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Quantitative Convergence and Stability of Seismic Inverse Problems.

Florian Faucher\textsuperscript{1},
Hélène Barucq\textsuperscript{1}, Henri Calandra\textsuperscript{2} and Guy Chavent\textsuperscript{1}.

Reconstruction Methods for Inverse Problems
Indam Workshop, Roma, Italy,
May 28\textsuperscript{th} – June 1\textsuperscript{st}, 2018.

\textsuperscript{1}Inria Bordeaux Sud-Ouest, Project-Team Magique-3D.
\textsuperscript{2}Total E& P
Seismic inverse problem

Reconstruction of subsurface Earth properties from seismic campaign: collection of wave propagation data at the surface.

- Reflection (back-scattered) partial data,
- only from the surface of the (large) domain,

nonlinear, ill-posed inverse problem.
Overview

1. Time-Harmonic Inverse Problem, FWI

2. Quantitative stability and convergence of FWI
   - Finite Curvature/Limited Deflection problem
   - Numerical convergence estimates
   - Numerical stability estimates

3. Numerical experiments

4. Conclusion
Plan

1. Time-Harmonic Inverse Problem, FWI
Time-harmonic wave equation

The forward problem wave equation depends on the medium:
The forward problem wave equation depends on the medium:

- acoustic isotropic \((c)\)

\[ (-\omega^2 c^{-2} - \Delta)p = 0, \]
**Time-harmonic wave equation**

The forward problem wave equation depends on the medium:

- **Acoustic isotropic** ($c$)
  \[ (-\omega^2 c^{-2} - \Delta) p = 0, \]

- **Elastic isotropic** ($\lambda, \mu, \rho$)
  \[ -\rho \omega^2 u - \nabla (\lambda \nabla \cdot u) - \nabla \cdot (\mu \left[ \nabla u + (\nabla u)^T \right]) = 0. \]
Time-harmonic wave equation

The forward problem wave equation depends on the medium:

- acoustic isotropic $(c)$
  \[ (-\omega^2 c^{-2} - \Delta) p = 0, \]

- elastic isotropic $(\lambda, \mu, \rho)$
  \[ -\rho \omega^2 u - \nabla (\lambda \nabla \cdot u) - \nabla \cdot (\mu [\nabla u + (\nabla u)^T]) = 0. \]

- anisotropy (stiffness tensor, $\rho$)
  \[ -\omega^2 \rho u - \nabla \cdot (C\varepsilon(u)) = 0. \]

Viscous behavior are considered with complex coefficients.
Full Waveform Inversion (FWI)

FWI provides a quantitative reconstruction of the subsurface by solving a minimization problem,

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \| F(m) - d \|^2.$$

- $d$ are the observed data,
- $F(m)$ represents the simulation using an initial model $m$:
  $$F : m \rightarrow \{ p(x_1), p(x_2), \ldots, p(x_{nrcv}) \}.$$
FWI, iterative minimization

Observations \rightarrow \text{Misfit functional } \mathcal{J} \rightarrow \text{Forward problem } F_\omega(m_k) \rightarrow \text{Initial model } m_0 \rightarrow k = 0 \rightarrow \text{Observations}
FWI, iterative minimization

Observations → Initial model $m_0$ → Forward problem $F_\omega(m_k)$ → Misfit functional $\mathcal{J}$ → Optimization procedure

1. Gradient
2. Search direction $s_k$
3. Line search $\alpha_k$

$k = k + 1$

update model $m_{k+1} = m_k + \alpha_k s_k$

Update $\omega$
FWI, iterative minimization

Observations → Misfit functional $\mathcal{J}$ → Forward problem $F_\omega(m_k)$ → Initial model $m_0$

Optimization procedure:
1. Gradient
2. Search direction $s_k$
3. Line search $\alpha_k$

$k = 0$

update model $m_{k+1} = m_k + \alpha_k s_k$

$k = k + 1$

Update $\omega$

Numerical methods:
- Forward problem resolution with Discontinuous Galerkin methods,
- parallel computation, HPC, large-scale optimization.

- Multi-frequency algorithm, **stability** and **convergence**.
Plan

2 Quantitative stability and convergence of FWI
   - Finite Curvature/Limited Deflection problem
   - Numerical convergence estimates
   - Numerical stability estimates
Helmholtz inverse problem from back-scattered *partial* data, quantitative reconstruction with iterative optimization.

\[ \min_{m \in M} J(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^{2}. \]

**Convergence radius**

- initial model needs to be within the radius of convergence.

**Stability**

- ensures reconstruction accuracy.
Helmholtz inverse problem from back-scattered partial data, quantitative reconstruction with iterative optimization.

\[
\min_{m \in M} J(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^2.
\]

**Convergence radius**
- initial model needs to be within the radius of convergence.

**Stability**
- ensures reconstruction accuracy.

**Numerical estimates** to guide the procedure
- frequency;
- geometry;
- parametrization;
- forward problem, . . .
Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- initial model needs to be within the radius of convergence.

Least squares minimization problem

\[
\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \| \mathcal{F}(m) - d \|^2.
\]

References:

M. V. de Hoop, L. Qiu, O. Scherzer
An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints
Numerische Mathematik 2015

G. Chavent
Springer 2010

G. Chavent and K. Kunisch
On weakly nonlinear inverse problems.
SIAM Journal on Applied Mathematics 1996
Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- initial model needs to be within the radius of convergence.

Least squares minimization problem

$$\min_{m \in M} J(m) = \frac{1}{2} \| F(m) - d \|^2.$$ 

Finite Curvature/Limited Deflection guarantees **uniqueness** of the solution and **unimodality**: no local minimum.

- M. V. de Hoop, L. Qiu, O. Scherzer
  An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints
  Numerische Mathematik 2015

- G. Chavent
  Springer 2010

- G. Chavent and K. Kunisch
  On weakly nonlinear inverse problems.
  SIAM Journal on Applied Mathematics 1996
Condition 1/2: Finite Curvature problem

\[ \forall m_0, \Delta_m \in M, \quad P : \ t \in [0, 1] \rightarrow F(m_0 + (2t - 1)\Delta_m). \]
Condition 1/2: Finite Curvature problem

\[ \forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m). \]

Finite Curvature controls the distance between data and attainable set,

\[ \text{dist}(d, \mathcal{F}) < \frac{\|P'(t)\|^2}{\|P''(t)\|}. \]

G. Chavent
Condition 2/2: Limited Deflection problem

\[ \forall m_0, \Delta m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m). \]
Condition 2/2: Limited Deflection problem

\[ \forall m_0, \Delta m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta m). \]

Limited Deflection property controls the attainable set and model space,

\[ \Theta(P) \leq \int_0^1 \frac{\|P''(t)\|}{\|P'(t)\|} \, dt. \]

FC/LD requires \( \Theta(P) \leq \frac{\pi}{2} \).

G. Chavent
Model space size estimation via Limited Deflection

For a given $m_0$, we estimate the maximal distance $\Delta_m$ that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

Larger size are required when missing a priori info.
Model space size estimation via Limited Deflection

For a given $m_0$, we estimate the maximal distance $\Delta_m$ that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

- $\Delta_m$ is estimated in the direction $\delta_k$,

$$
\|\Delta_{m}^{\delta_k}\| \leq \frac{\pi}{4} \frac{\| F'(m_0)(\delta_k) \|}{\| F''(m_0)(\delta_k, \delta_k)\|}.
$$

Larger size are required when missing a priori info.
Estimation of the basin of attraction

Numerical estimate of the size $\Delta m^\delta_k$

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$,
3. with the parametrization.
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{m}^{\delta_{k}}$

1. with direction $\delta_{k}$ (the geometry of the unknown),
2. with the frequency $\omega$,
3. with the parametrization.

**context:** Helmholtz equation with back-scattered data

\[ (-\omega^2 m - \Delta) p = 0, \]

$m_{0}$ is a smooth velocity

- reflection data (top free surface),
- absorbing conditions on the sides,
- piecewise constant $m$. 
Estimation of the basin of attraction

Numerical estimate of the size $\Delta^{\delta_k}_m$ with

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$. 
Estimation of the basin of attraction

Numerical estimate of the size $\Delta^{\delta_k}_m$ with

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$.

![Graph showing upper bound of $\Delta^{\delta_k}_m$]

- (a) direction $\delta_1$
- (b) direction $\delta_2$
- (c) direction $\delta_3$

Low frequencies increase the size. Reflecting objects complicate the procedure.
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{m}^{\delta_k}$ with

1. with direction $\delta_k$ (the geometry of the unknown),
2. with the frequency $\omega$.

Low frequencies increase the size.
Reflecting objects complicate the procedure.

(a) direction $\delta_1$
(b) direction $\delta_2$
(c) direction $\delta_3$
Complex frequency

\[ (-\omega^2 c^{-2} - \Delta) p = 0 \]

\[ -\omega^2 = (s + 2i\pi f)^2 \]

\[ s = 0: \text{Fourier domain } \omega^2 = (2\pi f)^2, \]

\[ f = 0: \text{Laplace domain } -\omega^2 = s^2. \]

- C. Shin and Y. H. Cha
  Waveform inversion in the laplace domain ; Waveform inversion in the laplace fourier domain

- W. Ha, S. Pyun, J. Yoo and C. Shin
  Acoustic full waveform inversion of synthetic land and marine data in the laplace domain ;
  Geophysical Prospecting 2010

- P. V. Petrov and G. A. Newman
  Three-dimensional inverse modelling of damped elastic wave propagation in the fourier domain.
  Geophysical Journal International 2014
Complex frequency

\[-\omega^2 = (s + 2i\pi f)^2\]

Selection of complex frequency.
Stability with frequency

2/ Stability

ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

\[ \| m_1 - m_2 \| \leq C \| \mathcal{F}(m_1) - \mathcal{F}(m_2) \|. \]

The stability constant \( C \) depends on the frequency and the number of unknowns.

---

1. **G. Alessandrini**
   - Stable determination of conductivity by boundary measurement
   - Applicable Analysis 1988

2. **E. Beretta, M. V. de Hoop, F. and O. Scherzer**
   - Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates.

3. **G. Alessandrini, M.V. de Hoop, R. Gaburro and E. Sincich**
   - Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data
Stability with frequency

2/ Stability

- ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

\[ \| m_1 - m_2 \| \leq C \| \mathcal{F}(m_1) - \mathcal{F}(m_2) \|. \]
Stability & Convergence estimates

From our estimates, we see different conditions for stability and convergence.

Quantitative estimates provide an initial, comprehensive, relation to guide the iterative procedure.
Plan

3 Numerical experiments
Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency

high stability constant.
Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency

high stability constant.

improve stability, *i.e.* resolution,

2./ Increase frequency

reduced radius of convergence.
Elastic Marmousi model

Multi-parameters inversion $17 \times 3.5\text{km}$, true models.

$$-\rho \omega^2 \mathbf{u} - \nabla (\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot (\mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) = 0.$$  

(a) P-wave speed $\sqrt{\frac{\lambda+2\mu}{\rho}}$  
(b) S-wave speed $\sqrt{\frac{\mu}{\rho}}$  
(c) Density
Elastic Marmousi model

Multi-parameters inversion $17 \times 3.5\text{km}$, starting models.

(a) P-wave speed

(b) S-wave speed

(c) Density
Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed

(b) True S-wave speed

(c) Initial P-wave speed

(d) Initial S-wave speed
Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed  
(b) True S-wave speed  
(c) 10Hz P-wave speed  
(d) 10Hz S-wave speed
Elastic Pluto models

Multi-parameters inversion $31 \times 7$km, true models.

(a) P-wave speed  
(b) S-wave speed  
(c) Density
Elastic Pluto models

Multi-parameters inversion $31 \times 7$km, starting models.

(a) P-wave speed

(b) S-wave speed

(c) Density
Elastic Pluto reconstruction $31 \times 7$km

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed
(b) True S-wave speed
(c) Initial P-wave speed
(d) Initial S-wave speed
Elastic Pluto reconstruction $31 \times 7$km

Frequency from 1 to 10Hz, unknown density, multi-level algorithm.

(a) True P-wave speed

(b) True S-wave speed

(c) 10Hz P-wave speed

(d) 10Hz S-wave speed
Elastic Pluto reconstruction $31 \times 7\text{km}$

Reminder from the convergence analysis.

Estimation of the basin of attraction

- Salt domes reduce the size of the radius of convergence.

Solution
- Start with low or complex frequency.
Elastic Pluto reconstruction $31 \times 7\text{km}$

Using complex frequencies, unknown density, multi-level algorithm.

(a) True P-wave speed
(b) True S-wave speed
(c) Initial P-wave speed
(d) Initial S-wave speed
Elastic Pluto reconstruction $31 \times 7\text{km}$

Using complex frequencies, unknown density, multi-level algorithm.

(a) True P-wave speed  
(b) True S-wave speed

(c) (0Hz,10) P-wave speed  
(d) (0Hz,10) S-wave speed
Elastic Pluto reconstruction $31 \times 7\text{km}$

Using complex frequencies, unknown density, multi-level algorithm.

(a) True P-wave speed
(b) True S-wave speed
(c) 10Hz P-wave speed
(d) 10Hz S-wave speed
Plan

4 Conclusion
Conclusion

Comprehensive FWI

- Quantitative stability and convergence analysis:
  - depending on the frequency and geometry,
  - depending on the methods.
- We can also quantify the noise robustness.

Perspectives

- Analytical relation to develop between the two components,
- obtain conditional progression in frequency from stability and convergence estimates.
Quantitative reconstruction method for inverse problem

- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

**Applications and ongoing investigations:**

Thank you
Quantitative reconstruction method for inverse problem

▶ Discontinuous Galerkin discretization in HPC framework,
▶ acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:
▶ Inverse scattering using obstacles positions,
▶ visco-elastic reconstruction: five unknowns,
▶ \( (\lambda, \mu, \rho, Q_\lambda, Q_\mu) \),
▶ unknown \( Q \) does not prevent \( \lambda \) recovery,
▶ procedure to recover attenuation parameter?
▶ helioseismology; anisotropy; model parametrization.
Quantitative reconstruction method for inverse problem

- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:
- Inverse scattering using obstacles positions,
- visco-elastic reconstruction: five unknowns,
- (\(\lambda, \mu, \rho, Q_\lambda, Q_\mu\)),
- unknown \(Q\) does not prevent \(\lambda\) recovery,
- procedure to recover attenuation parameter?

- helioseismology; anisotropy; model parametrization.

THANK YOU
APPENDIX
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta^{\delta_k}_{m}$ the problem components

1. the frequency $\omega$,
2. with direction $\delta_k$ (the geometry of the unknown),
3. with the parametrization.

The estimates allow a comparison of methods, in particular we want to compare the dependency of the optimization on the low frequencies.
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta^\delta_{m}$ the problem components

1. the frequency $\omega$,
2. with direction $\delta_k$ (the geometry of the unknown),
3. with the parametrization.

the MBTT method decomposes the model with a smooth part (propagator) and the reflectors:

$$c^{-2} = p + r = p + D\mathcal{F}^*(p)s.$$
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta^\delta_k$ the problem components

1. the frequency $\omega$,
2. with direction $\delta_k$ (the geometry of the unknown),
3. with the parametrization.

Comparison of convergence with parametrization,

$$c^{-2} = p + DF^*(p)s.$$
Quantitative convergence estimates with frequency

Frequency progression for iterative algorithm, \(-\omega^2 = (s + 2i\pi f)^2\).

Other application
- Single frequency gives high radius than range of frequencies,
- can be apply for other minimization problem,
- ...

Finite curvature indicates robustness to noise
- Low and complex frequencies more affected.

Stability
- Indicates the accuracy of the reconstruction.
Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

Starting available frequency is 4Hz.
Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

Starting available frequency is $4\text{Hz}$.

Using Fourier frequencies only: from 4 to $10\text{Hz}$.
Starting available frequency is 4Hz.

(a) Using Fourier frequencies only
(b) Using Complex frequencies from (4Hz, 7)