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Quantitative Convergence and Stability of Seismic Inverse Problems.

Florian Faucher\(^1\), Hélène Barucq\(^1\), Henri Calandra\(^2\) and Guy Chavent\(^1\).

Reconstruction Methods for Inverse Problems
Indam Workshop, Roma, Italy,
May 28\(^{\text{th}}\) – June 1\(^{\text{st}}\), 2018.

\(^1\)Inria Bordeaux Sud-Ouest, Project-Team Magique-3D.
\(^2\)Total E& P
Seismic inverse problem

Reconstruction of subsurface Earth properties from seismic campaign: collection of \textit{wave} propagation data at the surface.

- Reflection (back-scattered) partial data,
- only from the surface of the (large) domain,

\textbf{nonlinear, ill-posed inverse problem.}
Overview

1. Time-Harmonic Inverse Problem, FWI

2. Quantitative stability and convergence of FWI
   - Finite Curvature/Limited Deflection problem
   - Numerical convergence estimates
   - Numerical stability estimates

3. Numerical experiments

4. Conclusion
Plan

1. Time-Harmonic Inverse Problem, FWI
Time-harmonic wave equation

The forward problem wave equation depends on the medium:
The forward problem wave equation depends on the medium:

- **acoustic isotropic** \((c)\)

\[
(-\omega^2 c^{-2} - \Delta)p = 0,
\]
Time-harmonic wave equation

The forward problem wave equation depends on the medium:

- acoustic isotropic ($c$)
  \[ (-\omega^2 c^{-2} - \Delta) p = 0, \]

- elastic isotropic ($\lambda$, $\mu$, $\rho$)
  \[ -\rho \omega^2 u - \nabla (\lambda \nabla \cdot u) - \nabla \cdot (\mu [\nabla u + (\nabla u)^T]) = 0. \]
The forward problem wave equation depends on the medium:

- **acoustic isotropic** ($c$)

  
  \[ (-\omega^2 c^{-2} - \Delta) p = 0, \]

- **elastic isotropic** ($\lambda$, $\mu$, $\rho$)

  \[ -\rho \omega^2 u - \nabla (\lambda \nabla \cdot u) - \nabla \cdot (\mu [\nabla u + (\nabla u)^T]) = 0. \]

- **anisotropy** (stiffness tensor, $\rho$)

  \[ -\omega^2 \rho u - \nabla \cdot (C\varepsilon(u)) = 0. \]

Viscous behavior are considered with complex coefficients.
Full Waveform Inversion (FWI)

FWI provides a quantitative reconstruction of the subsurface by solving a minimization problem,

$$\min_{m \in M} \mathcal{J}(m) = \frac{1}{2} \| F(m) - d \|^2.$$ 

- $d$ are the observed data,
- $F(m)$ represents the simulation using an initial model $m$:

$$F : m \rightarrow \{ p(x_1), p(x_2), \ldots, p(x_{n_{rcv}}) \}.$$ 

- P. Lailly
  *The seismic inverse problem as a sequence of before stack migrations*
  *Conference on Inverse Scattering: Theory and Application, SIAM, 1983*

- A. Tarantola
  *Inversion of seismic reflection data in the acoustic approximation*
  *Geophysics, 1984*

- A. Tarantola
  *Inversion of travel times and seismic waveforms*
  *Seismic tomography, 1987*
FWI, iterative minimization

Observations $\rightarrow$ Initial model $m_0$ $\rightarrow$ Forward problem $F_\omega(m_k)$ $\rightarrow$ Misfit functional $\mathcal{J}$
FWI, iterative minimization

- **Observations**
- **Initial model** $m_0$
- **Forward problem** $F_\omega(m_k)$
- **Misfit functional** $\mathcal{J}$

**Optimization procedure**
1. Gradient
2. Search direction $s_k$
3. Line search $\alpha_k$

update model
$m_{k+1} = m_k + \alpha_k s_k$

Update $\omega$
FWI, iterative minimization

Observations → Initial model \( m_0 \) → Forward problem \( F_\omega(m_k) \) → Misfit functional \( \mathcal{J} \) → Optimization procedure:
1. Gradient
2. Search direction \( s_k \)
3. Line search \( \alpha_k \)

Update model:
\[ m_{k+1} = m_k + \alpha_k s_k \]

Update \( \omega \)

Numerical methods:
- Forward problem resolution with Discontinuous Galerkin methods,
- parallel computation, HPC, large-scale optimization.
- Multi-frequency algorithm, stability and convergence.
Plan

2 Quantitative stability and convergence of FWI
- Finite Curvature/Limited Deflection problem
- Numerical convergence estimates
- Numerical stability estimates
Helmholtz inverse problem from back-scattered partial data, quantitative reconstruction with iterative optimization.

\[
\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^2.
\]

**Convergence radius**
- Initial model needs to be within the radius of convergence.

**Stability**
- Ensures reconstruction accuracy.
Helmholtz inverse problem from back-scattered partial data, quantitative reconstruction with iterative optimization.

\[
\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^2.
\]

**Convergence radius**
- initial model needs to be within the radius of convergence.

**Stability**
- ensures reconstruction accuracy.

**Numerical estimates** to guide the procedure
- frequency;
- parametrization;
- geometry;
- forward problem, ...
Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- initial model needs to be within the radius of convergence.

Least squares minimization problem

\[
\min_{m \in \mathcal{M}} J(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^2.
\]

- M. V. de Hoop, L. Qiu, O. Scherzer
  An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints
  Numerische Mathematik 2015

- G. Chavent
  Springer 2010

- G. Chavent and K. Kunisch
  On weakly nonlinear inverse problems.
  SIAM Journal on Applied Mathematics 1996
Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- initial model needs to be within the radius of convergence.

Least squares minimization problem

\[
\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^2.
\]

Finite Curvature/Limited Deflection guarantees **uniqueness** of the solution and **unimodality**: no local minimum.

- M. V. de Hoop, L. Qiu, O. Scherzer
  An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints
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  SIAM Journal on Applied Mathematics 1996
Condition 1/2: Finite Curvature problem

\[ \forall m_0, \Delta m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta m). \]
Condition 1/2: Finite Curvature problem

\begin{align*}
\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] &\rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).
\end{align*}

Finite Curvature controls the distance between data and attainable set,

\[ \text{dist}(d, \mathcal{F}) < \frac{\|P'(t)\|^2}{\|P''(t)\|}. \]

G. Chavent


Springer 2010
Condition 2/2: Limited Deflection problem

\[ \forall m_0, \Delta m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta m). \]
Condition 2/2: Limited Deflection problem

\[ \forall m_0, \Delta m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta m). \]

Limited Deflection property controls the attainable set and model space,

\[ \Theta(P) \leq \int_0^1 \frac{\|P''(t)\|}{\|P'(t)\|} \, dt. \quad \text{FC/LD requires } \Theta(P) \leq \frac{\pi}{2}. \]

G. Chavent
Model space size estimation via Limited Deflection

For a given $m_0$, we estimate the maximal distance $\Delta_m$ that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

Larger size are required when missing a priori info.
Model space size estimation via Limited Deflection

For a given $m_0$, we estimate the maximal distance $\Delta m$ that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

- $\Delta m$ is estimated in the direction $\delta_k$,

$$
\| \Delta^\delta_k \| \leq \frac{\pi}{4} \frac{\| \mathcal{F}'(m_0)(\delta_k) \|}{\| \mathcal{F}''(m_0)(\delta_k, \delta_k) \|}.
$$

Larger size are required when missing a priori info.
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{\delta_k}^m$

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$,
3. with the parametrization.
Estimation of the basin of attraction

Numerical estimate of the size $\Delta^{\delta_k}$

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$,
3. with the parametrization.

**context:** Helmholtz equation with back-scattered data

$(-\omega^2 m - \Delta)p = 0,$

- reflection data (top free surface),
- absorbing conditions on the sides,
- piecewise constant $m$. 

$m_0$ is a smooth velocity
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{\delta_k}^m$ with

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$. 
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{m}^{\delta_{k}}$ with

1. with direction $\delta_{k}$ (the geometry of the unknown),
2. with the frequency $\omega$.

![Graph showing the upper bound of $\Delta_{m}^{\delta_{k}}$ as a function of frequency.](image)

(a) direction $\delta_{1}$
(b) direction $\delta_{2}$
(c) direction $\delta_{3}$

Low frequencies increase the size.
Reflecting objects complicate the procedure.
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{\delta_k}^{\delta_k}$ with

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$.

Low frequencies increase the size.

Reflecting objects complicate the procedure.
Inverse Problem
Stability & Convergence
Experiments
Conclusion

Complex frequency

\((-\omega^2 c^{-2} - \Delta)p = 0\)

\[\begin{align*}
\triangleright & \quad -\omega^2 = (s + 2i\pi f)^2 \\
\triangleright & \quad s = 0: \text{ Fourier domain } \omega^2 = (2\pi f)^2, \\
\triangleright & \quad f = 0: \text{ Laplace domain } -\omega^2 = s^2.
\end{align*}\]

C. Shin and Y. H. Cha
Waveform inversion in the laplace domain ; Waveform inversion in the laplace fourier domain

W. Ha, S. Pyun, J. Yoo and C. Shin
Acoustic full waveform inversion of synthetic land and marine data in the laplace domain ;
Geophysical Prospecting 2010

P. V. Petrov and G. A. Newman
Three-dimensional inverse modelling of damped elastic wave propagation in the fourier domain.
Geophysical Journal International 2014
Complex frequency

\[-\omega^2 = (s + 2i\pi f)^2\]

Selection of complex frequency.
Stability with frequency

2/ Stability

- ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

\[ \| m_1 - m_2 \| \leq C \| \mathcal{F}(m_1) - \mathcal{F}(m_2) \|. \]

The stability constant $C$ depends on the frequency and the number of unknowns.

- **G. Alessandrini**
  - Stable determination of conductivity by boundary measurement
  - Applicable Analysis 1988

- **E. Beretta, M. V. de Hoop, F. and O. Scherzer**
  - Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates.

- **G. Alessandrini, M.V. de Hoop, R. Gaburro and E. Sincich**
  - Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data
Stability with frequency

2/ Stability

- ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

\[ \| m_1 - m_2 \| \leq C \| \mathcal{F}(m_1) - \mathcal{F}(m_2) \|. \]
From our estimates, we see different conditions for stability and convergence.

Quantitative estimates provide an initial, comprehensive, relation to guide the iterative procedure.
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Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency

high stability constant.
Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

- largest radius of convergence,

1./ Start with low frequency

- high stability constant.

- improve stability, i.e. resolution,

2./ Increase frequency

- reduced radius of convergence.
Multi-parameters inversion $17 \times 3.5$km, true models.

$$-\rho \omega^2 \mathbf{u} - \nabla (\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot (\mu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]) = 0.$$ 

(a) P-wave speed $\sqrt{\frac{\lambda + 2\mu}{\rho}}$

(b) S-wave speed $\sqrt{\frac{\mu}{\rho}}$

(c) Density
Elastic Marmousi model

Multi-parameters inversion $17 \times 3.5\text{km}$, starting models.

(a) P-wave speed  
(b) S-wave speed  
(c) Density
Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed

(b) True S-wave speed

(c) Initial P-wave speed

(d) Initial S-wave speed
Elastic Marmousi reconstruction $17 \times 3.5km$

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed
(b) True S-wave speed
(c) 10Hz P-wave speed
(d) 10Hz S-wave speed
Elastic Pluto models

Multi-parameters inversion $31 \times 7$km, true models.

(a) P-wave speed

(b) S-wave speed

(c) Density
Elastic Pluto models

Multi-parameters inversion $31 \times 7\text{km}$, starting models.

(a) P-wave speed

(b) S-wave speed

(c) Density
Elastic Pluto reconstruction $31 \times 7$ km

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed
(b) True S-wave speed
(c) Initial P-wave speed
(d) Initial S-wave speed
Elastic Pluto reconstruction $31 \times 7\text{km}$

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed

(b) True S-wave speed

(c) 10Hz P-wave speed

(d) 10Hz S-wave speed
Elastic Pluto reconstruction $31 \times 7km$

Reminder from the convergence analysis.

Solution

- Start with low or complex frequency.
Elastic Pluto reconstruction $31 \times 7\text{km}$

Using complex frequencies, unknown density, multi-level algorithm.

(a) True P-wave speed  
(b) True S-wave speed

(c) Initial P-wave speed  
(d) Initial S-wave speed
Elastic Pluto reconstruction $31 \times 7$km

Using complex frequencies, unknown density, multi-level algorithm.

(a) True P-wave speed
(b) True S-wave speed

(c) (0Hz,10) P-wave speed
(d) (0Hz,10) S-wave speed
Elastic Pluto reconstruction $31 \times 7\text{km}$

Using complex frequencies, unknown density, multi-level algorithm.

(a) True P-wave speed

(b) True S-wave speed

(c) 10Hz P-wave speed

(d) 10Hz S-wave speed
Plan

4 Conclusion
Conclusion

Comprehensive FWI
- Quantitative stability and convergence analysis:
  - depending on the frequency and geometry,
  - depending on the methods.
- We can also quantify the noise robustness.

Perspectives
- Analytical relation to develop between the two components,
- obtain conditional progression in frequency from stability and convergence estimates.
Quantitative reconstruction method for inverse problem

- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

**Applications and ongoing investigations:**
Quantitative reconstruction method for inverse problem

- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:
- **Inverse scattering** using obstacles positions,
- **visco-elastic** reconstruction: five unknowns,
  - $(\lambda, \mu, \rho, Q_\lambda, Q_\mu)$,
  - unknown $Q$ does not prevent $\lambda$ recovery,
  - procedure to recover attenuation parameter?

- **helioseismology; anisotropy; model parametrization**.
Quantitative reconstruction method for inverse problem

- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:
- **Inverse scattering** using obstacles positions,
- **visco-elastic** reconstruction: five unknowns,
  - $(\lambda, \mu, \rho, Q_\lambda, Q_\mu)$,
  - unknown $Q$ does not prevent $\lambda$ recovery,
  - procedure to recover attenuation parameter?

- **helioseismology**, **anisotropy**, model **parametrization**.

[Images: (a) $\lambda$, (b) $Q_\lambda$]
APPENDIX
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_{\delta_k}^m$ the problem components

1. the frequency $\omega$,
2. with direction $\delta_k$ (the geometry of the unknown),
3. with the parametrization.

The estimates allow a comparison of methods, in particular we want to compare the dependency of the optimization on the low frequencies.
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_k^m$ the problem components

1. the frequency $\omega$,
2. with direction $\delta_k$ (the geometry of the unknown),
3. with the parametrization.

the MBTT method decomposes the model with a smooth part (propagator) and the reflectors:

$$c^{-2} = p + r = p + D\mathcal{F}^*(p)s.$$
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta^\delta_k$ the problem components

1. the frequency $\omega$,
2. with direction $\delta_k$ (the geometry of the unknown),
3. with the parametrization.

Comparison of convergence with parametrization,

$$c^{-2} = p + D\mathcal{F}^*(p)s.$$
Quantitative convergence estimates with frequency progression for iterative algorithm, \(-\omega^2 = (s + 2i\pi f)^2\).

Other application
- Single frequency gives high radius than range of frequencies,
- can be apply for other minimization problem,
- ...

Finite curvature indicates robustness to noise
- Low and complex frequencies more affected.

Stability
- Indicates the accuracy of the reconstruction.
Starting available frequency is 4Hz.
Starting available frequency is 4Hz.

Using Fourier frequencies only: from 4 to 10Hz
Starting available frequency is 4Hz.

(a) Using Fourier frequencies only

(b) Using Complex frequencies from (4Hz, 7)