First and second order rational solutions to the Johnson equation and rogue waves

Pierre Gaillard

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P. Gaillard,
Université de Bourgogne,
Institut de mathématiques de Bourgogne,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

Abstract

Rational solutions to the Johnson equation are constructed as a quotient of two polynomials in x, y and t depending on several real parameters. We obtain an infinite hierarchy of rational solutions written in terms of polynomials of degrees $2N(N+1)$ in x, and t, $4N(N+1)$ in y, depending on $2N-2$ real parameters for each positive integer N. We construct explicit expressions of the solutions in the cases $N=1$ and $N=2$ which are given in the following. We study the evolution of the solutions by constructing the patterns of their modulus in the (x, y) plane, and this for different values of parameters.

Key Words : Johnson equation, Fredholm determinants, wronskians, rational solutions, rogue waves.

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1 Introduction

We consider the Johnson equation (J) which can be written in the form

$$(u_t + 6uu_x + u_{xxx} + \frac{u}{2t})_x - 3\frac{u_{yy}}{t^2} = 0,$$

where subscripts x, y and t denote partial derivatives.
This equation first appears first in 1980, in a paper written by Johnson [1]. It gives the description of waves surfaces in shallow incompressible fluids [2]-[3]. This equation was widely accepted, and was later derived for internal waves in a stratified medium [4]. The physical model of this equation have the same degree

of universality as the Kadomtsev-Petviashvili equation [5].
The first solutions were constructed in 1980 by Johnson [1]. Various research were led and more general solutions of this equation were obtained, for example, some special solutions were found in [6]. In 2007, some important classes of solutions were obtained by using the Darboux transformation [7]. This equation is dissipative and there is no soliton-like solution with a linear front localized along straight lines in the (x, y) plane. The Johnson equation allows explaining the existence of the horseshoe like solitons and multisoliton solutions quite naturally. Other extensions have been considered as for example the elliptic case [8].
In this paper, we study rational solutions of the Johnson equation. We presents multi parametric families of rational solutions as a quotient of two polynomials in x, y and t depending on several real parameters. These solutions presented belong to an infinite hierarchy of rational solutions written in terms of polynomials of degrees $2N(N+1)$ in x, t and $4N(N+1)$ in y, depending on $2N-2$ real parameters for each positive integer N. The study here is limited to the simplest cases where $N = 1$ and $N = 2$.
The patterns of the modulus of the solutions in the (x, y) plane for different values of time t and parameters are studied.

2 First order rational solutions

We consider the Johnson equation

$$(u_t + 6uu_x + u_{xxx} + \frac{u}{2t})_x - 3\frac{u_{yy}}{t^2} = 0, \tag{1}$$

We have the following result at order $N = 1$:

Theorem 2.1 *The function v defined by*

$$v(x, y, t) = -2\,\frac{|n(x, y, t)|^2}{d(x, y, t)^2} \tag{2}$$

where

$$n(x, y, t) = 144\,x^2 + (6912\,t + 24\,y^2t)x + 108 + 576\,iyt + y^4t^2 + 82944\,t^2 \tag{3}$$

and

$$d(x, y, t) = 144\,x^2 + (6912\,t + 24\,y^2t)x - 36 + y^4t^2 + 82944\,t^2, \tag{4}$$

is a rational solution to the Johnson equation (1), quotient of two polynomials $|n(x, y, t)|^2$ and $d(x, y, t)^2$ of degree 4 in x, t and 8 in y.

Proof We have to replace the expression of the solution given by (2) and check that (1) is verified.

□

The modulus of the solution for different values of t are represented in the plane (x, y). It can be seen that as t tends to 0, the front of the solution contracts and is concentrated in a small strip along the y axis.
When the time t grows, we get peaks on special curves.

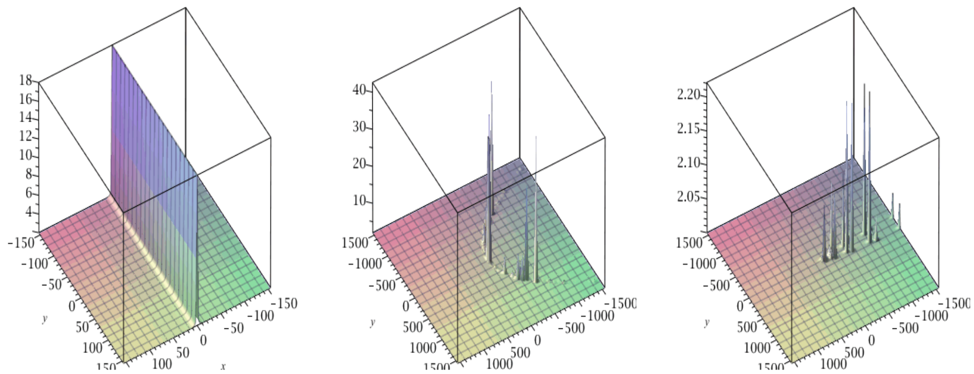

Figure 1. Solution of order 1 to (1), on the left for $t = 0$; in the center for $t = 0,01$; on the right for $t = 1$.

3 Second order rational solutions depending on 2 parameters

The Johnson equation defined by (1) is always considered. For this second order, we get the rational solutions given by :

Theorem 3.1 *The function v defined by*

$$v(x, y, t) = -2\,\frac{|n(x, y, t)|^2}{d(x, y, t)^2} \tag{5}$$

is a rational solution to the Johnson equation (1), quotient of two polynomials $|n(x, y, t)|^2$ and $d(x, y, t)^2$ of degree 12 in x, t and 24 in y depending on 2 real parameters a_1, b_1.

Polynomials n and d are respectively given by
$n(x, y, t) = A(x, y, t) + iB(x, y, t)$, $d(x, y, t) = C(x, y, t) + iD(x, y, t)$ with
$A(x, y, t) = \sum_{k=0}^{6} a_k(y, t)x^k$, $\quad B(x, y, t) = \sum_{k=0}^{6} b_k(y, t)x^k$,
$C(x, y, t) = \sum_{k=0}^{6} c_k(y, t)x^k$, $\quad D(x, y, t) = \sum_{k=0}^{6} d_k(y, t)x^k$,

$\mathbf{a}_6 = -2985984$, $\quad \mathbf{a}_5 = (-1492992\, y^2 - 429981696)t$, $\quad \mathbf{a}_4 = 17915904\, tyb_1 + (-311040\, y^4 - 143327232\, y^2 - 25798901760)t^2 + 11197440\, {a_1}^2 + 2239488\, {b_1}^2 - 6718464$, $\quad \mathbf{a}_3 = (-34560\, y^6 - 17915904\, y^4 - 5159780352\, y^2 - 825564856320)t^3 + (5971968\, y^3 b_1 + 1719926784\, yb_1)t^2 - 17915904\, a_1 b_1 + (3732480\, y^2 {a_1}^2 + 746496\, y^2 {b_1}^2 - 2239488\, y^2 - 71663616\, ya_1 + 1074954240\, {a_1}^2 + 214990848\, {b_1}^2 - 931627008)t$, $\quad \mathbf{a}_2 = (-2160\, y^8 - 995328\, y^6 - 358318080\, y^4 - 82556485632\, y^2 - 14860167413760)t^4 -$

$$2799360\,a_1^4-3359232\,a_1^2b_1^2-559872\,b_1^4+(746496\,y^5b_1+286654464\,y^3b_1+61917364224\,yb_1)t^3+(466560\,y^4a_1^2+93312\,y^4b_1^2-279936\,y^4-17915904\,y^3a_1+214990848\,y^2a_1^2+35831808\,y^2-5159780352\,ya_1+38698352640\,a_1^2+7739670528\,b_1^2-43858132992)t^2+8398080\,a_1^2+8398080\,b_1^2+(-4478976\,y^2a_1b_1-26873856\,ya_1^2b_1-8957952\,yb_1^3+100776960\,yb_1-1289945088\,a_1b_1)t+8398080, \quad \mathbf{a}_1=(-72\,y^{10}-20736\,y^8-11943936\,y^6-3439853568\,y^4-495338913792\,y^2-142657607172096)t^5+(41472\,y^7b_1+11943936\,y^5b_1+3439853568\,y^3b_1+990677827584\,yb_1)t^4+4478976\,a_1^3b_1+4478976\,a_1b_1^3+(25920\,y^6a_1^2+5184\,y^6b_1^2-15552\,y^6-1492992\,y^5a_1+13436928\,y^4a_1^2-4478976\,y^4b_1^2+25380864\,y^4-573308928\,y^3a_1+3869835264\,y^2a_1^2-1289945088\,y^2b_1^2+2149908480\,y^2-123834728448\,ya_1+619173642240\,a_1^2+123834728448\,b_1^2-866843099136)t^3+(-373248\,y^4a_1b_1-4478976\,y^3a_1^2b_1-1492992\,y^3b_1^3+16796160\,y^3b_1-1289945088\,ya_1^2b_1-429981696\,yb_1^3+3117367296\,yb_1-30958682112\,a_1b_1)t^2+(-466560\,y^2a_1^4-559872\,y^2a_1^2b_1^2-93312\,y^2b_1^4+1399680\,y^2a_1^2+1399680\,y^2b_1^2+17915904\,ya_1^3+53747712\,ya_1b_1^2-134369280\,a_1^4-161243136\,a_1^2b_1^2-26873856\,b_1^4+1399680\,y^2-67184640\,ya_1+618098688\,a_1^2+188116992\,b_1^2+1048080384)t, \quad \mathbf{a}_0=(-y^{12}-248832\,y^8-20639121408\,y^4-570630428688384)t^6+46656\,a_1^6+139968\,a_1^4b_1^2+139968\,a_1^2b_1^4+46656\,b_1^6+(864\,y^9b_1+143327232\,y^5b_1+5944066965504\,yb_1)t^5+(540\,y^8a_1^2+108\,y^8b_1^2-324\,y^8-41472\,y^7a_1+248832\,y^6a_1^2-248832\,y^6b_1^2+1327104\,y^6-11943936\,y^5a_1+89579520\,y^4a_1^2+17915904\,y^4b_1^2+89579520\,y^4-3439853568\,y^3a_1+20639121408\,y^2a_1^2-20639121408\,y^2b_1^2-990677827584\,ya_1+3715041853440\,a_1^2+743008370688\,b_1^2-6191736422400)t^4-279936\,a_1^4-2519424\,a_1^2b_1^2-2239488\,b_1^4+(-10368\,y^6a_1b_1-186624\,y^5a_1^2b_1-62208\,y^5b_1^3+699840\,y^5b_1+8957952\,y^4a_1b_1-71663616\,y^3a_1^2b_1+23887872\,y^3b_1^3-179159040\,y^3b_1+2579890176\,y^2a_1b_1-15479341056\,ya_1^2b_1-5159780352\,yb_1^3+16769286144\,yb_1-247669456896\,a_1b_1)t^3+(-19440\,y^4a_1^4-23328\,y^4a_1^2b_1^2-3888\,y^4b_1^4+58320\,y^4a_1^2+58320\,y^4b_1^2+1492992\,y^3a_1^3+4478976\,y^3a_1b_1^2-8957952\,y^2a_1^4+8957952\,y^2b_1^4+58320\,y^4-5598720\,y^3a_1-22394880\,y^2a_1^2-165722112\,y^2b_1^2+429981696\,ya_1^3+1289945088\,ya_1b_1^2-1612431360\,a_1^4-1934917632\,a_1^2b_1^2-322486272\,b_1^4+107495424\,ya_1+9997074432\,a_1^2-322486272\,b_1^2+13436928000)t^2-734832\,a_1^2-734832\,b_1^2+(373248\,y^2a_1^3b_1+373248\,y^2a_1b_1^3+1119744\,ya_1^4b_1+2239488\,ya_1^2b_1^3+1119744\,yb_1^5-28553472\,ya_1^2b_1-33032448\,yb_1^3+107495424\,a_1^3b_1+107495424\,a_1b_1^3-35271936\,yb_1+429981696\,a_1b_1)t+2099520$$

$$\mathbf{b}_6=5971968, \quad \mathbf{b}_5=(2985984\,y^2+859963392)t, \quad \mathbf{b}_4=,(622080\,y^4+286654464\,y^2+51597803520)t^2-11197440\,a_1^2-2239488\,b_1^2+(-17915904\,yb_1+35831808\,y)t+13436928, \quad \mathbf{b}_3=(69120\,y^6+35831808\,y^4+10319560704\,y^2+1651129712640)t^3+(-5971968\,y^3b_1+11943936\,y^3-1719926784\,yb_1+3439853568\,y)t^2+17915904\,a_1b_1+(-3732480\,y^2a_1^2-746496\,y^2b_1^2+4478976\,y^2+71663616\,ya_1-1074954240\,a_1^2-214990848\,b_1^2+1863254016)t, \quad \mathbf{b}_2=(4320\,y^8+1990656\,y^6+716636160\,y^4+165112971264\,y^2+29720334827520)t^4+2799360\,a_1^4+3359232\,a_1^2b_1^2+559872\,b_1^4+(-746496\,y^5b_1+1492992\,y^5-286654464\,y^3b_1+573308928\,y^3-61917364224\,yb_1+123834728448\,y)t^3+(-466560\,y^4a_1^2-93312\,y^4b_1^2+559872\,y^4+17915904\,y^3a_1-214990848\,y^2a_1^2-71663616\,y^2+5159780352\,ya_1-38698352640\,a_1^2-7739670528\,b_1^2+87716265984)t^2-8398080\,a_1^2-8398080\,b_1^2+(4478976\,y^2a_1b_1+26873856\,ya_1^2b_1+8957952\,yb_1^3-100776960\,yb_1+1289945088\,a_1b_1+53747712\,y)t-16796160, \quad \mathbf{b}_1=(144\,y^{10}+41472\,y^8+23887872\,y^6+6879707136\,y^4+990677827584\,y^2+285315214344192)t^5+(-41472\,y^7b_1+82944\,y^7-11943936\,y^5b_1+23887872\,y^5-3439853568\,y^3b_1+6879707136\,y^3-990677827584\,yb_1+1981355655168\,y)t^4-4478976\,a_1^3b_1-4478976\,a_1b_1^3+(-25920\,y^6a_1^2-5184\,y^6b_1^2+31104\,y^6+1492992\,y^5a_1-13436928\,y^4a_1^2+4478976\,y^4b_1^2-50761728\,y^4+573308928\,y^3a_1-3869835264\,y^2a_1^2+1289945088\,y^2b_1^2-4299816960\,y^2+123834728448\,ya_1-619173642240\,a_1^2-123834728448\,b_1^2+1733686198272)t^3+(373248\,y^4a_1b_1+4478976\,y^3a_1^2b_1+1492992\,y^3b_1^3-16796160\,y^3b_1+1289945088\,ya_1^2b_1+429981696\,yb_1^3+8957952\,y^3-3117367296\,yb_1+30958682112\,a_1b_1-859963392\,y)t^2+(466560\,y^2a_1^4+559872\,y^2a_1^2b_1^2+93312\,y^2b_1^4-1399680\,y^2a_1^2-1399680\,y^2b_1^2-17915904\,ya_1^3-53747712\,ya_1b_1^2+134369280\,a_1^4+161243136\,a_1^2b_1^2+26873856\,b_1^4-2799360\,y^2+67184640\,ya_1-618098688\,a_1^2-188116992\,b_1^2-2096160768)t, \quad \mathbf{b}_0=(2\,y^{12}+497664\,y^8+41278242816\,y^4+1141260857376768)t^6-46656\,a_1^6-139968\,a_1^4b_1^2-139968\,a_1^2b_1^4-46656\,b_1^6+(-864\,y^9b_1+1728\,y^9-143327232\,y^5b_1+286654464\,y^5-5944066965504\,yb_1+11888133931008\,y)t^5+(-540\,y^8a_1^2-108\,y^8b_1^2+648\,y^8+41472\,y^7a_1-248832\,y^6a_1^2+248832\,y^6b_1^2-2654208\,y^6+11943936\,y^5a_1-89579520\,y^4a_1^2-17915904\,y^4b_1^2-179159040\,y^4+3439853568\,y^3a_1-20639121408\,y^2a_1^2+20639121408\,y^2b_1^2+990677827584\,ya_1-3715041853440\,a_1^2-743008370688\,b_1^2+12383472844800)t^4+279936\,a_1^4+2519424\,a_1^2b_1^2+2239488\,b_1^4+(10368\,y^6a_1b_1+186624\,y^5a_1^2b_1+62208\,y^5b_1^3-699840\,y^5b_1-8957952\,y^4a_1b_1+71663616\,y^3a_1^2b_1-23887872\,y^3b_1^3+373248\,y^5+179159040\,y^3b_1-2579890176\,y^2a_1b_1+15479341056\,ya_1^2b_1+5159780352\,yb_1^3-16769286144\,yb_1+247669456896\,a_1b_1-51597803520\,y)t^3+(19440\,y^4a_1^4+23328\,y^4a_1^2b_1^2+3888\,y^4b_1^4-58320\,y^4a_1^2-58320\,y^4b_1^2-1492992\,y^3a_1^3-4478976\,y^3a_1b_1^2+8957952\,y^2a_1^4-8957952\,y^2b_1^4-116640\,y^4+5598720\,y^3a_1+22394880\,y^2a_1^2+165722112\,y^2b_1^2-429981696\,ya_1^3-1289945088\,ya_1b_1^2+1612431360\,a_1^4+1934917632\,a_1^2b_1^2+322486272\,b_1^4-107495424\,ya_1-9997074432\,a_1^2+322486272\,b_1^2-26873856000)t^2+734832\,a_1^2+734832\,b_1^2+(-373248\,y^2a_1^3b_1-373248\,y^2a_1b_1^3-1119744\,ya_1^4b_1-2239488\,ya_1^2b_1^3-1119744\,yb_1^5+28553472\,ya_1^2b_1+33032448\,yb_1^3-107495424\,a_1^3b_1-107495424\,a_1b_1^3+35271936\,yb_1-429981696\,a_1b_1-33592320\,y)t-4199040$$

$$\mathbf{c}_6=-2985984, \quad \mathbf{c}_5=(-1492992\,y^2-429981696)t, \quad \mathbf{c}_4=17915904\,tyb_1+(-311040\,y^4-143327232\,y^2-25798901760)t^2+11197440\,a_1^2+2239488\,b_1^2+2239488, \quad \mathbf{c}_3=(-34560\,y^6-17915904\,y^4-5159780352\,y^2-825564856320)t^3+(5971968\,y^3b_1+1719926784\,yb_1)t^2+(3732480\,y^2a_1^2+746496\,y^2b_1^2+746496\,y^2+1074954240\,a_1^2+214990848\,b_1^2-71663616)t, \quad \mathbf{c}_2=(-2160\,y^8-995328\,y^6-358318080\,y^4-82556485632\,y^2-14860167413760)t^4-2799360\,a_1^4-3359232\,a_1^2b_1^2-559872\,b_1^4+(746496\,y^5b_1+286654464\,y^3b_1+61917364224\,yb_1)t^3+(466560\,y^4a_1^2+93312\,y^4b_1^2+93312\,y^4+214990848\,y^2a_1^2+$$

$35831808\,y^2 + 38698352640\,{a_1}^2 + 7739670528\,{b_1}^2 - 12899450880)t^2 - 5038848\,{a_1}^2 - 5038848\,{b_1}^2 +$
$(-26873856\,y{a_1}^2b_1 - 8957952\,y{b_1}^3 - 6718464\,yb_1)t - 5038848, \quad \mathbf{c}_1 = (-72\,y^{10} - 20736\,y^8 - 11943936\,y^6 -$
$3439853568\,y^4 - 495338913792\,y^2 - 142657607172096)t^5 + (41472\,y^7b_1 + 11943936\,y^5b_1 + 3439853568\,y^3b_1 +$
$990677827584\,yb_1)t^4 + (25920\,y^6{a_1}^2 + 5184\,y^6{b_1}^2 + 5184\,y^6 + 13436928\,y^4{a_1}^2 - 4478976\,y^4{b_1}^2 +$
$7464960\,y^4 + 3869835264\,y^2{a_1}^2 - 1289945088\,y^2{b_1}^2 - 3009871872\,y^2 + 619173642240\,{a_1}^2 + 123834728448\,{b_1}^2 -$
$371504185344)t^3 + (-4478976\,y^3{a_1}^2b_1 - 1492992\,y^3{b_1}^3 - 1119744\,y^3b_1 - 1289945088\,y{a_1}^2b_1 - 429981696\,y{b_1}^3 -$
$2042413056\,yb_1)t^2 + (-466560\,y^2{a_1}^4 - 559872\,y^2{a_1}^2{b_1}^2 - 93312\,y^2{b_1}^4 - 839808\,y^2{a_1}^2 - 839808\,y^2{b_1}^2 -$
$134369280\,{a_1}^4 - 161243136\,{a_1}^2{b_1}^2 - 26873856\,{b_1}^4 - 839808\,y^2 - 26873856\,{a_1}^2 - 456855552\,{b_1}^2 -$
$456855552)t, \quad \mathbf{c}_0 = (-y^{12} - 248832\,y^8 - 20639121408\,y^4 - 570630428688384)t^6 + 46656\,{a_1}^6 +$
$139968\,{a_1}^4{b_1}^2 + 139968\,{a_1}^2{b_1}^4 + 46656\,{b_1}^6 + (864\,y^9b_1 + 143327232\,y^5b_1 + 5944066965504\,yb_1)t^5 +$
$(540\,y^8{a_1}^2 + 108\,y^8{b_1}^2 + 108\,y^8 + 248832\,y^6{a_1}^2 - 248832\,y^6{b_1}^2 + 331776\,y^6 + 89579520\,y^4{a_1}^2 +$
$17915904\,y^4{b_1}^2 + 161243136\,y^4 + 20639121408\,y^2{a_1}^2 - 20639121408\,y^2{b_1}^2 - 82556485632\,y^2 + 3715041853440\,{a_1}^2 +$
$743008370688\,{b_1}^2 - 3219702939648)t^4 + 279936\,{a_1}^4 + 839808\,{a_1}^2{b_1}^2 + 559872\,{b_1}^4 + (-186624\,y^5{a_1}^2b_1 -$
$62208\,y^5{b_1}^3 - 46656\,y^5b_1 - 71663616\,y^3{a_1}^2b_1 + 23887872\,y^3{b_1}^3 + 107495424\,y^3b_1 - 15479341056\,y{a_1}^2b_1 -$
$5159780352\,y{b_1}^3 - 45148078080\,yb_1)t^3 + (-19440\,y^4{a_1}^4 - 23328\,y^4{a_1}^2{b_1}^2 - 3888\,y^4{b_1}^4 - 34992\,y^4{a_1}^2 -$
$34992\,y^4{b_1}^2 - 8957952\,y^2{a_1}^4 + 8957952\,y^2{b_1}^4 - 34992\,y^4 - 22394880\,y^2{a_1}^2 + 49268736\,y^2{b_1}^2 -$
$1612431360\,{a_1}^4 - 1934917632\,{a_1}^2{b_1}^2 - 322486272\,{b_1}^4 + 35831808\,y^2 + 2257403904\,{a_1}^2 - 8062156800\,{b_1}^2 -$
$14941863936)t^2 + 944784\,{a_1}^2 + 944784\,{b_1}^2 + (1119744\,y{a_1}^4b_1 + 2239488\,y{a_1}^2{b_1}^3 + 1119744\,y{b_1}^5 -$
$1679616\,y{a_1}^2b_1 + 11757312\,y{b_1}^3 + 11757312\,yb_1)t + 419904$

$\mathbf{d}_6 = 5971968, \quad \mathbf{d}_5 = (2985984\,y^2 + 859963392)t, \quad \mathbf{d}_4 = -17915904\,tyb_1 + (622080\,y^4 + 286654464\,y^2 +$
$51597803520)t^2 - 11197440\,{a_1}^2 - 2239488\,{b_1}^2 - 4478976, \quad \mathbf{d}_3 = (69120\,y^6 + 35831808\,y^4 + 10319560704\,y^2 +$
$1651129712640)t^3 + (-5971968\,y^3b_1 - 1719926784\,yb_1)t^2 + (-3732480\,y^2{a_1}^2 - 746496\,y^2{b_1}^2 - 1492992\,y^2 -$
$1074954240\,{a_1}^2 - 214990848\,{b_1}^2 + 143327232)t, \quad \mathbf{d}_2 = (4320\,y^8 + 1990656\,y^6 + 716636160\,y^4 +$
$165112971264\,y^2 + 29720334827520)t^4 + 2799360\,{a_1}^4 + 3359232\,{a_1}^2{b_1}^2 + 559872\,{b_1}^4 + (-746496\,y^5b_1 -$
$286654464\,y^3b_1 - 61917364224\,yb_1)t^3 + (-466560\,y^4{a_1}^2 - 93312\,y^4{b_1}^2 - 186624\,y^4 - 214990848\,y^2{a_1}^2 -$
$71663616\,y^2 - 38698352640\,{a_1}^2 - 7739670528\,{b_1}^2 + 25798901760)t^2 + 5038848\,{a_1}^2 + 5038848\,{b_1}^2 +$
$(26873856\,y{a_1}^2b_1 + 8957952\,y{b_1}^3 + 6718464\,yb_1)t + 10077696, \quad \mathbf{d}_1 = (144\,y^{10} + 41472\,y^8 + 23887872\,y^6 +$
$6879707136\,y^4 + 990677827584\,y^2 + 285315214344192)t^5 + (-41472\,y^7b_1 - 11943936\,y^5b_1 - 3439853568\,y^3b_1 -$
$990677827584\,yb_1)t^4 + (-25920\,y^6{a_1}^2 - 5184\,y^6{b_1}^2 - 10368\,y^6 - 13436928\,y^4{a_1}^2 + 4478976\,y^4{b_1}^2 -$
$14929920\,y^4 - 3869835264\,y^2{a_1}^2 + 1289945088\,y^2{b_1}^2 + 6019743744\,y^2 - 619173642240\,{a_1}^2 - 123834728448\,{b_1}^2 +$
$743008370688)t^3 + (4478976\,y^3{a_1}^2b_1 + 1492992\,y^3{b_1}^3 + 1119744\,y^3b_1 + 1289945088\,y{a_1}^2b_1 + 429981696\,y{b_1}^3 +$
$2042413056\,yb_1)t^2 + (466560\,y^2{a_1}^4 + 559872\,y^2{a_1}^2{b_1}^2 + 93312\,y^2{b_1}^4 + 839808\,y^2{a_1}^2 + 839808\,y^2{b_1}^2 +$
$134369280\,{a_1}^4 + 161243136\,{a_1}^2{b_1}^2 + 26873856\,{b_1}^4 + 1679616\,y^2 + 26873856\,{a_1}^2 + 456855552\,{b_1}^2 +$
$913711104)t, \quad \mathbf{d}_0 = 2\,t^6y^{12} - 864\,t^5y^9b_1 + 497664\,t^6y^8 - 540\,t^4y^8{a_1}^2 - 108\,t^4y^8{b_1}^2 - 216\,t^4y^8 -$
$248832\,t^4y^6{a_1}^2 + 248832\,t^4y^6{b_1}^2 - 143327232\,t^5y^5b_1 + 186624\,t^3y^5{a_1}^2b_1 + 62208\,t^3y^5{b_1}^3 + 41278242816\,t^6y^4 -$
$663552\,t^4y^6 - 89579520\,t^4y^4{a_1}^2 - 17915904\,t^4y^4{b_1}^2 + 19440\,t^2y^4{a_1}^4 + 23328\,t^2y^4{a_1}^2{b_1}^2 + 3888\,t^2y^4{b_1}^4 +$
$46656\,t^3y^5b_1 + 71663616\,t^3y^3{a_1}^2b_1 - 23887872\,t^3y^3{b_1}^3 - 322486272\,t^4y^4 - 20639121408\,t^4y^2{a_1}^2 +$
$20639121408\,t^4y^2{b_1}^2 + 34992\,t^2y^4{a_1}^2 + 34992\,t^2y^4{b_1}^2 + 8957952\,t^2y^2{a_1}^4 - 8957952\,t^2y^2{b_1}^4 - 5944066965504\,t^5yb_1 -$
$107495424\,t^3y^3b_1 + 15479341056\,t^3y{a_1}^2b_1 + 5159780352\,t^3y{b_1}^3 - 1119744\,ty{a_1}^4b_1 - 2239488\,ty{a_1}^2{b_1}^3 -$
$1119744\,ty{b_1}^5 + 1141260857376768\,t^6 + 165112971264\,t^4y^2 - 3715041853440\,t^4{a_1}^2 - 743008370688\,t^4{b_1}^2 +$
$69984\,t^2y^4 + 22394880\,t^2y^2{a_1}^2 - 49268736\,t^2y^2{b_1}^2 + 1612431360\,t^2{a_1}^4 + 1934917632\,t^2{a_1}^2{b_1}^2 +$
$322486272\,t^2{b_1}^4 - (2985984\,y^2 + 859963392)tx^5 - 46656\,{a_1}^6 - 139968\,{a_1}^4{b_1}^2 - 139968\,{a_1}^2{b_1}^4 -$
$46656\,{b_1}^6 + 45148078080\,t^3yb_1 + 1679616\,ty{a_1}^2b_1 - 11757312\,ty{b_1}^3 + (2985984\,ty^2 + 859963392\,t)x^5 +$
$6439405879296\,t^4 - 71663616\,t^2y^2 - 2257403904\,t^2{a_1}^2 + 8062156800\,t^2{b_1}^2 + (622080\,t^2y^4 + 286654464\,t^2y^2 -$
$17915904\,tyb_1 + 51597803520\,t^2 - 11197440\,{a_1}^2 - 2239488\,{b_1}^2 - 4478976)x^4 - (-17915904\,tyb_1 +$
$(622080\,y^4 + 286654464\,y^2 + 51597803520)t^2 - 11197440\,{a_1}^2 - 2239488\,{b_1}^2 - 4478976)x^4 - 279936\,{a_1}^4 -$
$839808\,{a_1}^2{b_1}^2 - 559872\,{b_1}^4 - 11757312\,tyb_1 + (69120\,t^3y^6 + 35831808\,t^3y^4 - 5971968\,t^2y^3b_1 + 10319560704\,t^3y^2 -$
$3732480\,ty^2{a_1}^2 - 746496\,ty^2{b_1}^2 - 1719926784\,t^2yb_1 + 1651129712640\,t^3 - 1492992\,ty^2 - 1074954240\,t{a_1}^2 -$
$214990848\,t{b_1}^2 + 143327232\,t)x^3 - ((69120\,y^6 + 35831808\,y^4 + 10319560704\,y^2 + 1651129712640)t^3 +$
$(-5971968\,y^3b_1 - 1719926784\,yb_1)t^2 + (-3732480\,y^2{a_1}^2 - 746496\,y^2{b_1}^2 - 1492992\,y^2 - 1074954240\,{a_1}^2 -$
$214990848\,{b_1}^2 + 143327232)t)x^3 + 29883727872\,t^2 - ((4320\,y^8 + 1990656\,y^6 + 716636160\,y^4 + 165112971264\,y^2 +$
$29720334827520)t^4 + 2799360\,{a_1}^4 + 3359232\,{a_1}^2{b_1}^2 + 559872\,{b_1}^4 + (-746496\,y^5b_1 - 286654464\,y^3b_1 -$
$61917364224\,yb_1)t^3 + (-466560\,y^4{a_1}^2 - 93312\,y^4{b_1}^2 - 186624\,y^4 - 214990848\,y^2{a_1}^2 - 71663616\,y^2 -$
$38698352640\,{a_1}^2 - 7739670528\,{b_1}^2 + 25798901760)t^2 + 5038848\,{a_1}^2 + 5038848\,{b_1}^2 + (26873856\,y{a_1}^2b_1 +$
$8957952\,y{b_1}^3 + 6718464\,yb_1)t + 10077696)x^2 + (4320\,t^4y^8 + 1990656\,t^4y^6 - 746496\,t^3y^5b_1 + 716636160\,t^4y^4 -$
$466560\,t^2y^4{a_1}^2 - 93312\,t^2y^4{b_1}^2 - 286654464\,t^3y^3b_1 + 165112971264\,t^4y^2 - 186624\,t^2y^4 - 214990848\,t^2y^2{a_1}^2 -$
$61917364224\,t^3yb_1 + 26873856\,ty{a_1}^2b_1 + 8957952\,ty{b_1}^3 + 29720334827520\,t^4 - 71663616\,t^2y^2 - 38698352640\,t^2{a_1}^2 -$
$7739670528\,t^2{b_1}^2 + 2799360\,{a_1}^4 + 3359232\,{a_1}^2{b_1}^2 + 559872\,{b_1}^4 + 6718464\,tyb_1 + 25798901760\,t^2 +$
$5038848\,{a_1}^2 + 5038848\,{b_1}^2 + 10077696)x^2 - 944784\,{a_1}^2 - 944784\,{b_1}^2 - ((144\,y^{10} + 41472\,y^8 + 23887872\,y^6 +$
$6879707136\,y^4 + 990677827584\,y^2 + 285315214344192)t^5 + (-41472\,y^7b_1 - 11943936\,y^5b_1 - 3439853568\,y^3b_1 -$
$990677827584\,yb_1)t^4 + (-25920\,y^6{a_1}^2 - 5184\,y^6{b_1}^2 - 10368\,y^6 - 13436928\,y^4{a_1}^2 + 4478976\,y^4{b_1}^2 -$
$14929920\,y^4 - 3869835264\,y^2{a_1}^2 + 1289945088\,y^2{b_1}^2 + 6019743744\,y^2 - 619173642240\,{a_1}^2 - 123834728448\,{b_1}^2 +$
$743008370688)t^3 + (4478976\,y^3{a_1}^2b_1 + 1492992\,y^3{b_1}^3 + 1119744\,y^3b_1 + 1289945088\,y{a_1}^2b_1 + 429981696\,y{b_1}^3 +$
$2042413056\,yb_1)t^2 + (466560\,y^2{a_1}^4 + 559872\,y^2{a_1}^2{b_1}^2 + 93312\,y^2{b_1}^4 + 839808\,y^2{a_1}^2 + 839808\,y^2{b_1}^2 +$
$134369280\,{a_1}^4 + 161243136\,{a_1}^2{b_1}^2 + 26873856\,{b_1}^4 + 1679616\,y^2 + 26873856\,{a_1}^2 + 456855552\,{b_1}^2 +$
$913711104)t)(x) + (144\,t^5y^{10} + 41472\,t^5y^8 - 41472\,t^4y^7b_1 + 23887872\,t^5y^6 - 25920\,t^3y^6{a_1}^2 - 5184\,t^3y^6{b_1}^2 -$
$11943936\,t^4y^5b_1 + 6879707136\,t^5y^4 - 10368\,t^3y^6 - 13436928\,t^3y^4{a_1}^2 + 4478976\,t^3y^4{b_1}^2 - 3439853568\,t^4y^3b_1 +$

$$4478976\, t^2 y^3 {a_1}^2 b_1 + 1492992\, t^2 y^3 {b_1}^3 + 990677827584\, t^5 y^2 - 14929920\, t^3 y^4 - 3869835264\, t^3 y^2 {a_1}^2 + 1289945088\, t^3 y^2 {b_1}^2 + 466560\, t y^2 {a_1}^4 + 559872\, t y^2 {a_1}^2 {b_1}^2 + 93312\, t y^2 {b_1}^4 - 990677827584\, t^4 y b_1 + 1119744\, t^2 y^3 b_1 + 1289945088\, t^2 y {a_1}^2 b_1 + 429981696\, t^2 y {b_1}^3 + 285315214344192\, t^5 + 6019743744\, t^3 y^2 - 619173642240\, t^3 {a_1}^2 - 123834728448\, t^3 {b_1}^2 + 839808\, t y^2 {a_1}^2 + 839808\, t y^2 {b_1}^2 + 134369280\, t {a_1}^4 + 161243136\, t {a_1}^2 {b_1}^2 + 26873856\, t {b_1}^4 + 2042413056\, t^2 y b_1 + 743008370688\, t^3 + 1679616\, t y^2 + 26873856\, t {a_1}^2 + 456855552\, t {b_1}^2 + 913711104\, t)(x) - 839808$$

Proof it is sufficient to replace the expression of the solution given by (5) in (1) and check that the relation is verified.

□

The modulus of the solution for different values of t is represented in the plane (x, y). It can be seen that when the value of t increases, the maximum of the modulus of the solution decreases rapidly. We observe also the formation of horseshoes waves when time grows. The horseshoe multisoliton solutions correspond very well to real waves observed in thin films of shallow water. Waves with crossed parabolic profiles are also observed.

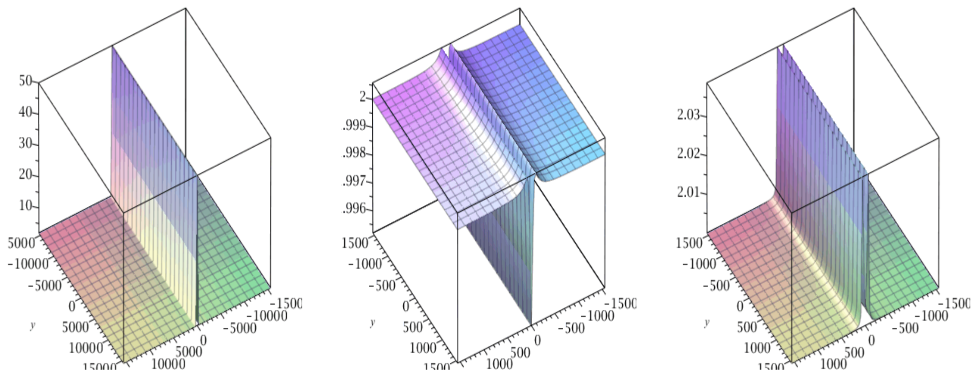

Figure 2. Solution of order 2 to (1) for $t = 0$, on the left $a_1 = 0$, $b_1 = 0,$; in the center $a_1 = 100$, $b_1 = 0,$; on the right $a_1 = 0$, $b_1 = 100$.

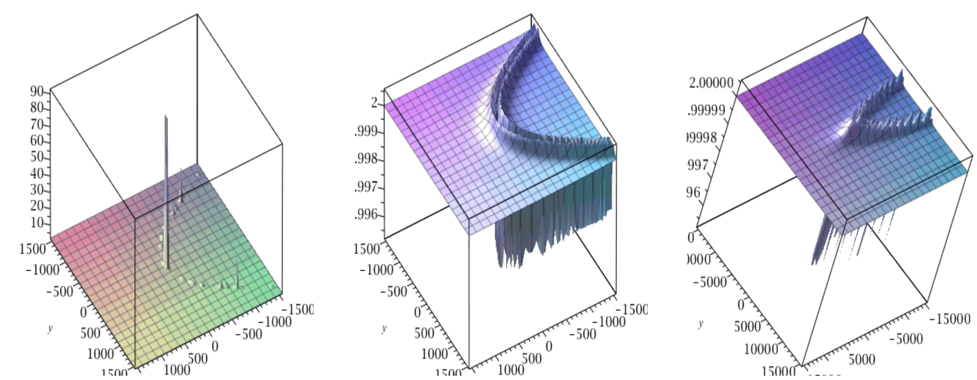

Figure 3. Solution of order 2 to (1) for $t = 0, 01$, on the left $a_1 = 0$, $b_1 = 0$; in the center $a_1 = 10^2$, $b_1 = 0$; on the right $a_1 = 10^3$, $b_1 = 0$.

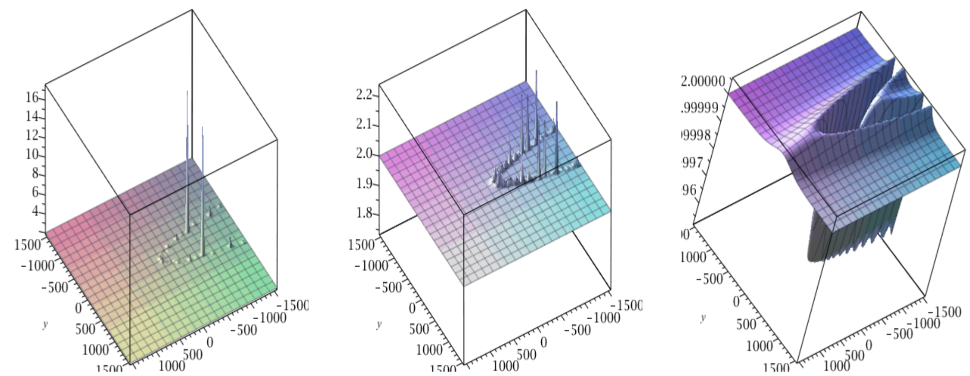

Figure 4. Solution of order 2 to (1) for $t = 0, 1$; on the left $a_1 = 0$, $b_1 = 0$; in the center $a_1 = 10$, $b_1 = 0$; on the right $a_1 = 10^3$, $b_1 = 0$.

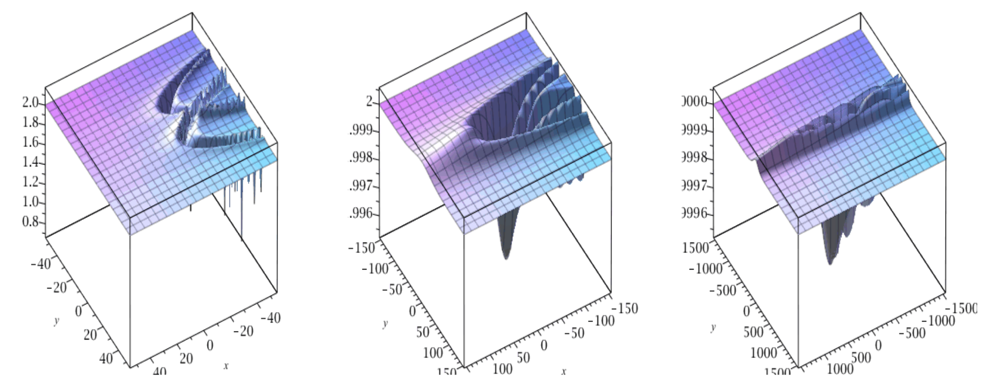

Figure 5. Solution of order 2 to (1) for $t = 1$; on the left $a_1 = 0$, $b_1 = 0$; in the center $a_1 = 10^2$, $b_1 = 0$; on the right $a_1 = 10^3$, $b_1 = 0$.

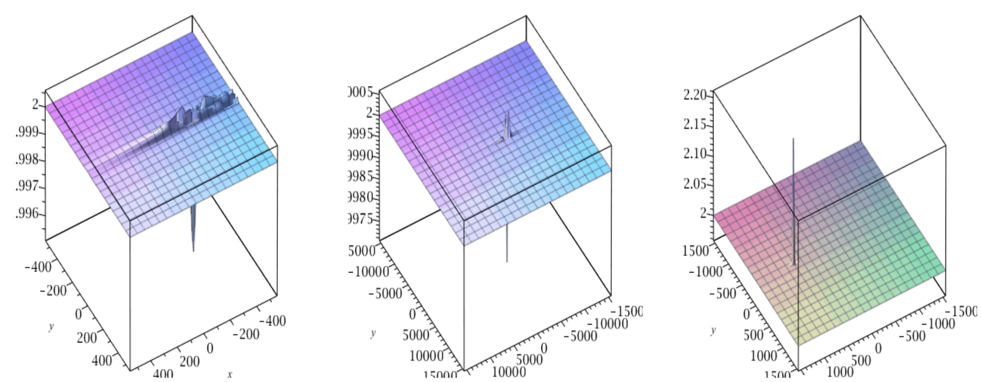

Figure 6. Solution of order 2 to (1); on the left for $t = 10$, $a_1 = 10^2$, $b_1 = 0$; in the center for $t = 100$, $a_1 = 10^3$, $b_1 = 0$; on the right for $t = 100$, $a_1 = 10^3$, $b_1 = 10^3$.

It is remarkable to note that coefficients of maximum degree in x, y, t are respectively the same ones, which explains the asymptotic behavior of solutions v : $\lim_{t\to\infty} v(x,y,t) = 2$, $\lim_{x\to\pm\infty} v(x,y,t) = 2$, $\lim_{y\to\pm\infty} v(x,y,t) = 2$.

4 Conclusion

Rational solutions to the Johnson equation of order 1 and 2 have been constructed here.
The following asymptotic behavior has been highlighted : $\lim_{t\to\infty} v(x,y,t) = 2$, $\lim_{x\to\pm\infty} v(x,y,t) = 2$, $\lim_{y\to\pm\infty} v(x,y,t) = 2$.
In a forthcoming publication we will to give the general method to construct rational solutions to the Johnson equation at order N.
We will show that these solutions can be written as a ratio of two polynomials of degrees $2N(N+1)$ in x, and t, $4N(N+1)$ in y, depending on $2N-2$ parameters.
Moreover we will give a representation of these solutions in terms of Fredholm determinants, namely for every integer N these solutions can be written as the ratio of Fredholm determinants of order $2N$ depending on $2N-1$ parameters.
We will construct also these solutions in terms of of wronskians, i.e., they can be expressed as a quotient of wronskians of order $2N$ depending on $2N-1$ parameters.
We will present a proof of these present results and show the link with previous papers [14]-[16]-[32].
The method described in the present paper provides a powerful tool to get explicit solutions to the Johnson equation and to understand the behavior of rogue waves in the case of this equation.

References

[1] R.E. Johnson, "Water waves and Kortewegde Vries equations", J. Fluid Mech., V. 97, N. 4, 1980, pp 701719

[2] R.E. Johnson, "A Modern Introduction to the Mathematical Theory of Water Waves", Cambridge University Press, Cambridge, 1997

[3] M. J. Ablowitz, "Nonlinear Dispersive Waves : Asymptotic Analysis and Solitons", Cambridge University Press, Cambridge, 2011

[4] V.D. Lipovskii, "On the nonlinear internal wave theory in fluid of finite depth", Izv. Akad. Nauka., V. 21, N. 8, 1985, pp 864871

[5] B.B. Kadomtsev, V.I. Petviashvili, "On the stability of solitary waves in weakly dispersing media", Sov. Phys. Dokl., V. 15, N. 6, 1970, pp 539-541

[6] V.I. Golinko, V.S. Dryuma, Yu.A. Stepanyants, "Nonlinear quasicylindrical waves: Exact solutions of the cylindrical Kadomtsev- Petviashvili equa-

tion", in Proc. 2nd Int. Workshop on Nonlinear and Turbulent Processes in Physics, Kiev, Harwood Acad., Gordon and Breach, 1984, pp 13531360

[7] C. Klein, V.B. Matveev, A.O. Smirnov, "Cylindrical Kadomtsev-Petviashvili equation: old and new results", Theor. Math. Phys., V. 152, N. 2, 2007, pp 1132-1145

[8] K. R. Khusnutdinova, C. Klein, V.B. Matveev, A.O. Smirnov, "On the integrable elliptic cylindrical K-P equation," Chaos, V. 23, 2013, pp 013126-1-15

[9] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, Max-Planck-Institut für Mathematik, MPI 02-31, V. **161**, 2002

[10] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, Lett. Math. Phys., V. **68**, 77-90, 2004

[11] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, Lett. Math. Phys., V. **89**, 1-12, 2009

[12] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, J. Phys A : Math. Theor., V. **42**, 404409-1-16, 2009

[13] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, Eur. Phys. J. Spe. Top., V. **185**, 247-258, 2010

[14] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, J. Phys. A : Meth. Theor., V. **44**, 435204-1-15, 2011

[15] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, J. Math. Sciences : Adv. Appl., V. **13**, N. 2, 71-153, 2012

[16] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, J. Math. Phys., V. **54**, 013504-1-32, 2013

[17] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, J. Mod. Phys., V. **4**, N. 4, 246-266, 2013

[18] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, J. Math., V. **2013**, ID 645752, 1-10, 2013

[19] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, J. Math., V. **2013**, 1-111, 2013

[20] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, J. Theor. Appl. Phys., V. **7**, N. **45**, 1-6, 2013

[21] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, J. Math. Phys., V. **54**, 073519-1-22, 2013

[22] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, Phys. Rev. E, V. **88**, 042903-1-9, 2013

[23] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, Phys. Scripta, V. **89**, 015004-1-7, 2014

[24] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, Commun. Theor. Phys., V. **61**, 365-369, 2014

[25] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, J. Of Phys. : Conf. Ser., V. **482**, 012016-1-7, 2014

[26] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, Int. J. Mod. Phys. C, V. **26**, N. 2, 1550016-1-14, 2014

[27] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, J. Math. Phys., V. **5**, 093506-1-12, 2014

[28] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, J. Phys. : Conf. Ser., V. **574**, 012031-1-5, 2015

[29] P. Gaillard, Tenth Peregrine breather solution of the NLS, Ann. Phys., V. **355**, 293-298, 2015

[30] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation Phys. Lett. A, V. **379**, 1309-1313, 2015

[31] P. Gaillard, Other 2N-2 parameters solutions to the NLS equation and 2N+1 highest amplitude of the modulus of the N-th order AP breather, J. Phys. A: Math. Theor., V. **48**, 145203-1-23, 2015

[32] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, Adv. Res., V. **4**, N. **5**, 346-364, 2015

[33] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, Jour. Phys. : Conf. Ser., V. **633**, 012106-1-6, 2016

[34] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, Journal of Theoretical and Applied Physics, V. **10**,1-7, 2016

[35] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, Commun. Theor. Phys, V. **65**, N. **2**, 136-144, 2016

[36] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, Annals Of Physics, V. **367**, 1-5, 2016

[37] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, Adv. Res., V. **10**, 83-89, 2016

[38] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, Theor. And Math. Phys., V. **189**, N. **1**, 1440-1449, 2016

[39] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, Jour. of Math. Phys., V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016

[40] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, Jour. Of Bas. And Appl. Res. Int., V. **21**, N. 3, 130-139, 2017

[41] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N2$ parameters, Int. Jour. of Appl. Sci. And Math., V. **4**, N. 3, 60-70, 2017

[42] P. Gaillard, Families of Rational Solutions of Order 5 to the KPI Equation depending on 8 Parameters, New Hor. in Math. Phys., V. 1, N. 1, 26-31, 2017

[43] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, Jour. Of Bas. And Appl. Res. Int., V. **21**, N. 2, 92-98, 2017

[44] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, Int. Jour. of Adv. Res. in Phys. Sci., V. **4**, N. 11, 24-30, 2017