrid PDE-ODE system which may serve as a model for the motion of an overhead crane with a flexible cable. The dynamics of the flexible cable is assumed to be described by the wave equation with constant coefficients. Using a nonlinear feedback law inspired by those given by Haimo in [12] for a second-order ODE, we prove that a finite-time stabilization occurs for the full system platform + cable. The global well-posedness of the system is also established by using the theory of nonlinear semigroups.

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1. INTRODUCTION

The stabilization of hybrid PDE-ODE systems has attracted the attention of the control community since several decades. In [2], the authors derived and investigated a model for the dynamics of a motorized platform of mass M moving along an horizontal bench. A flexible (and nonstretching) cable of length L was attached to the platform and was holding a load mass m. Assuming that the transversal and angular displacements were small and that the acceleration of the load mass could be neglected with respect to the gravity, they obtained the following system:

$$z_{tt} - (a(x)z_x)_x = 0, \tag{1.1}$$

$$z_x(0,t) = 0, \tag{1.2}$$

$$z(L,t) = X_p(t), \tag{1.3}$$

$$\ddot{X}_p(t) = \lambda(az_x)(L,t) + \frac{v}{M}, \tag{1.4}$$

where

$$a(x) := gx + \frac{gm}{\rho}, \tag{1.5}$$

$$\lambda := \frac{(m+\rho L)g}{Ma(L)}. \tag{1.6}$$

In above system, x denotes the curvilinear abscissa (i.e. the arclength) along the cable, $z = z(x,t)$ is the horizontal displacement at time t of the point on the cable of curvilinear abscissa x, X_p is the abscissa of the platform, ρ the mass per unit length of the cable, and v the force applied to the platform. As usual, $z_{tt} = \partial^2 z/\partial t^2$, $z_{xx} = \partial^2 z/\partial x^2$ etc., and $\ddot{X}_p = d^2X_p/dt^2$.

When $m \gg \rho L$, then $gm/\rho \gg gx$ for $x \in (0,L)$ and it can be assumed that the function $a = a(x)$ is constant. Note that, with a clockwise orientation of the plane, $z_x(L,t) \approx \theta(t)$, where θ denotes the angular deviation of the cable with respect to the vertical axis at the curvilinear abscissa $x = L$ (i.e. at the connection point to the platform), which is supposed to be measured (see Fig. 1). After using some scaling and the following intermediate feedback law

$$v = Mu - (m+\rho L)gz_x(L,t), \tag{1.7}$$

we obtain the simplified system

$$z_{tt} - z_{xx} = 0, \qquad (x,t) \in (0,L) \times (0,+\infty), \tag{1.8}$$
$$z_x(0,t) = 0, \qquad t \in (0,+\infty), \tag{1.9}$$
$$z(L,t) = X_p(t), \qquad t \in (0,+\infty), \tag{1.10}$$
$$\ddot{X}_p(t) = u, \qquad t \in (0,+\infty). \tag{1.11}$$

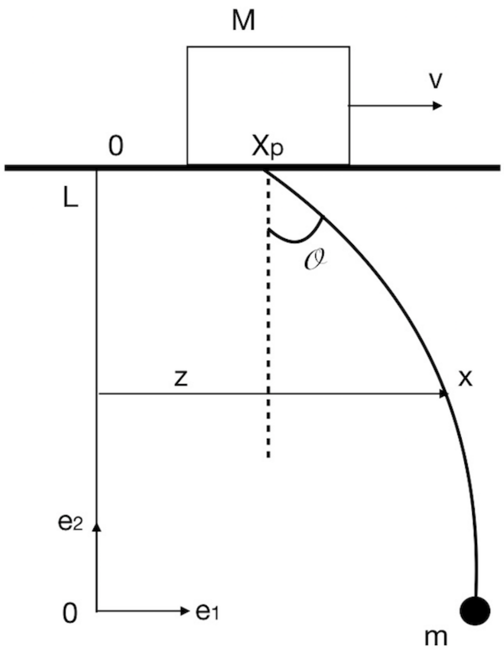

FIGURE 1. The overhead crane with flexible cable

An asymptotic (but not exponential) stabilization of (1.1)-(1.4) was established in [2], while an exponential stabilization was derived after in [3] by using the cascaded structure of the system and a backstepping approach. A similar result was obtained for system (1.8)-(1.11) (but with Dirichlet boundary conditions) in [19]. The dynamics of the load mass was taken into account in [17].

The backstepping approach is a powerful tool for the design of stabilizing controllers in the context of finite dimensional systems (see for example [21]), but the cascaded structure of flexible mechanical systems coupling ODE and PDE is also a useful property in regard to stabilization, as for the overhead crane with flexible cable. We also refer the reader to [8], where the authors proposed a class of nonlinear asymptotically stabilizing boundary feedback laws for a rotating body-beam without natural damping. Let us also mention that in [4] the authors considered the case of a variable length flexible cable.

In [3], if we restrict ourselves to system (1.8)-(1.11), the authors considered the linear feedback law

$$u = -K^{-1}\Big(z_{xt}(L,t) + kz_t(L,t)\Big) - \mu\Big(\dot{X}_p + K^{-1}\big(z_x(L,t) + kz(L,t)\big)\Big) \tag{1.12}$$

where $k, K > 0$ and $\mu > K/2$ are some constants, and proved that system (1.8)-(1.12) is exponentially stable.

Here, we consider instead of the linear feedback law (1.12) the nonlinear feedback law

$$u = -z_{xt}(L,t) - \Big(z_t(L,t) + z_x(L,t)\Big)^{\beta} - \Big(z(L,t) + \int_0^L z_t(\xi,t)d\xi\Big)^{\alpha}, \tag{1.13}$$

where the constants α and β are such that

$$0 < \beta < 1 \ \text{ and } \ \alpha > \frac{\beta}{2-\beta}\cdot \tag{1.14}$$

In (1.13), we have set $x^{\alpha} := \mathrm{sign}(x)|x|^{\alpha}$ for $x \in \mathbb{R}$ and $\alpha > 0$, where

$$\mathrm{sign}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

It is well known that for $\alpha \in (0,1)$, the function $x \to x^{\alpha}$ is continuous on $\mathbb{R}$, but not Lipschitz continuous around 0, and that the ODE $\dot{x} = -x^{\alpha}$ is finite-time stable; that is, the equilibrium $x = 0$ is stable and any trajectory of $\dot{x} = -x^{\alpha}$ reaches 0 in finite time. Haimo proved in [12] that for α and β as in (1.14), the second order ODE

$$\ddot{x} = -\dot{x}^{\beta} - x^{\alpha} \tag{1.15}$$

is finite-time stable. Note that (1.14) is satisfied when e.g. $\alpha = 1$ and $0 < \beta < 1$.

Remark 1.1. *It should be noticed that the finite-time stability of* (1.15) *holds as well in the limit case* $0 < \alpha < 1$ *and* $\beta = \frac{2}{1+\alpha^{-1}}$. *Indeed, the first-order system equivalent to* (1.15) *reads*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_2^{\beta} - x_1^{\alpha}. \end{aligned}$$

It is easy to see that this system is homogeneous of negative *degree* $k := \frac{\alpha-1}{2}$ *with respect to the dilation* $\delta_\varepsilon^r(x_1,x_2) := (\varepsilon x_1, \varepsilon^{\frac{\alpha+1}{2}} x_2)$, *a property which implies its finite-time stability (see e.g.* [5, 6]*).*

The aim of the paper is to prove the finite-time stability result of theorem 1.1:

Theorem 1.1. *System* (1.8)-(1.11) *with the feedback law* (1.13) *for* $\alpha = 1$ *and* $\beta \in (0,1)$ *is finite-time stable.*

From now on, we assume that

$$\alpha = 1.$$

Obviously, before studying the stability properties of the closed loop system, we have to investigate the wellposedness of system (1.8)-(1.11) and (1.13).

The paper is scheduled as follows. In section 2 we derive an abstract evolution equation describing the dynamics of the system platform + cable and we investigate its wellposedness. The finite-time stability of the system is next established in Section 3. Finally, we give some illustrative simulation results and words of conclusion in section 4.

2. Well-posedness of the system

2.1. **Introduction of the nonlinear operator.** We intend to put the system (1.8)-(1.11) and (1.13) in the form

$$w_t + \mathcal{A}w = 0, \quad t \in (0, +\infty), \tag{2.1}$$

with $w = (z, v, b, \eta)$, where z and v are functions of $x \in (0, L)$ and $t \geq 0$ and b, η are real functions of $t \geq 0$. More precisely, introduce the Hilbert space

$$\mathcal{H} := \{w = (z, v, b, \eta) \in H^1(0, L) \times L^2(0, L) \times \mathbb{R} \times \mathbb{R};\ z(L) = b\}$$

endowed with the scalar product

$$(w^1, w^2) := \int_0^L [z_x^1(x) z_x^2(x) + v^1(x) v^2(x)] dx + b^1 b^2 + \eta^1 \eta^2, \quad w^i = (z^i, v^i, b^i, \eta^i),\ i = 1, 2,$$

where $H^n(0, L)$ denotes the classical Sobolev space of (classes of) functions on $(0, L)$ whose derivatives of order at most n are square integrable. We denote by $\|w\| = (w, w)^{\frac{1}{2}}$ the associated Hilbertian norm. Let

$$D(\mathcal{A}) := \{w = (z, v, b, \eta) \in H^2(0, L) \times H^1(0, L) \times \mathbb{R} \times \mathbb{R}; \quad z(L) = b,\ z_x(0) = 0,\ \eta = z_x(L) + v(L)\}$$

and for $w = (z, v, b, \eta) \in D(\mathcal{A})$

$$\mathcal{A}w := \Big(-v, -z_{xx}, z_x(L) - \eta, b + \int_0^L v(\xi) d\xi + \eta^\beta \Big).$$

Finally, a solution $w = (z, v, b, \eta)$ of (2.1) solves

$$z_t = v, \tag{2.2}$$
$$v_t = z_{xx}, \tag{2.3}$$
$$\dot{b} = -z_x(L, t) + \eta, \tag{2.4}$$
$$\dot{\eta} = -\big(b + \int_0^L v(\xi, t) d\xi\big) - \eta^\beta. \tag{2.5}$$

Obviously, (2.2)-(2.3) is nothing but the first-order system representing the wave equation (1.8), (2.4) is equivalent to the condition

$$\eta(t) = z_x(L, t) + z_t(L, t) = z_x(L, t) + v(L, t). \tag{2.6}$$

Note that (2.6) is satisfied when $w(t) \in D(\mathcal{A})$. Finally, introducing the quantity

$$\phi := b + \int_0^L v(x) dx \tag{2.7}$$

which is meaningful whenever $w=(z,v,b,\eta)\in\mathcal{H}$, we notice that for $w=w(t)$ solving (2.1), we have that $\phi=\phi(t)$ solves

$$\dot\phi=\dot b+\int_0^L v_t(x,t)dx=\dot b+\int_0^L z_{xx}(x,t)dx=\dot b+z_x(L,t)=\eta \tag{2.8}$$

where we used (1.9) and (2.4). Thus (2.5) can be rewritten as

$$\ddot\phi=-\dot\phi^\beta-\phi. \tag{2.9}$$

Using (2.6), we obtain also the expression of the nonlinear feedback law (1.13):

$$u=\ddot X_p=z_{tt}(L,t)=-z_{xt}(L,t)-\Big(z_t(L,t)+z_x(L,t)\Big)^\beta-\Big(z(L,t)+\int_0^L z_t(\xi,t)d\xi\Big)=-\dot\theta-\dot\phi^\beta-\phi, \tag{2.10}$$

that is, (1.11) holds with the feedback law u given by (1.13) for $\alpha=1$ and $\beta\in(0,1)$.

The first main result in this paper is concerned with the wellposedness of (2.1).

Theorem 2.1. *Let $\beta\in(0,1)$ and let $\mathcal{A}$ be as above.*
1. For any $w^0=(z^0,v^0,b^0,\eta^0)\in D(\mathcal{A})$ and any $T>0$, there exists a unique solution $w=w(t)$ of the Cauchy problem

$$\begin{aligned} w_t+\mathcal{A}w &= 0, \quad t\in[0,T], &(2.11)\\ w(0) &= w^0 &(2.12)\end{aligned}$$

such that $w\in W^{1,\infty}([0,T],\mathcal{H})$, $w(t)\in D(\mathcal{A})$ for all $t\in[0,T]$, the map $t\in[0,T]\to\mathcal{A}w(t)\in\mathcal{H}$ is weakly continuous, and the map $t\in[0,T]\mapsto w(t)\in\mathcal{H}$ is weakly differentiable.
2. There exists a number $\lambda=\lambda(\beta)>0$ such that for any $w^0,\hat w^0\in D(\mathcal{A})$, with corresponding solutions $w,\hat w$, it holds

$$\|w(t)-\hat w(t)\|\le e^{\lambda t}\|w^0-\hat w^0\|\quad \forall t\in[0,+\infty). \tag{2.13}$$

3. There is a nonlinear semigroup $(S(t))_{t\ge0}$ on $\mathcal{H}$ such that for $w^0\in D(\mathcal{A})$, $w(t):=S(t)w^0$ is the strong solution of (2.11)-(2.12). The estimate (2.13) is still valid when $w^0,\hat w^0\in\mathcal{H}$.

2.2. **Proof of Theorem 2.1.** We shall use the semigroup theory developed in [7, 14, 18] for equations involving (nonlinear) operators that are maximal monotone (or accretive). As the operator $\mathcal{A}$ is not monotone for the Hilbertian norm in $\mathcal{H}$, but some translate of it is, we perform the change of variables $\tilde w:=e^{-\lambda t}w$. We decompose the operator $\mathcal{A}$ into its linear and its nonlinear part:

$$\mathcal{A}w:=A_0w+Fw,\quad w=(z,v,b,\eta)\in D(\mathcal{A})$$

with

$$A_0w=(-v,-z_{xx},z_x(L)-\eta,b+\int_0^L v(\xi)d\xi),\quad Fw=(0,0,0,\eta^\beta).$$

Then $\tilde w_t=-\lambda e^{-\lambda t}w+e^{-\lambda t}w_t$ so that

$$\tilde w_t+(A_0+\lambda I)\tilde w+e^{(\beta-1)\lambda t}F\tilde w=0,\quad \tilde w(0)=w^0. \tag{2.14}$$

We set

$$A(t)\tilde w:=(A_0+\lambda I)\tilde w+e^{(\beta-1)\lambda t}F\tilde w,\qquad \text{for } \tilde w\in D(A(t)):=D(\mathcal{A}).$$

We aim to apply the well-posedness theory developed in [14, Section 3 p. 513] to the Cauchy problem

$$\tilde w_t+A(t)\tilde w=0,\quad \tilde w(0)=w^0. \tag{2.15}$$

We have to check that the following assumptions (required in [14]) are satisfied:

(H1): The domain of $A(t)$ is independent of t.

(H2): There is some constant $C = C(T)$ such that for $w \in D(\mathcal{A})$ and $s,t \in [0,T]$,

$$\|A(t)w - A(s)w\| \le C|t-s|(1+\|w\|+\|A(s)w\|); \tag{2.16}$$

(H3): For any $t \in [0,T]$, the (nonlinear) operator $A(t)$ is maximal monotone.

We know that **(H1)** is satisfied. For **(H2)**, we notice that

$$\begin{aligned}
\|A(t)w - A(s)w\| &= \|\big(e^{(\beta-1)\lambda t} - e^{(\beta-1)\lambda s}\big)Fw\| \\
&= |e^{(\beta-1)\lambda(t-s)} - 1|\, e^{(\beta-1)\lambda s}|\eta|^\beta \\
&\le (1-\beta)\lambda e^{(1-\beta)\lambda T}|t-s|(\|A(s)w\| + |\lambda\eta + b + \int_0^L v(\xi)d\xi|) \\
&\le C|t-s|(\|A(s)w\|+\|w\|)
\end{aligned}$$

with $C := (1-\beta)\lambda e^{(1-\beta)\lambda T}(1+\lambda)$, where we used Cauchy-Schwarz inequality for the integral term. The verification that **(H3)** is satisfied for λ large enough is divided into two lemmas.

Lemma 2.1. *Assume that $\lambda \ge 1$. Then for all $t \in [0,T]$, the operator $A(t)$ is monotone; that is,*

$$(A(t)w_1 - A(t)w_2, w_1 - w_2) \ge 0 \quad \forall w_1, w_2 \in D(\mathcal{A}). \tag{2.17}$$

Proof. Pick any $w_1, w_2 \in D(\mathcal{A})$ and set $w := w_1 - w_2$. Then

$$(A(t)w_1 - A(t)w_2, w_1 - w_2) = ((A_0 + \lambda I)w, w) + e^{(\beta-1)\lambda t}(Fw_1 - Fw_2, w_1 - w_2).$$

For the second term, we notice that

$$e^{(\beta-1)\lambda t}(Fw_1 - Fw_2, w_1 - w_2) = e^{(\beta-1)\lambda t}(\eta_1^\beta - \eta_2^\beta)(\eta_1 - \eta_2) \ge 0,$$

for $\beta > 0$. For the first term, integrating by parts gives

$$\begin{aligned}
(A_0 w, w) &= \int_0^L [-v_x z_x - z_{xx} v]dx + (z_x(L) - \eta)b + \big(b + \int_0^L v(\xi)d\xi\big)\eta \\
&= -[z_x v]_0^L + z_x(L)b + \eta\int_0^L v(\xi)d\xi \\
&= z_x(L)(z_x(L) - \eta + b) + \eta\int_0^L v(\xi)d\xi
\end{aligned} \tag{2.18}$$

where we used the relation $\eta = z_x(L) + v(L)$ in the last line. It follows from Young inequality that

$$\begin{aligned}
|z_x(L)(b-\eta)| &\le z_x^2(L) + \frac{1}{4}(b-\eta)^2 \le z_x^2(L) + \frac{1}{2}(b^2+\eta^2), \\
|\eta\int_0^L v(\xi)d\xi| &\le \frac{1}{2}\left(\eta^2 + \int_0^L v(\xi)^2 d\xi\right).
\end{aligned}$$

Combined with (2.18), this yields for $\lambda \ge 1$

$$((A_0 + \lambda I)w, w) \ge (\lambda - 1)\eta^2 + (\lambda - \frac{1}{2})b^2 + (\lambda - \frac{1}{2})\int_0^L v(\xi)^2 d\xi + \lambda\int_0^L z_x(x)^2 dx \ge 0.$$

It follows that (2.17) holds for $\lambda \ge 1$. □

It remains to prove that $A(t)$ is *maximal* monotone.

Lemma 2.2. *Assume that* $\lambda \geq 1$. *Then for all* $\mu > 0$ *and* $t \in [0,T]$

$$R(I+\mu A(t)) = \mathcal{H}, \tag{2.19}$$

where $R(I+\mu A(t)) := \{w^0 \in \mathcal{H};\ \exists w \in D(\mathcal{A}),\ w^0 = (I+\mu A(t))w\}$ *is the range of the operator* $I+\mu A(t)$.

Proof. Note first that (2.19) is equivalent to $R(A_0+(\mu^{-1}+\lambda)I+e^{(\beta-1)\lambda t}F) = \mathcal{H}$. It is thus sufficient to prove that $R(A_0+\lambda I+e^{(\beta-1)\lambda t}F) = \mathcal{H}$ whenever $\lambda > 1$. Fix $\lambda > 1$ and $t \in [0,T]$. Pick any $w^0 = (z^0, v^0, b^0, \eta^0) \in \mathcal{H}$. We have to prove that there exists $w = (z, v, b, \eta) \in D(\mathcal{A})$ such that $(A_0+\lambda)w + e^{(\beta-1)\lambda t}Fw = w^0$; that is,

$$-v+\lambda z = z^0, \tag{2.20}$$
$$-z_{xx}+\lambda v = v^0, \tag{2.21}$$
$$-\eta+z_x(L)+\lambda b = b^0, \tag{2.22}$$
$$b+\int_0^L v(\xi)d\xi + e^{(\beta-1)\lambda t}\eta^\beta+\lambda\eta = \eta^0. \tag{2.23}$$

The requirement that $w = (z, v, b, \eta) \in D(\mathcal{A})$ imposes also the conditions

$$(z, v, b, \eta) \in H^2(0,L)\times H^1(0,L)\times\mathbb{R}\times\mathbb{R}, \tag{2.24}$$
$$z(L) = b, \tag{2.25}$$
$$z_x(0) = 0, \tag{2.26}$$
$$\eta = z_x(L)+v(L). \tag{2.27}$$

First, we note that (2.20)-(2.21) is equivalent to

$$v = \lambda z - z^0, \tag{2.28}$$
$$-z_{xx}+\lambda^2 z = \lambda z^0+v^0. \tag{2.29}$$

Next, we see that (2.22) & (2.27) are equivalent to (2.27) and

$$v(L) = \lambda b - b^0 \tag{2.30}$$

and that (2.30) follows from (2.25), (2.28) and the fact that $z^0(L) = b^0$, as $w^0 \in \mathcal{H}$. Thus, it is sufficient to find $w = (z, v, b, \eta)$ fulfilling the conditions (2.23)-(2.29).

Because of the many coupling terms present in the system, we first solve the subsystem (2.24)-(2.29) for a given value of $b \in \mathbb{R}$, and next check that the condition (2.23) can be satisfied for a certain value of b.

More precisely, for given $b \in \mathbb{R}$, we first solve the elliptic equation (2.29) together with the boundary conditions (2.25)-(2.26). We obtain some function $z \in H^2(0,L)$. Next, the function v and the number η are determined thanks to the conditions (2.27)-(2.28). Finally, the number b is determined by solving the *nonlinear* equation (2.23).

Pick first any $b \in \mathbb{R}$. To solve (2.29) together with (2.25) and (2.26), we set $z = b+\hat{z}$ and search $\hat{z} \in V := \{y \in H^1(0,L);\ y(L) = 0\}$ as a solution of the variational problem:

$$\int_0^L [\hat{z}_x y_x + \lambda^2 \hat{z} y]\,dx = \int_0^L (\lambda z^0 + v^0 - \lambda^2 b) y\,dx \quad \forall y \in V. \tag{2.31}$$

A straightforward application of Lax-Milgram lemma shows that the problem (2.31) has a unique solution $\hat{z} \in V$, which is also in $H^2(0,L)$. Therefore, for any given $b \in \mathbb{R}$, we obtain a solution $z \in H^2(0,L)$ of (2.25), (2.26) and (2.29). Moreover, we can write $z = z^1 + bz^2$ where the functions $z^i \in H^2(0,L)$ $(i=1,2)$ do not depend on b (but depend on λ). Then v and η are given by (2.28) and (2.27), respectively. It remains to prove that (2.23) is satisfied for a convenient choice of $b \in \mathbb{R}$. We note that the number η, resp. the expression $b + \int_0^L v(\xi)d\xi + \lambda\eta$, is affine in b, so that we can write for some constants a_i, $i = 1, \dots, 4$,

$$\eta = a_1 b + a_2, \quad b + \int_0^L v(\xi)d\xi + \lambda\eta = a_3 b + a_4.$$

Then, the nonlinear equation (2.23) reads

$$f(b) := e^{(\beta-1)\lambda t}(a_1 b + a_2)^\beta + a_3 b + a_4 = \eta^0. \tag{2.32}$$

If $a_3 = 0$, the only solution of (2.32) is

$$b = \frac{1}{a_1}\left([e^{(1-\beta)\lambda t}(\eta^0 - a_4)]^{\frac{1}{\beta}} - a_2\right).$$

If $a_3 \neq 0$, since $\lim_{b\to\pm\infty} \operatorname{sign}(a_3) f(b) = \pm\infty$, the intermediate value theorem yields the existence of (at least) one solution $b \in \mathbb{R}$ of (2.32). □

We infer from Lemmas 2.1 and 2.2 that the operator $A(t)$ is maximal monotone. Then the assertion 1. in Theorem 2.1 follows from [14, Theorems 1 and 2]. Note that for any given $w_1^0, w_2^0 \in D(\mathcal{A})$, if we denote by $\tilde{w}_1$ and $\tilde{w}_2$ the corresponding solutions of (2.15), then by [14, Theorem 2] we have that

$$\|\tilde{w}_1(t) - \tilde{w}_2(t)\| \leq \|w_1^0 - w_2^0\|, \quad \forall t \geq 0.$$

Then the functions $w_1(t) := e^{\lambda t}\tilde{w}_1(t)$ and $w_2(t) := e^{\lambda t}\tilde{w}_2(t)$, which solve the Cauchy problem (2.11)-(2.12) for the initial data w_1^0 and w_2^0, respectively, satisfy the estimate (2.13). The assertion 2. in Theorem 2.1 follows from (2.13) and the fact that $D(\mathcal{A})$ is dense in $\mathcal{H}$. The proof of Theorem 2.1 is complete. □

Once the wellposedness of the system is established, we have to pay some attention to the regularity of the feedback law in (1.13). Indeed, for practical reasons (e.g. if one wishes to do some numerics), it is desirable that the control input u be a function, typically in $L^2(0,T)$, and not merely a distribution. Fortunately this is true, but this does not follow from the above theory.

For the solution w of (2.11)-(2.12) issuing from $w^0 \in D(\mathcal{A})$, we have that for all $T > 0$,

$$w = (z, v, b, \eta) \in W^{1,\infty}(0,T,\mathcal{H}), \quad \mathcal{A}w \in C_w([0,T],\mathcal{H}), \tag{2.33}$$

where $C_w([0,T],\mathcal{H})$ denotes the space of *weakly* continuous functions from $[0,T]$ to $\mathcal{H}$. From the definitions of $\mathcal{H}$ and of $\mathcal{A}$, we infer that

$$z \in W^{1,\infty}(0,T,H^1(0,L)) \cap C_w([0,T],H^2(0,L)) \tag{2.34}$$

$$v \in W^{1,\infty}(0,T,L^2(0,L)) \cap C_w([0,T],H^1([0,L]). \tag{2.35}$$

Note that the regularity of z and v depicted in (2.34)-(2.35) is not sufficient to assert that $z_{tt}(L,.) = v_t(L,.) \in L^2(0,T)$, or that $z_{tx}(L,.) = v_x(L,.) \in L^2(0,T)$. Such properties, however, are true and come from a classical hidden regularity for the solutions of the wave equation [15, 16].

Corollary 2.1. *Let $w^0 \in D(\mathcal{A})$ and let $w(t) = (z, v, b, \eta)(t)$ be the corresponding solution of* (2.11)-(2.12). *Then $z_{tt}(L,.) = v_t(L,.) \in L^2(0,T)$ for all $T > 0$. In particular, all the terms in* (1.13) *belong to $L^2(0,T)$ for all $T > 0$.*

Proof. Note first that $v = z_t$ (the weak derivative of z) satisfies

$$v_{tt} - v_{xx} = 0, \tag{2.36}$$
$$v_x(0,t) = 0, \tag{2.37}$$
$$v(L,t) = \dot{b} \tag{2.38}$$

We infer from (2.35) that $v_x \in C_w([0,T],L^2(0,L))$, from (2.34) that $v_t = z_{tt} = z_{xx} \in C_w([0,T],L^2(0,L))$ and $z_x(L,.) \in C^0([0,T])$, so that $\dot{b} = \eta - z_x(L,.) \in C^0([0,T])$.

This regularity is not sufficient to justify the integrations by parts we wish to do, and thus we smooth out the function v by using a convolution in time.

Pick any function $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$, $\rho(-t) = \rho(t) \geq 0$ for all $t \in \mathbb{R}$ and $\rho(t) = 0$ for all $t \in \mathbb{R} \setminus (-1,1)$. For any given $h > 0$, let $\rho_h(t) := \frac{1}{h}\rho(\frac{t}{h})$ for $t \in \mathbb{R}$ and

$$v_h(x,t) := \int_{-h}^{h} v(x,t-s)\rho_h(s)\,ds, \quad (x,t) \in [0,L] \times [h+\infty).$$

Then v_h satisfies

$$v_{h,tt} - v_{h,xx} = 0, \tag{2.39}$$
$$v_{h,x}(0,t) = 0, \tag{2.40}$$
$$v_h(L,t) = \int_{-h}^{h} \dot{b}(t-s)\rho_h(s)ds, \tag{2.41}$$

so that $v_h \in C^2([h,T-h],L^2(0,L)) \cap C^0([h,T-h],H^2(0,L))$. Multiplying each term in (2.39) by the Morawetz multiplier $xv_{h,x}$, we obtain after some integrations by parts that

$$\int_h^{T-h} (v_{h,x}^2(L,t) + v_{h,t}^2(L,t))\,dt = \int_h^{T-h}\int_0^L (v_{h,x}^2 + v_{h,t}^2)\,dxdt + \left[\int_0^L xv_{h,x}v_{h,t}dx\right]_h^{T-h}. \tag{2.42}$$

Using the fact that $\int_{-\infty}^{\infty} \rho_h(t)dt = 1$, we have that

$$\int_h^{T-h}\int_0^L (v_{h,x}^2 + v_{h,t}^2)\,dxdt \leq \int_0^T\int_0^L (v_x^2 + v_t^2)\,dxdt \leq 2T(\|v_x\|^2_{L^\infty(0,T,L^2(0,L))} + \|v_t\|^2_{L^\infty(0,T,L^2(0,L))}). \tag{2.43}$$

On the other hand, we have that

$$\left|\left[\int_0^L xv_{h,x}v_{h,t}dx\right]_h^{T-h}\right| \leq 2L\|v_x\|_{L^\infty(0,T,L^2(0,L))}\|v_t\|_{L^\infty(0,T,L^2(0,L))}.$$

We conclude that

$$\int_h^{T-h} (v_{h,x}^2(L,t) + v_{h,t}^2(L,t))\,dt \leq (2T+L)(\|v_x\|^2_{L^\infty(0,T,L^2(0,L))} + \|v_t\|^2_{L^\infty(0,T,L^2(0,L))}).$$

Thus $\|v_{h,t}(L,.)\|_{L^2(h,T-h)} \leq K$ for some constant $K > 0$. From well-known properties of convolution, we know that $v_h(L,t) \to \dot{b} = v(L,t)$ uniformly w.r.t. t on any segment $[\varepsilon, T-\varepsilon]$, and hence $v_{h.t}(L,.) \to v_t(L,.)$ in the distributional sense on $(\varepsilon, T-\varepsilon)$ for any $\varepsilon \in (0,T/2)$. We conclude that $v_t \in L^2(0,T)$ with $\|v_t\|_{L^2(0,T)} \leq K$. Using (2.10) and (2.34), we infer that $z_{xt}(L,.) \in L^2(0,T)$. □

3. Finite stability of the complete system

The finite-time stability of the system *platform + cable* is the second main result in this paper.

Theorem 3.1. *The system (2.11)-(2.12) is finite-time stable. More precisely, there exists a nondecreasing function $T:(0,+\infty)\to(0,+\infty)$ such that for all $R>0$ and all $w^0=(z^0,v^0,b^0,\eta^0)\in\mathcal{H}$ with $\|w^0\|\le R$, the solution $w(t)=S(t)w^0$ of (2.11)-(2.12) satisfies*

$$w(t) = 0, \quad \forall t\in[T(R),+\infty), \tag{3.1}$$

$$\|w(t)\| \le e^{\lambda t}\|w^0\|, \quad \forall t\in[0,T(R)]. \tag{3.2}$$

Proof. Assume first that $w^0\in D(\mathcal{A})$ and let ϕ be as in (2.7). Then $\dot\phi=\eta$ by (2.8) and (2.9) holds. Since system (2.9) is finite-time stable (see [12]), there is a nondecreasing function $\tau:(0,+\infty)\to(0,+\infty)$ with such that $(\phi(t),\eta(t))=(0,0)$ for all $t\ge\tau(R)$, if $|\phi(0)|+|\eta(0)|\le R$. Note that the last assumption is fulfilled, for

$$|\phi(0)|+|\eta(0)|\le|b^0|+\|v^0\|_{L^2(0,L)}+|\eta^0|\le\|w^0\|\le R.$$

Note that z satisfies

$$z_{tt}-z_{xx} = 0, \quad (x,t)\in(0,L)\times(0,+\infty), \tag{3.3}$$

$$\eta(t)=z_x(L,t)+z_t(L,t) = 0, \quad \forall t\ge\tau(R), \tag{3.4}$$

$$z_x(0,t) = 0, \quad \forall t\ge 0, \tag{3.5}$$

$$\phi(t)=z(L,t)+\int_0^L z_t(\xi,t)\,d\xi = 0, \quad \forall t\ge\tau(R). \tag{3.6}$$

Note that (3.4) is a *transparent boundary condition* for the wave equation, which allows the waves to leave the domain without bounce at $x=L$.

Introducing as e.g. in [1, 20, 9] the Riemann invariants $s:=z_t+z_x$ and $d:=z_t-z_x$ which solve by (3.3) the transport equations $s_t-s_x=0$ and $d_t+d_x=0$, respectively, we infer from (3.4) that $s(x,t)=0$ for $x\in(0,L)$ and $t\ge\tau(R)+L$, which, combined with (3.5), yields $d(0,t)=0$ for $t\ge\tau(R)+L$. It follows that $d(x,t)=0$ for $x\in(0,L)$ and $t\ge T(R):=\tau(R)+2L$. We conclude that $z_t=z_x=0$ for $x\in(0,L)$ and $t\ge T(R)$. Combined with (3.6), this yields $z(L,t)=0$ for $t\ge T(R)$, and thus $z(x,t)=0$ for $x\in(0,L)$ and $t\ge T(R)$.

The estimate (3.2) follows from (2.13) (with $\hat w^0=0$). The estimates (3.1)-(3.2) are extended to the general case ($w^0\in\mathcal{H}$) by using again (2.13) and the density of $D(\mathcal{A})$ in $\mathcal{H}$. This achieves also the proof of Theorem 1.1. □

Remark 3.1. (1) *Even if $\tau(0^+)=0$, it is only expected that $T(0^+)=2L$.*

(2) *Given $(z^0,v^0)\in H^2(0,L)\times H^1(0,L)$ and $X_p^0\in\mathbb{R}$ with $X_p^0=z^0(L)$ and $z_x^0(0)=0$, we have to set $b^0=X_p^0$ and $\eta^0=z_x^0(L)+v^0(L)$ to apply Theorem 2.1 and obtain a (strong) solution $w(t)\in D(\mathcal{A})$ for all $t\ge 0$. Another choice of η^0 would give a solution $w(t)$ solely in $\mathcal{H}$. More generally, given $(z^0,v^0)\in H^1(0,L)\times L^2(0,L)$ and $X_p^0\in\mathbb{R}$ with $X_p^0=z^0(L)$, we still have to set $b^0=X_p^0$ but we can pick any $\eta^0\in\mathbb{R}$. The choice $\eta^0=0$ is allowed. With this choice, we obtain by Theorem 2.1 a solution $w\in C([0,+\infty),\mathcal{H})$ for which the conclusion of Theorem 3.1 is valid. Note that w^0 is then bounded in $\mathcal{H}$ when (z^0,v^0) is bounded in $H^1(0,L)\times L^2(0,L)$.*

4. SIMULATION RESULTS AND CONCLUSION

In order to simulate the simplified system (1.8)-(1.11) in closed-loop with the two control laws (1.12) and (1.13), we have used a modal decomposition and obtained a linear truncated finite dimensional state formulation of the system similar to the one described in [2]. More precisely, introducing

$$\bar{z}(x,t) = z(x,t) - X_p(t) = z(x,t) - z(L,t) \tag{4.1}$$

leads to homogeneous boundary conditions

$$\bar{z}(L,t) = 0 \text{ and } \bar{z}_x(0,t) = 0. \tag{4.2}$$

It can be shown that the solution $\bar{z}(x,t)$ can be written as follows:

$$\bar{z}(x,t) = \sum_{i=0}^{+\infty} \alpha_i(t)\psi_i(x) \tag{4.3}$$

where the $\psi_i(x)$, $i = 0,\cdots,N$ are given by:

$$\psi_i(x) = A_i \sin(\omega_i x) + B_i \cos(\omega_i x) \text{ with } \omega_i = \frac{(2i+1)\pi}{2L}\,,\ A_i = 0\,,\ B_i = \sqrt{\frac{1}{\int_0^L \cos^2(\omega_i x)dx}} = \sqrt{2/L}, \tag{4.4}$$

and the $\alpha_i(t)$, $i = 0,\cdots,N$ satisfy:

$$\ddot{\alpha}_i + \omega_i^2 \alpha_i = -K_i \ddot{X}_p \text{ with } K_i = (-1)^i \frac{2}{\pi}\frac{\sqrt{2L}}{2i+1}. \tag{4.5}$$

The modes ω_i and the constants A_i, B_i and K_i have been obtained writing the boundary conditions and the orthonormalization conditions:

$$\int_0^L \psi_i^2(x)dx = 1\,,\ \int_0^L \psi_i\psi_j(x)dx = 0 \text{ for } i \neq j, \tag{4.6}$$

since the ψ_i's are known to constitute an orthonormal Hilbert basis in $L^2(0,L)$ (see e.g. [10]).

Then, the finite truncated dynamical system with state vector $X = (X_p, \alpha_0, \cdots, \alpha_N, \dot{X}_p, \dot{\alpha}_0, \cdots, \dot{\alpha}_N)'$, $N+1$ being the number of modes ω_i, has been respectively simulated in closed-loop with the asymptotic stabilizing control law (1.12) and the finite-time stabilizing control law (1.13).

4.1. **The asymptotic stabilization.**

In this subsection, the linear control law (1.12) is used, which gives:

$$u = -K^{-1}\left(\dot{\theta} + k\dot{X}_p\right) - \mu\left(\dot{X}_p + K^{-1}(\theta + kX_p)\right), \tag{4.7}$$

where the variable $X_p = z(L,t)$ (the position of the platform) and $\theta = z_x(L,t)$ (the angular deviation of the cable) and their time-derivatives are supposed to be measured, so that u is a *linear boundary feedback law*.

Remark 4.1. *Let us point out that $\theta = z_x(L,t) = \bar{z}_x(L,t)$. Using (4.3) and (4.4), it can be easily checked that θ (respectively $\dot{\theta}$) is an infinite linear combination of the α_i (respectively $\dot{\alpha}_i$) as follows:*

$$\begin{cases} \theta = \sum_{i=0}^{+\infty} \alpha_i(t) \frac{\partial \psi_i}{\partial x}(L) = \sum_{i=0}^{+\infty} c_i \alpha_i \\ \dot{\theta} = \sum_{i=0}^{+\infty} \dot{\alpha}_i(t) \frac{\partial \psi_i}{\partial x}(L) = \sum_{i=0}^{+\infty} c_i \dot{\alpha}_i \\ \text{with } c_i = -B_i(-1)^i \omega_i \text{ with } B_i = \sqrt{2/L},\ \forall i = 0,\cdots,N. \end{cases} \tag{4.8}$$

For simulation purposes, θ and $\dot{\theta}$ are estimated using the truncated expressions of (4.8) *where* $i = 0, \cdots, N$.

The behavior of the simplified system (1.8)-(1.11) in closed-loop with this boundary linear feedback law is displayed on Figure 2 with $N+1 = 10$ modes, $L = 1\,m$ and with the following gains: $K = 5$, $k = 4$, $\mu = 4$. The sampled period is equal to $1\,ms$ and the simulation was performed during $100\,s$, using an implicit Euler numerical scheme as suggested in [11]. The evolution of the variable X_p (resp. θ) is displayed at the top-left (resp. top-right) of Figure 2. The control law u is plotted at the bottom-left.

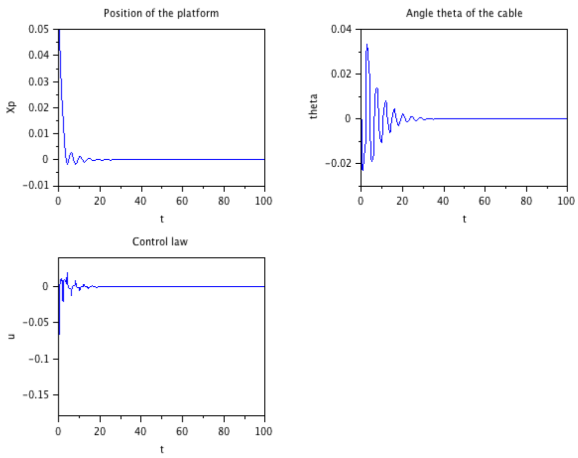

FIGURE 2. The overhead crane in closed-loop with the boundary linear feedback law

4.2. **The finite-time stabilization.** In this subsection, the linear control law (1.13) is used, which gives:

$$u = -\dot{\theta} - \left(\dot{X}_p + \theta\right)^{\beta} - \left(X_p + \int_0^L z_t(\xi,t)d\xi\right)^{\alpha}, \tag{4.9}$$

where the variables $X_p = z(L,t)$ (the position of the platform), $\theta = z_x(L,t)$ (the angular deviation of the cable) and their time-derivatives are supposed to be measured. Compared to the linear boundary control (4.7), the non linear control (4.9) is a distributed one due to the term $\int_0^L z_t(\xi,t)d\xi$. Using (4.1), (4.3), and (4.4), this integral term can be written:

$$\int_0^L z_t(\xi,t)d\xi = L\dot{X}_p + \sqrt{2/L}\sum_{i=0}^{N}\frac{(-1)^i\dot{\alpha}_i}{\omega_i}. \tag{4.10}$$

As expected from Theorem 3.1, the behavior of the simplified system (1.8)-(1.11) in closed-loop with this distributed nonlinear feedback law with $\alpha = 1$ and $\beta = \frac{1}{2}$ produces a finite-time stabilization as displayed in Figure 3. In this simulation, we have chosen as for the linear boundary feedback law: $N+1 = 10$ modes, $L = 1\,m$, the sampled period equal to $1\,ms$ and the simulation was performed during $100\,s$, using an implicit Euler numerical scheme. The evolution of the variable X_p (resp. θ) is displayed at the top-left (resp. top-right) of Figure 2. The variable ϕ defined by (2.7) and satisfying (2.9) is plotted at the bottom-left, while the control law u is plotted at the bottom-right.

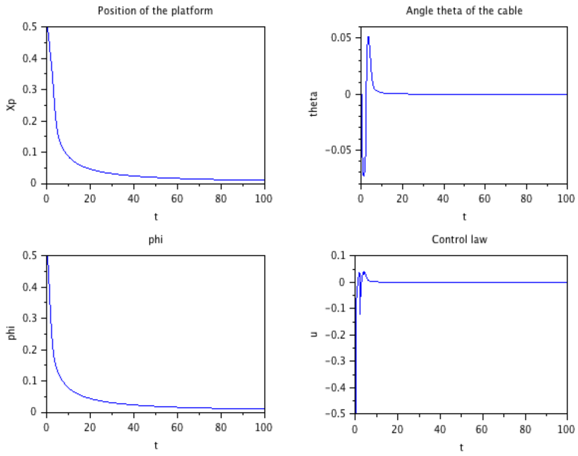

FIGURE 3. The overhead crane in closed-loop with the nonlinear finite-time feedback law with $\alpha = 1$ and $\beta = 1/2$

The variable ϕ has a similar behavior to the one of X_p. The case $\alpha = \dfrac{1}{2}$ and $\beta = \dfrac{1}{2}$ is illustrated in Figure 4. As expected, the variables are stabilized to zero *faster* than in the previous Figure 3 (corresponding to $\alpha = 1$ and $\beta = 1/2$).

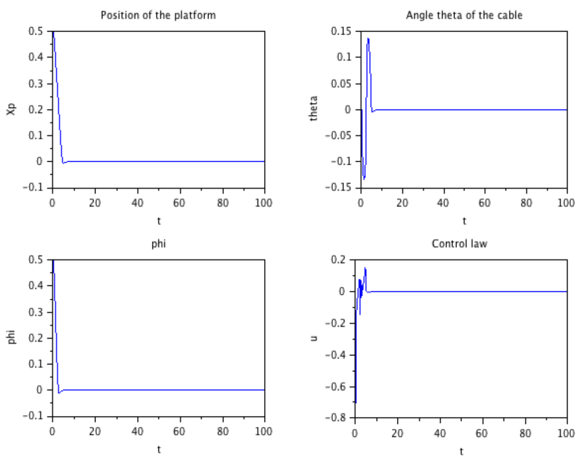

FIGURE 4. The overhead crane in closed-loop with the nonlinear finite-time feedback law with $\alpha = 1/2$ and $\beta = 1/2$

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