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A GENERATING FUNCTION FOR RANKIN-COHEN BRACKETS

MICHAEL PEVZNER

ABSTRACT. We give a closed formula for the Rankin–Cohen formal covariant quantization in terms of generating functions of Jacobi polynomials.

The idea to interpret the quantization procedure as a formal deformation of the commutative algebra structure of classical observables comes back to the celebrated work of F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer ([2, 3]). Apart from physical considerations motivated by the perturbation theory this approach is originated from the semi-classical analysis of the Weyl operator calculus which is a way to associate with a reasonable function, say $f \in L^2(\mathbb{R}^{2n})$, an operator $\text{Op}_W(f)$ defined on $L^2(\mathbb{R}^n)$ by

$$(1) \quad (\text{Op}_W(f)u)(x) = \hbar^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} f\left(\frac{x+y}{2}, \eta\right) e^{\frac{2i\pi}{\hbar}\langle x-y, \eta \rangle} u(y) dy d\eta \quad u \in L^2(\mathbb{R}^n).$$

Notice that the composition of two operators $\text{Op}_W(f_1) \circ \text{Op}_W(f_2)$ is given in terms of their Weyl symbols f_1 and f_2 (without loss of generality one may assume $f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2n})$) as follows:

$$(2) \quad f_1 \star f_2(X) = \text{Op}_W^{-1}(\text{Op}_W(f_1) \circ \text{Op}_W(f_2))$$

$$(3) \quad = (2/\hbar)^{2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} f_1(Y) f_2(Z) e^{-4i\pi\hbar^{-1}\omega(Y-X, Z-X)} dY dZ$$

Using spectral theory we get another, fully equivalent, expression for the composition of symbols

$$(4) \quad f_1 \star f_2(X) = \text{Rest}_{|X=Y=Z, \exp(i\pi L)}(f_1(Y) \cdot f_2(Z)).$$

where $X = (x, \eta), Y = (y, \eta), Z = (z, \zeta) \in \mathbb{R}^{2n}$, $\omega(X, Y) = -\langle x, \eta \rangle + \langle y, \xi \rangle$ is the standard symplectic form on $\mathbb{R}^{2n}$ and

$$(5) \quad i\pi L = \frac{\hbar}{4i\pi} \sum_{j=1}^n \left(-\frac{\partial^2}{\partial y_j \partial \zeta_j} + \frac{\partial^2}{\partial z_j \partial \eta_j} \right).$$

Expanding the exponential into a series, the expression (4) gives rise to the Moyal formula

$$(6) \quad (f_1 \star f_2)(x, \xi) \\ = \sum \frac{(-1)^{|\alpha|}}{\alpha! \beta!} \left(\frac{\hbar}{4i\pi} \right)^{|\alpha|+|\beta|} \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial \xi} \right)^\beta f_1(x, \xi) \left(\frac{\partial}{\partial x} \right)^\beta \left(\frac{\partial}{\partial \xi} \right)^\alpha f_2(x, \xi),$$

which can be understood as an asymptotic expansion of $(f_1 \star f_2)$ with respect to the rescaling parameter $\hbar$ and yields the prototypical example of a deformation quantization or a $\star$ -product.

The exponential function arising in the expression (4) plays an important role in the whole construction and underlines the very nature of the Weyl calculus which can be understood as an extension of the Schrödinger representation of the Heisenberg Lie algebra $\mathfrak{h}_n$ to the group algebra of the Heisenberg group.

Both approaches, the operator calculus and the formal deformation have their own interests and merits. The aim of this short note is to give an explicit formula analogous to (4) for the covariant deformation quantization of the upper half-plane based on the family of Rankin–Cohen bi-differential operators.

We notice that the first two terms of (6) are respectively the usual point-wise product of functions $f_1 \cdot f_2$ and their image $\{f_1, f_2\}$ by the Poisson bracket associated with the standard (constant coefficient) symplectic form $\omega(\cdot, \cdot)$ on $\mathbb{R}^{2n}$.

The fact that any smooth Poisson structure on a real manifold M defines a formal associative product on $C^\infty(M)$ is a deep result of cohomological nature that follows from Kontsevich' formality theorem. In presence of symmetries, that is, under the assumption that the manifold M is a homogeneous space, the construction of a covariant deformation is even more involved and different techniques were recently developed for appropriate classes of symmetry groups (see for instance [5, 11, 12, 13, 18, 19] and the literature quoted therein).

One such a nontrivial but handable example is provided by the covariant quantization of the Lobachevsky upper half plane $H_2 = \{z = x + iy \in \mathbb{C} : y > 0\} \simeq SL(2, \mathbb{R})/SO(2, \mathbb{R})$. One may mention at least four different constructions. Namely,

- (1) A. Unterberger and J. Unterberger (*cf.* [21]) developed an $SL(2, \mathbb{R})$ -equivariant analog of the Weyl operator calculus which introduces, in the spirit of (2), a covariant noncommutative product of functions on H_2 via the composition of operators they correspond to.

It turns out that in an appropriate setting such a product can be expressed in terms of Rankin-Cohen brackets defined for any $a \in \mathbb{N}$ and $f_j \in B^{k_j}(H_2) :=$

$\mathcal{O}(H_2) \cap L^2(H_2, y^{k_j-2} dx dy)$, with $k_j \in \mathbb{N} \setminus \{0\}$, $j = 1, 2$ by:

$$(7) \quad \mathcal{RC}_{k_1, k_2}^{k_3}(f_1, f_2)(z) := \sum_{\ell=0}^a (-1)^\ell \binom{k_1 + a - 1}{\ell} \binom{k_2 + a - 1}{a - \ell} f_1^{(a-\ell)}(z) f_2^{(\ell)}(z),$$

where $f^{(n)}(z) = \frac{d^n}{dz^n} f(z)$, and $k_3 := k_1 + k_2 + 2a$ (see [6]).

More precisely, the associativity of the natural composition of operators implies that the formula

$$(8) \quad f_1 \# f_2 = \sum_{a \in \mathbb{N}} c_{k_1, k_2}^{k_3} \mathcal{RC}_{k_1, k_2}^{k_1+k_2+2a}(f_1, f_2),$$

defined for $f_j \in B^{k_j}(H_2)$, $k_j \in \mathbb{N} \setminus \{0\}$, $j = 1, 2$, where $c_{k_1, k_2}^{k_3}$ are explicit constants obtained from the multiplication table of reproducing kernels of Hilbert spaces $B^{k_j}(H_2)$, $k_j \in \mathbb{N} \setminus \{0\}$ for $j = 1, 2, 3$ (see [21, Theorem 4.2]), gives rise to an associative, noncommutative product on H_2 deforming the point-wise product $\mathcal{RC}_{k_1, k_2}^{k_1+k_2}$ in the direction of the Poisson bracket $\mathcal{RC}_{k_1, k_2}^{k_1+k_2+2}$.

- (2) P. Cohen, Yu. Manin and D. Zagier (*cf.* [7]) followed the opposite way and having imposed *de facto* the associativity constraint on formal expressions of the form

$$(9) \quad f_1 \star_\alpha f_2 := \sum_{a \in \mathbb{N}} \hbar^a \alpha_{k_1, k_2}^{k_3} \mathcal{RC}_{k_1, k_2}^{k_1+k_2+2a}(f_1, f_2)$$

found, up to a rescaling of coefficients $\alpha_{k_1, k_2}^{k_3}$, the same expression for the deformed product as in (8).

The fact that the authors looked for formal deformations given in terms of Rankin–Cohen brackets is not an *ad hoc* requirement: one shows (*e.g.* [10]) that such bi-differential operators are the only, up to a scalar multiple, operators intertwining the tensor product of $SL(2, \mathbb{R})$ representations in $B^{k_1}(H_2)$ and $B^{k_2}(H_2)$ with its irreducible component isomorphic to $B^{k_3}(H_2)$ (such representations are precisely the holomorphic discrete series representations of the Lie group $SL(2, \mathbb{R})$).

Motivated by the work [7] A. Connes and H. Moscovici gave an intrinsic explanation of the associativity of deformations of type (8) in terms of Hopf algebras of transversal geometry ([8]). The authors also emphasized that putting all the combinatorial coefficients $\alpha_{k_1, k_2}^{k_3}$ to be equal to one leads, as predicted by W. Eholzer and T. Ibukiyama (*cf.* [9]), to an associative product:

$$(10) \quad f_1 \star_1 f_2 := \sum_{a \in \mathbb{N}} \hbar^a \mathcal{RC}_{k_1, k_2}^{k_1+k_2+2a}(f_1, f_2).$$

- (3) One may also capture the associativity of a formal deformation through the Yang–Baxter equation and, in the covariant setting, the fusion matrices for

Verma modules. This construction is due to A. Alekseev and A. Lachowska [1] and the link with Rankin–Cohen deformations (8) was established by M. Medina-Luna in [17].

- (4) Yet another method is based on the Fedosov construction adapted by P. Bieliavsky, X. Tang and Y. Yao [4] to the natural symplectic structure of H_2 .

However none of these methods exhibits a closed formula of type (4) for the covariant quantization of the Lobachevsky upper half-plane H_2 .

Investigating the general structure of intertwining operators for discretely decomposable branching rules of infinite dimensional representations of real reductive Lie groups (the decomposition $B^{k_1}(H_2) \otimes B^{k_2}(H_2) \simeq \widehat{\oplus}_{a \in \mathbb{N}} B^{k_1+k_2+2a}(H_2)$ into irreducible components of the tensor product of two holomorphic discrete series representations of $SL(2, \mathbb{R})$ mentioned above is a first example of this paradigm) it was recently pointed out (cf [16, (9.9)]) that Rankin–Cohen brackets are related to the classical Jacobi polynomials:

$$(11) \quad \begin{aligned} P_\ell^{\alpha, \beta}(t) &= \frac{\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha + 1)\ell!} {}_2F_1\left(-\ell, \alpha + \beta + \ell + 1; \alpha + 1; \frac{1-t}{2}\right) \\ &= \frac{\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha + \beta + \ell + 1)} \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{\Gamma(\alpha + \beta + \ell + m + 1)}{\Gamma(\alpha + m + 1)\ell!} \left(\frac{t-1}{2}\right)^m, \end{aligned}$$

where ${}_2F_1(a, b; c; z)$ denotes the Gauss hypergeometric function that can be defined, for instance, for $z \in \mathbb{C}: |z| < 1$ by a power series $= \sum_{n \in \mathbb{N}} \frac{(a)_n (b)_n}{(c)_n n!} z^n$ (here $(x)_n$ denotes the rising Pochhammer symbol).

Namely, define a homogeneous polynomial in two variables obtained from the classical Jacobi polynomial $P_\ell^{\alpha, \beta}(t)$ of degree ℓ by the following inflation procedure:

$$(12) \quad \begin{aligned} P_\ell^{\alpha, \beta}(x, y) &:= y^\ell P_\ell^{\alpha, \beta}\left(2\frac{x}{y} + 1\right) \\ &= \frac{\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha + \beta + \ell + 1)} \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{\Gamma(\alpha + \beta + \ell + m + 1)}{\Gamma(\alpha + m + 1)\ell!} x^m y^{\ell-m}. \end{aligned}$$

Then, for every $a \in \mathbb{N}$ the following identity on holomorphic bidifferential operators holds:

$$(13) \quad \mathcal{RC}_{k_1, k_2}^{k_1+k_2+2a} = \text{Rest} \Big|_{z_1=z_2=z} \circ (-1)^a P_a^{k_1-1, 1-k_1-k_2-2a} \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right).$$

Of course, once known, the above formula (13) is elementary to check. The general method, referred to as the *F-method* and developed in [15], gives a general framework for constructing all differential symmetry breaking operators for discretely decomposable branching rules of real reductive Lie groups and explains the “raison d’être”

of the formula (13) interpreting the covariance property of the Rankin–Cohen bi-differential operators $\mathcal{RC}_{k_1, k_2}^{k_3}$ as a constraint on their symbols, a constraint which turns out to be in this case the Gauss hypergeometric equation.

The formula (13) can be expressed in a more symmetric way.

Lemma 1. *Let $\tilde{P}_\ell^{\alpha, \beta}(x, y) = (-1)^\ell (x + y)^\ell P_\ell^{\alpha, \beta}\left(\frac{y-x}{x+y}\right)$. Then, for any $a \in \mathbb{N}$:*

$$\mathcal{RC}_{k_1, k_2}^{k_1+k_2+2a} = \text{Rest} \Big|_{z_1=z_2=z} \tilde{P}_a^{k_1-1, k_2-1} \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right),$$

Proof. According to the first Kummer's relation for the hypergeometric function (see for instance [14, 8.962]):

$$\begin{aligned} P_\ell^{\alpha, \beta}(t) &= \frac{(-1)^\ell \Gamma(\ell + 1 + \beta)}{\ell! \Gamma(1 + \beta)} {}_2F_1 \left(\ell + \alpha + \beta + 1, -\ell; 1 + \beta; \frac{1+t}{2} \right) \\ &= \frac{\Gamma(\ell + 1 + \beta)}{\ell! \Gamma(1 + \beta)} \left(\frac{t-1}{2} \right)^\ell {}_2F_1 \left(-\ell, -\ell - \alpha; 1 + \beta; \frac{1+t}{t-1} \right) \\ &= \frac{\Gamma(\ell + 1 + \alpha)}{\ell! \Gamma(1 + \alpha)} \left(\frac{t+1}{2} \right)^\ell {}_2F_1 \left(-\ell, -\ell - \beta; 1 + \alpha; \frac{t-1}{t+1} \right) \end{aligned}$$

we get

$$P_\ell^{\alpha, \beta}(t) = \left(\frac{1+t}{2} \right)^\ell P_\ell^{\alpha, -\alpha-\beta-2\ell-1} \left(\frac{3-t}{1+t} \right),$$

and therefore

$$P_\ell^{k_1-1, 1-k_1-k_2-2a}(1-2s) = (1-s)^\ell P_\ell^{k_1-1, k_2-1} \left(\frac{1+s}{1-s} \right),$$

what concludes the proof together with the definition (12). $\square$

Let us recall ((cf [20, (4.4.5)]) that the generating series of Jacobi polynomials is given by :

$$(14) \quad \sum_{\ell \geq 0} P_\ell^{\alpha, \beta}(t) z^\ell = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta},$$

where $R = r(t, z) = \sqrt{1-2tz+z^2}$. This leads us to the following conclusion.

Theorem 2. *Consider the following function of two variables (x, y) depending on a formal parameter $\hbar$:*

$$F_{\alpha, \beta}^{\hbar}(x, y) = \frac{\left(1 + \hbar(x+y) + \sqrt{1 + 2(y-x)\hbar + \hbar^2(x+y)^2} \right)^{-\alpha} \left(1 - \hbar(x+y) + \sqrt{1 + 2(y-x)\hbar + \hbar^2(x+y)^2} \right)^{-\beta}}{2^{-\alpha-\beta} \sqrt{1 + 2(y-x)\hbar + \hbar^2(x+y)^2}}$$

Then the formal Rankin–Cohen covariant quantization $\star_1$ of the upper half-plane H_2 is given by

$$\begin{aligned} f_1 \star_1 f_2(z) &= \sum_{a \in \mathbb{N}} \hbar^a \mathcal{RC}_{k_1, k_2}^{k_1+k_2+2a}(f_1, f_2) \\ &= \text{Rest}|_{x=y=z} \circ F_{k_1, k_2}^{\hbar} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) (f_1(x) \cdot f_2(y)), \end{aligned}$$

where $f_{k_j} \in B^{k_j}(H_2)$, $k_j \in \mathbb{N} \setminus \{0\}$, $j = 1, 2$.

Proof. The formula (14) for the generating function of Jacobi polynomials implies that

$$\sum_{\ell \geq 0} (-1)^\ell P_\ell^{\alpha, \beta} \left(\frac{y-x}{x+y} \right) (x+y)^\ell \hbar^\ell = F_{\alpha, \beta}^{\hbar}(x, y).$$

Together with Lemma 1 this observation concludes the proof. $\square$

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M. PEVZNER. LABORATOIRE DE MATHÉMATIQUES DE REIMS, FRE 2011, UNIVERSITÉ DE REIMS CHAMPAGNE–ARDENNE, F–51687, REIMS, FRANCE & ISC J.-V. PONCELET, CNRS UMI 2615, `PEVZNER@UNIV-REIMS.FR`.