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SUFFICIENT OPTIMALITY CONDITIONS FOR BILINEAR OPTIMAL CONTROL OF THE LINEAR DAMPED WAVE EQUATION

FRANZ BETHKE AND AXEL KRÖNER

Abstract. In this paper we discuss sufficient optimality conditions for an optimal control problem for the linear damped wave equation with the damping parameter as the control. We address the case that the control enters quadratic in the cost function as well as the singular case that the control enters affine. For the non-singular case we consider strong and weak local minima, in the singular case we derive sufficient optimality conditions for weak local minima. Thereby, we take advantage of the Goh transformation applying techniques recently established in Aronna, Bonnans, and Kröner [*Math. Program. 168(1):717–757, 2018*] and [*INRIA research report, 2017*]. Moreover, a numerical example for the singular case is presented.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \leq 3$ with sufficiently smooth boundary $\partial\Omega$ and $T > 0$. We consider optimal control problems for the damped linear wave equation in which the control u enters as a damping parameter, i.e. equations of type

$$\begin{cases} \ddot{y}(t,x) - \Delta y(t,x) = u(t)b(x)\dot{y}(t,x) + f(t,x) \;\text{ in } (0,T)\times\Omega, \\ y(t,x) = 0 \;\text{ on } (0,T)\times\partial\Omega, \quad y(0,x) = y_{0,1}(x) \;\text{ in } \Omega, \quad \dot{y}(0,x) = y_{0,2}(x) \;\text{ in } \Omega \end{cases} \tag{1.1}$$

with suitable chosen f, b and initial data $y_{0,i}$, $i = 1,2$. In one space dimension this equation describes the motion of a string with forced damping and additional source term in the space-time domain. In the damping term we have a bilinear coupling of the control and the velocity.

We consider optimal control problems for (1.1) with cost function

$$\begin{cases} J(u,y) := \dfrac{\beta_1}{2}\|y - y_{d,1}\|^2_{L^2(0,T;L^2(\Omega))} + \dfrac{\beta_2}{2}\|y(T) - y^2_{dT,1}\|^2_{L^2(\Omega)} \\ \qquad + \dfrac{\beta_3}{2}\|\dot{y} - y_{d,2}\|^2_{L^2(0,T;H^{-1}(\Omega))} + \dfrac{\beta_4}{2}\|\dot{y}(T) - y^2_{dT,2}\|^2_{H^{-1}(\Omega)} \\ \qquad + \displaystyle\int_0^T \alpha_1 u(t) + \frac{\alpha_2}{2}u(t)^2 \mathrm{d}t \quad \text{subject to (1.1) and } u \in \mathcal{U}_{ad}, \end{cases} \tag{1.2}$$

with $\mathcal{U}_{ad} := \{u \in L^2(0,T) : u_m \leq u(t) \leq u_M, \text{ a.e. in } (0,T)\}$ the set of admissible controls, given constant control bounds $0 \leq u_m < u_M$, and sufficiently smooth desired states $y_{d,i}\colon (0,T)\times\Omega \to \mathbb{R}$, $y_{dT,i}\colon \Omega \to \mathbb{R}$ for $i = 1,2$, $\alpha_1 \in \mathbb{R}$, $\alpha_2 \geq 0$, and $\beta_j \geq 0$, $j = 1,\ldots,4$.

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The main contribution of this paper is the application of techniques developed in Aronna, Bonnans, and Kröner [4, 2] to the optimal control problem of the damped wave equation given in (1.2). We derive sufficient optimality conditions for weak and strong local minima in the case $\alpha_2 > 0$ as well as for weak local minima in the singular case $\alpha_1 = 0$. For the non-sigular case we rely on techniques developed in [2], where control problems for the Schrödinger equation are considered. In the singular case the classical techniques of the calculus of variations do not lead to the formulation of second-order sufficient optimality conditions. By applying the Goh transformation [19] (introduced in the context of optimal control of ordinary differential equations) and following ideas in [4], we derive sufficient optimality conditions guaranteeing weak quadratic growth. Thereby, a commutator plays an important role. However, this commutator is a differential operator of qualitative difference to the one considered in [4]; while in the latter reference singular optimal control problems for wave equations with bilinear coupling of control and state are considered which leads to a zero-order operator, here the control is coupled bilinearly with the velocity which gives a second-order differential operator for the commutator. Consequently, different regularity properties are necessary to obtain compactness results. We will restrict the presentation to differences to the setting developed in [4]. At the end, a numerical example for the case $\alpha_1 = \alpha_2 = 0$ is presented.

The first extension of the Goh calculus to optimal control problems to PDEs was done in Bonnans [10], where a semilinear heat equation with scalar control was considered and then further extended in [4] to problems for strongly continuous semigroups. In [2] the approach was transferred to a complex setting and applied to optimal control problems for the linear Schrödinger equation.

To complete the list of references we mention some applications of the Goh transformation in the context of ordinary differential equations, see, e.g., [20, 19, 18, 1].

In the context of optimal control of PDEs there exist only a few papers on sufficient optimality conditions for control-affine problems, see Bergounioux and Tiba [8], Tröltzsch [27], Bonnans and Tiba [13], and Casas, Wachsmuth, and Wachsmuth [15].

For bilinear optimal control of wave equations, see, e.g., Lenhart and Protopopescu [22] where an identification problem for a coefficient in the wave equation via optimal control is considered, and Sonawane [25] where a bilinear optimal control problem for a vibrating string is analyzed. Furthermore, we refer to Lasiecka and Triggiani [21] and Bales and Lasiecka [5] for optimal control of wave equations in a semigroup setting.

For results on controllability of the damped wave equation see, e.g., [28] and for stabilization, e.g., [16, 30, 17].

The paper is organized as follows: In Section 2 the damped wave equation is introduced. In Section 3 existence of optimal controls is verified. In Section 4 we derive first- and second-order necessary optimality conditions. In Section 5 sufficient optimality conditions for the non-singular case ($\alpha_2 > 0$), in Section 6 for the singular case ($\alpha_2 = 0$) are derived, and in Section 7 we present a numerical example for the singular case.

Notation: For given Hilbert space $\mathcal{H}$, with norm $\|\cdot\|_{\mathcal{H}}$, we denote by $\mathcal{H}^*$ its topological dual and by $\langle h^*, h\rangle_{\mathcal{H}}$ the duality product between $h \in \mathcal{H}$ and $h^* \in \mathcal{H}^*$. For the inner product we write $(\cdot,\cdot)_{\mathcal{H}}$. We omit the index $\mathcal{H}$ if there is no ambiguity. By $\mathcal{L}(\mathcal{H})$ we denote the space of linear and continuous operators on $\mathcal{H}$. If $\mathcal{A}$ is a linear (possibly unbounded) operator from $\mathcal{H}$ into itself, its adjoint operator is denoted by $\mathcal{A}^*$. We use the standard notation for Lebesgue and Sobolev spaces. The Euclidean norm is denoted by $|\cdot|$ and for diagonal matrices $D \in \mathbb{R}^{n\times n}$ we use the notation $\operatorname{diag}(d_1, \dots, d_n)$.

2. The damped wave equation

Let $\Omega \subset \mathbb{R}^n$, $n \le 3$ be a bounded domain with smooth boundary $T > 0$, and $H := L^2(\Omega)$, $V := H_0^1(\Omega)$. We define the unbounded operator $\mathbf{T}^{-1} := -\Delta$ in H with domain $\operatorname{dom}(\mathbf{T}^{-1}) := H^2(\Omega) \cap H_0^1(\Omega)$ as

$$\mathbf{T}^{-1}\colon \operatorname{dom}(\mathbf{T}^{-1}) \subset H \to H. \tag{2.1}$$

Then $\mathbf{T}\colon H \to H$ is a bounded operator. Let

$$\mathcal{H} := H \times V^*. \tag{2.2}$$

In the following we will consider for given $b \in L^\infty(\Omega)$, $u \in L^1(0,T)$, $f \in L^1(0,T;V^*)$, and $(y_{0,1}, y_{0,2}) \in \mathcal{H}$ the general second-order hyperbolic equation

$$\begin{cases} \ddot{y}_1(t,x) - \Delta y_1(t,x) = f(t,x) + u(t)b(x)\dot{y}_1(t,x) & \text{in } (0,T)\times\Omega,\\ y_1(0,x) = y_{0,1}(x), \quad \dot{y}_1(0,x) = y_{0,2}(x) & \text{in } \Omega,\\ y_1(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{2.3}$$

Setting $y_2(t) := \dot{y}_1(t)$, we can reformulate the state equation formally as a first-order system in time given by

$$\dot{y} + \mathcal{A}y = \mathcal{F} + u\mathcal{B}y \quad t \in (0,T), \quad y(0) = y_0, \tag{2.4}$$

with

$$\mathcal{A} := \begin{pmatrix} 0 & -\mathrm{id} \\ \mathbf{T}^{-1} & 0 \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad \mathcal{F} := \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad y_0 := \begin{pmatrix} y_{0,1} \\ y_{0,2} \end{pmatrix}. \tag{2.5}$$

From the Hille Yosida theorem we can derive by classical arguments that $\mathcal{A}$ is the generator of a contraction semigroup $e^{-t\mathcal{A}}$ with $\operatorname{dom}(\mathcal{A}) = H_0^1 \times L^2(\Omega) \subset \mathcal{H}$. We define the *mild solution* of (2.4) as the function $y \in C(0,T;\mathcal{H})$ such that, for all $t \in [0,T]$:

$$y(t) = e^{-t\mathcal{A}}y_0 + \int_0^t e^{-(t-s)\mathcal{A}}\big(\mathcal{F}(s) + u(s)\mathcal{B}y(s)\big)\mathrm{d}s. \tag{2.6}$$

The existence of a mild solution follows by a fixed-point argument, cf. Ball, Marsden, and Slemrod [7]. Furthermore, we have the following estimate; cf. [4, Thm. 2].

Theorem 2.1. *There exists $c > 0$ such that the solution y of (2.6) satisfies*

$$\|y\|_{C([0,T];\mathcal{H})} \le c\left(\|y_0\|_{\mathcal{H}} + \|f\|_{L^1(0,T;H^{-1}(\Omega))} + \|\mathcal{B}\|_{\mathcal{L}(\mathcal{H})}\|u\|_{L^1(0,T)}\right)e^{c\|u\|_{L^1(0,T)}}. \tag{2.7}$$

The dual semigroup on $\mathcal{H}^*$ is well-defined (see [4, 23]) and generated by $\mathcal{A}^* = -\mathcal{A}$ with $\operatorname{dom}(\mathcal{A}^*) = \operatorname{dom}(\mathcal{A})$, identifying $\mathcal{H}$ with $\mathcal{H}^*$(cf. Appendix A).

From Ball [6] we recall that any mild solution of (2.4) coincides with a *weak solution*, i.e. $y \in \mathcal{Y} := C(0,T;\mathcal{H})$ satisfies $y(0) = y_0$ and, for any $\phi \in \operatorname{dom}(\mathcal{A}^*)$, the function $t \mapsto \langle \phi, y(t)\rangle$ is absolutely continuous over $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi, y(t)\rangle + \langle \mathcal{A}^*\phi, y(t)\rangle = \langle \phi, \mathcal{F} + u(t)\mathcal{B}y(t)\rangle, \quad \text{for a.a. } t \in [0,T]. \tag{2.8}$$

In the following we denote for given $u \in \mathcal{U}$ the corresponding mild solution by $y[u]$ and its components by $y_1[u]$ and $y_2[u]$.

We introduce the *linearized state equation* at a point $(\hat{y}, \hat{u})$ with $\hat{y} = y[\hat{u}], \hat{u} \in \mathcal{U}$ for given $v \in \mathcal{U}$ given by

$$\dot{z}(t) + \mathcal{A}z(t) = \hat{u}(t)\mathcal{B}z(t) + v(t)\mathcal{B}\hat{y}(t); \quad z(0) = 0, \tag{2.9}$$

to be understood in the sense of mild solutions. It is easily checked that the equation (2.9) has a unique solution denoted by $z[v]$, and using the implicit function theorem, that the mapping $u \mapsto y[u]$ from $\mathcal{U}$ to $\mathcal{Y}$ is of class C^∞ with $Dy[u]v = z[v]$.

We recall the notion of *restriction property* from [4, Def. 2].

Definition 2.2. Let W be a Hilbert space with norm $\|\cdot\|_W$ and continuous inclusion in $\mathcal{H}$. Assume that the restriction of $e^{-t\mathcal{A}}$ to W has image in W, and that it is a continuous semigroup over this space. We let $\mathcal{A}'$ denote its associated generator, and $e^{-t\mathcal{A}'}$ the associated semigroup. Then we have that $\mathrm{dom}(\mathcal{A}') \subset \mathrm{dom}(\mathcal{A})$, and $\mathcal{A}'$ is the restriction of $\mathcal{A}$ to $\mathrm{dom}(\mathcal{A}')$. We have that

$$\|e^{-t\mathcal{A}'}\|_{\mathcal{L}(W)} \le c_{\mathcal{A}'} e^{\lambda_{\mathcal{A}'} t} \tag{2.10}$$

for some constants $c_{\mathcal{A}'}$ and $\lambda_{\mathcal{A}'}$. Denote by $\mathcal{B}'$ the restriction of $\mathcal{B}$ to W, which is supposed to have image in W and to be continuous in the topology of W, that is,

$$\mathcal{B}' \in \mathcal{L}(W). \tag{2.11}$$

In this case we say that W has the *restriction property.*

Lemma 2.3. *Let W have the restriction property, $y_0 \in W$, and $f \in L^1(0,T;W)$ hold. Then the solution of (2.4) satisfies $y \in C(0,T;W)$ and the mapping $u \mapsto y[u]$ is of class C^∞ from $L^1(0,T)$ to $C(0,T;W)$.*

Proof. See [4, Lem. 1]. □

This allows to prove higher regularity. Let

$$E_1 := H_0^1(\Omega) \times L^2(\Omega), \tag{2.12}$$

$$H_{0,1}^2 := H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega). \tag{2.13}$$

Hypothesis 2.1. We assume

$$y_0 \in E_1, \quad b \in L^\infty(\Omega), \quad f \in L^2(0,T;L^2(\Omega)). \tag{2.14}$$

Lemma 2.4. *The space E_1 has the restriction property with restricted semigroup $\mathcal{A}'$ and domain* $\mathrm{dom}(\mathcal{A}') := H_{0,1}^2$.

Proof. We refer to Pazy [23].

□

3. The optimal control problem

Let q and q_T be continuous quadratic forms over $\mathcal{H}$, with associated symmetric and continuous operators

$$Q, Q_T \in \mathcal{L}(\mathcal{H}); \;\; q(y) := (Qy, y); \;\; q_T(y) := (Q_T y, y). \tag{3.1}$$

Given

$$y_d \in L^\infty(0,T;\mathcal{H}); \quad y_{dT} \in \mathcal{H}, \tag{3.2}$$

we introduce the cost function

$$J(u,y) := \tfrac{1}{2}\int_0^T q(y(t)-y_d(t))\mathrm{d}t+\tfrac{1}{2}q_T(y(T)-y_{dT})+\int_0^T (\alpha_1 u(t)+\frac{\alpha_2}{2}u(t)^2)\mathrm{d}t \tag{3.3}$$

with $\alpha_1 \in \mathbb{R}$, and $\alpha_2 \geq 0$; this includes in particular the case (1.2). Then, recalling the set of admissible controls

$$\mathcal{U}_{ad} = \{u \in \mathcal{U} : u_m \leq u(t) \leq u_M, \ \text{ a.e. on } (0,T)\}; \tag{3.4}$$

with $\mathcal{U} := L^2(0,T)$ and defining the reduced cost by $F(u) := J(u, y[u])$ the optimization problem reads as

$$\operatorname*{Min}_{u\in\mathcal{U}_{ad}} F(u). \tag{P}$$

For optimal control problems of type (P) we consider weak local minima. We call $\hat{u} \in \mathcal{U}_{\mathrm{ad}}$ a weak local minimum if there exists an $\varepsilon > 0$ such that

$$F(\hat{u}) \leq F(u) \quad \text{for all } u \text{ in } \mathcal{U}_{\mathrm{ad}} \text{ with } \|u-\hat{u}\|_{L^\infty(0,T)} \leq \varepsilon. \tag{3.5}$$

To obtain existence we need a compactness result.

Lemma 3.1. *Let for given control $u \in \mathcal{U}$ and initial condition $y_0 \in E_1$ the function $y = y[u]$ denote the corresponding solution of (2.4). Then the mapping*

$$\mathcal{U} \to L^2(0,T;H), \quad u \mapsto by_2[u], \tag{3.6}$$

is compact.

Proof. We have

$$\mathcal{U} \to L^2(0,T;H), \quad u \mapsto by_2[u], \tag{3.7}$$

with $y[u]$ being the solution of

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 - \Delta y_1 = uby_2 + f. \end{cases} \tag{3.8}$$

We check the compactness hypothesis. We have

$$\begin{aligned} y[u] &\in C(0,T;E_1), \\ \dot{y}[u] &\in L^1(0,T;\mathcal{H}); \end{aligned} \tag{3.9}$$

the second inclusion in (3.9) follows from (2.8) and estimate (2.7) with E_1 instead of $\mathcal{H}$. Since E_1 is compactly embedded in $\mathcal{H}$, we conclude by Aubin's Lemma in the variant given in [26, Remark 2.1, p. 189]; cf. also [24, p. 37]. □

Corollary 3.2. Problem (P) has a solution.

Proof. The existence follows by classical arguments using the compactness result from Lemma 3.1; cf. [2, Thm. 2.15]. □

4. First and second-order necessary optimality conditions

The costate equation is given in $\mathcal{H}$ as

$$-\dot{p} + \mathcal{A}^* p = Q(y - y_d) + u\mathcal{B}^* p; \quad p(T) = Q_T(y(T) - y_{dT}). \tag{4.1}$$

The corresponding mild solution is well-defined in $C(0,T;\mathcal{H})$ (cf. the comments on the dual semigroup in Section 2) and is denoted by $p[u] = p$.

Next, for given $u \in \mathcal{U}$ we introduce

$$\Lambda(t) := \alpha_1 + \alpha_2 u(t) + y_2[u](t)b(x)p_2[u](t). \tag{4.2}$$

Since $u \mapsto F(u)$ is of class C^∞ by the implicit function theorem, we have the representation

$$DF(u)v = \int_0^T \Lambda(t)v(t)\mathrm{d}t \quad \text{for all } v \in \mathcal{U}. \tag{4.3}$$

By standard arguments we obtain the necessary optimality: The contact sets are

$$\begin{aligned} I_m(u) \quad &:= \{t \in (0,T) : u(t) = u_m\}; \\ I_M(u) \quad &:= \{t \in (0,T) : u(t) = u_M\}. \end{aligned} \tag{4.4}$$

and we have the following proposition.

Proposition 4.1. Let $\hat{u} \in \mathcal{U}_{ad}$ be a weak local minimum of (P). Then, up to a set of measure zero, there holds

$$\{t;\ \Lambda(t) > 0\} \subset I_m(\hat{u}), \quad \{t;\ \Lambda(t) < 0\} \subset I_M(\hat{u}). \tag{4.5}$$

To formulate second-order conditions we introduce the second variation of the Lagrangian $\mathcal{Q}\colon C(0,T;\mathcal{H}) \times \mathcal{U} \to \mathbb{R}$ by

$$\mathcal{Q}(z,v) := \int_0^T \Big(\|z(t)\|^2_{\mathcal{H}} + \alpha_2 v(t)^2 + 2v(t)(p_2(t), z_2(t))_{V^*} \Big)\mathrm{d}t + \|z(T)\|^2_{\mathcal{H}}. \tag{4.6}$$

Given a feasible control u, the critical cone is defined as

$$C(u) := \left\{ \begin{matrix} v \in \mathcal{U} \mid \Lambda(t)v(t) = 0 \text{ a.e. on } [0,T], \\ v(t) \geq 0 \text{ a.e. on } I_m(u),\ v(t) \leq 0 \text{ a.e. on } I_M(u) \end{matrix} \right\}. \tag{4.7}$$

Then we can formulate the second-order necessary optimality conditions.

Theorem 4.2. *Let $\hat{u}$ be a weak local minimum of (P). Then we have,*

$$\mathcal{Q}(z[v], v) \geq 0 \quad \textit{for all } v \in C(\hat{u}). \tag{4.8}$$

Proof. See [4, Thm. 6]. □

5. Second-order sufficient optimality conditions for the non-singular case $\alpha_2 > 0$

Next, we formulate sufficient optimality conditions for weak as well as strong local minima. Therefore we introduce a positive definiteness condition for the second variation of the Lagrangian: Let $\alpha_0 > 0$ be such that

$$\mathcal{Q}(z[v], v) \geq \alpha_0 \|v(t)\|^2_{L^2(0,T)} \quad \text{for all } v \in C(\hat{u}). \tag{5.1}$$

Theorem 5.1. *Let $\hat{u} \in \mathcal{U}$ satisfy the first order optimality conditions in Proposition 4.1, and let the positive definiteness condition (5.1) hold. Then $\hat{u}$ is a weak local minimum of problem (P) that satisfies the quadratic growth condition.*

Proof. It follows by an adaption of [11, Thm. 4.3] or Casas and Tröltzsch [14]. □

We introduce the notion of strong local minima.

Definition 5.2. A control $\hat{u} \in \mathcal{U}_{ad}$ is a strong local minimum if there exists $\varepsilon > 0$ such that, for all $u \in \mathcal{U}_{ad}$ and $\|y[u] - y[\hat{u}]\|_{C(0,T;\mathcal{H})} < \varepsilon$ we have $F(\hat{u}) \leq F(u)$.

Definition 5.3. A control $\hat{u} \in \mathcal{U}_{ad}$ satisfies the *quadratic growth condition for strong solutions* if there exists $\sigma > 0$ and $\varepsilon > 0$ such that for any feasible control u:

$$F(\hat{u}) + \sigma\|u - \hat{u}\|^2_{L^2(0,T)} \leq F(u), \quad \text{whenever } \|y[u] - y[\hat{u}]\|_{C(0,T;\mathcal{H})} < \varepsilon. \tag{5.2}$$

Theorem 5.4. *Let $\hat{u} \in \mathcal{U}_{ad}$ satisfy the first order necessary optimality condition (4.5), and the positive definiteness condition (5.1). Then $\hat{u}$ is a strong solution that satisfies the above quadratic growth condition.*

Proof. The proof follows the same ideas as in the complex setting presented in [2, Thm. 5.10] based on the decomposition principle and using techniques by Bonnans and Osmolovskiĭ [12]. □

6. Second-order sufficient optimality conditions for the singular case $\alpha_2 = 0$

Here, we reformulate the second-order necessary optimality conditions for weak local minima by a transformed quadratic form, apply the Goh transformation, and derive sufficient optimality conditions.

Given $\hat{u} \in \mathcal{U}$, let $\hat{y} = y[\hat{u}]$ and $\hat{p} = p[\hat{u}]$ be the associated state and costate.

Hypothesis 6.1. In the sequel we assume (additional to Hypothesis 2.1) that the costate $\hat{p}$ is in $C(0,T;E_1)$.

Remark 6.1. If $\mathcal{Q}(y - y_d) \in L^2(0,T;E_1)$ and $\mathcal{Q}_T(y - y_{dT}) \in E_1$ this can be guaranteed by the semigroup property. For $\mathcal{Q}$ and $\mathcal{Q}_T$ given by cost functions as in (1.2) and $y_d \in E_1$ this is the case.

The space $E_1 \subset \mathcal{H}$ with continuous inclusion has the restriction property (see Definition 2.2). Using for the restriction of $\mathcal{A}$ and $\mathcal{B}$ to E_1 the same notation we have by the regularity of b given in Hypothesis 2.1 that

$$\mathcal{B}^k \operatorname{dom}(\mathcal{A}) \subset \operatorname{dom}(\mathcal{A}), \quad (\mathcal{B}^k)^* \operatorname{dom}(\mathcal{A}^*) \subset \operatorname{dom}(\mathcal{A}^*), \quad k = 1, 2. \tag{6.1}$$

This allows to define the following operators with domains $\operatorname{dom}(\mathcal{A})$ and $\operatorname{dom}(\mathcal{A}^*)$:

$$[\mathcal{A}, \mathcal{B}^k] := \mathcal{A}\mathcal{B}^k - \mathcal{B}^k_2\mathcal{A}, \quad [(\mathcal{B}^k)^*, \mathcal{A}^*] := (\mathcal{B}^k)^*\mathcal{A}^* - \mathcal{A}^*(\mathcal{B}^k)^*. \tag{6.2}$$

Let us define for $k = 1, 2$

$$M_k y := \overline{[\mathcal{A}, \mathcal{B}^k]} y \tag{6.3}$$

considering the closure of the operator in E_1. Then we have

$$M_k = \begin{pmatrix} 0 & -b^k \\ -b^k\mathbf{T}^{-1} & 0 \end{pmatrix}; \quad [M_k, \mathcal{B}] = \begin{pmatrix} 0 & -b^{k+1} \\ b^{k+1}\mathbf{T}^{-1} & 0 \end{pmatrix}; \quad k = 1, 2. \tag{6.4}$$

We observe that here the commutator is a second-order, self-adjoint differential operator (cf. Appendix A), i. e.

$$M_k^* = M_k. \tag{6.5}$$

Thus, using the fact that $\hat{p} \in C(0,T;E_1)$ (see Hypothesis 6.1) we have in particular

$$M_k^*\hat{p} \in C(0,T;\mathcal{H}). \tag{6.6}$$

To illustrate the dependencies we restrict the presentation to cost functions of type (1.2), nevertheless corresponding statements can be derived for general cost functions as given in (3.3). We define the space

$$W := L^2(0,T;E_1) \cap C([0,T];\mathcal{H}) \times L^2(0,T) \times \mathbb{R} \tag{6.7}$$

and introduce a quadratic form over W by

$$\widehat{\mathcal{Q}}(\xi,w,h) := \widehat{\mathcal{Q}}_T(\xi,h) + \widehat{\mathcal{Q}}_a(\xi,w) + \widehat{\mathcal{Q}}_b(w), \tag{6.8}$$

where

$$\widehat{\mathcal{Q}}_b(w) := \int_0^T w^2(t)R(t)\mathrm{d}t \tag{6.9}$$

and

(6.10)
$$\begin{aligned}
\widehat{\mathcal{Q}}_T(\xi,h) &:= \beta_2 \left\|\xi_1(T)\right\|_H^2 + \beta_4 \left\|\xi_2(T) + hb\hat{y}_2(T)\right\|_{V^*}^2 + h(\hat{p}_2(T), b\xi_2(T))_{V^*} \\
&\quad + h^2(\hat{p}(T), b^2\hat{y}_2(T))_{V^*}, \\
\widehat{\mathcal{Q}}_a(\xi,w) &:= \int_0^T \Big(\beta_1 \left\|\xi_1\right\|_H^2 + \beta_3 \left\|\xi_2\right\|_{V^*}^2\Big)\,\mathrm{d}t + \int_0^T 2w\beta_3(\xi_2, b\hat{y}_2)_{V^*}\mathrm{d}t, \\
&\quad + \int_0^T 2w\Big(\beta_3(\hat{y}_2 - y_{d,2}, b\xi_2)_{V^*} + (b\hat{p}_2,\xi_1)_H - (b\mathbf{T}^{-1}\hat{p}_2,\xi_2)_{V^*}\Big)\mathrm{d}t, \\
R(t) &:= \beta_3 \left\|b\hat{y}_2\right\|_{V^*}^2 + \beta_3(\hat{y}_2 - y_{d,2}, b^2\hat{y}_2)_{V^*} + (\hat{p}_2, b^2(f + \mathbf{T}^{-1}\hat{y}_1))_{V^*} \\
&\quad - (\hat{p}_1, b^2\hat{y}_2)_H.
\end{aligned}$$

We will show that the quadratic form $\widehat{\mathcal{Q}}$ is positive semidefinit on a transformed critical cone. Let $PC_2(\hat{u})$ be the closure in the $L^2 \times \mathbb{R}$–topology of the set

$$PC(\hat{u}) := \{(w,h) \in W^{1,\infty}(0,T) \times \mathbb{R}, \dot{w} \in C(\hat{u});\ w(0) = 0,\ w(T) = h\}. \tag{6.11}$$

Theorem 6.2. *Let Hypothesis 2.1 and 6.1 hold and let the cost function be given by (1.2). Let $\hat{u} \in \mathcal{U}_{ad}$ be a weak local minimum of (P) and z solution of the linearized state equation at $(\hat{u}, y[\hat{u}])$. Furthermore, let for given $v \in \mathcal{U}$ the functions (w,ξ) be defined by the Goh transformation*

$$w(t) := \int_0^t v(s)\mathrm{d}s, \quad \xi := z - w\mathcal{B}\hat{y}. \tag{6.12}$$

(i) Then ξ is the mild solution of

$$\left\{\begin{aligned} \dot{\xi}_1 - \xi_2 &= wb\hat{y}_2, \\ \dot{\xi}_2 - \Delta\xi_1 &= \hat{u}b\xi_2 - wb\Delta\hat{y}_1 - wbf \end{aligned}\right. \tag{6.13}$$

with $\xi(0) = 0$.

(ii) We have

$$\widehat{\mathcal{Q}}(\xi,w) = \mathcal{Q}(z,v). \tag{6.14}$$

(iii) In particular, the transformed second-order variation is positive semidefinit on the transformed critical cone, i.e.

$$\widehat{\mathcal{Q}}(\xi,w) \geq 0 \quad \textit{for all } (w,h) \in PC_2(\hat{u}). \tag{6.15}$$

Proof. (i) By a direct calculation (cf. [4, (3.28)]) we obtain formally

$$\dot{\xi} + \mathcal{A}\xi = \hat{u}\mathcal{B}\xi - w\mathcal{B}\mathcal{F} - w[\mathcal{A},\mathcal{B}]\hat{y}, \quad \xi(0) = 0. \tag{6.16}$$

We check well-posedness. Since by the regularity assumption on the data in Hypothesis 2.1 we have $\hat{y} \in C(0,T;E_1)$ and since the commutator is a second-order differential operator we derive $\xi \in C(0,T;\mathcal{H})$. Furthermore, (6.16) is, in the case under consideration, equivalent to (6.13).

(ii) We apply [4, Thm. 7] to obtain (6.14). We have to verify the conditions [4, (3.23)], namely: (6.1), which is satisfied, and

$$\begin{cases} \text{(a)} & \text{for } k = 1,2: M_k \text{ has a continuous extension to } E_1, \\ \text{(b)} & \text{for } k = 1,2: \ M_k^*\hat{p} \in L^\infty(0,T;\mathcal{H}), \\ \text{(c)} & \hat{y} \in L^2(0,T;E_1); \ \ [M_1,\mathcal{B}]\hat{y} \in L^\infty(0,T;\mathcal{H}). \end{cases} \tag{6.17}$$

In (6.4) we have seen that the commutator M_k is a second-order differential operator having a continuous extension to E_1 which gives (a). Condition (b) results from the regularity of the costate $\hat{p} \in C(0,T;E_1)$, cf. Hypothesis 6.1. Finally, we have $\hat{y} \in C(0,T;E_1)$ by Hypothesis 2.1 so $[M_1,\mathcal{B}]\hat{y} \in C(0,T;\mathcal{H})$ which gives (c).

(iii) This follows from (4.8) and (ii), cf. [4, Lem. 6]. □

Remark 6.3. In comparison to [2], where the control is coupled bilinearly with the state and not the velocity component, the right hand side of the equation in ξ given in (6.13) is of qualitativ difference, since it involves the Laplacian of y_1.

Definition 6.4 (Singular arc)**.** A control $u \in \mathcal{U}_{ad}$ is said to have a *singular arc* over (t_1,t_2), with $0 \le t_1 < t_2 \le T$, if, for all $\theta \in (0, \frac{1}{2}(t_2 - t_1))$, there exists $\varepsilon > 0$ such that

$$u(t) \in [u_m + \varepsilon, u_M - \varepsilon], \quad \text{for a.a. } t \in (t_1 + \theta, t_2 - \theta). \tag{6.18}$$

We may also say that (t_1,t_2) is a singular arc itself. We call (t_1,t_2) a *lower boundary arc* if $u(t) = u_m$ for a.a. $t \in (t_1,t_2)$, and an *upper boundary arc* if $u(t) = u_M$ for a.a. $t \in (t_1,t_2)$. We sometimes simply call them boundary arcs. We say that a boundary arc (c,d) is *initial* if $c = 0$, and *final* if $d = T$.

Let

$$E_2 := H_0^2(\Omega) \times H_0^1(\Omega). \tag{6.19}$$

Hypothesis 6.2. We assume

$$y_0 \in E_2; \quad f \in L^\infty(0,T;H_0^1(\Omega)), \quad b \in W^{1,\infty}(\Omega). \tag{6.20}$$

Corollary 6.5. Let Hypothesis 6.2 hold and $\hat{u} \in \mathcal{U}_{ad}$ be a weak local minimum for problem (P). Then,

$$\text{the mapping } w \mapsto \xi[w] \text{ is compact from } L^2(0,T) \text{ to } L^2(0,T;\mathcal{H}). \tag{6.21}$$

Let (t_1,t_2) be a singular arc. Then $R \in L^\infty(0,T;\mathcal{H})$ defined in (6.10) satisfies

$$R(t) \ge 0 \quad \text{for a.a. } t \in (t_1,t_2). \tag{6.22}$$

Proof. We know $\xi \in C(0,T;\mathcal{H})$. Since E_2 has the restriction property (by the Hille-Yosida theorem) [23, Cor. 3.8] we have by Hypothesis 6.2 and Lemma 2.3 that the first component y_1 of the solution of (2.4) is in $C(0,T;H_0^2(\Omega))$. Consequently, the right hand side of equation (6.13) is in $C(0,T;E_1)$. Similarly as in the proof of Lemma 3.1 we further obtain $\dot{\xi}[w] \in L^1(0,T;\mathcal{H})$ and thus the compactness property in (6.21).

Using (6.21) we obtain that the terms of $\widehat{\mathcal{Q}}$ where ξ is involved are weakly continuous. We conclude as in [4, Cor. 5]. □

To formulate second-order sufficient optimality conditions we make the following hypotheses.

Hypothesis 6.3. We assume

(1) *Regular data:* Let Hypotheses 2.1, 6.1, and 6.2 hold and additionally,

$$f \in C(0,T;H^{-1}(\Omega)), \quad y_d \in C(0,T;\mathcal{H}). \tag{6.23}$$

(2) *finite structure:*

$$\left\{ \begin{array}{l} \text{there are finitely many boundary and singular maximal arcs} \\ \text{and the closure of their union is } [0,T], \end{array} \right. \tag{6.24}$$

(3) *strict complementarity* for the control constraint (note that Λ is a continuous function of time)

$$\left\{ \begin{array}{l} \Lambda \text{ has nonzero values over the interior of each boundary arc, and} \\ \text{at time 0 (resp. } T\text{) if an initial (resp. final) boundary arc exists,} \end{array} \right. \tag{6.25}$$

(4) letting $\mathcal{T}_{BB}$ denote the *set of bang-bang junctions,* we assume

$$R(t) > 0, \quad t \in \mathcal{T}_{BB}. \tag{6.26}$$

Let

$$\widehat{PC}_2(\hat{u}) := \left\{ \begin{array}{l} (w,h) \in L^2(0,T) \times \mathbb{R};\ w \text{ is constant over boundary arcs,} \\ w = 0 \text{ over an initial boundary arc,} \\ \text{and } w = h \text{ over a terminal boundary arc.} \end{array} \right\}. \tag{6.27}$$

Proposition 6.6. Let $\hat{u} \in \mathcal{U}_{ad}$ satisfy (6.24) and (6.25). Then $PC_2(\hat{u})$ defined before (6.11) can be characterized as

$$PC_2(\hat{u}) = \{w \in \widehat{PC}_2(\hat{u})\ :\ w \text{ is continuous on bang-bang junctions}\} \tag{6.28}$$

Proof. See [1, Proposition 4]. □

We say that $\hat{u}$ satisfies a *weak quadratic growth condition* if there exists $\beta > 0$ such that for any $u \in \mathcal{U}_{ad}$, setting $v := u - \hat{u}$ and $w(t) := \int_0^T v(s)\mathrm{d}s$, we have

$$F(u) \geq F(\hat{u}) + \beta(\|w\|^2_{L^2(0,T)} + w(T)^2), \quad \text{if } \|v\|_{L^1(0,T)} \text{ is small enough.} \tag{6.29}$$

The word 'weak' refers to the fact that the growth is obtained for the L^2–norm of w, and not of v.

Similar as in [4, Thm. 8] (taking into account [3]) we obtain the following statement.

Theorem 6.7. *Let $\hat{u} \in \mathcal{U}_{ad}$, the cost function be given by (1.2), and let Hypothesis 6.3 hold.*

(a) *If $\hat{u}$ satisfies the necessary optimality conditions (4.1) and if there exists $\alpha > 0$ such that*

$$\widehat{\mathcal{Q}}(\xi[w],w,h) \geq \alpha(\|w\|^2_{L^2(0,T)} + h^2), \quad \textit{for all } (w,h) \in \widehat{PC}_2(\hat{u}), \tag{6.30}$$

then the quadratic growth condition (6.29) is satisfied.

(b) If $\hat{u}$ is a weak minimum and the quadratic growth condition (6.29) is satisfied, then

$$\widehat{\mathcal{Q}}(\xi[w], w, h) \geq \alpha(\|w\|^2_{L^2(0,T)} + h^2), \quad \text{for all } (w,h) \in PC_2(\hat{u}), \tag{6.31}$$

holds.

Proof. We have to check:

$$\text{(i)} \quad \mathcal{B}^2 f \in C(0,T;\mathcal{H}); \;\; y_d \in C(0,T;\mathcal{H}); \quad \text{(ii)} \quad M_k^* \hat{p} \in C(0,T;\mathcal{H}), \;\; k = 1,2. \tag{6.32}$$

Then we conclude by [4, Thm. 8]. (i) follows immediately from (6.23); (ii) results from the fact that $\hat{p} \in C(0,T;E_1)$, cf. Hypothesis 6.1. □

Remark 6.8. When $\hat{u}$ has no bang-bang switch, the cones $PC_2(\hat{u})$ and $\widehat{PC_2}(\hat{u})$ coincide and, therefore, the necessary and sufficient conditions have no gap.

7. Numerical example

We present a numerical example for the singular case $\alpha_1 = \alpha_2 = 0$. Let $\Omega = (0,1)$, $T = 1$, and for $(t,x) \in (0,T) \times \Omega$ we set

$$\begin{aligned} y_{0,1} &= \sin^2(\pi x); & y_{0,2} &= 0; \\ f &= 0; & b &= 1; \\ y_{d,1} &= 0; & y_{d,2} &= \tfrac{1}{10}\sin(2\pi t)\sin(\pi x); \\ Q &= \mathrm{id}; & Q_T &= 0; \\ u_m &= 0; & u_M &= 10. \end{aligned} \tag{7.1}$$

For the approximation in space we use a spectral basis $\{\sqrt{2}\sin(k\pi x) \mid 1 \leq k \leq N\}$ with $N = 10$ and for the approximation in time an implicit Euler scheme with 1000 time steps. The numerical simulation was performed with Bocop [9] which uses IPOPT [29]. In Fig. 1 we see the control is first on the lower bound, then a singular arc appears, and then it is on the upper bound. The computed control is stable with respect to time and space discretization.

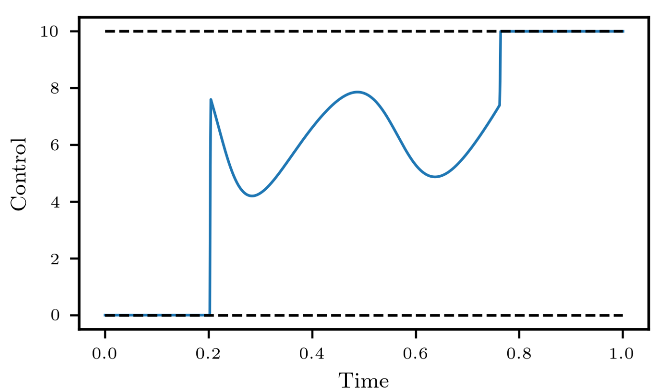

FIGURE 1. Optimal control with singular arc.

Appendix A. On the adjoint equation

We consider the derivation of the costate equation for problem (P) with cost function (1.2).

Lemma A.1. *Given constants $c_i \in \mathbb{R}$, $i = 1, \dots, 4$ and the operator*

$$N := \begin{pmatrix} c_1 \mathrm{id} & c_2 \mathrm{id} \\ c_3 \mathbf{T}^{-1} & c_4 \mathrm{id} \end{pmatrix} : V \times H \subset \mathcal{H} \to \mathcal{H}. \tag{A.1}$$

Identifying $\mathcal{H}$ with its dual $\mathcal{H}^*$ the adjoint operator N^* is given by

$$N^* = \begin{pmatrix} c_1 \mathrm{id} & c_3 \mathrm{id} \\ c_2 \mathbf{T}^{-1} & c_4 \mathrm{id} \end{pmatrix} : V \times H \subset \mathcal{H} \to \mathcal{H}. \tag{A.2}$$

Proof. Given y and p in $V \times H$. A direct calculation shows

$$\begin{cases} (c_2 y_2, p_1)_H = (y_2, c_2 \mathbf{T}^{-1} p_1)_{V^*}, \\ (c_3 \mathbf{T}^{-1} y_1, p_2)_{V^*} = (y_1, c_3 p_2)_H. \end{cases} \tag{A.3}$$

This proves the claim. □

Corollary A.2. (i) The adjoint operators of $\mathcal{A}$ and M_k defined in (2.5) and (6.4) are given as

$$\mathcal{A}^* = -\mathcal{A}, \quad M_K^* = M_k \quad \text{for } k = 1, 2. \tag{A.4}$$

(ii) The costate equation (4.1) for problem (P) with cost function (1.2) and $\beta_2 = \beta_4 = 0$ is given as

$$\begin{cases} -\dot{p}_1 + p_2 = \beta_1 (y_1 - y_{d,1}); & p_1(T) = y_1(T) - y_{dT,1}; \\ -\dot{p}_2 + \Delta p_1 = ubp_2 + \beta_3 (y_2 - y_{d,2}); & p_2(T) = y_2(T) - y_{dT,2}. \end{cases} \tag{A.5}$$

Proof. (i) Setting $c_1 = c_4 = 0$, and $c_2 = -1$, $c_3 = 1$ resp. $c_2 = c_3 = -b^k$ shows the result.

(ii) Setting $c_1 = 0$, $c_2 = -1$, $c_3 = 1$ and $c_4 = -ub$ we obtain

$$N^* = (\mathcal{A} - u\mathcal{B})^* = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathbf{T}^{-1} & ub\mathrm{id} \end{pmatrix}. \tag{A.6}$$

Thus with $Q = \mathrm{diag}(\beta_1, \beta_3)$ (cf. (3.1)) we conclude. □

Remark A.3. Formally differentiating the second equation in time, assuming sufficient regularity, and resubstitution leads to

$$\begin{aligned} &\ddot{p} - \Delta p = -(ub\dot{p} + \dot{u}bp) - \beta_1 \Delta (y_1 - y_{d,1}) - \beta_3 (\dot{y}_2 - \dot{y}_{d,2}); \\ &p(0, \cdot) = y_1(T) - y_{dT,1}, \quad p_t(0, \cdot) = y_2(T) - y_{dT,2}, \end{aligned} \tag{A.7}$$

where p takes the role of p_2.

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References

1. M.S. Aronna, J.F. Bonnans, and B.S. Goh, *Second order analysis of control-affine problems with scalar state constraint*, Math. Program. **160** (2016), no. 1-2, Ser. A, 115–147.
2. M.S. Aronna, J.F. Bonnans, and A. Kröner, *Optimal control of PDEs in a complex space setting; application to the Schrödinger equation*, Research report, INRIA, 2017.
3. ______, *Erratum to the paper "Optimal Control of Infinite Dimensional Bilinear Systems: Application to the Heat and Wave Equations"*, accepted, Math. Program., 2018.
4. ______, *Optimal control of infinite dimensional bilinear systems: application to the heat and wave equations*, Math. Program. **168** (2018), no. 1-2, Ser. B, 717–757.
5. L. Bales and I. Lasiecka, *Continuous finite elements in space and time for the nonhomogeneous wave equation*, Comput. Math. Appl. **27** (1994), no. 3, 91–102.
6. J.M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proc. Amer. Math. Soc. **63** (1977), no. 2, 370–373.
7. J.M. Ball, J. Marsden, and M. Slemrod, *Controllability for distributed bilinear systems*, SIAM Journal on Control and Optimization **20** (1982), no. 4, 575–597.
8. M. Bergounioux and D. Tiba, *General optimality conditions for constrained convex control problems*, SIAM J. Control Optim. **34** (1996), no. 2, 698–711.
9. F. Bonnans, V. Grélard, and P. Martinon, *Bocop, the optimal control solver, open source toolbox for optimal control problems 2011*, Tech. report.
10. J.F. Bonnans, *Optimal control of a semilinear parabolic equation with singular arcs*, Optim. Methods Softw. **29** (2014), no. 2, 964–978.
11. J.F. Bonnans and P. Jaisson, *Optimal control of a parabolic equation with time-dependent state constraints*, SIAM J. Control Optim. **48** (2010), no. 7, 4550–4571.
12. J.F. Bonnans and N.P. Osmolovskiĭ, *Second-order analysis of optimal control problems with control and initial-final state constraints*, J. Convex Anal. **17** (2010), no. 3-4, 885–913.
13. J.F. Bonnans and D. Tiba, *Control problems with mixed constraints and application to an optimal investment problem*, Math. Rep. (Bucur.) **11(61)** (2009), no. 4, 293–306.
14. E. Casas and F. Tröltzsch, *Second-Order Optimality Conditions for Weak and Strong Local Solutions of Parabolic Optimal Control Problems*, Vietnam J. Math. **44** (2016), no. 1, 181–202.
15. E. Casas, D. Wachsmuth, and G. Wachsmuth, *Second-order analysis and numerical approximation for bang-bang bilinear control problems*, preprint: arXiv:1707.06880 (2017).
16. S. Cox and E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana Univ. Math. J. **44** (1995), no. 2, 545–573.
17. B. Dehman, G. Lebeau, and E. Zuazua, *Stabilization and control for the subcritical semilinear wave equation*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 4, 525–551.
18. H. Frankowska and D. Tonon, *The Goh necessary optimality conditions for the Mayer problem with control constraints*, J. Control Decis. (2013), 538–543.
19. B.S. Goh, *Necessary conditions for singular extremals involving multiple control variables*, SIAM J. Control **4** (1966), 716–731.
20. H.J. Kelley, *A second variation test for singular extremals*, AIAA Journal **2** (1964), 1380–1382.
21. I. Lasiecka and R. Triggiani, *Control theory for partial differential equations: Continuous and approximation theories, vol. 1 and vol. 2*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Philadelphia, 2000.
22. S. Lenhart and V. Protopopescu, *Identification problem for a wave equation via optimal control*, pp. 79–84, Springer US, Boston, MA, 1999.
23. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
24. Jacques Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.
25. R. B. Sonawane, A. Kumar, and S.B. Nimse, *Optimal control for a vibrating string with variable axial load and damping gain*, IFAC Proceedings Volumes **47** (2014), no. 1, 75 – 81, 3rd International Conference on Advances in Control and Optimization of Dynamical Systems (2014).
26. Roger Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition. MR 1846644
27. F. Tröltzsch, *Regular Lagrange multipliers for control problems with mixed pointwise control-state constraints*, SIAM J. Optim. **15** (2004/05), no. 2, 616–634.

28. J. Vancostenoble, *Exact controllability of a damped wave equation with distributed controls*, Acta Math. Hungar. **89** (2000), no. 1-2, 71–92.
29. A. Wächter and L.T. Biegler, *On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming*, Mathematical Programming Series A **106** (2006), 25–57.
30. E. Zuazua, *Exponential decay for the semilinear wave equation with locally distributed damping*, Comm. Partial Differential Equations **15** (1990), no. 2, 205–235.

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