Adaptive PSK Modulation Scheme in the Presence of Phase Noise

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***Abstract*—Phase noise is one of the major impairments affecting severely performance of millimeter-wave systems. This paper addresses the problem of link adaption for coherent and non-coherent phase modulated signals subject to Gaussian and Wiener phase noise. We first derive closed-form approximations of the bit error rate. Then, in contrast to usual link adaptation techniques, we propose a simple scheme exploiting estimations of not only the signal-to-noise ratio but also of the phase noise variance, which is essential to achieve reliable communications.**

***Index Terms*—Millimeter wave, Phase noise, Phase shift keying, Adaptive Systems, Performance analysis, Channel estimation**

I. Introduction

Future 5G millimeter-wave (mmWave) bands should lead to user throughputs above 1 Gbit/s. However, additional breakthrough technologies are necessary to reach this requirement. Therefore, for the first time in the telecommunication history, bands above 90 GHz are studied for future communications [1]. In particular, mmWave systems are critically impacted by phase noise (PN) due to the poor performance of high frequency oscillators [2]. This has motivated extensive work on PN estimation and compensation [3], [4]. Further, achieving high-rate communications over wireless links in the mmWave domain demands tremendous amount of power, and so, high-efficiency and wide-bandwidth power amplifiers [5]. Therefore, coherent Phase Shift Keying (PSK) modulations are highly valuable since they demonstrate a constant envelope property, and thus, offer an efficient use of power amplifiers. Yet, PSK modulations are highly sensitive to phase related impairments: Carrier Frequency Offset (CFO), PN. Differential PSK (DPSK) has been introduced as a more robust scheme. At the expense of a noise enhancement, DPSK enables non-coherent communications [6]. Prior work in [7] has confronted the robustness of PSK and DPSK when affected by a Tikohonov PN. Setting the modulation scheme to maximize spectral efficiency while maintaining robustness is correspondingly the motivation behind link adaption. Such adaptive systems are now implemented in most of practical systems [8] and rely on a channel quality estimation to set the modulation.

Contributions: In this paper, we seek to improve the design of inherently robust communication systems impacted by PN. First, we assume a high Signal-to-Noise Ratio (SNR) and a Gaussian PN to derive a closed-form approximation of the Bit Error Rate (BER) for any M-ary PSK. Considering that the superposition of a Gaussian and a Wiener process is a more realistic PN model [9], [10], we derive and confront the BER performance of PSK and DPSK for such PN. Furthermore, in contrast to usual link adaptation techniques, we propose a simple scheme exploiting estimations of not only the SNR but also of the PN variance, which proves to be truly essential to achieve reliable communications. We hence derive the appropriate Maximum Likelihood (ML) estimators of the thermal and phase noise variances. Eventually, we propose a statistical test to determine whether the coherent or non-coherent modulation is the most robust one and should be employed.

Organization: The remainder of this paper is structured as follows. Section II introduces a brief description of the channel and PN models. Section III is devoted to the performance analysis of PSK and DPSK impacted by Gaussian and Wiener PN. Finally, Section IV derives the appropriate channel estimation to implement the link adaptation scheme.

II. System Model

A. Channel model

Considering a single carrier communication system with a perfectly mitigated channel, the received symbol at time instant k is defined by

$$r_k = s_k \cdot e^{j\phi_k} + n_k, \tag{1}$$

where s is the modulated symbol from constellation $\mathcal{C}$ with average symbol energy E_s, ϕ is the oscillator PN and n represents independent identically distributed samples of a zero-mean complex Additive White Gaussian Noise (AWGN) with variance $2\sigma_n^2$. A PSK constellation with modulation order M is defined by $\mathcal{C} = \{\sqrt{E_s}\exp\left(j\frac{2\pi}{M}\cdot i\right) | i = 1, ..., M\}$. In the case of a DPSK, information is encoded in the phase difference from one symbol to another denoted $\delta\theta_k = \underline{/s_k} - \underline{/s_{k-1}}$. We denote by $\underline{/s}$ and $|s|$ the phase and amplitude of a symbol s.

B. Phase noise model

Oscillators PN is generated from the transformation of amplitude fluctuations into phase fluctuations [11], such that it describes a cumulative random process [9], [12]. If only white noise sources are considered[1], the oscillator PN ϕ may thus be modeled by the superposition of a Wiener (Gaussian random-walk) process ϕ_w and a Gaussian one ϕ_g [9]. This may be expressed by

$$\begin{aligned} \phi_k &= \phi_{w,k} + \phi_{g,k}, & \phi_{g,k} &\sim \mathcal{N}(0, \sigma_g^2), \\ \phi_{w,k} &= \phi_{w,k-1} + \delta\phi_{w,k}, & \delta\phi_{w,k} &\sim \mathcal{N}(0, \sigma_w^2), \end{aligned} \tag{2}$$

where σ_w^2 and σ_g^2 denote respectively the variances of the Wiener PN increment $\delta\phi_w$ and the Gaussian PN. Besides mathematical convenience [13], [14], the Gaussian distribution is also a relevant PN model. When considering wide bandwidth systems[2], the oscillator noise floor represents the

[1]Flicker noise is diregarded in this study.
[2]Typically the case for mmWave systems

greatest contribution to the overall PN [10], such that the Wiener PN becomes negligible compared to the Gaussian one, *i.e.* $\sigma_g^2 >> \sigma_w^2$. Thereby, Section III-A and IV-A use the Gaussian PN model to pursue a simple analytical analysis while Section III-B and IV-B extend it to the sum of Wiener and Gaussian PN.

III. Performance Analysis

A. PSK over Gaussian phase noise

To begin with, let us assume that the channel is dominated by Gaussian PN. Regarding symbol-by-symbol detection, it is known that the ML criterion minimizes the Symbol Error Probability (SEP) [6], [15]. In order to design such optimum receiver, let us derive the likelihood function, which may be rewritten as

$$p(r_k|s_k) = p\Big(\,|r_k|\,,\underline{\angle r_k}\;\Big|\;|s_k|\,,\underline{\angle s_k}\Big). \tag{3}$$

By definition, the AWGN is invariant under rotation such that, for simplicity of notation, n may stand for $n \cdot e^{j\theta}$ for any θ fixed. The received symbol should be studied through its amplitude and phase as

$$\begin{aligned} |r_k| &= \Big|(|s_k| + n_k) \cdot e^{j(\underline{\angle s_k} + \phi_k)}\Big| \\ &= \sqrt{(|s_k| + \Re(n_k))^2 + \Im(n_k)^2} \\ &\simeq \sqrt{E_s} + \Re(n_k) \end{aligned} \tag{4}$$

$$\begin{aligned} \underline{\angle r_k} &= \arg[(|s_k| + n_k) \cdot e^{j(\underline{\angle s_k} + \phi_k)}] \\ &= \underline{\angle s_k} + \phi_k + \arctan\left(\frac{\Im(n_k)}{|s_k| + \Re(n_k)}\right) \\ &\simeq \underline{\angle s_k} + \phi_k + \frac{\Im(n_k)}{\sqrt{E_s}}. \end{aligned} \tag{5}$$

These first-order approximations are tight for a high SNR scenario [16], [17]. By Eq. (2), the likelihood function follows a bivariate Gaussian distribution and is thus expressed by

$$p(r_k|s_k) = \frac{\exp\left(-\frac{1}{2}\left(\frac{(|r_k| - \sqrt{E_s})^2}{\sigma_n^2} + \frac{(\underline{\angle r_k} - \underline{\angle s_k})^2}{\sigma_g^2 + \sigma_n^2/E_s}\right)\right)}{2\pi\sqrt{\sigma_n^2(\sigma_g^2 + \sigma_n^2/E_s)}}. \tag{6}$$

This expression, originally derived in [14], can be exploited in our study to approximate the BER of a PSK. With respect to the ML decision rule for a PSK, the detected symbol is

$$\begin{aligned} \hat{s_k} &= \underset{s\,\in\,\mathcal{C}}{\text{argmax}}\; p(r_k|s_k) \\ &= \underset{s\,\in\,\mathcal{C}}{\text{argmin}}\; (\underline{\angle r_k} - \underline{\angle s_k})^2, \end{aligned} \tag{7}$$

which enables to derive the SEP as follows

$$\begin{aligned} P_{se} &= 1 - \Pr(\hat{s_k} = s_k|s_k) \\ &= 1 - p\Big(-\frac{\pi}{M} < \underline{\angle r_k} - \underline{\angle s_k} < \frac{\pi}{M}\Big) \\ &= 2Q\Big(\frac{\pi}{M\sqrt{\sigma_g^2 + \sigma_n^2/E_s}}\Big). \end{aligned} \tag{8}$$

To directly relate the BER to the SEP, we shall make two commonly used assumptions. First, the bit labeling of the constellation satisfies a Gray coding. Furthermore, misdetections only occur on the nearest neighbors of the sent symbol. Finally, the BER of a PSK affected by Gaussian PN is approximated by

$$P^g_{be,\text{PSK}} \simeq \frac{2}{\log_2(M)} Q\left(\frac{\pi}{M\sqrt{\sigma_g^2 + \frac{1}{2\cdot\log_2(M)}\frac{1}{E_b/N_0}}}\right), \tag{9}$$

where the average bit energy E_b satisfies $E_s = E_b \log_2(M)$ and the noise power spectral density is defined by $N_0 = 2\sigma_n^2$. The BER performance of PSK for arbitrary orders of modulation and PN variances are illustrated in Fig. 1. As exhibited, the closed-form expression holds accurate approximations of the BER at high SNR.

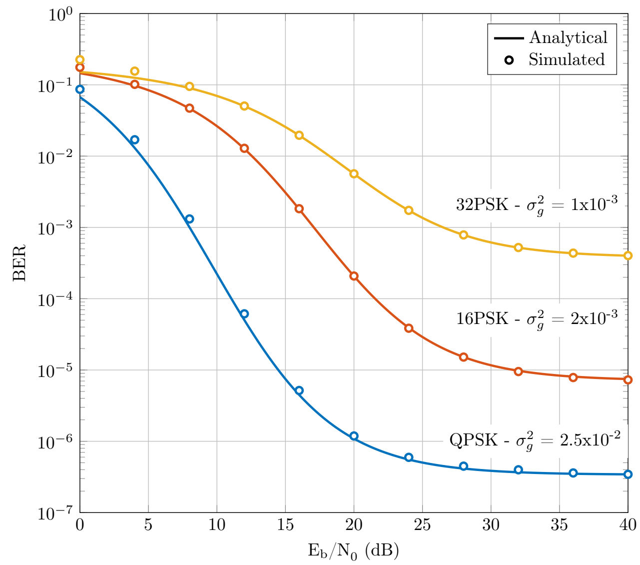

Fig. 1: Analytical and simulated BER performance of M-ary PSK affected by Gaussian PN with variance σ_g^2

B. DPSK over Wiener phase noise

DPSK enables a non-coherent demodulation at the receiver. As a consequence, estimation of the carrier phase is not required. Moreover, a differential modulation is not subject to the cumulative nature of the PN, and only slightly to the CFO. This robustness is achieved at the expense of a BER performance degradation. Yet, the stronger these impairments, the more valuable it seems to opt for differential encoding. The pending question is when DPSK is more robust than PSK. Therefore, let us derive the performance of PSK and DPSK when the PN is described by the sum of a Wiener and a Gaussian PN. The optimum decision for a DPSK receiver [6] is given by

$$\hat{\delta\theta} = \underset{\delta\theta\,\in\,\mathcal{C}}{\text{argmin}}\; \Big((\underline{\angle r_k} - \underline{\angle r_{k-1}}) - \delta\theta\Big)^2. \tag{10}$$

With regard to the PN model in Eq. (2) and to the high SNR approximation in Eq. (5), we have

$$\underline{\angle r_k} - \underline{\angle r_{k-1}} \simeq \delta\theta_k + \delta\phi_{w,k} + \phi_{g,k} - \phi_{g,k-1} + \frac{\Im(n_k - n_{k-1})}{\sqrt{Es}}, \tag{11}$$

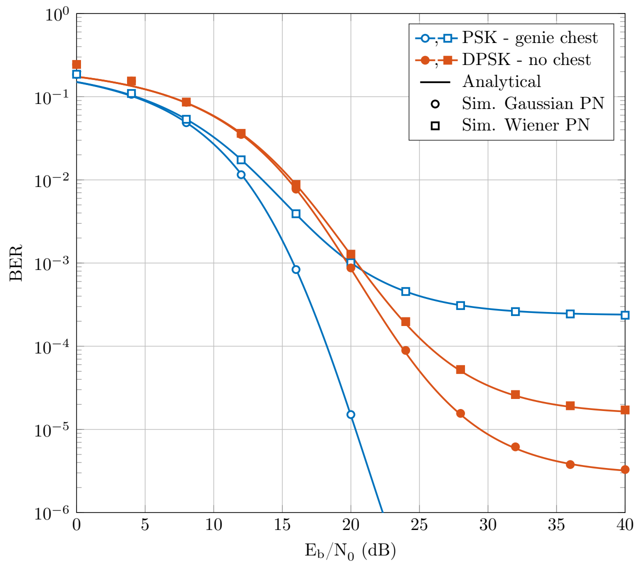

Fig. 2: Analytical and simulated BER performance of 16PSK and 16DPSK for Gaussian PN ($\sigma_g^2 = 10^{-3}$, $\sigma_w^2 = 0$) and Wiener PN ($\sigma_g^2 = 10^{-3}$, $\sigma_w^2 = \sigma_g^2/2$)

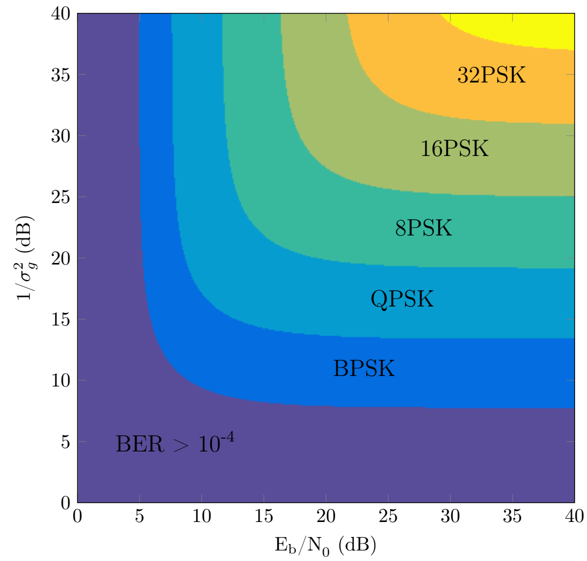

Fig. 3: Modulation Scheme Regions for M-ary PSK: highest modulation order M achieving BER $< 10^{-4}$ as a function of E_b/N_0 and the inverse of PN variance $1/\sigma_g^2$

which is directly equivalent to

$$\begin{gathered}\angle r_k - \angle r_{k-1} \simeq \delta\theta_k + \phi_{eq,k} \ , \\ \text{where } \phi_{eq} \sim \mathcal{N}(0, \sigma_w^2 + 2\sigma_g^2 + 2\sigma_n^2/E_s).\end{gathered} \tag{12}$$

Thereupon, the BER for a DPSK subject to Gaussian and Wiener PN may be approximated by

$$P_{be,\text{DPSK}}^{w} \simeq \frac{2}{\log_2(M)} Q\left(\frac{\pi}{M\sqrt{\sigma_w^2 + 2\sigma_g^2 + \frac{1}{E_b/N_0}\frac{1}{\log_2(M)}}}\right). \tag{13}$$

To establish a comparison with the PSK modulation, we consider a *genie estimation* at the PSK receiver. With one pilot symbol, the receiver is able to estimate and compensate perfectly the Wiener process. The pilot period is defined by T, such that if $k = n \cdot T$ then $\phi_{w,k+1} = \delta\phi_{w,k}$. The estimation problem in presence of Wiener PN exceeds the scope of this paper, but is addressed in [4]. Though the derivation is not developed here, the BER for a PSK in this case is given by

$$\begin{aligned} P_{be,\text{PSK}}^{w} \simeq & \frac{1}{T-1}\sum_{k=1}^{T-1}\frac{2}{\log_2(M)} \\ & \cdot Q\left(\frac{\pi}{M\sqrt{k\cdot\sigma_w^2 + \sigma_g^2 + \frac{T/(T-1)}{E_b/N_0}\frac{1}{2\cdot\log_2(M)}}}\right). \end{aligned} \tag{14}$$

Figure 2 confronts the performance of a 16PSK to the ones of a 16DPSK for Gaussian and Wiener PN. The PSK receiver presents a pilot density of 10% with a *genie estimation*. Conversely, the DPSK receiver does not implement any channel estimation. Figure 2 shows that analytical expressions are tight to the simulation results. Moreover, Fig. 2 confirms an aforementioned intuition. The DPSK presents a loss in BER performance, still, when considering stronger Wiener PN then it is more robust than the PSK.

IV. Link Adaption

A. *Estimation of thermal and phase noise variances*

To improve spectral efficiency, practical communication systems implement adaptive modulation scheme [8]. Namely, the modulation order is inferred from a SNR estimation. In the case of a channel dominated by Gaussian PN, the analytical approximation of the BER in Eq. (9) enables to determine the greatest value of M while maintaining the error rate below a fixed target. Figure 3 depicts the modulation scheme regions for M-ary PSK, *i.e.* the highest modulation order M achieving BER $< 10^{-4}$ given E_b/N_0 and σ_g^2. It is shown that the performance is highly related to the PN variance, which should necessarily be considered in the link adaptation to maintain robustness.

Such link adaptation requires an estimation of σ_n^2 and σ_g^2. This may be realized by inserting N pilot samples possibly distributed and known from the receiver denoted $\boldsymbol{s} = (s_1, s_2, ..., s_N)$ and by deriving the appropriate estimators. From Eq. (6), the joint likelihood function of the N received samples $\boldsymbol{r} = (r_1, ..., r_N)$ is given by

$$p_N(\boldsymbol{r}|\boldsymbol{s}, \sigma_n^2, \sigma_g^2) = \prod_{k=1}^{N} p(r_k|s_k, \sigma_n^2, \sigma_g^2). \tag{15}$$

We derive the ML estimate of the covariance matrix for the multivariate normal distribution [15]. It yields here the ML estimators of σ_n^2 and σ_g^2 expressed by

$$\begin{aligned} \hat{\sigma}_n^2 &= \frac{1}{N}\sum_{k=1}^{N}(|r_k| - \sqrt{E_s})^2, \\ \hat{\sigma}_g^2 &= \frac{1}{N}\sum_{k=1}^{N}(\angle r_k - \angle s_k)^2 - \frac{\hat{\sigma}_n^2}{E_s}. \end{aligned} \tag{16}$$

Writing these estimators as $\chi^2(N)$ distributions enables us to quantify their performance in terms of bias and dispersion respectively with the means and variances given by

$$\begin{cases} \mathrm{E}[\hat{\sigma}_n^2] = \sigma_n^2 \\ \mathrm{V}[\hat{\sigma}_n^2] = \frac{2\sigma_n^4}{N} \end{cases}, \quad \begin{cases} \mathrm{E}[\hat{\sigma}_g^2] = \sigma_g^2 \\ \mathrm{V}[\hat{\sigma}_g^2] = \frac{2(\sigma_g^2+\sigma_n^2/E_s)^2}{N} + \frac{2\sigma_n^4}{NE_s} \end{cases}. \tag{17}$$

Since these estimators are unbiased and the regularity conditions clearly satisfied in Eq. (6), we may compare the estimators to their Cramer-Rao Lower Bounds (CRLB) [18] written as

$$\mathrm{V}\left[\hat{\sigma}_n^2\right] \geq \frac{2\sigma_n^4}{N}, \quad \mathrm{V}\left[\hat{\sigma}_g^2\right] \geq \frac{2(\sigma_g^2 + \sigma_n^2/E_s)^2}{N}. \tag{18}$$

It follows easily that the thermal noise variance estimator is *efficient* in the sense of achieving the CRLB *cf.* [15]. As for the PN variance, the estimator is tight to the CRLB but does not reach it. However, both estimators demonstrate the smallest Mean Square Error (MSE) among any unbiased estimators. From Eq. (15), it is straightforward that the joint density p_N belongs to the exponential family such that $S = (\sum_{k=1}^{N}(|r_k| - \sqrt{E_s})^2, \sum_{k=1}^{N}(\angle r_k - \angle s_k)^2)$ is a complete sufficient statistic for parameter (σ_n^2, σ_g^2). By the Lehman-Scheffé theorem, the unbiased estimators $(\hat{\sigma}_n^2, \hat{\sigma}_g^2)$, defined upon S, are respectively the unique Minimum-Variance Unbiased Estimators (MVUE) of σ_n^2 and σ_g^2.

B. Is the phase noise a cumulative process?

As illustrated in Fig. 2, the DPSK should be favored at the detriment of PSK only if the cumulative nature of the PN is non-negligible. Hence, the link adaptation problem can be expressed as a simple detection problem: is the cumulative process of the phase noise strong enough to advantage a differential modulation. Let us formulate the corresponding binary hypothesis test as

$$\begin{aligned} H_0 &: \text{Gaussian PN}, && \text{if } \sigma_w^2/\sigma_g^2 < \lambda \text{ fixed}, \\ H_1 &: \text{Wiener PN}, && \text{otherwise}. \end{aligned} \tag{19}$$

The link adaptation decision δ is then expressed by

$$\delta(\boldsymbol{r}) = \begin{cases} \text{PSK}, & \text{if } S(\boldsymbol{r}) < \varphi(\lambda) \\ \text{DPSK}, & \text{otherwise} \end{cases}, \tag{20}$$

where $\boldsymbol{r} = (r_1, ..., r_N)$ is a received sequence of N consecutive pilots, S denotes a sufficient statistic of the received pilots sample and φ is a test threshold function of λ. Fortuitously, this problem has been extensively studied in the financial field under a different name: the random walk hypothesis for stock market prices [19]. Numerous statistical tests have been designed, yet, one in particular has retained most of the attention: the variance ratio test [19], [20]. This test relies on the fact that the variance of the random walk increments increases linearly with p the sampling interval. Applied to our context, this is expressed by

$$V[\phi_{k+p} - \phi_k] = p \cdot \sigma_w^2 + 2\sigma_g^2, \tag{21}$$

which is denoted further by $\sigma_{\Delta p}^2$. Letting $\sigma_w^2 = \lambda \cdot \sigma_g^2$ and summing over p, the statistical test becomes

$$S(\boldsymbol{r}) = \sum_{p=1}^{M} \frac{\hat{\sigma}_{\Delta p}^2}{\hat{\sigma}_{\Delta 1}^2} < \varphi(\lambda) = \frac{\lambda \cdot M(M+1) + 4M}{4\lambda + 2}, \tag{22}$$

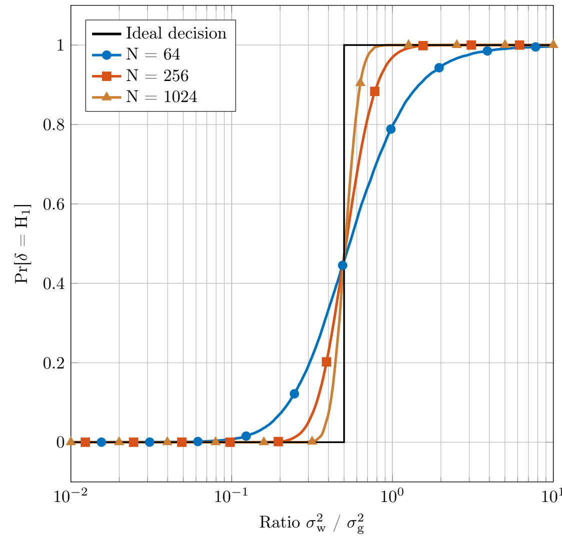

Fig. 4: Mean probability of taking decision $\delta(\boldsymbol{r}) = H_1$ when $\lambda = 1/2$ for different lengths of pilot sequence N

where $\hat{\sigma}_{\Delta p}^2$ is the estimator of $\sigma_{\Delta p}^2$ and evaluated by

$$\hat{\sigma}_{\Delta p}^2 = \frac{1}{N-p} \sum_{k=1}^{N-p} \Big((\angle r_{k+p} - \angle s_{k+p}) - (\angle r_k - \angle s_k) \Big)^2. \tag{23}$$

The upper bound of summation M satisfies $1 < M < N$, and is optimized heuristically[3]. Figure 4 depicts the mean probability of taking decision $\delta(\boldsymbol{r}) = H_1$ and thus use DPSK. Several pilot lengths are considered and λ is set to $1/2$ as in Fig. 2 . As exhibited, the longer the pilot sequence, the more accurate the test. The performance degradation caused by an incorrect decision should be evaluated to properly specify the length of the training sequence. Due to space limitation, it is not carried out here but using the proposed framework, it is straightforward to derive.

V. Conclusion

In this paper, we have addressed the problematic of link adaption for phase modulated signals affected by PN. We have first considered PSK modulations impacted by a Gaussian PN and provided the closed-form approximation of the BER. Optimum ML estimators of thermal and phase noise variances have been derived to propose a link adaptation scheme. Exploiting estimations of both the SNR and the PN variance proved to be essential to maintain robustness. Then, the PN has been described with a more realistic model: a sum of a Gaussian and a Wiener process. Performance of PSK and DPSK were compared in order to determine when the use of a differential modulation is beneficial. Accordingly, we proposed a simple detection test, inspired from the financial field, to perform the link adaptation and exploit the most robust modulation between PSK and DPSK. Nevertheless, these results remain an analytical analysis and it is of particular interest to investigate this adaptation problem under practical implementation considerations.

[3]The upper bound M in Eq. 22 must be large enough to yield a robust test. However, as the sampling interval p gets bigger, the dispersion of $\hat{\sigma}_{\Delta p}^2$ in Eq. 23 increases. Thus, if M is too large, the test is defined upon unreliable estimations.

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