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# Decidability of the Riemann Hypothesis

Frederick Ira Moxley III

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**Abstract** The Hamiltonian of a quantum mechanical system has an affiliated spectrum, and in order for this spectrum to be locally observable, the Hamiltonian should be Hermitian. Non-Hermitian Hamiltonians can be observed non-locally via parity, i.e. by taking the expectation value of the Wigner distribution evaluated at the origin in phase space. Studies such as these quantum nonlocality analogies have led to the Bender-Brody-Müller (BBM) conjecture, which involves a non-Hermitian Hamiltonian eigenequation whose eigenvalues are the nontrivial zeros of the Riemann zeta function. Herein it is shown from symmetrization of the BBM Hamiltonian that the eigenvalues are not locally observable, i.e. *the analytic continuation of the Riemann zeta function is not an analytically computable function at  $\sigma = 1/2$* . In the present case, the Riemann zeta function is analogous to chaotic quantum systems, as the harmonic oscillator is for integrable quantum systems. As such, herein we perform a symmetrization procedure of the BBM Hamiltonian to obtain a Hermitian Hamiltonian using a similarity transformation, and provide a trivial analytical expression for the eigenvalues of the results using Green's functions. A nontrivial expression for the eigensolution of the eigenequation is also obtained. A Gelfand triplet is then used to ensure that the eigensolution is well defined. The holomorphicity of the resulting eigenvalue spectrum is demonstrated, and it is shown that that the expectation value of the Hamiltonian operator is periodically zero such that the nontrivial zeros of the Riemann zeta function are not observable, i.e., the Riemann Hypothesis is not decidable. Moreover, a second quantization of the resulting Schrödinger equation is performed, and a convergent solution for the nontrivial zeros of the analytic continuation of the Riemann zeta function is obtained. Finally, from the holomorphicity of the eigensolution it is shown that the real part of every nontrivial zero of the Riemann zeta function exists at  $\sigma = 1/2$ , and a general solution is obtained by performing an invariant similarity transformation.

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Hearne Institute for Theoretical Physics, Department of Physics & Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA

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## 1 Introduction

The unification of number theory with quantum mechanics has been the subject of many research investigations [1–5]. It has been proven that an infinitude of prime numbers exist [6]. In Ref. [7,8], it was shown that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeros of the Riemann zeta function. If the Riemann Hypothesis is correct [9], the zeros of the Riemann zeta function can be considered as the spectrum of an operator  $\hat{R} = \hat{I}/2 + i\hat{H}$ , where  $\hat{H}$  is a self-adjoint Hamiltonian operator [5,10], and  $\hat{I}$  is identity. Hilbert proposed the Riemann Hypothesis as the eighth problem on a list of significant mathematics problems [11]. Although the BBM Hamiltonian is pseudo-Hermitian [12], it is consistent with the Berry-Keating conjecture [13–15], which states that when  $\hat{x}$  and  $\hat{p}$  commute, the Hamiltonian reduces to the classical  $H = 2xp$ . Berry, Keating, and Connes proposed a classical Hamiltonian in order to map the Riemann zeros to a Hamiltonian spectrum. Recently, the classical Berry-Keating Hamiltonians were quantized, and were shown to smoothly approximate the Riemann zeros [16,17]. This reformulation was found to be physically equivalent to the Dirac equation in Rindler spacetime [18]. Herein, the eigenvalues of the BBM Hamiltonian are taken to be the imaginary parts of the nontrivial zeroes of the analytical continuation of the Riemann zeta function

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad (1)$$

where the complex number  $s = \sigma + it = |s|\exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$ , and  $\Re(s) > 0$ . The idea that the imaginary parts of the nontrivial zeros of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [19]. Hilbert and Pólya asserted that the nontrivial zeros of Eq. (1) can be considered as the spectrum of a self-adjoint operator in a suitable Hilbert space. The Hilbert-Pólya conjecture has also found applications in quantum field theories [20]. The Riemann Hypothesis states that the nontrivial zeros of Eq. (1) on  $0 \leq \sigma < 1$  have real part equal to  $1/2$  [9,21]. In Ref. [22], Hardy proved that infinitely many zeros are located at  $\sigma = 1/2$ . According to the Prime Number Theorem [23,24], no zeros of Eq. (1) can exist at  $\sigma = 1$ . The paper is organized as follows: In Sec. 2 we present a Schrödinger equation whose eigenvalues are identical to those of the BBM Hamiltonian, i.e. the nontrivial zeros of the Riemann zeta function, and evaluate the convergence of the expression by studying the orthonormalization constraint on the density. A self-adjoint Hamiltonian is derived from the BBM Hamiltonian using a similarity transformation [25,26], and a second quantization of the resulting Schrödinger equation is then performed to obtain the equations of motion. Moreover, we study the holomorphic eigenvalues of the Riemann zeta

function by taking the expectation values of the resulting Schrödinger equation. We show that at  $\sigma = 1/2$ , the real part of every nontrivial zero of the analytic continuation of the Riemann zeta function is not decidable. Finally we obtain a general solution to the Riemann zeta Schrödinger equation by performing a similarity transformation in Sec. 3, and make concluding remarks in Sec. 4.

### 1.1 Preliminaries

**Definition 11** The complex valued function (eigenstate)  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) : \mathbb{X} \rightarrow \mathbb{C}$  is measurable if  $\mathbb{E}$  is a measurable subset of the measure space  $\mathbb{X}$  and for each real number  $r$ , the sets  $\{x \in \mathbb{E} : \phi_\sigma(x) > r\}$  and  $\{x \in \mathbb{E} : \phi_t(x) > r\}$  are measurable for  $\sigma, t \in \mathbb{R}$  [27].

**Definition 12** Let  $\phi_s$  be a complex-valued eigenstate on a measure space  $\mathbb{X}$ , and  $\phi_s = \phi_\sigma + i\phi_t$ , with  $\phi_\sigma$  and  $\phi_t$  real. Therefore,  $\phi_s$  is measurable iff  $\phi_\sigma$  and  $\phi_t$  are measurable (Ibid.).

**Definition 13** Suppose  $\mu$  is a measure on the measure space  $\mathbb{X}$ , and  $\mathbb{E}$  is a measurable subset of the measure space  $\mathbb{X}$ , and  $\phi_s$  is a complex-valued eigenstate on  $\mathbb{X}$ . It follows that  $\phi_s \in (\mathcal{H} = \mathcal{L}(\mu))$  on  $\mathbb{E}$ , and  $\phi_s$  is complex square-integrable, if  $\phi_s$  is measurable and (Ibid.)

$$\int_{\mathbb{E}} |\phi_s| d\mu < +\infty. \quad (2)$$

**Definition 14** The complex valued function (eigenstate)  $\phi_s = \phi_\sigma + i\phi_t$  defined on the measurable subset  $\mathbb{E}$  is said to be integrable if  $\phi_\sigma$  and  $\phi_t$  are integrable for  $\sigma, t \in \mathbb{R}$ , where  $\mu$  is a measure on the measure space  $\mathbb{X}$ . The Lebesgue integral of  $\phi_s$  is defined by (Ibid.)

$$\int_{\mathbb{E}} \phi_s d\mu = \int_{\mathbb{E}} \phi_\sigma d\mu + i \int_{\mathbb{E}} \phi_t d\mu. \quad (3)$$

**Definition 15** Let  $\mathbb{X}$  be a measure space, and  $\mathbb{E}$  be a measurable subset of  $\mathbb{X}$ . Given the complex eigenstate  $\phi_s$ , then  $\phi_s \in (\mathcal{H} = \mathcal{L}^2(\mu))$  on  $\mathbb{E}$  if  $\phi_s$  is Lebesgue measurable and if

$$\int_{\mathbb{E}} |\phi_s|^2 d\mu < +\infty, \quad (4)$$

such that  $\phi_s$  is square-integrable. For  $\phi_s \in (\mathcal{H} = \mathcal{L}^2(\mu))$  we define the  $\mathcal{L}^2$ -norm of  $\phi_s$  as

$$\|\phi_s\|_2 = \left( \int_{\mathbb{E}} |\phi_s|^2 d\mu \right)^{1/2}, \quad (5)$$

where  $\mu$  is the measure on the measure space  $\mathbb{X}$  (Ibid.).

**Definition 16** Let  $\mathbb{X}$  be a measure space, and  $\mathbb{E}$  be a measurable subset of  $\mathbb{X}$ . Given the complex eigenstate  $\phi_s$ , then  $\phi_s \in (\mathcal{H} = \mathcal{L}^p(\mu))$  on  $\mathbb{E}$  if  $\phi_s$  is Lebesgue measurable and if

$$\int_{\mathbb{E}} |\phi_s|^p d\mu < +\infty, \quad (6)$$

such that  $\phi_s$  is  $p$ -integrable. For  $\phi_s \in (\mathcal{H} = \mathcal{L}^p(\mu))$  we define the  $\mathcal{L}^p$ -norm of  $\phi_s$  as

$$\|\phi_s\|_p = \left( \int_{\mathbb{E}} |\phi_s|^p d\mu \right)^{1/p}, \quad (7)$$

where  $\mu$  is the measure on the measure space  $\mathbb{X}$  (Ibid.).

**Definition 17** A rigged Hilbert space (i.e., a Gelfand triplet [28]) is a triplet  $(\Phi, \mathcal{H}, \Phi^*)$ , where  $\Phi$  is a dense subspace of  $\mathcal{H}$  and  $\Phi^*$  is its continuous dual space.

**Definition 18** In the theory of computation, an observable is called decidable, or effective, if and only if its behavior is given by a computable function [29].

**Definition 19** Observables, e.g.  $\hat{x}$  and  $\hat{p}$  of a system, are represented in quantum mechanics by self-adjoint operators (which we will not notationally distinguish from the observables themselves). If there exists an observable  $C$  such that  $C = \alpha\hat{x} + \beta\hat{p}$ , and if  $\langle\hat{x}\rangle$  and  $\langle\hat{p}\rangle$  denote the expectation values of  $\hat{x}$  and  $\hat{p}$  respectively, then  $\langle C \rangle = \alpha\langle\hat{x}\rangle + \beta\langle\hat{p}\rangle$  is the expectation value of  $C$ . According to Heisenberg's uncertainty principle, if the observables corresponding to two quantities  $\hat{x}$  and  $\hat{p}$  do not commute, i.e.  $[\hat{x}, \hat{p}] \neq 0$ , both quantities cannot simultaneously be measured to arbitrary accuracy [30].

**Definition 110** A linear operator  $\hat{H}$  is Hermitian (self-adjoint) if it is defined on a linear everywhere-dense set  $\mathcal{D}(\hat{H})$  in a Hilbert space  $\mathcal{H}$  coinciding with its adjoint operator  $\hat{H}^\dagger$ , that is, such that  $\mathcal{D}(\hat{H}) = \mathcal{D}(\hat{H}^\dagger)$  and

$$\langle\hat{H}x, y\rangle = \langle x, \hat{H}y\rangle \quad (8)$$

for every  $x, y \in \mathcal{D}(\hat{H})$  [31–33].

## 2 Riemann Zeta Schrödinger Equation

We consider the eigenvalues of the Hamiltonian

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}), \quad (9)$$

where  $\hat{p} = -i\hbar\partial_x$ ,  $\hbar = 1$ , and  $\hat{x} = x$ . For the Hamiltonian operator as given by Eq. (9), the Hilbert space is  $\mathcal{H} = \mathcal{L}^{p=2}[1, \infty)$ . In Refs. [7, 8], it is conjectured

that if the Riemann Hypothesis is correct, the eigenvalues of Eq. (9) are non-degenerate. Next, we let  $|\Psi_s(x)\rangle$  be an eigenstate of Eq. (9) with an eigenvalue  $t = i(2s - 1)$ , such that

$$\hat{H} |\Psi_s(x)\rangle = t |\Psi_s(x)\rangle, \quad (10)$$

and  $x \in \mathbb{R}^+$ ,  $s \in \mathbb{C}$ . The system is described by a Hilbert space

$$\mathcal{H} = \bigotimes_{j=1}^n \mathcal{H}_j, \quad (11)$$

from the tensor product of infinite dimensional Fock spaces  $\mathcal{H}_j$ . These Fock spaces are annihilated, and created, respectively by  $\hat{a}_j$  and  $\hat{a}_j^\dagger$ , where

$$\hat{a}_j = \frac{1}{\sqrt{2}}(x_j + \hbar\partial_{x_j}) \quad (12a)$$

$$\hat{x}_j = (\hat{a}_j + \hat{a}_j^\dagger) \quad (12b)$$

$$\hat{p}_j = (\hat{a}_j - \hat{a}_j^\dagger)/i \quad (12c)$$

for the canonical coordinates  $\hat{x}_j, \hat{p}_j$ . As such, the Bose commutation relations are satisfied

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}. \quad (13)$$

Letting  $\hat{\Phi} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)$  denote the vector of canonical coordinates, we then obtain the canonical commutation relations in symplectic form

$$[\hat{\Phi}_j, \hat{\Phi}_k] = 2i\omega_{jk} = 2i \bigotimes_{j=1}^n \omega, \quad (14)$$

where  $\omega_{jk}$  is an antisymmetric matrix, i.e.,  $\omega = -\omega^T$  [34]. For non-normalized eigenvectors  $|\Psi_s(x)\rangle$  of the quadrature operators  $\{\hat{x}_j\}$

$$\hat{x}_j |\Psi_s(x)\rangle = x_j |\Psi_s(x)\rangle, \quad (15)$$

where  $x \in \mathbb{R}^n$  for  $(j = 1, \dots, n)$ , i.e.  $|\Psi_s(x)\rangle$  is an eigenstate of the operator  $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$  and  $\hat{x}_j$  is multiplication by  $x_j$ . Similarly, for non-normalized eigenvectors  $|\Psi_s(x)\rangle$  of the quadrature operators  $\{\hat{p}_j\}$

$$\hat{p}_j |\Psi_s(x)\rangle = -i\hbar\partial_{x_j} |\Psi_s(x)\rangle, \quad (16)$$

where  $|\Psi_s(x)\rangle$  is an eigenstate of the operator  $\hat{p} = (\hat{p}_1, \dots, \hat{p}_n)$  and  $\hat{p}_j$  is the operation  $-i\hbar\partial_{x_j}$ . Solutions to Eq. (10) are given by the analytic continuation of the Hurwitz zeta function

$$\begin{aligned} |\Psi_s(x)\rangle &= -\zeta(s, x+1) \\ &= -\Gamma(1-s) \frac{1}{2\pi i} \oint_C \frac{z^{s-1} e^{(x+1)z}}{1-e^z} dz, \end{aligned} \quad (17)$$

on the positive half line  $x \in \mathbb{R}^+$  with eigenvalues  $i(2s-1)$ ,  $s \in \mathbb{C}$ ,  $\Re(s) \leq 1$ , the contour  $C$  is a loop around the negative real axis, and  $\Gamma$  is the Euler gamma function for  $\Re(s) > 0$

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx. \quad (18)$$

As  $-\langle \Psi_s(x=1) \rangle$  is  $1 - \zeta(s^*)$ , this implies that  $s$  belongs to the discrete set of nontrivial zeros of the Riemann zeta function when  $s^* = \sigma - it = |s| \exp(-i\theta)$ . As  $-\langle \Psi_s(x=-1) \rangle$  is  $\zeta(s)$ , this implies that  $s$  belongs to the discrete set of nontrivial zeros of the Riemann zeta function when  $s = \sigma + it = |s| \exp(i\theta)$  and  $\sigma = 1/2$ . Herein we demonstrate that at the orthonormalization constraint  $x = \pm 1$ ,  $\sigma$  must always be equal to  $1/2$ , and  $t = 2\pi n$  such that  $\theta$  is periodically equal to zero. However, we are interested in the case when  $x \geq 1$ , so we will focus on the positive-valued orthonormalization  $x = 1$ . From inserting Eq. (10) into Eq. (9), we have the relation

$$\frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) |\Psi_s(x)\rangle = t |\Psi_s(x)\rangle. \quad (19)$$

Given that Eq. (9) is not Hermitian, it is useful to symmetrize the system. This can be accomplished by letting

$$\begin{aligned} |\phi_s(x)\rangle &= [1 - \exp(-\partial_x)] |\Psi_s(x)\rangle, \\ &= \hat{\Delta} |\Psi_s(x)\rangle \\ &= |\Psi_s(x)\rangle - |\Psi_s(x-1)\rangle, \end{aligned} \quad (20)$$

and defining a shift operator

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x). \quad (21)$$

For  $s > 0$  the only singularity of  $\zeta(s, x)$  in the range of  $0 \leq x \leq 1$  is located at  $x = 0$ , behaving as  $x^{-s}$ . More specifically,

$$\zeta(s, x+1) = \zeta(s, x) - \frac{1}{x^s}, \quad (22)$$

with  $\zeta(s, x)$  finite for  $x \geq 1$  [35]. As such, it can be seen from Eq. (20) that the Berry-Keating eigenfunction [13, 14]

$$\begin{aligned} |\phi_s(x)\rangle &= \frac{1}{x^s} \\ &= \exp\left(\ln(x)(-\sigma - it)\right) \\ &= \exp\left(-\sigma \ln(x) - it \ln(x)\right) \\ &= \exp\left(-\sigma \ln(x)\right) \left(\cos(t \ln(x)) - i \sin(t \ln(x))\right) \\ &= x^{-\sigma} \left(\cos(t \ln(x)) - i \sin(t \ln(x))\right). \end{aligned} \quad (23)$$

Furthermore, the distributional orthonormality relation at  $x = 1$  is satisfied such that [37]

$$\langle \phi_s | \phi_{s'} \rangle = \delta_{ss'}. \quad (24)$$

Upon inserting Eq. (20) into Eq. (19) we obtain

$$-i[x\partial_x + \partial_x x] |\phi_s(x)\rangle = t |\phi_s(x)\rangle. \quad (25)$$

Let  $\mathcal{H}$  be a Hilbert space, and from Eq. (25) we have the Hamiltonian operator

$$\begin{aligned} \hat{H} &= -i\hbar [x\partial_x + \partial_x x] \\ &= -i\hbar [2x\partial_x + 1], \end{aligned} \quad (26)$$

for  $x \in \mathbb{R}$  acting in  $\mathcal{H}$ , such that

$$\langle \hat{H}x, y \rangle = \langle x, \hat{H}y \rangle \quad \forall x, y \in \mathcal{D}(\hat{H}). \quad (27)$$

For the Hamiltonian operator as given by Eq. (26), the Hilbert space is  $\mathcal{H} = \mathcal{L}^{p=2}[1, \infty)$  [38, 39, 37]. Restricting  $x \in \mathbb{R}^+$ , Eq. (26) is then written

$$\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}, \quad (28)$$

where  $s \in \mathbb{C}$ , and  $x \in \mathbb{R}^+$ . For the Hamiltonian operator as given by Eq. (28), the Hilbert space is  $\mathcal{H} = \mathcal{L}^{p=2}(-\infty, -1] \cup [1, \infty)$ . We then impose on Eq. (28) the following minimal requirements, such that its domain is not too artificially restricted.

- i  $\hat{H}$  is a symmetric (Hermitian) linear operator;
- ii  $\hat{H}$  can be applied on all functions of the form

$$g(x, s) = P(x, s) \exp\left(-\frac{x^2}{2}\right), \quad (29)$$

where  $P$  is a polynomial of  $x$  and  $s$ . Here, it should be pointed out that  $\hat{H} = \hat{T} + \hat{V}$ , and from Eq. (26), it can be seen that  $\hat{T} = -2i\hbar x\partial_x$ ,  $\hat{V} = -i\hbar$ . From (ii),  $\hat{V}g(x, s)$  must belong to the Hilbert space  $\mathcal{H} = \mathcal{L}^2$  defined over the space  $x \geq 1$ . This is guaranteed as  $|-i\hbar| \leq \hbar$  where  $\hbar$  is the reduced Planck constant or Dirac constant, (Planck's constant multiplied by an imaginary number is strictly bounded, i.e. strictly less than infinity). The domain  $\mathcal{D}_{\hat{V}}$  of the potential energy  $\hat{V}$  consists of all  $\phi \in \mathcal{H}$  for which  $\hat{V}\phi \in \mathcal{H}$ . As such,  $\hat{V}$  is self-adjoint. It is not necessary to specify the domain of Eq. (28), as it is only necessary to admit that Eq. (28) is defined on a certain  $\mathcal{D}_{\hat{H}}$  such that (i) and (ii) are satisfied. If we denote by  $\mathcal{D}_1$  the set of all functions in Eq. (29), then (ii) implies that  $\mathcal{D}_{\hat{H}} \supseteq \mathcal{D}_1$ . By letting  $\hat{H}_1$  be the contraction of  $\hat{H}$  with domain  $\mathcal{D}_1$ , i.e.,  $\hat{H}$  is an extension of  $\hat{H}_1$ , and letting  $\tilde{H}_1$  be the closure of  $\hat{H}_1$ , it can be seen that  $\tilde{H}_1$  is self-adjoint. Since  $\hat{H}$  is symmetric and  $\hat{H} \supseteq \tilde{H}_1$ , i.e.,  $\hat{H}$  is an extension of  $\tilde{H}_1$ , it follows that  $\tilde{H} = \tilde{H}_1$  and  $\hat{H}$  is essentially self-adjoint, where  $\tilde{H}$  is the unique self-adjoint extension [40]. Other



than eigenfunctions  $\phi_s(x)$  in configuration space as seen in Eq. (23), it is useful to represent eigenfunctions in momentum space  $\phi_s(p)$ . The transformation between configuration space eigenfunctions and momentum space eigenfunctions can be obtained via Plancherel transforms [41], where the one-to-one correspondence  $\phi_s(x) \rightleftharpoons \phi_s(p)$  is linear and isometric.

## 2.1 Green's function

In order to obtain eigenstates that are orthonormal when  $x \neq 1$ , as seen in Eq. (24), we begin by writing Eq. (28) as the eigenvalue equation

$$-2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x) = t\phi_s(x). \quad (30)$$

*Remark 1* Solutions to Eq. (30) are symmetric about the origin, i.e.,  $x \in (-\infty, -1] \cup [1, \infty)$ , and subject to the singularity at  $\phi_s(x=0) = 0$  [35].

Dividing by  $-2i\hbar$  on both sides and rearranging the terms, we obtain

$$\phi_s' + \frac{1}{x} \frac{t}{2i\hbar} \phi_s = -\frac{1}{2x} \phi_s. \quad (31)$$

This can be written as

$$\phi_s' + k^2 = Q, \quad (32)$$

where

$$k \equiv \sqrt{\frac{t}{2i\hbar x}}, \quad (33)$$

and

$$Q \equiv -\frac{1}{2x} \phi_s. \quad (34)$$

Therefore, we can express Eq. (30) as

$$(\partial_x + k^2)\phi_s = Q. \quad (35)$$

In order to solve an inhomogeneous differential equation such as Eq. (35), we can find a Green's function that uses a delta function source, viz.,

$$(\partial_x + k^2)G(x) = \delta(x), \quad (36)$$

where the delta potential is given by [36]

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x) dx = 1. \quad (37)$$

It then follows from Eq. (36) that we can express  $\phi_s$  as an integral to obtain  $Q(x)$ , i.e.,

$$\phi_s(x) = \int_{\mathbb{R}^n} G(x - x_0)Q(x_0)d^n x_0, \quad (38)$$

and it must satisfy

$$\begin{aligned} (\partial_x + k^2)\phi_s(x) &= \int_{\mathbb{R}^n} [(\partial_x + k^2)G(x - x_0)]Q(x_0)d^n x_0 \\ &= \int_{\mathbb{R}^n} \delta(x - x_0)Q(x_0)d^n x_0 = Q(x). \end{aligned} \quad (39)$$

In order to obtain the Green's function  $G(x)$  such that a solution to Eq. (36) can be obtained, we take the Fourier transform which turns the differential equation into an algebraic one, like

$$G(x) = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)g(\omega)d\omega, \quad (40)$$

where  $g(\omega)$  is the projection, and  $\exp(i\omega x)$  is the complete basis set. Upon inserting Eq. (40) into Eq. (36), we obtain

$$(\partial_x + k^2)G(x) = \frac{1}{\sqrt{2\pi}} \int g(\omega)(\partial_x + k^2) \exp(i\omega x)d\omega = \delta(x). \quad (41)$$

However, since

$$\partial_x \exp(i\omega x) = i\omega \exp(i\omega x), \quad (42)$$

and

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)d\omega, \quad (43)$$

Eq. (36) can be expressed as

$$\frac{1}{\sqrt{2\pi}} \int (i\omega + k^2) \exp(i\omega x)g(\omega)d\omega = \frac{1}{\sqrt{2\pi}} \int \exp(i\omega x)d\omega, \quad (44)$$

where

$$g(\omega) = \frac{1}{\sqrt{2\pi}(i\omega + k^2)}. \quad (45)$$

Hence we have poles at

$$k = \pm\sqrt{i\omega}. \quad (46)$$

Now consider the contour integral

$$\frac{1}{\sqrt{2\pi}} \int_C f(z)dz = \frac{1}{\sqrt{2\pi}} \int_C \frac{\exp(izx)}{(iz + k^2)} dz. \quad (47)$$

Since  $\exp(izx)$  is an entire function, Eq. (47) has singularities only at the poles, as given in Eq. (46), i.e.,  $z = ik^2$ . As  $f(z)$  is

$$\frac{\exp(izx)}{(iz + k^2)} = \frac{\exp(izx)}{i} \frac{1}{(z - ik^2)}, \quad (48)$$

the residue of  $f(z)$  at  $z = ik^2$  is

$$\text{Res}_{z=ik^2} f(z) = \frac{\exp(-k^2x)}{i}. \quad (49)$$

According to the residue theorem, we then obtain

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_C f(z) dz &= \frac{2\pi i}{\sqrt{2\pi}} \text{Res}_{z=ik^2} f(z) \\ &= \sqrt{2\pi} \exp(-k^2x) = G(x). \end{aligned} \quad (50)$$

Hence, the most general solution to Eq. (36) is

$$\phi_s(x) = \sqrt{2\pi} \int_{\mathbb{R}^n} \exp(-k^2x_0) \left( -\frac{1}{2x_0} \phi_s(x_0) \right) d^n x_0. \quad (51)$$

From Eq. (23) it can be seen that  $\phi_s(x_0) = x_0^{-s}$ . As such,

$$\begin{aligned} \phi_s(x) &= -\sqrt{2\pi} \int_{\mathbb{R}^n} \exp(-k^2x_0) \left( \frac{x_0^{-s-1}}{2} \right) d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \frac{\exp(-k^2x_0)}{x_0^{s+1}} d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \frac{\exp\left(-\frac{tx_0}{2i\hbar x}\right)}{x_0^{s+1}} d^n x_0 \\ &= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \cos\left(\frac{tx_0}{2\hbar x}\right) \frac{1}{x_0^{s+1}} d^n x_0 - i\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \sin\left(\frac{tx_0}{2\hbar x}\right) \frac{1}{x_0^{s+1}} d^n x_0, \end{aligned} \quad (52)$$

Moreover, by using Eq. (23) it can be seen that

$$\begin{aligned} \int_{\mathbb{R}^n} \cos\left(\frac{tx_0}{2\hbar x}\right) \frac{1}{x_0^{s+1}} d^n x_0 &= \int_{\mathbb{R}^n} \cos\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \cos(t \ln(x_0)) d^n x_0 \\ &\quad - i \int_{\mathbb{R}^n} \cos\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \sin(t \ln(x_0)) d^n x_0, \end{aligned} \quad (53)$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} \sin\left(\frac{tx_0}{2\hbar x}\right) \frac{1}{x_0^{s+1}} d^n x_0 &= \int_{\mathbb{R}^n} \sin\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \sin(t \ln(x_0)) d^n x_0 \\ &\quad + i \int_{\mathbb{R}^n} \sin\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \cos(t \ln(x_0)) d^n x_0. \end{aligned} \quad (54)$$

Since  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$ , it can be seen that

$$\begin{aligned}
\phi_\sigma(x) &= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \cos\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \cos\left(t \ln(x_0)\right) d^n x_0 \\
&\quad - \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \sin\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \sin\left(t \ln(x_0)\right) d^n x_0 \\
&= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} x_0^{-\sigma-1} \cos\left(ix_0 k^2 - t \log(x_0)\right) d^n x_0 \\
&= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} x_0^{-\sigma-1} \left[ \cosh\left(k^2 x_0\right) \cos\left(t \log(x_0)\right) \right. \\
&\quad \left. + i \sinh\left(k^2 x_0\right) \sin\left(t \log(x_0)\right) \right] d^n x_0, \tag{55}
\end{aligned}$$

and

$$\begin{aligned}
\phi_t(x) &= \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \cos\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \sin\left(t \ln(x_0)\right) d^n x_0 \\
&\quad - \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} \sin\left(\frac{tx_0}{2\hbar x}\right) \frac{x_0^{-\sigma}}{x_0} \cos\left(t \ln(x_0)\right) d^n x_0 \\
&= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} x_0^{-\sigma-1} \sin\left(ix_0 k^2 - t \log(x_0)\right) d^n x_0 \\
&= -\sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} x_0^{-\sigma-1} \left[ -\cosh\left(k^2 x_0\right) \sin\left(t \log(x_0)\right) \right. \\
&\quad \left. + i \sinh\left(k^2 x_0\right) \cos\left(t \log(x_0)\right) \right] d^n x_0. \tag{56}
\end{aligned}$$

Here, we can use the identities

$$\cos\left(t \log(x_0)\right) = \frac{1}{2} x_0^{-it} + \frac{1}{2} x_0^{it}, \tag{57}$$

and

$$\sin\left(t \log(x_0)\right) = \frac{i}{2} x_0^{-it} - \frac{i}{2} x_0^{it}, \tag{58}$$

to rewrite Eqs. (55)-(56) as

$$\begin{aligned}
\phi_\sigma(x) &= -\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} x_0^{-\sigma-1} \cos\left(t \log(x_0)\right) \exp(-k^2 x_0) dx_0 \\
&= -\frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{-k^2 x_0} (1 + x_0^{2it}) x_0^{-\sigma-it-1} dx_0, \tag{59}
\end{aligned}$$

and

$$\begin{aligned}
\phi_t(x) &= \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} x_0^{-\sigma-1} \sin\left(t \log(x_0)\right) \exp(-k^2 x_0) dx_0 \\
&= -\frac{1}{2} i \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{-k^2 x_0} (-1 + x_0^{2it}) x_0^{-\sigma-it-1} dx_0. \tag{60}
\end{aligned}$$

Taking  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$ , we arrive at the expression using Eq. (33)

$$\begin{aligned}
\phi_s(x) &= \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \left(-e^{-k^2 x_0}\right) x_0^{-\sigma-it-1} dx_0 \\
&= \frac{\sqrt{\frac{\pi}{2}} (k^* - ik) e^{-\frac{1}{2}\pi(t+3i\sigma)} (e^{2\pi t} - e^{2i\pi\sigma}) (-k^4)^{\frac{1}{2}(\sigma+it)} \Gamma(-it - \sigma)}{2k^*} \\
&= \frac{\sqrt{\pi} 2^{-\sigma-it-\frac{3}{2}} e^{-\frac{1}{2}\pi(t+3i\sigma)} (e^{2\pi t} - e^{2i\pi\sigma}) \left(t - x\sqrt{\frac{t^2}{x^2}}\right) \Gamma(-it - \sigma) \left(\frac{t^2}{x^2}\right)^{\frac{1}{2}(\sigma+it)}}{t} \\
&= 0 \quad \forall x \in \mathbb{R}_{\geq 1}^+. \tag{61}
\end{aligned}$$

Hence, the nontrivial zeros of the Riemann zeta function can be considered as the spectrum of an operator  $\hat{R} = \hat{I}/2 + i\hat{H}$ , where  $\hat{H}$  is a self-adjoint Hamiltonian operator [5, 10], and  $\hat{I}$  is identity, such that

$$\begin{aligned}
\langle \hat{R} \rangle &= \hat{I}/2 + i\langle \hat{H} \rangle \\
&= \hat{I}/2 \tag{62}
\end{aligned}$$

and the eigenvalues  $\langle \hat{H} \rangle = t$  are not observable, as seen from Eq. (30).

*Remark 2* In case the reader considers Eq. (61) a trivial solution, from Eq. (31) it can be seen that by taking  $y = \phi_s$ ,

$$y' + \frac{1}{x} \left( \frac{1}{2} + \frac{t}{2i\hbar} \right) y = 0, \tag{63}$$

such that a nontrivial solution is admitted as

$$y = c_1 \frac{1}{x^s}, \tag{64}$$

where

$$\boxed{s = \frac{1}{2} + \frac{t}{2i\hbar}}, \tag{65}$$

and  $c_1$  is a constant.

## 2.2 Measure

**Theorem 1** *The eigenstate  $\phi_s(x) = x^{-s} : \mathbb{X} \rightarrow \mathbb{C}$  is measurable. That is,  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x)$  where  $\phi_\sigma, \phi_t : \mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$  are measurable for  $s = \sigma + it = |s| \exp(i\theta)$ , and  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$ .*

*Proof* Owing to the one-to-one correspondence obtained from Plancherel transforms between configuration space and momentum space eigenstates, it can be seen that

$$\begin{aligned}\phi_s(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(x) \exp(-ipx) dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}i\pi s\right) (\operatorname{sgn}(p) + 1) \sin(\pi s) \Gamma(1-s) |p|^{s-1} \\ &= \frac{i}{\sqrt{2\pi}} (\operatorname{sgn}(p) + 1) e^{\frac{1}{2}\pi(t-i\sigma)} \sinh\left(\pi(t-i\sigma)\right) \Gamma(-it-\sigma+1) |p|^{\sigma+it-1}, \\ &0 < \sigma < 1.\end{aligned}\tag{66}$$

and

$$\phi_s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(p) \exp(ipx) dp.\tag{67}$$

Since

$$\begin{aligned}\|\phi_s(x)\|_1 &\equiv \int_{-\infty}^{-1} |\phi_s(x)| dx + \int_{-1}^1 |\phi_s(x)| \delta(x) dx + \int_1^{\infty} |\phi_s(x)| dx \\ &= \int_{-\infty}^{-1} |\phi_s(p)| dp + \int_{-1}^1 |\phi_s(p)| \delta(p) dp + \int_1^{\infty} |\phi_s(p)| dp \equiv \|\phi_s(p)\|_1,\end{aligned}\tag{68}$$

from which

$$\|\phi_s(x)\|_1 = \|\phi_s(p)\|_1 = -\frac{1}{s\pi^{1/2}} \exp\left(\frac{1}{2}\pi\Im(s)\right) \sqrt{\sin^2(\pi s)} \sqrt{\Gamma(1-s)^2}.\tag{69}$$

It then follows that  $\phi_s$  is complex square-integrable, i.e.,

$$\phi_s(x) \in \mathcal{H} \iff \int_{\mathbb{E}} |\phi_s(x)| d\mu < +\infty.\tag{70}$$

**Theorem 2** *Let the complex valued eigenstate  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$  where  $s = \sigma + it = |s| \exp(i\theta)$ , and  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$ , and let the measurable subset  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ . The  $\mathcal{H} = \mathcal{L}^2$ -norm of the complex-valued eigenstate  $\phi_s = x^{-s}$  is  $\infty$ , i.e.,  $\phi_s$  is not  $p = 2$  integrable at  $\sigma = 1/2$ .*

*Proof* Owing to the one-to-one correspondence obtained from Plancherel transforms between configuration space and momentum space eigenstates, it can be seen that

$$\begin{aligned}\phi_s(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(x) \exp(-ipx) dx \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}i\pi s\right) (\operatorname{sgn}(p) + 1) \sin(\pi s) \Gamma(1-s) |p|^{s-1} \\ &= \frac{i}{\sqrt{2\pi}} (\operatorname{sgn}(p) + 1) e^{\frac{1}{2}\pi(t-i\sigma)} \sinh\left(\pi(t-i\sigma)\right) \Gamma(-it-\sigma+1) |p|^{\sigma+it-1}, \\ &0 < \sigma < 1.\end{aligned}\tag{71}$$

and

$$\phi_s(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_s(p) \exp(ipx) dp, \quad (72)$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\left(t \cdot \arg(x)\right) \cos\left(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\right), \quad (73)$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\left(t \cdot \arg(x)\right) \sin\left(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\right) \quad (74)$$

for  $x \in \mathbb{R}_{\geq 1}^+$ . Since

$$\|\phi_s\|_p = \left( \int_1^\infty |\phi_s(x)|^p dx \right)^{\frac{1}{p}}, \quad (75)$$

and<sup>1</sup>

$$\|\phi_s(p)\|_p = \left( \int_1^\infty |\phi_s(p)|^p dp \right)^{\frac{1}{p}}, \quad (76)$$

from which

$$\|\phi_s(p)\|_p = \|\phi_s(x)\|_p = \left( \frac{1}{p\sigma - 1} \right)^{\frac{1}{p}}. \quad (77)$$

It then follows that as  $\sigma \rightarrow 1/2$ ,

$$\|\phi_s(p)\|_p = \|\phi_s(x)\|_p = \left( \frac{1}{\frac{p}{2} - 1} \right)^{\frac{1}{p}}, \quad (78)$$

such that the  $\mathcal{L}^{p=2}$ -norm of  $\phi_s$  is of indeterminate form. Furthermore, it can be seen from

$$\lim_{p \rightarrow 2} \left( \frac{1}{\frac{p}{2} - 1} \right)^{\frac{1}{p}}, \quad (79)$$

and letting

$$y = \left( \frac{1}{\frac{p}{2} - 1} \right)^{\frac{1}{p}}, \quad (80)$$

---

<sup>1</sup> Here, the reader is cautioned not to confuse the  $\mathcal{L}^p$ -norm with the momentum  $p$ . It can easily be seen that the  $\mathcal{L}^p$ -norm of  $\phi_s$  is also of indeterminate form for  $x \in (-\infty, -1]$ . The  $\mathcal{L}^p$ -norm vanishes for  $x \in [-1, 1]$  owing to the Dirac delta (singularity) at the origin  $x = 0$  [36].

then

$$\begin{aligned}
\ln(y) &= \frac{1}{p} \ln\left(\frac{1}{\frac{p}{2} - 1}\right) \\
&= \frac{1}{p} \left(\ln(1) - \ln\left(\frac{p}{2} - 1\right)\right) \\
&= -\frac{1}{p} \ln\left(\frac{p}{2} - 1\right),
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\lim_{p \rightarrow 2} \ln(y) &= \lim_{p \rightarrow 2} \left(-\frac{1}{p} \ln\left(\frac{p}{2} - 1\right)\right) \\
&= \infty.
\end{aligned} \tag{82}$$

Exponentiating both sides, we obtain

$$\begin{aligned}
\exp\left[\lim_{p \rightarrow 2} \ln(y)\right] &= \lim_{p \rightarrow 2} \left[\exp\left(\ln(y)\right)\right] \\
&= \lim_{p \rightarrow 2} y = \exp(\infty) = \infty,
\end{aligned} \tag{83}$$

such that we obtain the infinite density [25]

$$\|\phi_s(p)\|_{p=2} = \|\phi_s(x)\|_{p=2} = \infty. \tag{84}$$

**Corollary 1** *Let  $\mathcal{H} = \mathcal{L}^2(-\infty, -1] \cup [1, \infty)$  and consider the Hamiltonian observable given by*

$$\hat{H}\phi_s(x) = -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x). \tag{85}$$

*Although the action of  $\hat{H}$  is in principle well-defined for all  $\phi_s(x) \in \mathcal{L}^2$ , there are functions which are in  $\mathcal{L}^2$ , but for which  $\hat{H}\phi_s(x)$  is no longer an element of  $\mathcal{L}^2$ , e.g., when  $\sigma = 1/2$ ,*

$$\begin{aligned}
\phi_{\frac{1}{2}+it}(x) &= \frac{e^{t \arg(x)} \cos\left(\frac{\arg(x)}{2} + \frac{1}{2}t \log(x^2)\right)}{\sqrt[4]{x^2}} \\
&\quad - \frac{ie^{t \arg(x)} \sin\left(\frac{\arg(x)}{2} + \frac{1}{2}t \log(x^2)\right)}{\sqrt[4]{x^2}}.
\end{aligned} \tag{86}$$

*Therefore the domain of  $\hat{H}$  is given by*

$$\begin{aligned}
\mathcal{D}(\hat{H}) &= \left\{ \phi_s(x) \in \mathcal{L}^2 : \int_{-\infty}^{-1} \left| -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x) \right|^2 dx \right. \\
&\quad + \int_{-1}^1 \left| -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x) \right|^2 \delta(x) dx \\
&\quad \left. + \int_1^{\infty} \left| -2i\hbar\sqrt{x}\partial_x\sqrt{x}\phi_s(x) \right|^2 dx < \infty \right\} \subset \mathcal{L}^2.
\end{aligned} \tag{87}$$



Similarly, the domain of  $\hat{H}^2$  is

$$\begin{aligned} \mathcal{D}(\hat{H}^2) = & \left\{ \phi_s(x) \in \mathcal{L}^2 : \int_{-\infty}^{-1} \left| \left( -2i\hbar\sqrt{x}\partial_x\sqrt{x} \right)^2 \phi_s(x) \right|^2 dx \right. \\ & + \int_{-1}^1 \left| \left( -2i\hbar\sqrt{x}\partial_x\sqrt{x} \right)^2 \phi_s(x) \right|^2 \delta(x) dx \\ & \left. + \int_1^{\infty} \left| \left( -2i\hbar\sqrt{x}\partial_x\sqrt{x} \right)^2 \phi_s(x) \right|^2 dx < \infty \right\} \subset \mathcal{D}(\hat{H}), \end{aligned} \quad (88)$$

etc. As such, we define the dense subspace of  $\mathcal{H}$  as

$$\Phi \equiv \bigcap_{n=0}^{\infty} \mathcal{D}(\hat{H}^n), \quad (89)$$

such that for every  $\phi_s(x) \in \Phi$ , the solution is well-defined at  $\sigma = 1/2$ .

Eqs. (66) and (67) are two vector representations of the same Hilbert space  $\mathcal{H} = \mathcal{L}^{p=2}(-\infty, -1] \cup [1, \infty)$ . From Eq. (26), it can be seen that

$$\hat{T} = -2i\hbar x \partial_x, \quad (90)$$

such that we define a multiplicative operator  $\hat{T}_0$  in momentum space  $(\hat{T}_0\Phi_s)(p) = \hat{T}_0(p)\Phi_s(p)$ , where

$$\hat{T}_0(p) = 2\hat{x}\hat{p}. \quad (91)$$

Here, it should be pointed out that as  $\hat{x} = i\hbar d/dp$ , as such Eq. (91) reduces to

$$\hat{T}_0(p) = 2i\hbar, \quad (92)$$

and Eq. (26) is then rewritten in momentum space as  $\hat{H}(p) = i\hbar$ . The domain  $\mathcal{D}_0$  of  $\hat{T}_0$  is defined as the set of all functions  $\phi_s(p) \in \mathcal{H}$  such that  $\hat{T}_0(p)\phi_s(p) \in \mathcal{H}$ . As such,  $\hat{T}_0$  is definitively self-adjoint. From Eq. (29) we have defined the set  $\mathcal{D}_1$  of functions in configuration space. From the Plancherel transform [41] of Eq. (29), we obtain the set  $\mathcal{D}_1$  of functions in momentum space having the form

$$G(p, s) = P(p, s) \exp\left(-\frac{p^2}{2}\right), \quad (93)$$

where  $P$  is a polynomial of  $p$  and  $s$ . Eqs. (66) and (67) are true if  $\phi_s(x) \in \mathcal{D}_1$  or  $\phi_s(p) \in \mathcal{D}_1$  and since  $\phi_s(p) \in \mathcal{D}_1 \rightarrow 0$  as  $p \rightarrow \infty$  then  $\mathcal{D}_1 \subseteq \mathcal{D}_0$ . Moreover, for  $\phi_s(x) \in \mathcal{D}_1$ ,  $\hat{T}_0$  coincides with Eq. (90) [40]. Using Eq. (66) and  $\hat{H}(p) = i\hbar$ , the eigenrelation

$$\hat{H}(p) |\Phi_s(p)\rangle = \lambda |\Phi_s(p)\rangle \quad (94)$$

is obtained. In order to find the expectation value for  $\hat{H}$  we take the complex conjugate of Eq. (94), set  $\hbar = 1$ , multiply by the eigenfunction  $\phi_s(p)$ , and then integrate over  $p$  to obtain

$$\int_{-\infty}^{\infty} \left( i \frac{e^{-\frac{1}{2}i\pi s} (\operatorname{sgn}(p) + 1) \sin(\pi s) \Gamma(1-s) |p|^{s-1}}{2\pi^{1/2}} \right)^* \left( \frac{e^{-\frac{1}{2}i\pi s} (\operatorname{sgn}(p) + 1) \sin(\pi s) \Gamma(1-s) |p|^{s-1}}{2\pi^{1/2}} \right) dp = \lambda^* \|\Phi_s\|_p, \quad (95)$$

where  $\lambda$  is the eigenvalue.

**Theorem 3** *Let the complex valued eigenstate  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$  where  $s = \sigma + it = |s| \exp(i\theta)$ , and  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$ , and let the measurable subset  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ . The following are equivalent for  $\sigma, t \in \mathbb{R}$ <sup>2</sup>*

1. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_\sigma(x) > r\}$  is measurable.
2. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_t(x) > r\}$  is measurable.
3. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_\sigma(x) \geq r\}$  is measurable.
4. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_t(x) \geq r\}$  is measurable.
5. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_\sigma(x) < r\}$  is measurable.
6. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_t(x) < r\}$  is measurable.
7. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_\sigma(x) \leq r\}$  is measurable.
8. For each real number  $r$ , the set  $\{x \in \mathbb{E} : \phi_t(x) \leq r\}$  is measurable.

*Proof* Note that the intersection of sets,

$$\{x \in \mathbb{E} : \phi_\sigma(x) \geq r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_\sigma(x) > r - \frac{1}{n}\}, \quad (96)$$

$$\{x \in \mathbb{E} : \phi_t(x) \geq r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_t(x) > r - \frac{1}{n}\}, \quad (97)$$

$$\{x \in \mathbb{E} : \phi_\sigma(x) > r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_\sigma(x) \geq r + \frac{1}{n}\}, \quad (98)$$

$$\{x \in \mathbb{E} : \phi_t(x) > r\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{E} : \phi_t(x) \geq r + \frac{1}{n}\}, \quad (99)$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\left(t \cdot \arg(x)\right) \cos\left(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\right), \quad (100)$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\left(t \cdot \arg(x)\right) \sin\left(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\right). \quad (101)$$

<sup>2</sup> <http://zeta.math.utsa.edu/mqr328/class/real2/>

**Theorem 4** Let  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$  be a measurable subset of the measure space  $\mathbb{X}$ . If the complex valued eigenstate  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$  where  $s = \sigma + it = |s| \exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$ , and  $\phi_\sigma(x)$ , and  $\phi_t$  are continuous a.e. on  $\mathbb{E}$ , then  $\phi_s(x)$  is measurable for  $\sigma, t \in \mathbb{R}$ .

*Proof* Let  $\mathcal{D}$  be the singleton  $\{0\}$  owing to the singularity at  $x = 0$  of  $\phi_s(x) = x^{-s}$ . Then  $\mu(\mathcal{D}) = 0$  and all of its subsets are measurable. Let  $r \in \mathbb{R}$  and note that

$$\{x \in \mathbb{E} : \phi_\sigma(x) > r\} = \{x \in \mathbb{E} - \mathcal{D} : \phi_\sigma(x) > r\} \cup \{x \in \mathcal{D} : \phi_\sigma(x) > r\} \quad (102)$$

where

$$\phi_\sigma(x) = (x^2)^{-\sigma/2} \exp\left(t \cdot \arg(x)\right) \cos\left(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\right), \quad (103)$$

and

$$\phi_t(x) = -(x^2)^{-\sigma/2} \exp\left(t \cdot \arg(x)\right) \sin\left(\sigma \cdot \arg(x) + \frac{t}{2} \log(x^2)\right). \quad (104)$$

Letting

$$C_\sigma = \{x \in \mathbb{E} - \mathcal{D} : \phi_\sigma(x) > r\}, \quad (105)$$

for each  $x \in C_\sigma$ , as  $\phi_\sigma(x)$  is continuous at  $x$ , we can find  $\delta_x > 0$  such that if  $y \in V_{\delta_x}(x)$  then  $\phi_\sigma(y) > r$ . It can be seen that  $\phi_\sigma(x)$  is measurable, since

$$C_\sigma = (\mathbb{E} - \mathcal{D}) \bigcap_{x \in C_\sigma} V_{\delta_x}(x). \quad (106)$$

Similarly, noting that

$$\{x \in \mathbb{E} : \phi_t(x) > r\} = \{x \in \mathbb{E} - \mathcal{D} : \phi_t(x) > r\} \cup \{x \in \mathcal{D} : \phi_t(x) > r\} \quad (107)$$

and letting

$$C_t = \{x \in \mathbb{E} - \mathcal{D} : \phi_t(x) > r\}, \quad (108)$$

for each  $x \in C_t$ , as  $\phi_t(x)$  is continuous at  $x$ , we can find  $\delta_x > 0$  such that if  $y \in V_{\delta_x}(x)$  then  $\phi_t(y) > r$ . It can be seen that  $\phi_t(x)$  is measurable since

$$C_t = (\mathbb{E} - \mathcal{D}) \bigcap_{x \in C_t} V_{\delta_x}(x). \quad (109)$$

Let  $\{\phi_s\} = \{\phi_\sigma\} + i\{\phi_t\}$  be a sequence of functions defined on the measure space  $\mathbb{X} \rightarrow \mathbb{C}$ . Denoting

$$\sup_s \phi_s(x) = \sup\{\phi_s(x) : s \in \mathbb{C}\} \quad (110)$$

and

$$\limsup_s \phi_s(x) = \lim_s \left( \sup_{k \geq s} \phi_k(x) \right), \quad (111)$$

it can be seen that

$$\limsup_s \phi_s(x) = \inf_s \left( \sup_{k \geq s} \phi_k(x) \right). \quad (112)$$

Similarly, from

$$\inf_s \phi_s(x) = \inf \{ \phi_s(x) : s \in \mathbb{C} \} \quad (113)$$

and

$$\liminf_s \phi_s(x) = \lim_s \left( \inf_{k \geq s} \phi_k(x) \right), \quad (114)$$

it can be seen that

$$\inf_s \phi_s(x) = - \sup_s \left( - \phi_s(x) \right), \quad (115)$$

and

$$\liminf_s \phi_s(x) = - \limsup_s \left( - \phi_s(x) \right). \quad (116)$$

**Theorem 5** *Let the sequence of measurable eigenstates  $\{\phi_s\} = \{\phi_\sigma\} + i\{\phi_t\}$  be defined on the measure space  $\mathbb{X} \rightarrow \mathbb{C}$ . For the sequence of measurable eigenstates  $\{\phi_\sigma\} : \mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$*

$$g(x) = \sup_\sigma \phi_\sigma(x), \quad (117)$$

and

$$h(x) = \limsup_\sigma \phi_\sigma(x), \quad (118)$$

such that  $g$  and  $h$  are measurable for  $x \in \mathbb{E}$ .

*Proof* For any  $r \in \mathbb{R}$ , we obtain

$$\{x \in \mathbb{E} : g(x) > r\} = \bigcup_\sigma \{x \in \mathbb{E} : \phi_\sigma(x) > r\}. \quad (119)$$

From Eqs. (112) and (115)-(116), this implies that  $h$  is also measurable.

**Corollary 2** *Let  $\phi_\sigma$  be a sequence of measurable eigenstates defined on the measure space  $\mathbb{X}$ , and  $\phi_\sigma : \mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ . Since  $\{\phi_\sigma\}$  converges pointwise to  $\phi_\sigma$  a.e. on  $\mathbb{E}$ , then  $\phi_\sigma$  is measurable.*

**Theorem 6** *Let the sequence of measurable eigenstates  $\{\phi_s\} = \{\phi_\sigma\} + i\{\phi_t\}$  be defined on the measure space  $\mathbb{X} \rightarrow \mathbb{C}$ . For the sequence of measurable eigenstates  $\{\phi_t\} : \mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$*

$$g(x) = \sup_t \phi_t(x), \quad (120)$$

and

$$h(x) = \limsup_t \phi_t(x), \quad (121)$$

such that  $g$  and  $h$  are measurable for  $x \in \mathbb{E}$ .

*Proof* For any  $r \in \mathbb{R}$ , we obtain

$$\{x \in \mathbb{E} : g(x) > r\} = \bigcup_t \{x \in \mathbb{E} : \phi_t(x) > r\}. \quad (122)$$

From Eqs. (112) and (115)-(116), this implies that  $h$  is also measurable.

**Corollary 3** *Let  $\phi_t$  be a sequence of measurable eigenstates defined on the measure space  $\mathbb{X}$ , and  $\phi_t : \mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ . Since  $\{\phi_t\}$  converges pointwise to  $\phi_t$  a.e. on  $\mathbb{E}$ , then  $\phi_t$  is measurable.*

**Corollary 4** *Let  $\phi_s = \phi_\sigma + i\phi_t$  be a sequence of measurable eigenstates defined on the measure space  $\mathbb{X} \rightarrow \mathbb{C}$ . Since  $\{\phi_\sigma\}$  converges pointwise to  $\phi_\sigma$  a.e. on  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ , and  $\{\phi_t\}$  converges pointwise to  $\phi_t$  a.e. on  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ , then  $\phi_s$  is measurable.*

### 2.3 Expectation Value of the Observable

**Definition 21** *The Riemann zeta Schrödinger equation is*

$$-\hbar \partial_s |\Psi_s(x)\rangle = i \left[ \hat{\Delta}^{-1} \hat{x} \hat{p} \hat{\Delta} + \hat{\Delta}^{-1} \hat{p} \hat{x} \hat{\Delta} \right] |\Psi_s(x)\rangle, \quad (123)$$

where  $\hat{\Delta} = 1 - \exp(-\partial_x)$ ,  $\hat{x} = x$ ,  $\hat{p} = -i\hbar \partial_x$ ,  $\hbar = 1$ ,  $x \in \mathbb{R}^+ \geq 1$  owing to the difference operator  $\hat{\Delta} |\Psi_s(x)\rangle$ , and  $s \in \mathbb{C}$ .

Upon inserting Eq. (20) into Eq. (123) for  $x \in \mathbb{R}^+$ , we obtain the symmetrized Riemann zeta Schrödinger equation, i.e.,

$$\begin{aligned} \partial_s |\phi_s(x)\rangle &= 1/2(\partial_\sigma - i\partial_t) |\phi_s(x)\rangle \\ &= -\frac{2}{\hbar} \sqrt{x} \partial_x \sqrt{x} |\phi_s(x)\rangle. \end{aligned} \quad (124)$$

**Theorem 7** *Let the complex-valued eigenstate*

$\phi_s(x)$

$$= \frac{\sqrt{\pi} 2^{-\sigma-it-\frac{3}{2}} e^{-\frac{1}{2}\pi(t+3i\sigma)} (e^{2\pi t} - e^{2i\pi\sigma}) \left(t - x \sqrt{\frac{t^2}{x^2}}\right) \Gamma(-it - \sigma) \left(\frac{t^2}{x^2}\right)^{\frac{1}{2}(\sigma+it)}}{t}, \quad (125)$$

where  $s = \sigma + it = |s| \exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$ ; and let the measurable subset of the measure space  $\mathbb{X}$  be  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ , for the Hamiltonian operator  $\hat{H} = -2i\hbar \sqrt{x} \partial_x \sqrt{x}$ . The eigenstate is symmetric about the origin  $x = 0$ .

*Proof* Let  $|\phi_s(x)\rangle$  be an eigenstate of  $\hat{H}$  with eigenvalue  $t$ , i.e.,

$$\hat{H} |\phi_s(x)\rangle = t |\phi_s(x)\rangle. \quad (126)$$

In order to find the expectation value of  $\hat{H}$  we multiply  $\hat{H}$  by the eigenstate, take the complex conjugate, and then multiply the result by the eigenstate and integrate over  $\mathbb{E}$  to obtain

$$\begin{aligned} 2i \int_{\mathbb{E}} \left( \sqrt{x} \partial_x \sqrt{x} \phi_s(x) \right)^* \phi_s(x) dx &= t^* \int_{\mathbb{E}} \phi_s^*(x) \phi_s(x) dx \\ &= t^* \|\phi\|. \end{aligned} \quad (127)$$

Integrating by parts on the LHS then gives

$$\begin{aligned} -2i \left( \|\phi\| + \int_{-\infty}^{-1} \phi_s^*(x) x \frac{d}{dx} \phi_s(x) dx + \int_{-1}^1 \phi_s^*(x) x \frac{d}{dx} \phi_s(x) \delta(x) dx \right. \\ \left. + \int_1^{\infty} \phi_s^*(x) x \frac{d}{dx} \phi_s(x) dx \right) &= t^* \|\phi\|. \end{aligned} \quad (128)$$

Carrying out the integration on the LHS we obtain

$$\int_{-2\pi n}^0 \int_{-\infty}^{-1} \phi_s^*(x) x \frac{d}{dx} \phi_s(x) dx dt = \int_0^{2\pi n} \int_1^{\infty} \phi_s^*(x) x \frac{d}{dx} \phi_s(x) dx dt = 0 \quad \forall n. \quad (129)$$

Hence it can be seen that

$$\int_{-2\pi n}^0 \int_{-\infty}^{-1} \phi_s^*(x) \phi_s(x) dx dt = \int_0^{2\pi n} \int_1^{\infty} \phi_s^*(x) \phi_s(x) dx dt = 0 \quad \forall n. \quad (130)$$

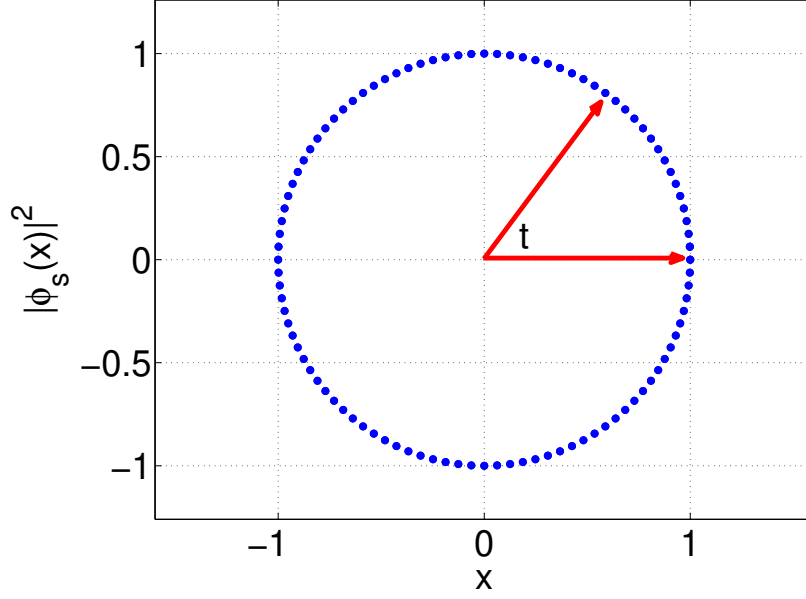
**Theorem 8** *Let the complex-valued eigenstate  $\phi_s(x) = \phi_\sigma(x) + i\phi_t(x) = x^{-s}$  where  $s = \sigma + it = |s| \exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$ , and let the measurable subset of the measure space  $\mathbb{X}$  be  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$ . For the Hamiltonian operator  $\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}$ , all of the eigenvalues  $t$  occur at  $|\sigma| = 1/2$  with  $\hbar = 1$ .*

*Proof* Let  $|\phi_s(x)\rangle$  be an eigenstate of  $\hat{H}$  with eigenvalue  $t$ , i.e.,

$$\hat{H} |\phi_s(x)\rangle = t |\phi_s(x)\rangle. \quad (131)$$

In order to find the expectation value of  $\hat{H}$  we multiply  $\hat{H}$  by the eigenstate, take the complex conjugate, and then multiply the result by the eigenstate and integrate over  $\mathbb{E}$  to obtain

$$\begin{aligned} 2i \int_{\mathbb{E}} \left( \sqrt{x} \partial_x \sqrt{x} \phi_s(x) \right)^* \phi_s(x) dx &= t^* \int_{\mathbb{E}} \phi_s^*(x) \phi_s(x) dx \\ &= t^* \|\phi\|. \end{aligned} \quad (132)$$



**Fig. 1** Plot of the density  $|\phi_s(x)|^2$ , where  $s = |s| \exp(i \arctan(t/\sigma)) = 1/2 - \log(x)/2$ , Eq. (136). Parity symmetry is exhibited about the origin, as  $\langle II \rangle = \pi W(0,0)/2$  [43]. The density is normalized when  $x \cos(t) = 1$  (color online).

Integrating by parts on the LHS then gives

$$\begin{aligned}
 -2i \left( \|\phi\| + \int_{-\infty}^{-1} \phi_s^*(x) x \frac{d}{dx} \phi_s(x) dx + \int_{-1}^1 \phi_s^*(x) x \frac{d}{dx} \phi_s(x) \delta(x) dx \right. \\
 \left. + \int_1^{\infty} \phi_s^*(x) x \frac{d}{dx} \phi_s(x) dx \right) = t^* \|\phi\|. \quad (133)
 \end{aligned}$$

Carrying out the integration on the LHS we obtain

$$2i(-1)^{-2\sigma} \left( (-1)^{2\sigma} + 1 \right) (\sigma + it) = (2\sigma - 1)(t^* + 2i) \|\phi\|. \quad (134)$$

Hence it can be seen that

$$\boxed{|\sigma| = \frac{1}{2} \forall t.} \quad (135)$$

## 2.4 Convergence

**Theorem 9** For the symmetrized Riemann zeta Schrödinger equation, i.e.,  $\hbar\partial_s |\phi_s(x)\rangle = -2\sqrt{x}\partial_x \sqrt{x} |\phi_s(x)\rangle$ , the complex-valued eigenstate  $|\phi_s(x)\rangle = x^{-s}$  where  $s = \sigma + it = |s|\exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$  normalizes at  $x \cos(t) = 1$ , i.e., the density  $|\phi_s(x)|^2 = 1$ .

*Proof* In order to obtain convergent solutions to the unsymmetric Riemann zeta Schrödinger Eq. (123), it can be seen that upon inserting Eq. (20) into the symmetric Eq. (124), we obtain

$$\begin{aligned} s &= |s|\exp\left(i\arctan(t/\sigma)\right) \\ &= \frac{1}{2} - \frac{\log(x)}{2}. \end{aligned} \quad (136)$$

Hence at  $x = 1$ , such that at  $|\sigma| = 1/2$  in agreement with Eq. (135)

$$t = 2\pi n, \quad (137)$$

where  $n \in \mathbb{Z}$  and  $t \in \mathbb{R}$ . This condition is required such that the density is normalized in agreement with Eq. (84), i.e.,

$$\begin{aligned} \|\phi_s\|_2 &= \sum_m \sum_n \hat{b}_n(s) \hat{b}_m^\dagger(s) \langle \phi_m | \phi_n \rangle \\ &= \sum_n |\hat{b}_n(s)|^2 \\ &= 1. \end{aligned} \quad (138)$$

Here it should be pointed out that by taking Eqs. (65) and (136) and inserting them into Eq. (23) gives the eigenequation relation

$$\frac{1}{x^{\frac{1}{2} - \frac{\log(x)}{2}}} = \frac{1}{x^{\frac{1}{2} + \frac{t}{2i\hbar}}}. \quad (139)$$

Hence we obtain the eigenfunction

$$\begin{aligned} \phi_s(x) &= \frac{1}{x^s} \\ &= \left(e^{\frac{it}{\hbar}}\right)^{\frac{1}{2}[-1 + \log(e^{\frac{it}{\hbar}})]}. \end{aligned} \quad (140)$$

**Theorem 10** For the Bender-Brody-Müller equation [7, 8], i.e.,

$$\frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) |\Psi_s(x)\rangle = t |\Psi_s(x)\rangle, \quad (141)$$

the nontrivial zeros of the Riemann zeta function can be obtained from the analytic continuation of the Riemann zeta function, i.e. Eq. (1) at the normalization constraint  $x = \sec(t = 2\pi n) = 1$ , such that  $|\sigma| = 1/2 \forall t \in \mathbb{R}$  where  $s = \sigma + it = |s|\exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$ . The analytic continuation of the Riemann zeta function is not decidable at  $|\sigma| = 1/2$ ,  $\forall n \in \mathbb{Z}$ , i.e. the analytic continuation of the Riemann zeta function is not a computable function at  $|\sigma| = 1/2$ .



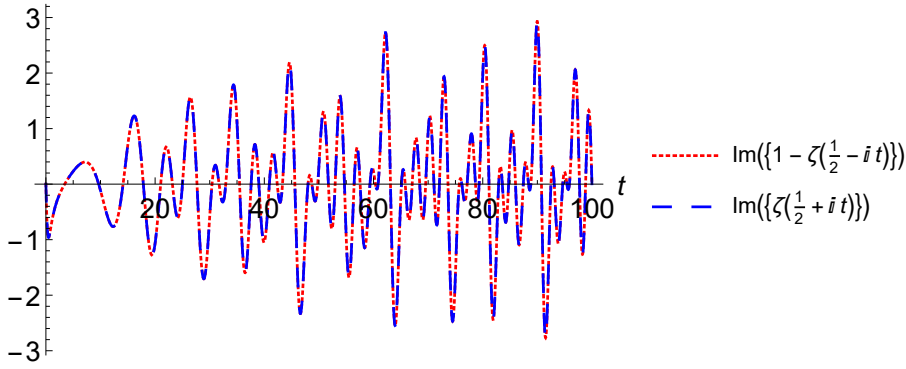
*Proof* At  $x = \sec(t = 2\pi n) = 1$ , the normalization constraint Eq. (138) is satisfied,  $\sigma = \frac{1}{2} - it$ , and Eq. (17) can be written

$$\begin{aligned}\Psi_s(x=1) &= -\zeta(s=1/2, 2) \\ &= -\Gamma(1/2) \frac{1}{2\pi i} \oint_C \frac{\sqrt{z} e^{2z}}{1-e^z} dz \\ &= 1 - \zeta(\sigma = \frac{1}{2} - it).\end{aligned}\tag{142}$$

where the contour  $C$  is about  $\mathbb{R}^-$ . From the analytic continuation relations of Eq. (1)

$$\begin{aligned}\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} &= \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \exp(-i \cdot t \ln(n))}{n^\sigma} \\ &= \frac{1}{1-2^{1-s}} \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(t \cdot \ln(n))}{n^\sigma} \right. \\ &\quad \left. - i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(t \cdot \ln(n))}{n^\sigma} \right],\end{aligned}\tag{143}$$

$$\begin{aligned}1 - \left( \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \right)^* &= 1 - \frac{1}{1-2^{1-s^*}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \exp(i \cdot t \ln(n))}{n^\sigma} \\ &= 1 - \frac{1}{1-2^{1-s^*}} \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(t \cdot \ln(n))}{n^\sigma} \right. \\ &\quad \left. + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(t \cdot \ln(n))}{n^\sigma} \right].\end{aligned}\tag{144}$$



**Fig. 2** Plot of the imaginary components of Eq. (1). Results are compared with Eq. (147) (color online).

$$\begin{aligned}
& \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-2^{-\sigma+1} \cos(t \log(2)) \cos(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{\cos(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-2^{-\sigma+1} \sin(t \log(2)) \sin(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-2^{-\sigma+1} \sin(t \log(2)) \cos(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{2^{-\sigma+1} \cos(t \log(2)) \sin(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-\sin(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2},
\end{aligned} \tag{145}$$

such that

$$\begin{aligned}
& 1 - \left( \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \right)^* \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{2^{-\sigma+1} \cos(t \log(2)) \cos(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-\cos(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-2^{-\sigma+1} \sin(t \log(2)) \sin(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-2^{-\sigma+1} \sin(t \log(2)) \cos(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{2^{-\sigma+1} \cos(t \log(2)) \sin(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2} \\
&+ i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\sigma} \cdot \frac{-\sin(t \ln(n))}{2^{-2\sigma+2} \sin^2(t \log(2)) + [1 - 2^{-\sigma+1} \cos(t \log(2))]^2}, \tag{146}
\end{aligned}$$

Owing to the periodicity of  $t = 2\pi n$  at  $x = \sec(t)$ , i.e. Eq. (137), it can be seen that

$$\Im \left[ \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \right] = \Im \left[ 1 - \left( \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \right)^* \right]. \tag{147}$$

Owing to Eq. (135), at  $|\sigma| = 1/2$  we obtain

$$\Im [\zeta(s)] = i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \cdot \frac{\sin(t \ln(n)) - \sqrt{2} \sin(t \log(\frac{n}{2}))}{2\sqrt{2} \cos(t \log(2)) - 3}. \tag{148}$$

However, since at  $|\sigma| = 1/2$  the eigenvalues  $t$  are not observable, i.e.,  $\langle \hat{H} \rangle = t = 0$ , we have

$$\begin{aligned}
\Im [\zeta(s)] &= i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \cdot \frac{\sin(2\pi n \overset{0}{\ln}(n)) - \sqrt{2} \sin(2\pi n \overset{0}{\log}(\frac{n}{2}))}{2\sqrt{2} \cos(2\pi n \overset{0}{\log}(2)) - 3} \\
&= 0 \quad \forall n \in \mathbb{Z}. \tag{149}
\end{aligned}$$

*Remark 3* It has been noted that there is a uniquely defined relation between prime numbers and the imaginary parts of the nontrivial Riemann zeros, independent of their real part [44].

*Remark 4* In the theory of computation, an observable is called *decidable*, or *effective*, if and only if its behavior is given by a computable function [29]. From Theorem 10, it can be seen that the *Riemann Hypothesis is not decidable*. If the Riemann hypothesis is undecidable, there is no proof it is false. If we find a non-trivial zero, that is a proof that it is false. Thus if it is undecidable there are no non-trivial zeros. This constitutes a proof the Riemann hypothesis is true [45].

*Remark 5* The Riemann Hypothesis states that the real part of all of the *nontrivial* zeros of the Riemann zeta function are located at  $\sigma = 1/2$  [9].

## 2.5 Second Quantization

**Theorem 11** *By representing the complex-valued eigenstate  $|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i|\phi_t(x)\rangle = x^{-s}$  where  $s = \sigma + it = |s| \exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$  as a linear combination of basis states, then the eigenspectrum of the Hamiltonian operator  $-2i\hbar\sqrt{x}\partial_x\sqrt{x}$  is not observable, i.e. zero, on the measure space  $\mathbb{E} \rightarrow (-\infty, -1] \cup [1, \infty)$  when  $|\sigma| = 1/2$  and  $\hbar = 1$ .*

*Proof* A standard way to introduce topology into the algebra of observables is to make them operators on a Hilbert space. In order to perform a second quantization [46], we can express the complex-valued eigenstate as a linear combination of basis states

$$|\phi_s(x)\rangle = \sum_{n \in \mathbb{Z}} \hat{b}_n(s) |\phi_n(x)\rangle, \quad (150)$$

where  $s = \sigma + it = |s| \exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$ ,  $s \in \mathbb{C}$ , and  $\sigma, t \in \mathbb{R}$ . As such, using Eqs. (23) and (135) we can rewrite Eq. (150) as

$$|\phi_s(x)\rangle = \sum_{n \in \mathbb{Z}} \hat{b}_n(s) x^{-\frac{1}{2} - in}. \quad (151)$$

From using this second quantization in Eq. (124), we find

$$\hbar \frac{d}{ds} \hat{b}_n(s) = -t_n \hat{b}_n(s). \quad (152)$$

We now find a Hamiltonian that yields Eq. (152) as the equation of motion, hence, we take

$$\begin{aligned} \langle \phi_{s'}(x) | \hat{H} | \phi_s(x) \rangle &= -2 \int_{-\infty}^{-1} \langle \phi_{s'}(x) | \sqrt{x} \partial_x \sqrt{x} | \phi_s(x) \rangle dx \\ &\quad - 2 \int_{-1}^1 \langle \phi_{s'}(x) | \sqrt{x} \partial_x \sqrt{x} | \phi_s(x) \rangle \delta(x) dx \\ &\quad - 2 \int_1^{\infty} \langle \phi_{s'}(x) | \sqrt{x} \partial_x \sqrt{x} | \phi_s(x) \rangle dx, \end{aligned} \quad (153)$$

as the expectation value. Upon substituting Eq. (151) into Eq. (153), we obtain the harmonic oscillator

$$\begin{aligned}
\langle \phi_m(x) | \hat{H} | \phi_n(x) \rangle &= -2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{-1} \frac{1}{x^{\frac{1}{2}-im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}+in}} dx \\
&\quad - 2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{-1}^1 \frac{1}{x^{\frac{1}{2}-im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}+in}} \delta(x) dx \\
&\quad - 2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_1^{\infty} \frac{1}{x^{\frac{1}{2}-im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}+in}} dx \\
&= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{b}_m^\dagger(s) \hat{b}_n(s) \langle m | \left( \frac{2n(\exp(\pi(n-m)) - 1)}{m-n} \right) | n \rangle,
\end{aligned} \tag{154}$$

for  $|m\rangle, |n\rangle = 1, 2, 3, \dots, \infty$ . Hence at  $m = n$ ,  $\langle n|n\rangle = \delta_{nn} = 1$  and

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = \sum_{n \in \mathbb{Z}} |\hat{b}_n(s)|^2 (-2\pi n). \tag{155}$$

In accordance with Eq. (135) and Eq. (138), at  $|\sigma| = 1/2$  and the zero periodicity of the eigenvalues  $t$ ,

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = 0. \tag{156}$$

Taking  $\hat{b}_n(s)$  as an operator, and  $\hat{b}_n^\dagger(s)$  as the adjoint, we obtain the usual properties:

$$\begin{aligned}
[\hat{b}_n(s), \hat{b}_m(s)] &= [\hat{b}_n^\dagger(s), \hat{b}_m^\dagger(s)] = 0, \\
[\hat{b}_n(s), \hat{b}_m^\dagger(s)] &= \delta_{nm}.
\end{aligned} \tag{157}$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned}
-\hbar \frac{d}{ds} \sum_{n \in \mathbb{Z}} \hat{b}_n(s) &= [\hat{b}_n(s), \hat{H}]_- \\
&= \sum_{m \in \mathbb{Z}} t_m \left( \hat{b}_n(s) \hat{b}_m^\dagger(s) \hat{b}_m(s) - \hat{b}_m^\dagger(s) \hat{b}_m(s) \hat{b}_n(s) \right) \\
&= \sum_{m \in \mathbb{Z}} t_m \left( \delta_{nm} \hat{b}_m(s) - \hat{b}_m^\dagger(s) \hat{b}_n(s) \hat{b}_m(s) - \hat{b}_m^\dagger(s) \hat{b}_m(s) \hat{b}_n(s) \right) \\
&= \sum_{m \in \mathbb{Z}} t_m \left( \delta_{nm} \hat{b}_m(s) + \hat{b}_m^\dagger(s) \hat{b}_m(s) \hat{b}_n(s) - \hat{b}_m^\dagger(s) \hat{b}_m(s) \hat{b}_n(s) \right) \\
&= \sum_{n \in \mathbb{Z}} \hat{b}_m^\dagger(s) \hat{b}_n(s) t_n.
\end{aligned} \tag{158}$$

The eigenvalues of  $\hat{H}$  are then periodically unobservable, i.e.,

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = 0. \tag{159}$$

From Eq. (158) it can be seen that

$$\begin{aligned} -\hbar \frac{d}{ds} \hat{b}_n &= 0, \\ -\hbar \frac{d}{ds} \hat{b}_n^\dagger &= -0. \end{aligned} \quad (160)$$

*Remark 6* Theorem 11 implies the Riemann hypothesis, as the spectrum of a Hermitian operator consists of real numbers as seen in Theorem 7, and 0 is a real number. This can be considered a “zero point spectrum.”

## 2.6 Holomorphicity

**Theorem 12** *The densely defined Hamiltonian operator  $\hat{H} = -2\sqrt{x}\partial_x\sqrt{x}$  on the Hilbert space  $\mathcal{H} = L^2(-\infty, -1] \cup [1, \infty)$  is symmetric (Hermitian) [47], for the complex-valued eigenstate  $|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i|\phi_t(x)\rangle = x^{-s}$  where  $s = \sigma + it = |s|\exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$  and  $\sigma, t \in \mathbb{R}$  when  $|\sigma| = 1/2$  and  $\hbar = 1$ .*

*Proof* By expressing the complex-valued eigenstate as a linear combination of basis states such that

$$|\phi_s(x)\rangle = \sum_{n \in \mathbb{Z}} \hat{b}_n(s) |\phi_n(x)\rangle, \quad (161)$$

where  $s \in \mathbb{C}$ ,  $s = \sigma + it = |s|\exp(i\theta)$ ,  $|s| = \sqrt{\sigma^2 + t^2}$ ,  $\theta = \arctan(t/\sigma)$ , and  $\sigma, t \in \mathbb{R}$ , it can be seen that by using Eq. (23) we can rewrite Eq. (161) as

$$|\phi_s(x)\rangle = \sum_{n \in \mathbb{Z}} \hat{b}_n(s) x^{-\frac{1}{2}-n}. \quad (162)$$

By taking the inner product

$$\begin{aligned} (\hat{H}\phi_n^*, \phi_m) &= -2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{-1} \frac{1}{x^{\frac{1}{2}+im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}-in}} dx \\ &\quad - 2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{-1}^1 \frac{1}{x^{\frac{1}{2}+im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}-in}} \delta(x) dx \\ &\quad - 2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_1^{\infty} \frac{1}{x^{\frac{1}{2}+im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}-in}} dx \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{b}_m^\dagger(s) \hat{b}_n(s) \langle m | \left( \frac{2n(\exp(\pi(n-m)) - 1)}{m-n} \right) | n \rangle, \end{aligned} \quad (163)$$

for  $|m\rangle, |n\rangle = 1, 2, 3, \dots, \infty$ . Hence at  $m = n$ ,  $\langle n|n\rangle = \delta_{nn} = 1$  and

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = \sum_{n \in \mathbb{Z}} |\hat{b}_n(s)|^2 (2\pi n). \quad (164)$$

In accordance with Eq. (135) and Eq. (138), at  $|\sigma| = 1/2$  and the zero periodicity of the eigenvalues  $t$ ,

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = 0. \quad (165)$$

Furthermore, by taking the inner product

$$\begin{aligned} (\phi_m^*, \hat{H} \phi_n) &= -2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{-1} \frac{1}{x^{\frac{1}{2}-im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}+in}} dx \\ &\quad - 2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_{-1}^1 \frac{1}{x^{\frac{1}{2}-im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}+in}} \delta(x) dx \\ &\quad - 2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \int_1^{\infty} \frac{1}{x^{\frac{1}{2}-im}} \sqrt{x} \partial_x \sqrt{x} \frac{1}{x^{\frac{1}{2}+in}} dx \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{b}_m^\dagger(s) \hat{b}_n(s) \langle m | \left( \frac{2n(\exp(\pi(n-m)) - 1)}{m-n} \right) | n \rangle, \end{aligned} \quad (166)$$

for  $|m\rangle, |n\rangle = 1, 2, 3, \dots, \infty$ . Hence at  $m = n$ ,  $\langle n | n \rangle = \delta_{nn} = 1$  and

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = \sum_{n \in \mathbb{Z}} |\hat{b}_n(s)|^2 (-2\pi n). \quad (167)$$

In accordance with Eq. (135) and Eq. (138), at  $|\sigma| = 1/2$ ,

$$\langle \phi_n(x) | \hat{H} | \phi_n(x) \rangle = 0. \quad (168)$$

Finally,

$$(\hat{H} \phi_n^*, \phi_m) = (\phi_m^*, \hat{H} \phi_n) = 2\pi n = 0 \quad \forall n \in \mathbb{Z}. \quad (169)$$

### 3 Similarity Solutions

Since Eq. (124), the Riemann zeta Schrödinger equation (RZSE) possesses symmetry about the origin  $x = 0$ , we then seek a similarity solution [48] of the form:

$$\phi_s(x) = x^\alpha f(\eta), \quad (170)$$

where  $\eta = s/x^\beta$ , and the RZSE becomes an ordinary differential equation (ODE) for  $f$ . As such, we consider Eq. (124), and introduce the transformation  $\xi = \epsilon^a x$ , and  $\tau = \epsilon^b s$ , so that

$$w(\xi, \tau) = \epsilon^c \phi(\epsilon^{-a} \xi, \epsilon^{-b} \tau), \quad (171)$$

where  $\epsilon \in \mathbb{R}$ , and  $\tau \in \mathbb{C}$ . From performing this change of variable we obtain

$$\begin{aligned} \frac{\partial}{\partial s} \phi &= \epsilon^{-c} \frac{\partial w}{\partial \tau} \frac{\partial \tau}{\partial s} \\ &= \epsilon^{b-c} \frac{\partial w}{\partial \tau}, \end{aligned} \quad (172)$$

and

$$\begin{aligned}
-2\sqrt{x}\frac{\partial}{\partial x}\sqrt{x}\phi &= -2\sqrt{x}\left(\frac{\partial\sqrt{x}}{\partial x}\phi + \sqrt{x}\frac{\partial\phi}{\partial x}\right) \\
&= -2\sqrt{x}\frac{1}{2\sqrt{x}}\phi - 2\sqrt{x}\sqrt{x}\frac{\partial\phi}{\partial x} \\
&= -\phi - 2x\frac{\partial\phi}{\partial x},
\end{aligned} \tag{173}$$

where

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \epsilon^{-c}\frac{\partial w}{\partial\xi}\frac{\partial\xi}{\partial x} \\
&= \epsilon^{a-c}\frac{\partial w}{\partial\xi}.
\end{aligned} \tag{174}$$

By using Eqs. (172)-(174) in Eq. (124), the RZSE is then written

$$\epsilon^{-c}\left[\epsilon^b\frac{\partial w}{\partial\tau} + w + 2\xi\frac{\partial w}{\partial\xi}\right] = 0, \tag{175}$$

and is invariant under the transformation  $\forall \epsilon$  if  $\epsilon^b = 2$ , i.e.,

$$\epsilon^{-c}\left[\frac{\epsilon^b}{2}\left(\frac{\partial w}{\partial\tau_{\Re}} - i\frac{\partial w}{\partial\tau_{\Im}}\right) + w + 2\xi\frac{\partial w}{\partial\xi}\right] = 0, \tag{176}$$

and

$$b = \frac{\log(2) + 2i\pi n}{\log(\epsilon)}, \quad \forall n \in \mathbb{Z}. \tag{177}$$

Therefore, it can be seen that since  $\phi$  solves the RZSE for  $x$  and  $s$ , then  $w = \epsilon^{-c}\phi$  solves the RZSE at  $x = \epsilon^{-a}\xi$ , and  $s = \epsilon^{-b}\tau$ . We now construct a group of independent variables such that

$$\begin{aligned}
\frac{\xi}{\tau^{a/b}} &= \frac{\epsilon^a x}{(\epsilon^b s)^{a/b}} \\
&= \frac{x}{s^{a/b}} \\
&= \eta(x, s),
\end{aligned} \tag{178}$$

and the similarity variable is then

$$\eta(x, s) = xs^{-\frac{a\log(\epsilon)}{\log(2)+2i\pi n}}. \tag{179}$$

Also,

$$\begin{aligned}
\frac{w}{\tau^{c/b}} &= \frac{\epsilon^c \phi}{(\epsilon^b s)^{c/b}} \\
&= \frac{\phi}{s^{c/b}} \\
&= \nu(\eta),
\end{aligned} \tag{180}$$



suggesting that we seek a solution of the RZSE with the form

$$\phi_s(x) = s^{\frac{c \log(\epsilon)}{\log(2) + 2i\pi n}} \nu(\eta). \quad (181)$$

Since the RZSE is invariant under the transformation, it is to be expected that the solution will also be invariant under the variable transformation. Taking  $a = c = \log^{-1}(\epsilon)$ , the partial derivatives transform like

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(x) &= \frac{\partial}{\partial s} \left( s^{\frac{1}{\log(2) + 2i\pi n}} \right) \nu(\eta) + \left( s^{\frac{1}{\log(2) + 2i\pi n}} \right) \nu'(\eta) \frac{\partial \eta}{\partial s} \\ &= \frac{s^{-1 + \frac{1}{\log(2) + 2i\pi n}}}{\log(2) + 2i\pi n} \left[ \nu(\eta) - \nu'(\eta) \right], \end{aligned} \quad (182)$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \phi_s(x) &= \left( s^{\frac{1}{\log(2) + 2i\pi n}} \right) \nu'(\eta) \frac{\partial \eta}{\partial x} \\ &= \nu'(\eta), \end{aligned} \quad (183)$$

where

$$\frac{\partial \eta}{\partial s} = -\frac{s^{-1}}{2i\pi n + \log(2)}, \quad (184)$$

and

$$\frac{\partial \eta}{\partial x} = s^{-\frac{1}{2i\pi n + \log(2)}}. \quad (185)$$

The RZSE then reduces to the ODE

$$\left[ s^{-1} + \log(2) + 2i\pi n \right] \nu(\eta) + \left[ -s^{-1} + 2\log(2)\eta + 4i\pi n\eta \right] \nu'(\eta) = 0, \quad \forall n \in \mathbb{Z}. \quad (186)$$

### 3.1 General Solution

The homogenous linear differential Eq. (186) is separable [49], viz.,

$$\frac{d\nu}{\nu} = \frac{2i\pi n + s^{-1} + \log(2)}{s^{-1} - 4i\pi n\eta - \eta \log(4)} d\eta. \quad (187)$$

Integrating on both sides, we obtain

$$\ln |\nu| = c_1 - \frac{\left( 2i\pi n + s^{-1} + \log(2) \right) \log \left( s^{-1} - 4i\pi n\eta - \eta \log(4) \right)}{4i\pi n + \log(4)}. \quad (188)$$

Exponentiating both sides,

$$|\nu| = \exp(c_1) \left( s^{-1} - 4i\pi n\eta - \eta \log(4) \right)^{-\frac{2i\pi n + s^{-1} + \log(2)}{4i\pi n + \log(4)}}. \quad (189)$$

Renaming the constant  $\exp(c_1) = C$  and dropping the absolute value recovers the lost solution  $\nu(\eta) = 0$ , giving the general solution to Eq. (186)

$$\nu_n(\eta) = C \left( s^{-1} - 4i\pi n\eta - \eta \log(4) \right)^{-\frac{2i\pi n + s^{-1} + \log(2)}{4i\pi n + \log(4)}}, \quad \forall n \in \mathbb{Z}, \forall C \in \mathbb{R} \quad (190)$$

By setting  $C = 1$ , and using Eqs. (179) and (181) in Eq. (190), we obtain the general solution to the RZSE Eq. (124), written

$$\phi_s(x) = s^{\frac{1}{\log(2)+2i\pi n}} \left[ \frac{1}{s} + s^{-\frac{1}{\log(2)+2i\pi n}} \left( -x \log(4) - 4i\pi n x \right) \right]^{-\frac{2\pi n s + i s \log(2) + i}{4\pi n s - i s \log(4)}}, \quad \forall n \in \mathbb{Z}. \quad (191)$$

## 4 Conclusion

In this study, we have discussed the decidability of the real part of every nontrivial zero of the analytic continuation of the Riemann zeta function. This was accomplished by developing a Riemann zeta Schrödinger equation and comparing it with the Bender-Brody-Müller conjecture in both configuration space and momentum space. A symmetrization procedure was implemented to study the eigenvalues of the system, and the expectation values were calculated from the resulting system to study the nontrivial zeros of the analytic continuation of the Riemann zeta function. It was found using Green's functions that the expectation value of the Hamiltonian operator for the eigenstates along the critical line  $\sigma = 1/2$  is also periodically zero such that the nontrivial zeros of the Riemann zeta function are not observable. A Gelfand triplet was implemented to ensure that the eigenvalues are well defined. Moreover, a second quantization procedure was performed for the Riemann zeta Schrödinger equation to obtain the equations of motion and an analytical expression for the eigenvalues. It was also demonstrated that the eigenvalues are holomorphic across the measurable subspace of the measure space. A normalized convergent expression for the analytic continuation of the nontrivial zeros of the Riemann zeta function was obtained, and a convergence test for the expression was performed demonstrating that the real part of every nontrivial zero of the Riemann zeta function exists at  $\sigma = 1/2$ . It was demonstrated from the orthogonality of the eigenstates that the Riemann Hypothesis is not decidable, i.e. *the analytic continuation of the Riemann zeta function is not an analytically computable function at  $\sigma = 1/2$* . Finally, a general solution to the Riemann zeta Schrödinger equation was found from performing an invariant similarity transformation.

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