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Approximation of variational problems with a convexity constraint by PDEs of Abreu type

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Abstract
Motivated by some variational problems subject to a convexity constraint, we consider an approximation using the logarithm of the Hessian determinant as a barrier for the constraint. We show that the minimizer of this penalization can be approached by solving a second boundary value problem for Abreu’s equation which is a well-posed nonlinear fourth-order elliptic problem. More interestingly, a similar approximation result holds for the initial constrained variational problem.

Keywords: Abreu equation, Monge-Ampère operator, calculus of variations with a convexity constraint.

MS Classification: 35G30, 49K30.

1 Introduction
Given Ω, a bounded, open, convex subset of \( \mathbb{R}^d \) with \( d \geq 2 \), \( F : \Omega \times \mathbb{R} \to \mathbb{R} \) strictly convex in its second argument, and \( \varphi \) a uniformly convex and smooth function defined in a neighbourhood of \( \Omega \), we are interested in the variational problem with a convexity constraint:

\[
\inf_{u \in S[\varphi, \Omega]} \mathcal{J}_0(u) := \int_{\Omega} F(x, u(x)) \, dx
\] 

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where $\mathcal{S}[\varphi, \Omega]$ consists of all convex functions on $\Omega$ which admit a convex extension by $\varphi$ in a neighbourhood of $\Omega$. This is a way to express in some weak sense the boundary conditions

$$u = \varphi \text{ and } \partial_\nu u \leq \partial_\nu \varphi \text{ on } \partial \Omega,$$

where $\nu$ denotes the outward normal to $\partial \Omega$ and $\partial_\nu$ denotes the normal derivative.

Due to the convexity constraint, it is really difficult to write a tractable Euler-Lagrange equation for (1.1) (see [7], [2]). One may therefore wish to construct suitable penalizations for the convexity constraint which force the minimizers to somehow remain in the interior of the constraint and thus to be a critical point of the penalized functional. Since the seminal work of Trudinger and Wang [9, 10] on the prescribed affine mean curvature equation, the regularity of convex solutions of fourth-order nonlinear PDEs which are Euler-Lagrange equations of convex functionals involving the Hessian determinant have received a lot of attention. In particular, the Abreu equation which corresponds to the logarithm of the Hessian determinant has been studied by Zhou [11] in dimension 2 and more recently by Chau and Weinkove [3] and Le [5, 6] in higher dimensions. What the well-posedness and regularity results of these references in particular suggest is that a penalization involving the logarithm of the Hessian determinant should act as a good barrier for the convexity constraint in problems like (1.1). This was indeed confirmed numerically at a discretized level, see [1].

Our goal is precisely to show that one can indeed approximate (1.1) by a suitable boundary value problem for the Abreu equation. To do so, we first introduce a penalized version of (1.1) with a small parameter $\varepsilon > 0$:

$$\inf_{v \in \mathcal{S}[\varphi, \Omega]} \mathcal{J}_\varepsilon(v) := \mathcal{J}_0(v) - \varepsilon \mathcal{F}_\Omega(v)$$

where, when $v \in \mathcal{S}[\varphi, \Omega]$ is smooth and strongly convex, (see section 2 for the definition for an arbitrary $v \in \mathcal{S}[\varphi, \Omega]$), $\mathcal{F}_\Omega(v)$ is defined by

$$\mathcal{F}_\Omega(v) := \int_\Omega \log(\det D^2 v).$$

Using the convexity of $\mathcal{J}_\varepsilon$ setting $f(x, u) := \partial_\nu F(x, u)$, one can easily see that if $u$ is smooth and uniformly convex up to $\partial \Omega$, and solves the first-boundary problem for Abreu equation

$$\varepsilon U^{ij} w_{ij} = f(x, u) \text{ in } \Omega, \quad u = \varphi \text{ and } \partial_\nu u = \partial_\nu \varphi \text{ on } \partial \Omega$$

(1.4)
where \( w := \det(D^2u)^{-1} \) and \( U \) denotes the cofactor matrix of \( D^2u \) then it is indeed the solution of (1.3). It turns out however that the second-boundary value problem (where instead of prescribing both values of \( u \) and \( \nabla u \) one rather prescribes \( u \) and \( \det(D^2u) \) on \( \partial\Omega \)) is much more well-behaved, see [3, 5, 6] and it was indeed used as an approximation for the affine Plateau problem in [9]. We shall also consider an extra approximation parameter and a second-boundary value problem on a larger domain and show that it approximates correctly not only (1.3) but also the initial problem (1.1) as the parameter converges to zero.

The paper is organized as follows. Section 2 gives some preliminaries. In section 3, we show a \( \Gamma \)-convergence result for \( J_\varepsilon \). In section 4, we consider an approximation by a second boundary value problem on a ball \( B \) containing \( \Omega \), with a further penalization \( \frac{1}{\delta}(u-\varphi) \) on \( B \setminus \Omega \), for which we prove existence and uniqueness of a smooth solution. In section 5 we show that when \( \delta \to 0 \), we recover the minimizer of the problem from section 3. Finally, we also show full convergence of the second boundary value problem to the initial constrained variational problem (1.1) when \( \delta = \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \), provided \( F \) satisfies a suitable uniform convexity assumption.

## 2 Preliminaries

In the sequel, \( \Omega \) will be an open, bounded and convex subset of \( \mathbb{R}^d \), \( d \geq 2 \). We are also given an open ball \( B \) containing \( \Omega \) and assume that the boundary datum \( \varphi \) satisfies for some \( \lambda > 0 \):

\[
\varphi \in C^{3,1}(\overline{B}), \; \varphi = 0 \text{ on } \partial B, \; D^2\varphi \geq \lambda \text{ id on } B. \tag{2.1}
\]

We then define \( \mathcal{S}[\varphi, \Omega] \) as the set of convex functions on \( \Omega \), which, once extended by \( \varphi \) on \( B \setminus \Omega \), are convex on \( B \). Note that elements of \( \mathcal{S}[\varphi, \Omega] \) coincide with \( \varphi \) on \( \partial \Omega \) and are Lipschitz continuous with Lipschitz constant at most \( \|\nabla \varphi\|_{L^\infty(B)} \) so that \( \mathcal{S}[\varphi, \Omega] \) is compact for the topology of uniform convergence.

Finally, we assume that the integrand \( F: (x, u) \in \Omega \times \mathbb{R} \mapsto F(x, u) \) in the definition of \( J_0 \) in (1.1) is measurable with respect to \( x \), strictly convex and differentiable with respect to \( u \) and such that that \( F(., 0) \in L^1(\Omega) \) and \( f(.,u) := \partial_u F(.,u) \) satisfies \( f(., u) \in L^\infty(\Omega) \) for every \( u \in \mathbb{R} \). These assumptions in particular guarantee that the convex functional \( J_0 \) is everywhere continuous and Gâteaux differentiable on \( \mathcal{S}[\varphi, \Omega] \).

Following [9, 10, 11], let us recall how to define \( \mathcal{S}_\Omega(v) \) for an arbitrary convex function \( v \) on \( \Omega \), first recall that the subdifferential of \( v \) at \( x \in \Omega \) is
given by
\[ \partial v(x) := \{ p \in \mathbb{R}^d : v(y) - v(x) \geq p \cdot (y - x), \forall y \in \Omega \}. \]
The Monge-Ampère measure of \( v \), denoted \( \mu[v] \) is then defined by
\[ \mu[v](A) := |\partial v(A)| \]
for every Borel subset \( A \) of \( \Omega \). From the seminal results of Alexandrov (see [4]), \( \mu[v] \) is indeed a Radon measure and \( v \mapsto \mu[v] \) is weakly continuous in the sense that whenever \( v_n \) are convex functions which locally uniformly converge to \( v \) then
\[ \limsup_{n} \mu[v_n](F) \leq \mu[v](F), \forall F \subset \Omega, \text{ closed}. \]
Decomposing the Monge-Ampère measure into its absolutely continuous part and its singular part (with respect to the Lebesgue measure \( \mathcal{L}^d \)) as
\[ \mu[v] = \mu_r[v] + \mu_s[v], \mu_r[v] \ll \mathcal{L}^d, \mu_s[v] \perp \mathcal{L}^d. \]
Thanks to Alexandrov’s theorem, \( v \) is differentiable twice a.e., at such points of twice differentiability, we denote by \( \partial^2 v \) its Hessian matrix, Trudinger and Wang proved in [9] that \( \det(\partial^2 v) \) is the density of \( \mu_r[v] \) with respect to \( \mathcal{L}^d \), and following their approach, one can define the functional \( F_\Omega \) by
\[ F_\Omega(v) := \int_{\Omega} \log(\det \partial^2 v(x))dx, \forall v \in \overline{S}[\varphi, \Omega]. \]
(2.2)
It is well-known that \( F_\Omega \) is a concave functional and we refer to [8, 9, 11] for a proof of the useful properties of \( F_\Omega \) recalled below in Lemmas 2.1 and 2.2
\[ F_\Omega(v) \leq C_{\Omega, \varphi} := |\Omega| \log \left( \frac{c_d \|
abla \varphi\|_{L^\infty}}{|\Omega|} \right), \forall v \in \overline{S}[\varphi, \Omega]. \]
(2.3)
As we shall also work on the larger domain \( B \), it will be also convenient to consider for every open subset \( \omega \) of \( B \) and every convex function \( u \) on \( B \) the concave functional
\[ F_\omega(v) := \int_\omega \log(\det \partial^2 v(x))dx. \]
(2.4)
Following the same lines as Lemma 6.4 in Trudinger-Wang [8], we also have:
Lemma 2.2. If $\omega$ is an open subset of $B$ with $\omega \subset \subset B$, then for every sequence of convex functions $u_n$ converging locally uniformly on $B$ to $u$, one has

$$\limsup_n \mathfrak{A}_\omega(u_n) \leq \mathfrak{A}_\omega(u).$$

3 Logarithmic penalization

Given $\varepsilon > 0$, we consider

$$(3.1) \quad \inf_{v \in \mathfrak{S}[\varphi, \Omega]} J_\varepsilon(v) := J_0(v) - \varepsilon \mathfrak{A}_\Omega(v).$$

Since $J_\varepsilon$ is strictly convex and lsc on the convex compact set $\mathfrak{S}[\varphi, \Omega]$, we immediately have:

Proposition 3.1. Problem $(3.1)$ admits a unique solution $v_\varepsilon$.

Arguing exactly as in [9, 11] by using Alexandrov’s maximum principle, one can show:

Lemma 3.2. Let $\varepsilon > 0$ and $v_\varepsilon$ be the solution of $(3.1)$ then $\mu_a[v_\varepsilon] = 0$ i.e. $\mu[v_\varepsilon]$ has no singular part.

Remark 3.3. Let us remark that Lemma 3.2 enables one to express $-\mathfrak{A}_\Omega(v_\varepsilon)$ in an alternative way as the entropy of the push-forward of the Lebesgue measure on $\Omega$ by $\nabla v_\varepsilon$. Also, thanks to Lemma 3.2 one can prove uniqueness of the solution of $(3.1)$ when $J_0$ is convex but not necessarily strictly convex.

In dimension 2, we actually even have a uniform local bound on $\det(\partial^2 v_\varepsilon)$:

Proposition 3.4. Let $d = 2$, $\varepsilon > 0$ and $v_\varepsilon$ be the solution of $(3.1)$, then $\mu[v_\varepsilon] = \det(\partial^2 v_\varepsilon) \in L^\infty_{\text{loc}}(\Omega)$.

Proof. It follows from Theorem 5.1 and Proposition 4.3 that $v_\varepsilon$ is the uniform limit as $\delta \to 0$ of a sequence of smooth functions $(v_\varepsilon^\delta)$ in $\mathfrak{S}[\varphi, \Omega]$ such that, for every open subset $\omega$ with $\omega \subset \subset \Omega$, $\|\det(D^2 v_\varepsilon^\delta)\|_{L^\infty(\omega)} \leq C$ where $C$ is a constant that depends on $\varepsilon$ and $\omega$ but not on $\delta$. By weak convergence of Monge-Ampère measures we deduce that $\det(\partial^2 v_\varepsilon) \in L^\infty_{\text{loc}}(\Omega)$.

Let us now state a $\Gamma$-convergence result for $J_\varepsilon$:
Proposition 3.5. The family of functionals $J_\varepsilon$ defined on $\overline{S}[\varphi, \Omega]$ equipped with the topology of uniform convergence $\Gamma$-converges to $J_0$ in particular $v_\varepsilon$ converges uniformly to the solution of (1.1).

Proof. Assume $u_\varepsilon$ is a family in $\overline{S}[\varphi, \Omega]$ that converges uniformly as $\varepsilon \to 0$ to $u$, thanks to (2.3) and Fatou’s Lemma, we have

$$\liminf_{\varepsilon} J_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon} (J_0(u_\varepsilon) - \varepsilon C_{\Omega, \varphi}) \geq J_0(u).$$

Given $u \in \overline{S}[\varphi, \Omega]$, we now look for a recovery sequence $u_\varepsilon \in \overline{S}[\varphi, \Omega]$ converging to $u$ and such that $\limsup_{\varepsilon} J_\varepsilon(u_\varepsilon) \leq J_0(u)$, we simply take

$$u_\varepsilon := (1 - \varepsilon)u + \varepsilon \varphi$$

since $\partial^2 u_\varepsilon \geq \varepsilon D^2 \varphi$ we have

$$\mathfrak{B}_\Omega(u_\varepsilon) \geq d|\Omega| \log(\varepsilon) + \int_{\Omega} \log(\det(D^2 \varphi))$$

with the convexity of $J_0$, we then have

$$\limsup_{\varepsilon} J_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon} ((1 - \varepsilon)J_0(u) + \varepsilon J_0(\varphi)) + O(\varepsilon \log(\varepsilon)) \leq J_0(u).$$

4 Second boundary value approximation

Having Proposition 3.5 in mind, we now fix the value of $\varepsilon$. Throughout this section, to simplify notations, we therefore take $\varepsilon = 1$ and we are interested in approximating the solution of

$$\inf_{v \in \overline{S}[\varphi, \Omega]} J_1(v) := \int_{\Omega} F(x, v(x))dx - \mathfrak{B}_\Omega(v), \quad (4.1)$$

by a second-boundary value problem for Abreu equation. More precisely given $\delta > 0$, we consider

$$U^{ij} w_{ij} = f_\delta(x, u), \quad \text{in } B, \ u = \varphi, \ w = \psi \text{ on } \partial B \quad (4.2)$$

where $\psi := \det((D^2 \varphi)^{-1})$ and

$$f_\delta(x, u) := \begin{cases} f(x, u) & \text{if } x \in \Omega \\ \frac{1}{\delta}(u - \varphi(x)) & \text{if } x \in B \setminus \Omega \end{cases}$$
and as before \( w = \det(D^2u)^{-1} \) and \( U \) is the cofactor matrix of \( D^2u \). In view of (4.2) and the definition of \( f_\delta \), it is natural to introduce the functional defined over convex functions on \( B \) by

\[
J_1^\delta(v) := \int_\Omega F(x, v(x))\,dx + \frac{1}{2\delta} \int_{B \setminus \Omega} (v - \varphi)^2 - \mathfrak{F}_B(v)
\]

where

\[
\mathfrak{F}_B(v) := \int_B \log(\det(\partial^2 v))
\]

so that

\[
J_1^\delta(v) = J_1(v) + \frac{1}{2\delta} \int_{B \setminus \Omega} (v - \varphi)^2 - \int_{B \setminus \Omega} \log(\det(\partial^2 v)). \tag{4.3}
\]

4.1 A priori estimates for the second boundary value problem

Following a similar convexity argument as in Lemma 2.2 in Chau and Weinkove [3], we first have

**Proposition 4.1.** Let \( u \) be a smooth and uniformly convex solution of (4.2), then

\[
\max_B |u| + \int_{\partial B} |\partial u|^d + \frac{1}{\delta} \int_{B \setminus \Omega} |u - \varphi|^2 \leq C \tag{4.4}
\]

for some constant \( C \) only depending on \( B, \|\varphi\|_{C^{1,1}(B)} \) and the constant \( \lambda \) in (2.1).

**Proof.** First observe that by convexity and (2.1), \( u < 0 \) in \( B \) and \( \partial u > 0 \) on \( \partial B \). Define \( \tilde{u} := \varphi, \tilde{U} \) as the cofactor matrix of \( D^2 \varphi, \tilde{w} := \det(D^2 \varphi)^{-1} \) and \( \tilde{f} := \tilde{U}^{ij} \tilde{w}_{ij} \) (whose \( L^\infty \) norm only depends on \( \|\varphi\|_{C^{1,1}(B)} \) and the constant \( \lambda \) in (2.1)) we have by the concavity of \( \mathfrak{F}_B, (4.2) \) and the monotonicity of
\begin{align*}
0 & \geq (\tilde{\mathcal{F}}_B'(u) - \tilde{\mathcal{F}}_B'(\tilde{u}))(u - \tilde{u}) \\
& = \int_B (U^{ij} w_{ij} - \tilde{U}^{ij} \tilde{w}_{ij})(u - \varphi) + \int_{\partial B} \psi(U^{ij} - \tilde{U}^{ij}) \partial_i (u - \varphi) \nu_j \\
& = \int_{\Omega} f(x, u)(u - \varphi) - \int_B \tilde{f}(u - \varphi) + \frac{1}{\delta} \int_{B\setminus\Omega} (u - \varphi)^2 \\
& + \int_{\partial B} \psi(U^{\nu\nu} - \tilde{U}^{\nu\nu}) \partial_{\nu} (u - \varphi) \\
& \geq \int_{\Omega} f(x, \varphi)(u - \varphi) - \int_B \tilde{f}(u - \varphi) + \frac{1}{\delta} \int_{B\setminus\Omega} (u - \varphi)^2 \\
& + \int_{\partial B} \psi(U^{\nu\nu} - \tilde{U}^{\nu\nu}) \partial_{\nu} (u - \varphi)
\end{align*}

where, in the last line, we have used the fact that \( \nabla u - \nabla \varphi = \partial_{\nu} (u - \varphi) \nu \) on \( \partial B \) and set \( U^{\nu\nu} = U^{\nu} \cdot \nu \), \( \tilde{U}^{\nu\nu} = \tilde{U}^{\nu} \cdot \nu \). Using the fact that \( f(x, \varphi), \tilde{f}, \varphi, \nabla \varphi \) and \( \tilde{\mathcal{U}} \) are bounded, we thus get

\[
\frac{1}{\delta} \int_{B\setminus\Omega} (u - \varphi)^2 + \int_{\partial B} \psi U^{\nu\nu} \partial_{\nu} u \leq C \left( 1 + \int_B |u| + \int_{\partial B} \partial_{\nu} u + \int_{\partial B} U^{\nu\nu} \right). \quad \text{(4.5)}
\]

Denoting by \( R \) the radius of \( B \) and by the same argument as in Lemma 2.2 in \cite{3}, one has

\[
U^{\nu\nu} = \frac{1}{R^{d-1}} \partial_{\nu} u^{d-1} + E \quad \text{with} \quad |E| \leq C(1 + \partial_{\nu} u^{d-2}) \quad \text{on} \quad \partial B. \quad \text{(4.6)}
\]

Moreover since \( u \) is convex and \( u = \varphi = 0 \) on \( \partial B \), one has

\[
\max_{B} |u| = -\min_{B} u \leq 2R \partial_{\nu} u(x) \quad \text{for all} \quad x \in \partial B. \quad \text{(4.7)}
\]

Putting together (4.5), (4.6), (4.7) and the fact that \( \inf_{\partial B} \psi > 0 \), we obtain

\[
\int_{\partial B} (\partial_{\nu} u)^d \leq C(1 + \int_{\partial B} (\partial_{\nu} u)^{d-1})
\]

which gives a bound on \( \| \partial_{\nu} u \|_{L^d(\partial B)} \) hence also on \( \max_B |u| \) by (4.7) so that finally the bound on \( \delta^{-1} \int_{B\setminus\Omega} (u - \varphi)^2 \) follows from the latter bounds and (4.5). \qed
4.2 Existence and uniqueness of a smooth uniformly convex solution

Thanks to Theorem 1.1 in [3], a Leray-Schauder degree argument and the a priori estimate (4.4), one easily deduces the following:

**Theorem 4.2.** For every $\delta > 0$, the second boundary value problem (4.2) admits a unique uniformly convex solution which is $W^{4,p}(B)$ for every $p \in [1, +\infty)$.

**Proof.** Let $D := \{ u \in C(\overline{B}), \| u \|_{C(\overline{B})} \leq C + 1 \}$ where $C$ is the constant from (4.4). For $t \in [0,1]$ and $u \in D$, it follows from Theorem 1.1 in [3] that there exists a unique $W^{4,p}$ for every $p \in [1, \infty)$ and uniformly convex solution of

$$V^{ij} w_{ij} = t f_\delta(x, u), w = \det(D^2 v)^{-1} \text{ in } B, \ v = \varphi, w = \psi \text{ on } \partial B \quad (4.8)$$

where $V$ denotes the cofactor matrix of $D^2 v$. We denote by $v = T_t(u)$ the solution of (4.8). Moreover, by Theorem 2.1 of [3], for every $\alpha \in (0,1)$ there are a priori bounds on $\| v \|_{C^\alpha}$ and on $\sup_B (\det(D^2 v) + \det(D^2 v)^{-1})$ that only depend on $C$, $\alpha$, $\delta$, $\| \varphi \|_{C^{2,1}}$ and the constant $\lambda$ in (2.1). Therefore $(t,u) \in [0,1] \times D \mapsto T_t(u)$ is continuous on $[0,1] \times D$ and $T_t$ is compact in $C(\overline{B})$ for every $t \in [0,1]$. Since $T_0$ is constant and by (4.4) it has a unique fixed point in $D$, again by (4.4), $T_t$ has no fixed point on $\partial D$, it thus follows from the Leray-Schauder Theorem that $T_1$ has a fixed point in $D$, this proves the existence claim for (4.2).

Finally, uniqueness follows from the same argument as in Lemma 7.1 from [10] where it is proven that two smooth solutions actually have the same gradient on $\partial B$ and then are the minimizers of the same strictly convex minimization problem hence coincide.

In dimension $d = 2$, following the argument of Remark 4.2 of Trudinger and Wang [9] and taking advantage of the fact that the right-hand side of the Abreu equation (4.2) does not depend on $\delta$ on $\Omega$, we have the following local bound (which we have used in the proof of Proposition 3.4):

**Proposition 4.3.** Let $d = 2$ and $u$ be the solution of (4.2) then for every open set $\omega \subset \subset \Omega$, $\| \det(D^2 u) \|_{L^\infty_{loc}(\omega)}$ is bounded independently of $\delta$.

**Proof.** Let $B_r := B_r(0) \subset \subset \Omega$, and observe that thanks to (1.4) both $\| f_\delta(\cdot, u(\cdot)) \|_{L^\infty(\Omega)} = \| f(\cdot, u(\cdot)) \|_{L^\infty(\Omega)}$ and $\| \nabla u \|_{L^\infty(\Omega)}$ are bounded independently of $\delta$. Define then $\eta(x) := \frac{1}{2}(r^2 - |x|^2)$ and consider $z := \log(w)$ -
\[2 \log(\eta) - \frac{1}{2} |\nabla u|^2,\] by construction \( z \) achieves its minimum at an interior point \( x_0 \) of \( B_r \). At such a point, we have

\[
\frac{\nabla w}{w} = 2 \frac{\nabla \eta}{\eta} + D^2 \nabla u. \tag{4.9}
\]

We also have

\[
z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + 2 \frac{\delta_{ij}}{\eta} + 2 \frac{\eta_i \eta_j}{\eta^2} - u_{ijk} u_k - u_{ik} u_{jk}, \tag{4.10}
\]

multiplying by \( wU = [D^2 u]^{-1} \), using \( wU z_{ij} \geq 0 \) at \( x_0 \), \( U^{ij} w_{ij} = f(x, u) \leq C \) and the identities

\[
w^{ij} u_{ik} u_{jk} = u_{ii} = \Delta u, \quad w^{ij} u_{ijk} u_k = -\frac{w_k}{w} u_k = -\frac{\nabla w}{w} \cdot \nabla u, \tag{4.11}
\]

(the second identity is classically obtained by first differentiating the relation \(- \log(w) = \log(\det D^2 u)\) and then taking the scalar product with \( \nabla u \)) as well as the fact that \( \text{Tr}(U) = \Delta u \) in dimension \( d = 2 \), we get

\[
0 \leq C - wU \frac{\nabla w}{w} \cdot \nabla w + 2 \frac{w}{\eta} \Delta u + 2 w U \frac{\nabla \eta}{\eta} \cdot \frac{\nabla \eta}{\eta} + \frac{\nabla w}{w} \cdot \nabla u - \Delta u. \tag{4.12}
\]

Using \( \text{(4.9)} \) and using again \( wU = [D^2 u]^{-1} \), we then obtain

\[
wU \frac{\nabla w}{w} \cdot \nabla w = 4 w U \frac{\nabla \eta}{\eta} \cdot \frac{\nabla \eta}{\eta} + D^2 \nabla u \cdot \nabla u + 4 \frac{\nabla \eta}{\eta} \cdot \nabla u \tag{4.13}
\]

and

\[
\frac{\nabla w}{w} \cdot \nabla u = 2 \frac{\nabla \eta}{\eta} \cdot \nabla u + D^2 \nabla u \cdot \nabla u. \tag{4.14}
\]

Replacing \( \text{(4.13)}, \text{ (4.14)} \) in \( \text{(4.12)} \), multiplying by \( \eta \) and rearranging gives

\[
\Delta u(\eta - 2w) \leq C \eta - 2 w U \nabla \eta \cdot \nabla \eta - 2 \nabla \eta \cdot \nabla u \leq C \eta + \|\nabla \eta\|_{L^\infty} \|\nabla u\|_{L^\infty(\Omega)} \leq C'. \tag{4.15}
\]

If \( \eta(x_0) \geq 4 w(x_0), \text{ (4.15)} \) gives \( \eta(x_0) \Delta u(x_0) \leq 2 C' \) and since \( \Delta u(x_0) w(x_0)^{1/2} \geq 2 \) we get the desired lower bound on the minimum of \( \eta^{-2} w \). In the remaining case \( w(x_0) \geq \frac{1}{4} \eta(x_0) \geq \frac{\eta^2(x_0)}{2r^2} \) and we reach the same conclusion. This gives a local lower bound on \( w \) i.e. the desired local upper bound on \( \det(D^2 u) \).
5 Convergence

5.1 Letting \( \delta \to 0 \) for fixed \( \varepsilon \)

In this paragraph, we fix \( \varepsilon \) (and thus normalize it to \( \varepsilon = 1 \) as we did in the whole of section 4).

**Theorem 5.1.** Let \( u_\delta \) be the unique smooth strictly convex solution of (4.2), then \( u_\delta \) converges uniformly on \( \Omega \) to the unique minimizer of (4.1) as \( \delta \to 0^+ \).

**Proof.** We already know from (4.4) that (possibly up to an extraction) \( u_\delta \) converges locally uniformly on \( B \) to some convex \( u \) and it also follows from (4.4) that \( u \in \mathcal{S}[\varphi, \Omega] \). Let \( v \in \mathcal{S}[\varphi, \Omega] \) (extended by \( \varphi \) on \( B \setminus \Omega \)), thanks to (4.2) and the convexity of \( J_1 \)

\[
J_1^\delta(v) - J_1^\delta(u_\delta) \geq \int_{\partial B} U_\delta^{nu} \psi \partial_n(u_\delta - \varphi)
\]

i.e.

\[
J_1(v) - J_1(u_\delta) \geq \frac{1}{2\delta} \int_{B \setminus \Omega} (u_\delta - \varphi)^2 + \int_{B \setminus \Omega} (\log(\det(D^2 \varphi)) - \log(\det(D^2 u_\delta)))
\]

\[
+ \int_{\partial B} U_\delta^{nu} \psi \partial_n(u_\delta - \varphi)
\]

\[
\geq \frac{1}{2\delta} \int_{B \setminus \Omega} (\log(\det(D^2 \varphi)) - \log(\det(D^2 u_\delta))) + \int_{\partial B} U_\delta^{nu} \psi \partial_n(u_\delta - \varphi).
\]

It follows from Lemma 5.2 below that

\[
\liminf_{\delta \to 0} \int_{B \setminus \Omega} (\log(\det(D^2 \varphi)) - \log(\det(D^2 u_\delta))) \geq 0.
\]

We now have to pay attention to the boundary term, we know from (4.6) that \( \theta_\delta := \psi U_\delta^{nu} \) satisfies \( 0 \leq \theta_\delta \leq C(1 + (\partial_n u_\delta)^{d-1}) \) so that thanks to (4.4), \( \theta_\delta \) is bounded in \( L^{\frac{d}{d-1}}(\partial B) \), up to an extraction we may therefore assume that it weakly converges in \( L^{\frac{d}{d-1}}(\partial B) \) to some nonnegative function \( \theta \). By convexity we also have that for \( \tau > 0 \)

\[
\partial_n u_\delta(x) \geq D_{\tau, \nu} u_\delta(x) := \frac{1}{\tau} \left( u_\delta(x - \tau \nu(x)) - u_\delta(x - 2\tau \nu(x)) \right), \quad \forall x \in \partial B
\]

For small fixed \( \tau > 0 \) note that \( D_{\tau, \nu} u_\delta \) is bounded independently of \( \delta \) thanks to (4.4) and that it converges as \( \delta \to 0 \) pointwise to \( D_{\tau, \nu} \varphi \), we thus have

\[
\liminf_{\delta \to 0} \int_{\partial B} \theta_\delta \partial_n(u_\delta - \varphi) \geq \liminf_{\delta \to 0} \int_{\partial B} \theta_\delta (D_{\tau, \nu} u_\delta - \partial_n \varphi)
\]

\[
= \int_{\partial B} \theta(D_{\tau, \nu} \varphi - \partial_n \varphi)
\]

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where in the last line we have passed to the limit using the fact that we have the product of a weakly convergent sequence with a strongly convergent sequence. Now letting $\tau \to 0$ and using the smoothness of $\varphi$, we deduce that

$$\liminf_{\delta \to 0} \int_{\partial B} U_\delta^{\nu \nu} \partial_\nu (u_\delta - \varphi) \geq 0.$$ 

Since $\mathcal{J}_1$ is lower semi-continuous thanks to Lemma 2.2, we can conclude that

$$\mathcal{J}_1(v) \geq \liminf_{\delta \to 0} \mathcal{J}_1(u_\delta) \geq \mathcal{J}_1(u)$$

so that $u$ solves (4.1) and by uniqueness of the minimizer there is in fact convergence of the whole sequence.

In the previous proof we have used:

**Lemma 5.2.** Let $u_\delta$ be the unique smooth strictly convex solution of (4.2) as before, then

$$\limsup_{\delta \to 0} \int_{B \setminus \Omega} \log(\det(D^2 u_\delta)) \leq \int_{B \setminus \Omega} \log(\det(D^2 \varphi)).$$

**Proof.** The key point here is the estimate $\int_B \Delta u_\delta = \int_{\partial B} \partial_\nu u_\delta \leq C$ which follows from (4.4). Let $\omega$ be an arbitrary Borel subset of $B$, we have (for some constant $C$ varying from a line to another):

$$\int_{\omega} \log(\det(D^2 u_\delta)) \leq C(\omega + \int_{\omega} \sqrt{\Delta u_\delta}) \leq C(\omega + \omega^{1/2} \left( \int_B \Delta u_\delta \right)^{1/2})$$

$$= C(\omega + \omega^{1/2} \left( \int_{\partial B} \partial_\nu u_\delta \right)^{1/2})$$

so that

$$\int_{\omega} \log(\det(D^2 u_\delta)) \leq C(\omega + \omega^{1/2}). \quad (5.1)$$

Take $0 < R' < R$ with $\Omega$ contained in $B_{R'}$ (recall $R$ is the radius of $B$), we then have, thanks to Lemma 2.2 the fact that $\log(\det(D^2 \varphi))$ is bounded.
\[ \limsup_{\delta \to 0} \mathcal{S}_{B \setminus \Omega}(u_{\delta}) = \limsup_{\delta \to 0} \mathcal{S}_{B \setminus \Omega}(u_{\delta}) \]
\[ \leq \limsup_{\delta \to 0} \mathcal{S}_{B \setminus \Omega}(u_{\delta}) + \limsup_{\delta \to 0} \mathcal{S}_{B \setminus B_{R'}^c}(u_{\delta}) \]
\[ \leq \mathcal{S}_{B \setminus \Omega}(\varphi) + C(|B \setminus B_{R'}| + |B \setminus B_{R'}|^{1/2}) \]
\[ \leq \mathcal{S}_{B \setminus \Omega}(\varphi) + C'(|B \setminus B_{R'}| + |B \setminus B_{R'}|^{1/2}). \]

The desired result follows by letting \( R' \) tend to \( R \).

### 5.2 Full convergence

We now take \( \delta = \delta_\varepsilon > 0 \) with
\[ \lim_{\varepsilon \to 0^+} \delta_\varepsilon = 0, \quad (5.2) \]
i.e. we only have a single small parameter \( \varepsilon \) and we consider the second-boundary value problem
\[ \varepsilon U_{ij}^{\varepsilon} w_{ij}^{\varepsilon} = g_\varepsilon(x, u_\varepsilon), \quad \text{in } B, \quad u_\varepsilon = \varphi, \quad w_\varepsilon = \psi \text{ on } \partial B \quad (5.3) \]
where \( \psi := \det((D^2 \varphi)^{-1}) \),
\[ g_\varepsilon(x, u) := \begin{cases} f(x, u) & \text{if } x \in \Omega \\ \frac{1}{\delta_\varepsilon}(u - \varphi(x)) & \text{if } x \in B \setminus \Omega \end{cases}, \]
\[ w_\varepsilon = \det(D^2 u_\varepsilon)^{-1} \text{ and } U_\varepsilon \text{ is the cofactor matrix of } D^2 u_\varepsilon. \]
We further assume that there is an \( \alpha > 0 \) such that
\[ (f(x, u) - f(x, v))(u - v) \geq \alpha (u - v)^2, \quad \forall (u, v) \in \mathbb{R}^d, \quad \text{a.e. } x \in \Omega \quad (5.4) \]
which amounts to say that the integrand \( F \) is uniformly convex in its second argument. Under these assumptions, we have a full convergence result:

**Theorem 5.3.** Let \( u_\varepsilon \) be the unique smooth strictly convex solution of (5.3), then \( u_\varepsilon \) converges uniformly on \( \Omega \) to the unique minimizer of (1.1) as \( \varepsilon \to 0^+ \).

**Proof. Step 1: a priori estimates.** The first step of the proof is similar to the proof of Proposition 4.1. Again define \( \tilde{u} := \varphi, \tilde{U} \) as the cofactor matrix
of $D^2\varphi$, $\tilde{w} := \det(D^2\varphi)^{-1}$ and $\tilde{f}_\varepsilon := \varepsilon \tilde{U}^{ij}\tilde{w}_{ij}$. We then have together with (5.4):

$$0 \geq \varepsilon(\tilde{F}_B(u_\varepsilon) - \tilde{F}_B(\tilde{u}))(u_\varepsilon - \tilde{u})$$

$$\geq \int_{\Omega} (f(x, \varphi) - \tilde{f}_\varepsilon)(u_\varepsilon - \varphi) + \alpha \int_{\Omega} (u_\varepsilon - \varphi)^2 + \frac{1}{\delta_\varepsilon} \int_{\partial B \setminus \Omega} (u_\varepsilon - \varphi)^2$$

$$+ \varepsilon \int_{\partial B} \psi(U^{\nu\nu}_\varepsilon - \tilde{U}^{\nu\nu})\partial_\nu(u_\varepsilon - \varphi)$$

thanks to the fact that $f(x, \varphi) - \tilde{f}_\varepsilon$ is bounded uniformly with respect to $\varepsilon$, using Young’s inequality and invoking (4.6), we get

$$\int_{\Omega} (u_\varepsilon - \varphi)^2 + \frac{1}{\delta_\varepsilon} \int_{\partial B \setminus \Omega} (u_\varepsilon - \varphi)^2 + \varepsilon \int_{\partial B} (\partial_\nu u_\varepsilon)^d \leq C. \quad (5.5)$$

**Step 2: convergence.** Thanks to (5.5), up to taking a subsequence of vanishing $\varepsilon_n$, we may assume that $u_\varepsilon$ converges locally uniformly in $B$ to some $u$ such that $u = \varphi$ in $B \setminus \Omega$ so that the restriction of $u$ to $\Omega$ belongs to $S[\varphi, \Omega]$. For every $v$ convex on $B$ such that $v = \varphi$ on $\partial B$, define

$$\tilde{J}_\varepsilon(v) := \int_{\Omega} F(x, v(x))dx + \frac{1}{2\delta_\varepsilon} \int_{\partial B \setminus \Omega} (v - \varphi)^2 - \varepsilon \int_{B} \log(\det(\partial^2 v)),$$

Let then $v \in S[\varphi, \Omega]$ (extended by $\varphi$ on $B \setminus \Omega$), we then have

$$\tilde{J}_\varepsilon(v) - \tilde{J}_\varepsilon(u_\varepsilon) \geq \varepsilon \int_{\partial B} \psi U^{\nu\nu}_\varepsilon \partial_\nu(u_\varepsilon - \varphi)$$

hence

$$J_0(v) \geq \lim inf_{\varepsilon} J_0(u_\varepsilon) + \lim inf_{\varepsilon} \varepsilon(\tilde{F}_B(v) - \tilde{F}_B(u_\varepsilon)) - \lim sup_{\varepsilon} \varepsilon \int_{\partial B} \psi U^{\nu\nu}_\varepsilon \partial_\nu \varphi.$$

Arguing as in the proof of Proposition 3.5, we may actually assume that $\tilde{F}_\Omega(v) > -\infty$ so that $\lim inf_{\varepsilon} \varepsilon\tilde{F}_B(v) \geq 0$. As for an upper bound for $\varepsilon\tilde{F}_B(u_\varepsilon)$ we use the fact that thanks to (3.5), we have $\int_{\partial B} \partial_\nu u_\varepsilon \leq C\varepsilon^{-1/d}$ and argue in a similar way as in the proof of Lemma 5.2 to obtain

$$\varepsilon\tilde{F}_B(u_\varepsilon) \leq C\varepsilon(1 + \int_{B} \det(D^2 u_\varepsilon)^{1/d}) \leq C\varepsilon(1 + \int_{\partial B} \partial_\nu u_\varepsilon) \leq C(\varepsilon + \varepsilon^{1-1/d}),$$

which yields

$$\lim inf_{\varepsilon} \varepsilon(\tilde{F}_B(v) - \tilde{F}_B(u_\varepsilon)) \geq 0.$$
Thanks to (4.6), we have
\[
\int_{\partial B} \psi U^\nu \partial_\nu \varphi \leq C \int_{\partial B} (1 + (\partial_\nu u_\varepsilon)^{d-1})
\]
but, thanks to (5.5) and Hölder’s inequality, we deduce
\[
\varepsilon \int_{\partial B} (\partial_\nu u_\varepsilon)^{d-1} \leq C \varepsilon^{\frac{1}{d}}
\]
so that
\[
\mathcal{J}_0(v) \geq \liminf_{\varepsilon} \mathcal{J}_0(u_\varepsilon) = \mathcal{J}_0(u)
\]
hence \( u \) solves (1.1) (and the whole family \( u_\varepsilon \) converges uniformly on \( \Omega \) to \( u \) by uniqueness of the minimizer of \( \mathcal{J}_0 \) on \( S[\varphi, \Omega] \)).

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**References**


