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ALGEBRAIC MODELS OF THE LINE IN THE REAL AFFINE PLANE

ADRIEN DUBOULOZ AND FRÉDÉRIC MANGOLTE

Abstract. We study smooth rational closed embeddings of the real affine line into the real affine plane, that is algebraic rational maps from the real affine line to the real affine plane which induce smooth closed embeddings of the real euclidean line into the real euclidean plane. We consider these up to equivalence under the group of birational automorphisms of the real affine plane which are diffeomorphisms of its real locus. We show that in contrast with the situation in the categories of smooth manifolds with smooth maps and of real algebraic varieties with regular maps where there is only one equivalence class up to isomorphism, there are non-equivalent smooth rational closed embeddings up to such birational diffeomorphisms.

Introduction

It is a consequence of the Jordan–Schoenflies theorem (see Remark 11 below) that every two smooth proper embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ are equivalent up to composition by diffeomorphisms of $\mathbb{R}^2$. Every algebraic closed embedding of the real affine line $\mathbb{A}^1_{\mathbb{R}}$ into the real affine plane $\mathbb{A}^2_{\mathbb{R}}$ induces a smooth proper embedding of the real locus $\mathbb{R}$ of $\mathbb{A}^1_{\mathbb{R}}$ into the real locus $\mathbb{R}^2$ of $\mathbb{A}^2_{\mathbb{R}}$. Given two such algebraic embeddings $f, g : \mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$, the famous Abhyankar-Moh Theorem [1], which is valid over any field of characteristic zero [23, § 5.4], asserts the existence of a polynomial automorphism ϕ of $\mathbb{A}^2_{\mathbb{R}}$ such that $f = \phi \circ g$. This implies in particular that the smooth proper embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ induced by f and g are equivalent under composition by a polynomial diffeomorphism of $\mathbb{R}^2$.

In this article, we consider a similar problem in a natural category intermediate between the real algebraic and the smooth ones. Our main objects of study consist of smooth embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ induced by rational algebraic maps $\mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ defined on the real locus of $\mathbb{A}^1_{\mathbb{R}}$ and whose restrictions to this locus induce smooth proper embeddings of $\mathbb{R}$ into $\mathbb{R}^2$. We call these maps *rational smooth embeddings*, and the question is the classification of these embeddings up to *birational diffeomorphisms* of $\mathbb{A}^2_{\mathbb{R}}$, that is, diffeomorphisms of $\mathbb{R}^2$ which are induced by birational algebraic endomorphisms of $\mathbb{A}^2_{\mathbb{R}}$ containing $\mathbb{R}^2$ in their domains of definition and admitting rational inverses of the same type.

A first natural working question in this context is to decide whether any rational smooth embedding is equivalent up to birational diffeomorphism to the standard regular closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$ as a linear subspace. Since every rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ uniquely extends (see Lemma 14) to a morphism $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ birational onto its image, a rational smooth embedding which can be rectified to a linear embedding by a birational diffeomorphism defines in particular a rational plane curve C that can be mapped onto a line by a birational automorphism of $\mathbb{P}^2_{\mathbb{R}}$. By classical results of Coolidge [6], Iitaka [14] and Kumar-Murthy [17], complex curves with this property are characterized by the negativity of the logarithmic Kodaira dimension of the complement of their proper transform in a minimal resolution of their singularities. Building on these ideas and techniques, we show the existence of *non-rectifiable* rational smooth embedding. In particular, we obtain the following result, see Theorem 26 for a stronger statement:

Theorem. *For every integer $d \geq 5$ there exists a non-rectifiable rational smooth embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$ whose associated projective curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a rational nodal curve of degree d.*

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The existence of non-rectifiable rational smooth embeddings motivates the search for weaker properties which can be satisfied by rational smooth embeddings. To this end we observe that the Abhyankar-Moh Theorem implies that the image of a regular closed embedding $\mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a real fiber of a structure of trivial $\mathbb{A}^1$-bundle $\rho : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ on $\mathbb{A}^2_{\mathbb{R}}$. In the complex case, this naturally leads to a "generalized Abhyankar-Moh property" for closed embeddings of the affine line in affine surfaces S equipped with $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{\mathbb{C}}$, i.e. morphisms $\pi : S \to \mathbb{A}^1_{\mathbb{C}}$ whose general fibers are affine lines, which was studied for certain classes of surfaces in [11]: the question there is whether the image of every regular closed embedding of $\mathbb{A}^1_{\mathbb{C}}$ in such a surface is an irreducible component of a fiber of an $\mathbb{A}^1$-fibration. The natural counterpart in our real birational setting consists in shifting the focus to the question whether the image of a rational smooth embedding is actually a fiber of an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ on a suitable real affine surface S birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$, but possibly non biregularly isomorphic to it. A rational smooth embedding with this property is said to be *biddable*.

Being a fiber of an $\mathbb{A}^1$-fibration on a surface birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$ imposes strong restrictions on the scheme-theoretic image $f_*(\mathbb{A}^1_{\mathbb{R}})$ of a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. We show in particular (Proposition 21) that the *real Kodaira dimension* [4] $\kappa_{\mathbb{R}}(\mathbb{A}^2_{\mathbb{R}} \setminus f_*(\mathbb{A}^1_{\mathbb{R}}))$ of the complement of the image has to be negative, with the consequence for instance that none of the rational smooth embedding mentioned in the theorem above is actually biddable.

In contrast, a systematic study of small degree embeddings (see in particular § 4.2) reveals that the images of these rational smooth embeddings are in a natural way smooth fibers of $\mathbb{A}^1$-fibrations on some *fake real planes*, a class of real birational models of $\mathbb{A}^2_{\mathbb{R}}$ introduced and studied in the series of papers [7, 8]. These are smooth real surfaces S non isomorphic to $\mathbb{A}^2_{\mathbb{R}}$ whose real loci are diffeomorphic to $\mathbb{R}^2$ and whose complexifications have trivial reduced rational singular homology groups. We show moreover that smooth rational embeddings $\mathbb{A}^2_{\mathbb{R}}$ of degree less than or equal to four are in fact rectifiable, except possibly for two special quartic cases for which we are unable to conclude.

The scheme of the article is the following: section 1 contains preliminaries on real algebraic varieties and a review of the structure of $\mathbb{A}^1$-*fibered algebraic models* of $\mathbb{R}^2$. In section 2, we introduce the precise notions of rectifiable and biddable rational smooth embeddings of the line in the plane and establish some basic properties. Section 3 is devoted to the construction of families of non-biddable rational smooth embeddings. Finally, section 4 contains a classification of rational smooth embeddings of degree less than or equal to four.

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1. Preliminaries

A real (resp. complex) algebraic variety is a geometrically integral scheme of finite type over the corresponding base field. A morphism between such varieties is a morphism of schemes over the corresponding base field. A complex algebraic variety X is said to be defined over $\mathbb{R}$ if there exists a real algebraic variety X' and an isomorphism of complex algebraic varieties between X and the complexification $X'_{\mathbb{C}} = X' \times_{\mathbb{R}} \mathbb{C}$ of X'.

1.1. **Euclidean topologies and birational diffeomorphisms.** The sets $X(\mathbb{R})$ and $X(\mathbb{C})$ of real and complex points of a real algebraic variety X are endowed in a natural way with the Euclidean topology, locally induced by the usual Euclidean topologies on $\mathbb{A}^n_{\mathbb{R}} \simeq \mathbb{R}^{2n}$ and $\mathbb{A}^n_{\mathbb{C}} \simeq \mathbb{C}^n$ respectively. Every morphism $f\colon X \to Y$ of real algebraic varieties induces a continuous map $f(\mathbb{R})\colon X(\mathbb{R}) \to Y(\mathbb{R})$ for the Euclidean topologies, which is an homeomorphism when f is an isomorphism. If X and Y are smooth, then $X(\mathbb{R})$ and $Y(\mathbb{R})$ can be further equipped with natural structures of smooth $\mathcal{C}^\infty$-manifolds, and the continuous map $f(\mathbb{R})$ induced by an isomorphism of real algebraic varieties is a diffeomorphism for these structures. See [18, Sections 1.4 and 2.4] for details.

Definition 1. Let $\alpha : X \dashrightarrow Y$ be a rational map between real algebraic varieties with non-empty real loci $X(\mathbb{R})$ and $Y(\mathbb{R})$.

a) We say that α is $\mathbb{R}$-*regular* if $X(\mathbb{R})$ is contained in its domain of definition.

b) We say that α is $\mathbb{R}$-*biregular* if it is $\mathbb{R}$-regular and birational and its inverse is also $\mathbb{R}$-regular. For simplicity, an $\mathbb{R}$-biregular birational map between smooth real algebraic varieties with non-empty real loci is called a *birational diffeomorphism*.

Note that when X and Y are smooth real varieties, a birational diffeomorphism between X and Y does restrict to a diffeomorphism between the real loci of X and Y equipped with their respective structure of smooth manifolds.

1.2. **Pairs, log-resolutions and SNC completions.** A *Simple Normal Crossing* (SNC) divisor on a smooth real or complex surface S is a curve B whose complexification $B_{\mathbb{C}}$ has smooth irreducible components and ordinary double points only as singularities. An SNC divisor B on a smooth complete surface V is said to be *SNC-minimal* if there does not exist any projective strictly birational morphism $\tau : V \to V'$ onto a smooth complete surface defined over the same base field, with exceptional locus contained in B and such that the image $\tau(B)$ of B is SNC.

A *smooth SNC pair* is a pair (V, B) consisting of a smooth projective surface V and an SNC divisor B, both defined over the considered base field.

A *smooth completion* of a smooth algebraic surface S is a smooth SNC pair (V, B) together with an isomorphism $V \setminus B \simeq S$, all defined over the considered base field. Such completions always exist as a consequence of classical completion and desingularization theorems [21, 24].

A *log-resolution* of a pair (V, D) consisting of a surface V and a reduced curve D on it is a projective birational morphism $\tau : V' \to V$ defined over the considered base field, such that V' is smooth and the union of the reduced total transform $\tau^{-1}(D)$ of D with the exceptional locus $\mathrm{Ex}(\tau)$ of τ is an SNC divisor B on V'. A log-resolution which is minimal with respect to the ordering by birational domination is called a *minimal log-resolution* of (V, D).

By a minimal log-resolution of a birational map between smooth complete surfaces, we mean a minimal log-resolution of its graph.

1.3. **$\mathbb{A}^1$-fibered algebraic models of $\mathbb{R}^2$.** In what follows, an *algebraic model* of $\mathbb{R}^2$ refers to a smooth geometrically integral real surface S with real locus $S(\mathbb{R})$ diffeomorphic to $\mathbb{R}^2$ and whose complexification has the rational homology type of a point, i.e. has trivial reduced homology groups $\tilde{H}_i(S_{\mathbb{C}}; \mathbb{Q})$ for every $i \geq 0$. It turns out that $\mathbb{R}^2$ admits many algebraic models non-isomorphic to $\mathbb{A}^2_{\mathbb{R}}$, called *fake real planes*, cf. [7, Definition 1.1]. A partial classification of these was established in [8] (see also [7]) according to their usual Kodaira dimension. In the present article, algebraic models S of negative Kodaira dimension are of particular interest to us due to the fact that they admit an $\mathbb{A}^1$-*fibration* $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$, that is, a surjective morphism whose generic fiber is isomorphic to the affine line over the function field of $\mathbb{A}^1_{\mathbb{R}}$. More precisely, we have the following characterization:

Theorem 2. ([8, Theorem 4.1]) *A smooth geometrically integral surface S defined over $\mathbb{R}$ is an algebraic model of $\mathbb{R}^2$ of Kodaira dimension $\kappa(S) = -\infty$ if and only if it admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ whose only singular fibers, if any, are geometrically irreducible real fibers of odd multiplicity, isomorphic to $\mathbb{A}^1_{\mathbb{R}}$ when equipped with their reduced structures.*

Remark 3. The multiplicities of the singular fibers of an $\mathbb{A}^1$-fibered algebraic model of $\mathbb{R}^2$ are intimately related to the topology of its complexification (see e.g. [20, Theorem 4.3.1 p. 231]). Namely, letting $m_1F_1, \ldots, m_rF_r$ be the singular fibers of π, where $F_i \simeq \mathbb{A}^1_{\mathbb{R}}$ and $m_i \geq 2$ for every $i = 1, \ldots, r$, and F be any fiber of π, we have

$$H_1(S_{\mathbb{C}}, \mathbb{Z}) = \bigoplus_{F_i} \mathbb{Z}_{m_i} \quad \text{and} \quad H_1((S \setminus F)_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z} \oplus \bigoplus_{F_i \neq F} \mathbb{Z}_{m_i}.$$

Given an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ on a smooth quasi-projective surface S, there exists a smooth projective completion (V, B) of S on which π extends to a morphism $\overline{\pi} : V \to \mathbb{P}^1_{\mathbb{R}}$ with generic fiber isomorphic to $\mathbb{P}^1$ over the function field of $\mathbb{P}^1_{\mathbb{R}}$, called a $\mathbb{P}^1$-fibration. For $\mathbb{A}^1$-fibered algebraic models of $\mathbb{R}^2$, we have the following more precise description, see [8, §4.1.1]:

Lemma 4. *A smooth $\mathbb{A}^1$-fibered algebraic model $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ admits a smooth projective completion (V, B) on which π extends to a $\mathbb{P}^1$-fibration $\overline{\pi} : V \to \mathbb{P}^1_{\mathbb{R}}$ for which the divisor B is a tree of*

rational curves that can be written in the form

$$B = F_\infty \cup H \cup \bigcup_{p \in \mathbb{A}^1_{\mathbb{R}}(\mathbb{R})} G_p$$

where:

a) $F_\infty \simeq \mathbb{P}^1_{\mathbb{R}}$ *is the fiber of* $\overline{\pi}$ *over* $\infty = \mathbb{P}^1_{\mathbb{R}} \setminus \mathbb{A}^1_{\mathbb{R}}$,

b) $H \simeq \mathbb{P}^1_{\mathbb{R}}$ *is a section of* $\overline{\pi}$,

c) G_p *is either empty if* $\pi^{-1}(p)$ *is a smooth fiber of* π *or a proper SNC-minimal subtree of* $\overline{\pi}^{-1}(p)$ *containing an irreducible component intersecting* H *otherwise.*

Furthermore, for every p *such that* $G_p \neq \emptyset$*, the support of* $\overline{\pi}^{-1}(p)$ *is a tree consisting of the union of* G_p *with the support of the closure in* V *of* $\pi^{-1}(p)$.

Only little is known so far towards the classification of the above $\mathbb{A}^1$-fibered algebraic models $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ up to birational diffeomorphism. At least, we have the following criterion:

Theorem 5. ([8, Theorem 4.9 and Corollary 4.10]) *Let* $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ *be an* $\mathbb{A}^1$*-fibered algebraic model of* $\mathbb{R}^2$*. If all but at most one real fibers of* π *are reduced then* S *is birationally diffeomorphic to* $\mathbb{A}^2_{\mathbb{R}}$.

Example 6. Let $m \geq 3$ be an odd integer and let S_m be the smooth surface in $\mathbb{A}^3_{\mathbb{R}}$ defined by the equation $u^2 z = v^m - u$. Since m is odd, the map $\mathbb{R}^2 \to S_m(\mathbb{R})$, $(u, z) \mapsto (u, \sqrt[m]{u^2 z + u}, z)$ is an homeomorphism, and since S_m is smooth, it follows that $S_m(\mathbb{R})$ is diffeomorphic to $\mathbb{R}^2$. On the other hand, the projection $\pi = \mathrm{pr}_u : S_m \to \mathbb{A}^1_{\mathbb{R}}$ is an $\mathbb{A}^1$-fibration restricting to a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_{\mathbb{R}} \setminus \{0\}$ and whose fiber $\pi^{-1}(0) \simeq \mathrm{Spec}(\mathbb{R}[v]/(v^m)[z])$ is isomorphic to $\mathbb{A}^1_{\mathbb{R}}$, of multiplicity m. It is then easy to check that $H_i((S_m)_{\mathbb{C}}; \mathbb{Z}) = 0$ for every $i \geq 2$, while $H_1((S_m)_{\mathbb{C}}; \mathbb{Z}) \simeq \mathbb{Z}_m$. Thus S_m is an algebraic model of $\mathbb{R}^2$ non biregularly isomorphic to $\mathbb{A}^2_{\mathbb{R}}$ but which is birationally diffeomorphic to it by the previous theorem.

2. Rational closed embeddings of lines

Given a rational map $f : X \dashrightarrow Y$ between real algebraic varieties, we denote by $f_*(X)$ the scheme theoretic image of the graph $\Gamma_f \subset X \times Y$ by the second projection.

Definition 7. Let X and Y be real algebraic varieties with non-empty real loci.

1) An $\mathbb{R}$-regular rational map $f : X \dashrightarrow Y$ is called an $\mathbb{R}$-*regular closed embedding* if it is birational onto its image and $f\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is a closed embedding of topological manifolds, whose image coincides with $(f_*(X))(\mathbb{R})$.

2) Two $\mathbb{R}$-regular closed embeddings $f : X \dashrightarrow Y$ and $g : X \dashrightarrow Y$ are called *equivalent* if there exists $\mathbb{R}$-biregular rational maps $\alpha : X \dashrightarrow X$ and $\beta : Y \dashrightarrow Y$ such that $\beta \circ f = g \circ \alpha$.

In the rest of the article, we focus on $\mathbb{R}$-regular closed embeddings of the affine line $\mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[t])$ into the affine plane $\mathbb{A}^2_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[x, y])$.

Definition 8. For simplicity, an $\mathbb{R}$-regular closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ into a smooth real algebraic variety is called a *rational smooth embedding.*

We begin with a series of examples illustrating this notion.

2.1. First examples and non-examples.

Example 9. The rational map

$$f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}},\ t \mapsto (t, \frac{t^3}{t^2+1})$$

is a rational smooth embedding. Indeed, since f is the graph of a rational function $\mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^1_{\mathbb{R}}$, defined everywhere except at the complex point of $\mathbb{A}^1_{\mathbb{R}}$ with defining ideal $(t^2+1) \subset \mathbb{R}[t]$, it is a locally closed embedding in the neighborhood of every real point of $\mathbb{A}^1_{\mathbb{R}}$. The scheme theoretic image $f_*(\mathbb{A}^1_{\mathbb{R}})$ of f is equal to the closed curve $U = \left\{x^3 - (x^2+1)y = 0\right\} \subset \mathbb{A}^2_{\mathbb{R}}$, and f induces an isomorphism $\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) \cong \mathbb{R} \xrightarrow{\cong} U(\mathbb{R}) \cong \mathbb{R}$.

The next two examples enlighten the role of the condition $f\,(X(\mathbb{R})) = (f_*(X))(\mathbb{R})$ in the definition of an $\mathbb{R}$-regular closed embedding.

Example 10. The rational map

$$f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}},\ t \mapsto (\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1})$$

is a locally closed embedding in the neighborhood of every real point of $\mathbb{R}$ but not a rational smooth embedding. Indeed, its scheme theoretic image is the conic $U = \left\{x^2 + y^2 = 1\right\}$ whose real locus is isomorphic to the circle $\mathbb{S}^1$, so that the map $\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) \cong \mathbb{R} \to f_*(\mathbb{A}^1_{\mathbb{R}})(\mathbb{R}) \cong \mathbb{S}^1$ induced by f is not an isomorphism.

Remark 11. In the previous example, the smooth embedding $f(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^2$ induced by f extends to an embedding $S^1 = \mathbb{R} \cup \{\infty\} \to \mathbb{R}^2$ thats maps the point ∞ onto the point $(-1, 0)$, in particular $f(\mathbb{R})$ is not a proper embedding hence is not equivalent under composition by a diffeomorphism of $\mathbb{R}^2$ to a linear embedding. In contrast, as mentioned in the introduction, it is a certainly well-known consequence of the Jordan-Schoenflies theorem that any two smooth proper embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ are equivalent under composition by diffeomorphisms of $\mathbb{R}^2$. Let us give a short argument because of a lack of appropriate reference. First observe that every smooth proper embedding $f : \mathbb{R} \to \mathbb{R}^2$ uniquely extends to an injective continuous map $\overline{f} : S^1 = \mathbb{R} \cup \{\infty\} \to S^2 = \mathbb{R}^2 \cup \{\infty\}$ such that $\overline{f}^{-1}(\infty) = \infty$. It follows that the closure in S^2 of the image C of f has a unique branch at the point ∞, which implies in turn that there exists a real number $R > 0$ such that for every disc D centered at the origin of $\mathbb{R}^2$ and of radius bigger than or equal to R, C intersects the boundary of D transversally in precisely two distinct points. Let $\{D_n\}_{n\in\mathbb{N}}$ be an exhaustion of $\mathbb{R}^2$ by a collection of such closed discs of strictly increasing radiuses. By the Jordan-Schoenflies theorem, there exists a diffeomorphism of D_0 which maps $C \cap D_0$ onto the horizontal diameter of D_0. We let h_0 be an extension of this diffeomorphism to a diffeomorphism of $\mathbb{R}^2$ which is the identity outside D_1. By construction, $f_0 = h_0 \circ f : \mathbb{R} \to \mathbb{R}^2$ is a smooth proper embedding and the intersection of its image with D_0 is equal to the horizontal diameter of D_0. Then by the Jordan-Schoenflies theorem again, we can find a diffeomorphism of D_2 restricting to the identity on D_0 and mapping $h_0(C) \cap D_2$ onto the horizontal diameter of D_2. We then let h_1 be an extension of this diffeomorphism to a diffeomorphism of $\mathbb{R}^2$ restricting to the identity outside D_3 to obtain a smooth proper embedding $f_1 = h_1 \circ f_0 : \mathbb{R} \to \mathbb{R}^2$ whose image intersects D_2 along the horizontal diameter of D_2. Iterating this procedure, we obtain a sequence of diffeomorphisms h_n, $n \geq 0$, of $\mathbb{R}^2$ restricting to the identity on D_{2n-2} and outside D_{2n+1} and a diffeomorphism $\varphi_n = h_n \circ \cdots \circ h_1 \circ h_0$ such that $\varphi_n \circ f : \mathbb{R} \to \mathbb{R}^2$ is a smooth proper embedding whose image intersects D_{2n} along the horizontal diameter of D_{2n}. Since by construction the sequence of diffeomorphisms $\varphi_n = h_n \circ \cdots \circ h_1 \circ h_0$, $n \geq 0$, converges on every compact subset of $\mathbb{R}^2$, we conclude that there exists a diffeomorphism φ of $\mathbb{R}^2$ such that the composition $\varphi \circ f : \mathbb{R} \to \mathbb{R}^2$ is the embedding of $\mathbb{R}$ as the horizontal coordinate axes.

Example 12. The morphism $f : \mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[t]) \to \mathbb{A}^2_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[x, y]),\ t \mapsto (t^2, t(t^2+1))$ induced by the normalization map of the nodal cubic $U \subset \mathbb{A}^2_{\mathbb{R}}$ defined by the equation $y^2 = x\,(x+1)^2$ is a locally closed immersion in a neighborhood of every real point of $\mathbb{A}^1_{\mathbb{R}}$ but not a rational smooth embedding. Indeed, $f(\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}))$ is a proper subset of $f_*(\mathbb{A}^1_{\mathbb{R}})(\mathbb{R}) = U\,(\mathbb{R}) = f(\mathbb{A}^1_{\mathbb{R}}(\mathbb{R})) \cup (-1, 0)$.

Note that the morphism $f(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^2, t(t^2 + 1))$ is a smooth proper embedding, which is thus equivalent in the category of smooth manifolds to a proper embedding as a linear subspace of $\mathbb{R}^2$. The following construction provides a diffeomorphism ψ of $\mathbb{R}^2$ such that the composition $\psi \circ f(\mathbb{R})$ is a proper embedding as the graph of a smooth function, from which it is straightforward to deduce a smooth proper embedding as a linear subspace. Let $h : \mathbb{R} \to [0, 1]$ be a smooth function such that $h(u) = 1$ if $u \geq 0$ and $h(u) = 0$ if $u \leq -\frac{1}{2}$. It then follows from the inverse function theorem that the map

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (y, (1 + yh(y))x)$$

is a diffeomorphism. Since by construction of h, we have $h(t^2) = 1$ for every $t \in \mathbb{R}$, φ maps the parametric curve $t \mapsto (t, t^2)$ onto the parametric curve $t \mapsto (t^2, (1 + t^2h(t^2))t) = (t^2, t + t^3)$. So $\varphi^{-1} \circ f(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^2$ is the smooth proper embedding defined by $t \mapsto (t, t^2)$.

2.2. **Rational embeddings of lines and projective rational plane curves.**

Notation 13. Unless otherwise stated, in the rest of the article, we identify $\mathbb{A}^1_{\mathbb{R}}$ with the open complement in $\mathbb{P}^1_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[u,v])$ of the $\mathbb{R}$-rational point $\{v=0\}$, and $\mathbb{A}^2_{\mathbb{R}}$ with the open complement in $\mathbb{P}^2_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[X,Y,Z])$ of the real line $L_\infty = \{Z=0\} \simeq \mathbb{P}^1_{\mathbb{R}}$.

Every rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ extends to a morphism $\overline{f} : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ birational onto its image $C \subset \mathbb{P}^2_{\mathbb{R}}$, such that $f_*(\mathbb{A}^1_{\mathbb{R}}) = \overline{f}_*(\mathbb{A}^1_{\mathbb{R}}) = C \cap \mathbb{A}^2_{\mathbb{R}}$ and $(f_*(\mathbb{A}^1_{\mathbb{R}}))(\mathbb{R}) = (\overline{f}_*(\mathbb{A}^1_{\mathbb{R}}))(\mathbb{R}) = (C \cap \mathbb{A}^2_{\mathbb{R}})(\mathbb{R})$. Conversely, given a rational curve $C \subset \mathbb{P}^2_{\mathbb{R}}$, the composition of the normalization morphism $\nu : \mathbb{P}^1_{\mathbb{R}} \to C$ with the inclusion $C \hookrightarrow \mathbb{P}^2_{\mathbb{R}}$ is a morphism $\overline{f} : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ birational onto its image C. The inverse image by ν of $C \cap L_\infty$ is a non-empty closed subset of $\mathbb{P}^1_{\mathbb{R}}$ defined over $\mathbb{R}$. Its complement is a smooth affine rational curve U on which $\overline{f}$ restricts to a proper morphism $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$. The following lemma characterizes rational planes curves C whose associated morphisms $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$ represent rational smooth embeddings $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$.

Lemma 14. *With the above notation, for a rational curve $C \subset \mathbb{P}^2_{\mathbb{R}}$, the following are equivalent:*

a) $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$ represents a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$.

b) C is smooth at every real point of $C \cap \mathbb{A}^2_{\mathbb{R}}$ and $U(\mathbb{R}) = \mathbb{R}$.

c) C is smooth at every real point of $C \cap \mathbb{A}^2_{\mathbb{R}}$ and the inverse image of $C \cap L_\infty$ in the normalization of C contains a unique real point.

Proof. Since $U(\mathbb{R}) = (\mathbb{P}^1_{\mathbb{R}} \setminus \overline{f}^{-1}(C \cap L_\infty))(\mathbb{R})$, we see that $U(\mathbb{R}) = \mathbb{R}$ if and only if $\overline{f}^{-1}(C \cap L_\infty)$ contains a unique real point. This proves the equivalence of b) and c). Now assume that $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$ represents a smooth rational embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. Since U is by definition the maximal affine open subset of $\mathbb{P}^1_{\mathbb{R}}$ on which $\overline{f}$ restricts to a morphism with image contained in $\mathbb{A}^2_{\mathbb{R}}$, it follows that U is the domain of definition of $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. This implies that $U(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{R}$. Since $f_*(\mathbb{A}^1_{\mathbb{R}}) = C \cap \mathbb{A}^2_{\mathbb{R}}$ and $f : \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = U(\mathbb{R}) \to \mathbb{A}^2_{\mathbb{R}}(\mathbb{R})$ is a proper embedding of topological manifolds with image equal to $(f_*(\mathbb{A}^1_{\mathbb{R}}))(\mathbb{R}) = (C \cap \mathbb{A}^2_{\mathbb{R}})(\mathbb{R})$ we conclude that C is smooth at every real point of $C \cap \mathbb{A}^2_{\mathbb{R}}$.

Conversely, assume that if C is smooth at every real point of $C' = C \cap \mathbb{A}^2_{\mathbb{R}}$ and that $\overline{f}^{-1}(C \cap L_\infty)$ contains a unique real point, say $p_\infty \in \mathbb{P}^1_{\mathbb{R}}$. Then U is an open subset of $\mathbb{P}^1_{\mathbb{R}} \setminus \{p_\infty\} \simeq \mathbb{A}^1_{\mathbb{R}}$ with real locus $U(\mathbb{R})$ equal to $\mathbb{A}^1_{\mathbb{R}}(\mathbb{R})$ and the morphism $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$, which is the composition of the normalization $\nu : U \to C'$ with the closed immersion $C' \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$, represents an $\mathbb{R}$-regular rational map $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ birational onto its image and such that $f_*(\mathbb{A}^1_{\mathbb{R}}) = C'$. Let $V \subset U$ be the inverse image by ν of the smooth locus C'_{reg} of C' and let $\tilde{\nu} : V \to C'_{\mathrm{reg}}$ be the induced isomorphism. Since C' is smooth at every real point, we have $C'_{\mathrm{reg}}(\mathbb{R}) = C'(\mathbb{R})$. This implies in turn that $V(\mathbb{R}) = U(\mathbb{R}) \simeq \mathbb{A}^1_{\mathbb{R}}(\mathbb{R})$ hence that $\tilde{\nu} : \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) \simeq V(\mathbb{R}) \to C'_{\mathrm{reg}}(\mathbb{R})$ is a diffeomorphism onto its image, which is equal to the closed submanifold $C'(\mathbb{R}) = (f_*(\mathbb{A}^1_{\mathbb{R}}))(\mathbb{R})$ of $\mathbb{A}^2_{\mathbb{R}}(\mathbb{R})$. This shows that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a rational smooth embedding. □

Example 15. Let $\overline{f} : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$, $[u:v] \mapsto \left[uv^2 : u^3 : u^2v + v^3\right]$ be the morphism induced by the normalization map of the irreducible cubic $C \subset \mathbb{P}^2_{\mathbb{R}}$ with equation $Z^2Y - X(X+Y)^2 = 0$. The intersection of C with L_∞ consists of two real points: $p_\infty = [0:1:0]$ and the unique singular point $p_s = [1:-1:0]$ of C, which is a real ordinary double point with non-real complex conjugate tangents, in other words, $\overline{f}^{-1}(p_s)$ is a $\mathbb{C}$-rational point of $\mathbb{P}^1_{\mathbb{R}}$. With the notation above, we have $U \simeq \mathrm{Spec}(\mathbb{R}[t]_{t^2+1})$ and $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$ is the morphism defined by

$$t \mapsto \left(\frac{t}{t^2+1}, \frac{t^3}{t^2+1}\right)$$

Clearly $U(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{R})$ and so $f_0 : U \to \mathbb{A}^2_{\mathbb{R}}$ defines an $\mathbb{R}$-regular map $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. Since $p_s \in C \cap L_\infty$, the affine part $C \cap \mathbb{A}^2_{\mathbb{R}} \simeq \left\{xy^2 + 2x^2y + x^3 - y = 0\right\}$ is a smooth curve, and so $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a rational smooth embedding.

2.3. **Biddable and rectifiable lines.**

Definition 16. A rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is called:

a) *rectifiable* if it is equivalent to the linear closed embedding $j_{\mathrm{lin}} : \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^2_{\mathbb{R}}$, $t \mapsto (t, 0)$.

b) *biddable* if there exists a birational diffeomorphism $\alpha : \mathbb{A}^2_{\mathbb{R}} \dashrightarrow S$ onto an $\mathbb{A}^1$-fibered algebraic model $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ such that the induced rational smooth embedding $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow S$ is a regular closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ as the support of a fiber of π.

Remark 17. Note that in case b) in Definition 16, we do require that $\alpha \circ f$ is in fact a morphism $\mathbb{A}^1_{\mathbb{R}} \to S$, ond not only a rational map. A rectifiable rational smooth embedding is biddable since for any birational diffeomorphism α of $\mathbb{A}^2_{\mathbb{R}}$ realizing the equivalence with j_{lin}, $\alpha \circ f$ coincides with the closed immersion of $\mathbb{A}^1_{\mathbb{R}}$ as the fiber of the trivial $\mathbb{A}^1$-bundle $\pi = \mathrm{pr}_2 : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ over $0 \in \mathbb{A}^1_{\mathbb{R}}(\mathbb{R})$.

The following lemma is an immediate reformulation of Definition 16:

Lemma 18. *For an a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ with associated irreducible rational curve $C = \overline{f}(\mathbb{P}^1_{\mathbb{R}}) \subset \mathbb{P}^2_{\mathbb{R}}$, the following hold:*

a) f is rectifiable if and only if there exists a birational endomorphism of $\mathbb{P}^2_{\mathbb{R}}$ that restricts to a birational diffeomorphism $\mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \xdashrightarrow{\sim} \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty$ and maps C onto a line $\ell \simeq \mathbb{P}^1_{\mathbb{R}}$.

b) f is biddable if and only if there exists a smooth pair (V, B) with a $\mathbb{P}^1$-fibration $\overline{\pi} : V \to \mathbb{P}^1_{\mathbb{R}}$ as in Lemma 4 such that $\pi = \overline{\pi}|_S : S = V \setminus B \to \mathbb{A}^1_{\mathbb{R}}$ is an $\mathbb{A}^1$-fibered model of $\mathbb{R}^2$, and a birational map $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow V$ restricting to a birational diffeomorphism $\mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \xdashrightarrow{\sim} S$ that maps C to an irreducible component of a fiber of $\overline{\pi}$.

Example 19. Let $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ be the rational smooth embedding with associated curve $C = \{Z^2Y - X(X+Y)^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ constructed in Example 15. The birational endomorphism of $\mathbb{P}^2_{\mathbb{R}}$ defined by

$$[X : Y : Z] \mapsto [(X+Y)\left((X+Y)^2 + Z^2\right) : Z^2Y - X(X+Y)^2 : Z\left((X+Y)^2 + Z^2\right)]$$

maps L_∞ onto itself and restricts to a birational diffeomorphism of $\mathbb{P}^2 \setminus L_\infty$. It contracts the union $\{(X+Y)^2 + Z^2 = 0\}$ of the two non-real complex conjugate tangents of C at its singular point $p_s = [1 : -1 : 0]$ onto the point $[0 : 1 : 0] \in L_\infty$, and maps C onto the line $\ell = \{Y = 0\}$. It follows that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is rectifiable, a birational diffeomorphism $\alpha : \mathbb{A}^2_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ such that $\alpha \circ f(t) = (t, 0)$ being given for instance by

$$(x, y) \mapsto (x + y, -x + \frac{x+y}{(x+y)^2 + 1}).$$

2.4. **Real Kodaira dimension: a numerical obstruction to biddability.**
Given an SNC divisor B on a real smooth projective surface V, we denote by $B_{\mathbb{R}} \subset B$ the union of all components of B which are defined over $\mathbb{R}$ and have infinite real loci. We say that B has no *imaginary loop* if for every two distinct irreducible components A and A' of $B_{\mathbb{R}}$ the intersection $A \cap A'$ is either empty or consists of real points only. If B contains imaginary loops, then the reduced total transform $\hat{B}$ of B in the blow-up $\tau : \hat{V} \to V$ of the set of non-real singular points of $B_{\mathbb{R}}$ is an SNC divisor with no imaginary loops for which τ restricts to an isomorphism $\hat{V} \setminus \hat{B} \simeq V \setminus B$ (see [4, Lemma 2.5]). This implies in particular that every smooth quasi-projective real surface S admits a smooth completion (V, B) such that that B has no imaginary loop.

Definition 20. The *real Kodaira dimension* $\kappa_{\mathbb{R}}(S)$ of a smooth quasi-projective real surface S is the Iitaka dimension [12] $\kappa(V, K_V + B_{\mathbb{R}})$ of the pair $(V, B_{\mathbb{R}})$, where (V, B) is any smooth completion of S such that that B has no imaginary loop and K_V is a canonical divisor on V

It is established in [4, Corollary 2.12] that the so-defined element $\kappa_{\mathbb{R}}(S) \in \{-\infty, 0, 1, 2\}$ is indeed independent on the choice of a smooth completion (V, B) of S such that B has no imaginary loop. Furthermore $\kappa_{\mathbb{R}}(S)$ is smaller than or equal to the usual Kodaira dimension $\kappa(S) = \kappa(V, K_V + B)$, with equality in the case where $B = B_{\mathbb{R}}$, and is an invariant of the isomorphism class of S up to birational diffeomorphisms [4, Proposition 2.13].

The following proposition then provides a simple numerical necessary condition for biddability of a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$:

Proposition 21. *Let $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ be a rational smooth embedding and let C be the image of the corresponding morphism $\overline{f} : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$. If f is biddable then $\kappa_{\mathbb{R}}(\mathbb{P}^2 \setminus (C \cup L_\infty)) = -\infty$.*

Proof. If f is biddable then by definition there exists a birational diffeomorphism $\alpha : \mathbb{A}^2_{\mathbb{R}} \dashrightarrow S$ to a smooth $\mathbb{A}^1$-fibered model $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ such that $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \to S$ is a closed embedding as the support of a fiber of π, say $\pi^{-1}(p)_{\mathrm{red}}$ for some point $p \in \mathbb{A}^1_{\mathbb{R}}(\mathbb{R})$. So $\kappa_{\mathbb{R}}(\mathbb{P}^2_{\mathbb{R}} \setminus (C \cup L_\infty)) = \kappa_{\mathbb{R}}(\mathbb{A}^2_{\mathbb{R}} \setminus C) = \kappa_{\mathbb{R}}(S \setminus \pi^{-1}(p))$ by invariance of the real Kodaira dimension under birational diffeomorphism.

Furthermore, since π restricts to an $\mathbb{A}^1$-fibration on $S\setminus\pi^{-1}(p)$, we have $\kappa_{\mathbb{R}}(\mathbb{A}^2_{\mathbb{R}}\setminus C)=\kappa_{\mathbb{R}}(S\setminus\pi^{-1}(p))\leq \kappa(S\setminus\pi^{-1}(p))=-\infty$. □

Question 22. *Is every rational smooth embedding $f:\mathbb{A}^1_{\mathbb{R}}\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ whose associated projective curve C satisfies $\kappa_{\mathbb{R}}(\mathbb{P}^2_{\mathbb{R}}\setminus(C\cup L_\infty))=-\infty$ biddable ?*

2.5. **Rectifiability of biddable lines: a sufficient criterion.** Recall that a rectifiable rational smooth embedding $f:\mathbb{A}^1_{\mathbb{R}}\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ is in particular biddable. In this section, we establish a sufficient criterion for a biddable rational smooth embedding to be rectifiable.

Let $f:\mathbb{A}^1_{\mathbb{R}}\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ be a biddable rational smooth embedding with associated projective curve $C=\overline{f}(\mathbb{P}^1_{\mathbb{R}})\subset\mathbb{P}^2_{\mathbb{R}}$. By Definition 16, there exists a birational diffeomorphism $\alpha:\mathbb{A}^2_{\mathbb{R}}\dashrightarrow S$ onto an $\mathbb{A}^1$-fibered algebraic model $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ such that the composition $\alpha\circ f:\mathbb{A}^1_{\mathbb{R}}\to S$ is a closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ as the support of a fiber of π. Letting (V,B) be a smooth $\mathbb{P}^1$-fibered completion of $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$ as in Lemma 4, the composition $\overline{\pi}\circ\alpha:\mathbb{P}^2_{\mathbb{R}}\dashrightarrow\mathbb{P}^1_{\mathbb{R}}$ is a pencil of rational curves defined by a one-dimensional linear system without fixed component $\mathcal{M}=(\overline{\pi}\circ\alpha)^{-1}_*|\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)|\subset|\mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(d)|$ for some $d\geq 1$, that has C as an irreducible component of one of its members. Every real member M of $\mathcal{M}$ can be written as $M=M_{\mathbb{R}}+M'$ where $M_{\mathbb{R}}$ is an effective divisor whose support is equal to the union of all irreducible components of M which are defined over $\mathbb{R}$ and have infinite real loci and where M' is an effective divisor defined over $\mathbb{R}$ whose support does not contain any irreducible component of M with infinite real locus.

Lemma 23. *With the above notation, the following hold:*

a) The linear system $\mathcal{M}$ has a real member M_∞ such that the support of $M_{\infty,\mathbb{R}}$ is equal to L_∞

b) For every real member $M\neq M_\infty$ of $\mathcal{M}$, the divisor $M_{\mathbb{R}}$ is an irreducible rational curve.

Proof. Let (V,B) be a smooth completion of S on which π extends to a $\mathbb{P}^1$-fibration $\overline{\pi}:V\to\mathbb{P}^1_{\mathbb{R}}$ as in Lemma 4. Since $\overline{\pi}$ restricts to an $\mathbb{A}^1$-fibration $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$, the real locus of the real member of $\mathcal{M}$ corresponding to $\overline{\pi}^{-1}(\mathbb{P}^1_{\mathbb{R}}\setminus\mathbb{A}^1_{\mathbb{R}})$ is equal to L_∞. This proves a). By Theorem 2, every real fibers of π contains a unique geometrically irreducible component, which is isomorphic to $\mathbb{A}^1_{\mathbb{R}}$ when equipped with its reduced structure. Since every geometrically irreducible component of a real member of $\mathcal{M}$ intersecting $\mathbb{A}^2_{\mathbb{R}}$ becomes a geometrically irreducible component of a real fiber of π, assertion b) follows. □

Proposition 24. *Let $f:\mathbb{A}^1_{\mathbb{R}}\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ be a biddable rational smooth embedding with associated projective curve $C=\overline{f}(\mathbb{P}^1_{\mathbb{R}})\subset\mathbb{P}^2_{\mathbb{R}}$ and let $\mathcal{M}\subset|\mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(d)|$ be the linear system associated to a birational diffeomorphism $\alpha:\mathbb{A}^2_{\mathbb{R}}\dashrightarrow S$ onto an $\mathbb{A}^1$-fibered algebraic model $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ such that $\alpha\circ f:\mathbb{A}^1_{\mathbb{R}}\to S$ is a closed immersion as the support of a fiber F of π. Assume that one of the following two conditions is satisfied:*

a) For every real member $M\neq M_\infty$ of $\mathcal{M}$, the curve $M_{\mathbb{R}}$ is irreducible and reduced.

b) There exists a unique real member $M_0\neq M_\infty$ of $\mathcal{M}$ such that $M_{0,\mathbb{R}}$ is non-reduced and C is the support of $M_{0,\mathbb{R}}$.

Then $f:\mathbb{A}^1_{\mathbb{R}}\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ is rectifiable.

Proof. For every real member M of $\mathcal{M}$, the curve $M_{\mathbb{R}}$ is a reduced component of M if and only if $\alpha(M_{\mathbb{R}}\cap\mathbb{A}^2_{\mathbb{R}})$ is a reduced component of a real fiber of π. So by Lemma 4, condition a) is equivalent to the property that all scheme theoretic fibers of $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$ are isomorphic to $\mathbb{A}^1$ over the corresponding residue fields. This implies by [16] that $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$ is isomorphic over $\mathbb{A}^1_{\mathbb{R}}$ to the trivial $\mathbb{A}^1$-bundle $\mathrm{pr}_2:\mathbb{A}^1_{\mathbb{R}}\times\mathbb{A}^1_{\mathbb{R}}\to\mathbb{A}^1_{\mathbb{R}}$. It follows that $\alpha\circ f:\mathbb{A}^1_{\mathbb{R}}\to S\simeq\mathbb{A}^2_{\mathbb{R}}$ is a closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ as a real fiber of pr_2, hence that f is rectifiable.

Now suppose that condition b) if satisfied. Then $\pi:S\to\mathbb{A}^1_{\mathbb{R}}$ is an $\mathbb{A}^1$-fibration with a unique degenerate fiber, say $\pi^{-1}(0)$, and $\alpha\circ f:\mathbb{A}^1_{\mathbb{R}}\to S$ is equal to the inclusion of $(\pi^{-1}(0))_{\mathrm{red}}\simeq\mathbb{A}^1_{\mathbb{R}}$ in S. By Theorem 5, S is birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$. In fact, a careful tracing of the construction of a specific birational diffeomorphism $\beta:S\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ described in [8, Section 4.2] shows that the composition of the inclusion $i_0:\mathbb{A}^1_{\mathbb{R}}\hookrightarrow S$ of $\pi^{-1}(0)_{\mathrm{red}}$ with β is a linear regular closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$. This implies that $\beta\circ\alpha\circ f:\mathbb{A}^1_{\mathbb{R}}\dashrightarrow\mathbb{A}^2_{\mathbb{R}}$ is a linear regular closed embedding, hence that f is rectifiable. □

Let S_m be a smooth affine surface with an $\mathbb{A}^1$-fibration $\pi_m : S_m \to \mathbb{A}^1_{\mathbb{R}}$ defined over $\mathbb{R}$ and admitting a unique degenerate real fiber, isomorphic to $\mathbb{A}^1_{\mathbb{R}}$, of odd multiplicity $m \geq 3$ (for instance, take S as in Example 6). By Theorem 5, S_m is an algebraic model of $\mathbb{R}^2$ birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$ but non-biregularly isomorphic to it. Let $\alpha^{-1} : S_m \xrightarrow{\sim} \mathbb{A}^2_{\mathbb{R}}$ be such a birational diffeomorphism and let $i : \mathbb{A}^1_{\mathbb{R}} \hookrightarrow S_m$ be the inclusion of a general smooth real fiber of π. The composition $f_m = \alpha^{-1} \circ i : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is then a biddable rational smooth embedding to which the criterion of Proposition 24 does not apply. We do not know whether such rational smooth embedding $f_m : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ are rectifiable or not.

Question 25. *Does there exist biddable but non-rectifiable rational smooth embeddings $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$? In particular, are the biddable rational smooth embedding $f_m : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ above rectifiable ?*

3. Families of non-biddable affine lines

In this section, we exhibit examples of non-biddable, hence in particular non-rectifiable, rational smooth embeddings $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated projective curves have any degree larger than or equal to 5.

Theorem 26. *For every integer $d \geq 5$ there exists a non-biddable rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated projective curve $C_d \subset \mathbb{P}^2_{\mathbb{R}}$ is a rational nodal curve of degree d.*

The proof given below proceeds in two steps. We first construct in Proposition 27 real rational curves $C_d \subset \mathbb{P}^2_{\mathbb{R}}$ having only ordinary nodes as singularities for which the inclusion $C_d \cap \mathbb{A}^2_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$ defines a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. We then show by direct computation in Lemma 29 that the real Kodaira dimension $\kappa_{\mathbb{R}}(\mathbb{P}^2 \setminus (C_d \cup L_\infty))$ (see § 2.4) is nonnegative, which implies by virtue of Proposition 21 that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is not biddable.

Proposition 27. *For any integer $d \geq 1$ there exists a singular real rational curve $C_d \subset \mathbb{P}^2_{\mathbb{R}}$ of degree d with only ordinary nodes as singularities with the following properties:*

1) If $d \equiv 1$ or 2 modulo 4, $Sing(C_d)$ consists of pairs of non-real complex conjugate ordinary nodes. Furthermore, the intersection of C_d with L_∞ contains a unique real point and is transversal at every other point.

2) If $d \equiv 0$ or 3 modulo 4, $Sing(C_d)$ consists of pairs of non-real complex conjugate ordinary nodes and a unique real node p_s with non-real complex conjugate tangents. Furthermore, the intersection of C_d with L_∞ contains p_s and a unique other real point, and is transversal at every other point.

Proof. The assertion is clear for $d = 1, 2$. We assume from now on that $d \geq 3$.

1) If $d \equiv 1, 2 \mod 4$, let $k = \frac{1}{2}(d-1)$ if d is odd or $k = \frac{1}{2}(d-2)$ if d is even. Observe that in any case, $k \neq 0$ is even, thus a general nodal rational complex curve $D_k \subset \mathbb{P}^2_{\mathbb{C}}$ of degree k has no real point, meaning that $D_k \cap \mathbb{P}^2_{\mathbb{R}}(\mathbb{R}) = \varnothing$. Let D_k be such a curve given by an homogeneous complex polynomial h of degree k. A well-chosen projection D_k to $\mathbb{P}^2_{\mathbb{C}}$ of the rational normal curve of degree k in $\mathbb{P}^k_{\mathbb{C}}$ should suit. In particular, D_k has $\frac{1}{2}(k-1)(k-2)$ non-real nodal points. The conjugated curve $\overline{D}_k$ of D_k defined by the vanishing of the conjugated polynomial of h intersects transversally D_k in k^2 points Thus $D_k \cup \overline{D}_k$ is a reducible curve of degree $d-1$ defined over $\mathbb{R}$ with $(k-1)(k-2)+k^2$ non-real nodal points. The union $\tilde{C}_d = D_k \cup \overline{D}_k \cup E$, where E is a general real line if d is odd or a general real conic with non-empty real locus if d is even, is a reducible curve of degree d defined over $\mathbb{R}$.

If d is odd, E meets $D_k \cup \overline{D}_k$ in $2k$ nodal points which are necessarily non-real, then $\tilde{C}_d$ has $k^2 - \frac{1}{2}k + 1$ pairs of non-real complex conjugate ordinary nodes. If d is even, E meets $D_k \cup \overline{D}_k$ in $4k$ nodal points then $\tilde{C}_d$ has $k^2 + \frac{1}{2}k + 1$ pairs of non-real complex conjugate ordinary nodes. Given any pair of conjugate nodes of $(D_k \cap E) \cup (\overline{D}_k \cap E)$, it follows from Brusotti Theorem [3, §5.5] that there exists a small perturbation of $\tilde{C}_d$ which realizes a local smoothing of the later pair of nodes but preserves all the others. The resulting real curve C_d is irreducible of degree d with $\frac{1}{4}(d-1)(d-2)$ pairs of non-real conjugate nodes. The genus formula then implies that C_d is geometrically rational. Furthermore, the real locus of C_d contains smooth real points derived from the real locus of E outside of $D_k \cup \overline{D}_k$. Thus C_d is rational. We now observe that the real locus of C_d is a simple closed curve in the real projective plane which is one-sided if and only if d is odd. If d is odd (resp. d is even) we deduce that there exists a real projective line L_∞ meeting transversally $C_d(\mathbb{R})$ in one point p and

transversally at every other point (resp. meeting tangentially $C_d(\mathbb{R})$ in one point p and transversally at every other point).

2) If $d \equiv 0, 3 \mod 4$, let $k = \frac{1}{2}(d-1)$ if d is odd or $k = \frac{1}{2}(d-2)$ if d is even. Observe that in any case, k is odd. As in the preceding case, let $D_k \subset \mathbb{P}^2_{\mathbb{C}}$ be a general nodal rational curve given by an homogeneous polynomial h of degree k. As k is odd, D_k meets $\mathbb{P}^2_{\mathbb{R}}(\mathbb{R})$ in a unique point, say p_s.

The same construction of a perturbation C_d of $(D_k \cup \overline{D}_k \cup E)$ as in the preceding case works except that the line L_∞ at the end of the construction is taken in the set of real lines passing through p_s. □

Example 28. Let us describe in more details the case $d = 5$. Here $k = 2$ and D_k is a smooth conic. Take for example the complex conic given by

$$x^2 + (1 - \frac{i}{3})y^2 + (1-i)z^2 = 0$$

Then $D_k \cap \mathbb{P}^2_{\mathbb{R}}(\mathbb{R}) = \varnothing$ and the quartic $D_k \cup \overline{D}_k$ has exactly four non-real singular points $[\pm\sqrt{2}, \pm i\sqrt{3}, 1]$ which are all nodes. Let E be the real line of equation $y = 0$. Then $\tilde{C}_d = D_k \cup \overline{D}_k \cup E$ is given by

$$2x^2y^3 + 2z^2x^2y + \frac{8}{3}y^3z^2 + yx^4 + \frac{10}{9}y^5 + 2z^4y = 0$$

The curve $\tilde{C}_d$ has exactly eight singular points $[\pm\sqrt{2}, \pm i\sqrt{3}, 1], [\pm\sqrt[4]{2}e^{\pm i\frac{3\pi}{8}}, 0, 1]$ which are all non-real nodes. Choosing a pair of conjugated nodes, say $[\sqrt[4]{2}e^{\pm i\frac{3\pi}{8}}, 0, 1]$ and perturbating them, we get a real quintic C_d with one oval as real locus and three pairs of non-real complex conjugate ordinary nodes. Thus C_d is a real rational curve.

The following lemma completes the proof of Theorem 26:

Lemma 29. *For a real nodal rational curve $C_d \subset \mathbb{P}^2_{\mathbb{R}}$ of degree $d \geq 5$ as in Proposition 27, we have $\kappa_{\mathbb{R}}(\mathbb{P}^2_{\mathbb{R}} \setminus (C_d \cup L_\infty)) \geq 0$.*

Proof. It is well-known that if $d \geq 6$, the complexification of $C = C_d$ cannot be mapped to a line by a birational automorphism of $\mathbb{P}^2_{\mathbb{C}}$. Indeed, since the proper transform of $C_{\mathbb{C}}$ in any resolution of its singularities $\sigma : X \to \mathbb{P}^2_{\mathbb{C}}$ has self-intersection lower than or equal to -2, $C_{\mathbb{C}}$ cannot be contracted to a point by a birational automorphism of $\mathbb{P}^2_{\mathbb{C}}$.

It then follows from a characterization attributed to Coolidge [6] (see also [17]) that the Kodaira dimension $\kappa(X, K_X + \tilde{C}_{\mathbb{C}})$ is non-negative. Let $\tau : V \to \mathbb{P}^2_{\mathbb{R}}$ be a log resolution of the pair $(\mathbb{P}^2_{\mathbb{R}}, C \cup L_\infty)$ defined over $\mathbb{R}$ restricting to an isomorphism over $\mathbb{P}^2_{\mathbb{R}} \setminus (C \cup L_\infty)$ and such that $B = \tau^{-1}(C \cup L_\infty)_{\mathrm{red}}$ has no imaginary loop. Since $\tilde{C}_{\mathbb{C}}$ is an irreducible component of $(B_{\mathbb{R}})_{\mathbb{C}}$, we have

$$\kappa_{\mathbb{R}}(\mathbb{P}^2_{\mathbb{R}} \setminus (C \cup L_\infty)) = \kappa(V_{\mathbb{C}}, K_{V_{\mathbb{C}}} + (B_{\mathbb{R}})_{\mathbb{C}}) \geq \kappa(V_{\mathbb{C}}, K_{V_{\mathbb{C}}} + \tilde{C}_{\mathbb{C}}) \geq 0.$$

In the remaining case $d = 5$, the above argument does not longer work since every rational nodal quintic can be mapped to a line by a birational diffeomorphism of $\mathbb{P}^2_{\mathbb{R}}$ (see Remark 30 below). Nevertheless, a proof can be derived essentially along the same lines as above from Iitaka's results [14, 15] on equivalence classes of pairs of complex irreducible plane curves up to birational automorphisms of $\mathbb{P}^2_{\mathbb{C}}$. But we find more enlightening to provide a direct and self-contained argument. So here the curve $C = C_5$ is a rational nodal quintic defined over $\mathbb{R}$ with only pairs of non-real complex conjugate ordinary nodes $\{q_i, \overline{q}_i\}_{i=1,2,3}$ as singularities, which intersects L_∞ transversally in a real point p and two pairs $\{p_j, \overline{p}_j\}_{j=1,2}$ of non-real complex conjugate points. Let $\sigma_1 : \mathbb{F}_1 \to \mathbb{P}^2_{\mathbb{R}}$ be the blow-up of p with exceptional divisor C_0. Denote by ℓ_i and $\overline{\ell}_i$ the proper transforms of the lines $[p, q_i]$ and $[p, \overline{q}_i]$ in $\mathbb{P}^2_{\mathbb{R}}$ and let $\mathbf{f} = \sum_{i=1}^3 \ell_i + \overline{\ell}_i$. Let $\sigma_2 : V \to \mathbb{F}_1$ be the blow-up of the pairs of points $\{q_i, \overline{q}_i\}$ with exceptional divisors E_i and $\overline{E}_i$ and of the pairs $\{p_j, \overline{p}_j\}$ with exceptional divisors F_j and $\overline{F}_j$. We let $\mathbf{E} = \sum_{i=1}^3 E_i + \overline{E}_i$, $\mathbf{F} = \sum_{j=1}^2 F_j + \overline{F}_j$, and we let $\mathbf{f}'$ and L' be the proper transform of $\mathbf{f}$ and L respectively. The composition $\tau = \sigma_1 \circ \sigma_2 : V \to \mathbb{P}^2_{\mathbb{R}}$ is a log-resolution of the pair $(\mathbb{P}^2_{\mathbb{R}}, C \cup L_\infty)$ defined over $\mathbb{R}$, restricting to an isomorphism over $\mathbb{P}^2_{\mathbb{R}} \setminus (C \cup L_\infty)$ and such that

$$B = \tau^{-1}(C \cup L_\infty)_{\mathrm{red}} = C' + C'_0 + L' + \mathbf{E} + \mathbf{F} = B_{\mathbb{R}} + \mathbf{E} + \mathbf{F}$$

is an SNC divisor without imaginary loop.

Set $\alpha = \frac{1}{3}$, $\lambda = \frac{2}{3}$, $\beta = \mu = 1$ so that $6\alpha + \beta = 3$ and $6\lambda + \mu = 5$. Letting ℓ be the proper transform on $\mathbb{F}_1$ of a general real line through p and L be the proper transform of L, we have

$$K_{\mathbb{F}_1} \sim -2C_0 - 3\ell \sim_{\mathbb{Q}} -2C_0 - \alpha\mathbf{f} - \beta L$$
$$C \sim 4C_0 + 5\ell \sim_{\mathbb{Q}} 4C_0 + \lambda\mathbf{f} + \mu L$$

where we identified C with its proper transform on $\mathbb{F}_1$. Then on V, we obtain

$$K_V \sim \sigma_2^*(K_{\mathbb{F}_1}) + \mathbf{E} + \mathbf{F} \sim_{\mathbb{Q}} -2C_0' - \alpha\mathbf{f}' - \beta L' + (1-\alpha)\mathbf{E} + (1-\beta)\mathbf{F}$$
$$C' \sim \sigma_2^*(C) - 2\mathbf{E} - \mathbf{F} \sim_{\mathbb{Q}} 4C_0' + \lambda\mathbf{f}' + \mu L' + (\lambda-2)\mathbf{E} + (\mu-1)\mathbf{F}$$

where C_0' and C' denote the proper transforms of C_0 and C respectively. We thus obtain

$$\begin{aligned} 2K_V + B_{\mathbb{R}} &\sim_{\mathbb{Q}} C_0' + (\lambda - 2\alpha)\mathbf{f}' + (\mu - 2\beta + 1)L' + (\lambda - 2\alpha)\mathbf{E} + (\mu - 2\beta + 1)\mathbf{F} \\ &\sim C_0' \end{aligned}$$

which implies that $\kappa_{\mathbb{R}}(\mathbb{P}^2_{\mathbb{R}} \setminus (C \cup L_\infty)) \geq 0$ as desired. □

Remark 30. As explained in the proof above, for every $d \geq 6$, the rational curve C_d constructed in Proposition 27 cannot be transformed into a line by a birational automorphism of $\mathbb{P}^2_{\mathbb{R}}$. This directly implies the weaker fact that the corresponding rational smooth embedding $\mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is not rectifiable. In contrast, there exists a birational diffeomorphism of $\mathbb{P}^2_{\mathbb{R}}$ that maps the nodal rational quintic C_5 to a line: it consists of the blow-up $\sigma : V \to \mathbb{P}^2_{\mathbb{R}}$ of the three pairs $\{q_i, \overline{q}_i\}_{i=1,2,3}$ of non-real complex conjugate nodes of C_5, followed by the contraction $\tau : V \to \mathbb{P}^2_{\mathbb{R}}$ of the proper transforms of the three pairs of non-real complex conjugate nonsingular conics passing through five of the six points blown-up. The proper transform of C_5 and L_∞ by $\tau \circ \sigma^{-1}$ are respectively a line and a quintic with three pairs of non-real complex conjugate nodes.

4. Classification up to degree four

In this section, we study the rectifiability and biddability of rational smooth embeddings $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated projective curves $C \subset \mathbb{P}^2_{\mathbb{R}}$ have degrees less than or equal to four.

4.1. Degree lower than or equal to 3.

Proposition 31. *Every rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ has degree ≤ 3 is rectifiable.*

Proof. If $\deg C = 1$ then C is a real line in $\mathbb{P}^2_{\mathbb{R}}$ and so f is rectifiable. If $\deg C = 2$, then C is a smooth conic, whose intersection with L_∞ must consists of a unique real point p_∞ by Lemma 14. It follows that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is actually a closed emdedding of $\mathbb{R}$-schemes. So $f : \mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$ is rectifiable by an $\mathbb{R}$-automorphism of $\mathbb{A}^2_{\mathbb{R}}$ by virtue of the Abhyankar-Moh Theorem over $\mathbb{R}$.

We now assume that C is a rational cubic in $\mathbb{P}^2_{\mathbb{R}}$. Then C has a unique singular point which is either an ordinary double point or a cusp of multiplicity two. Since this point is unique, it is a real point p_s of C, and it follows from Lemma 14 that $p_s \in C \cap L_\infty$. The intersection multiplicity of C and L_∞ at p_s is thus at least 2, and since $C \cdot L_\infty = 3$, it follows that $C \cap L_\infty$ contains at most another point distinct from p_s. This yields the following alternative:

a) $C \cap L_\infty = \{p_s\}$. Suppose that p_s is an ordinary double point of C. Then L_∞ is the tangent to one of the two analytic branches of C at p_s. The tangent to the other branch is then necessarily real, so that the inverse image of p_s in the normalization of C consists of two distinct real points, which is impossible by Lemma 14. Thus p_s is an ordinary cusp, with tangent L_∞. Then $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a closed embedding of $\mathbb{R}$-schemes, hence is rectifiable by the Abhyankar-Moh Theorem over $\mathbb{R}$.

b) $C \cap L_\infty$ consists of p_s and a second real point p_0 of C. Since the intersection multiplicity of C and L_∞ at p_0 is equal to one, it follows that p_0 is a smooth point of C at which C and L_∞ intersect transversally. The inverse of $C \cap L_\infty$ in the normalization of C thus contains the inverse image of p_0 as a real point. It then follows from Lemma 14 that p_s is an ordinary double point with non-real complex conjugate tangents T and $\overline{T}$. Example 15 provides an illustration of this situation in which the rational smooth embedding under consideration is rectifiable. Let us show that this holds in general. Let $\tau_1 : \mathbb{F}_1 \to \mathbb{P}^2_{\mathbb{R}}$ be the blow up of p_s with exceptional divisor E and let $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$ be the $\mathbb{P}^1$-bundle structure on $\mathbb{F}_1$ induced by the projection $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ from the point p_s. The proper

transform of L_∞ is a real fiber of ρ_1 while T and $\overline{T}$ are a pair of non-real complex conjugate fibers. The proper transform of C is a smooth rational curve which is a section of ρ_1 intersecting E in the pair of non-real complex conjugate points $\{q, \overline{q}\} = (T \cup \overline{T}) \cap E$. Let $\tau_2 : \mathbb{F}_1 \dashrightarrow \mathbb{F}_3$ be the elementary transformation defined over $\mathbb{R}$ obtained by blowing-up q and $\overline{q}$ and contracting the proper transforms of T and $\overline{T}$. The proper transform of C is then a section of $\rho_3 : \mathbb{F}_3 \to \mathbb{P}^1_{\mathbb{R}}$ which does not intersect the proper transform of E. The composition $\tau_2 \circ \tau_1^{-1} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{F}_3$ induces a birational diffeomorphism $\alpha : \mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \dashrightarrow \mathbb{F}_3 \setminus (E \cup L_\infty) \simeq \mathbb{A}^2_{\mathbb{R}}$ with the property that $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a closed embedding of $\mathbb{R}$-schemes which is thus rectifiable by the Abhyankar-Moh Theorem over $\mathbb{R}$. This implies in turn that f is rectifiable. □

4.2. **The quartic case.** Now we consider the case where the projective curve C associated to $f\colon \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a quartic. The types and configurations of singularities that can appear on, or infinitely near to, an irreducible rational complex plane quartic $C_{\mathbb{C}}$ are completely determined by the two classical formulas

$$g = \frac{(d-1)(d-2)}{2} - \sum_p \frac{1}{2} m_p(m_p - 1) \qquad \text{and} \qquad \sum_{q \mapsto p} m_q \leq m_p.$$

In the first one, p varies over all singularities of $C_{\mathbb{C}}$ including infinitely near ones and m_p denotes the multiplicity at p. In the second one q varies in the first infinitely near neighborhood of p. The above formulas permit nine possible types of singularities in the complex case. Some additional sub-types arise in the real case according to the type of branches of C at a given singular point, see [19, 9, 22], these are summarized in Table 1.

Classical Name	Modern Name	Multiplicity sequence
Ordinary Node	A_1, A_1^*	[2]
Ordinary Cusp	A_2	[2]
Tacnode	A_3,A_3^*	[2, 2]
Double Cusp	A_4	[2, 2]
Oscnode	A_5, A_5^*	[2, 2, 2]
Ramphoid Cusp	A_6	[2, 2, 2]
Ordinary triple point	D_4,D_4^*	[3]
Tacnode Cusp	D_5	[3]
Multiplicity 3 Cusp	E_6	[3]
Pair of imaginary ordinary nodes	$2A_1^i$	[2] and [2]
Pair of imaginary ordinary cusps	$2A_2^i$	[2] and [2]

TABLE 1. Possible singularity types of a real rational quartic.

The modern notation in Table 1 refers to Arnold's one [2], with the following additional convention introduced by Gudkov [10]:

- If there is no asterisk in the notation of a point then the point is real and all the branches centered in it are real.

- If there is one asterisk in the notation of a point then it is real and precisely two branches centered in this point are non-real complex conjugate.

- The upper index refers to a pair of non-real conjugate points of the given type.

The property for a real rational quartic C to be the associated projective curve of a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ imposes additional restrictions on its configuration of singular points:

Lemma 32. *The possible configurations of singularities of a real plane quartic C associated with a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ are the following (see Table 2):*

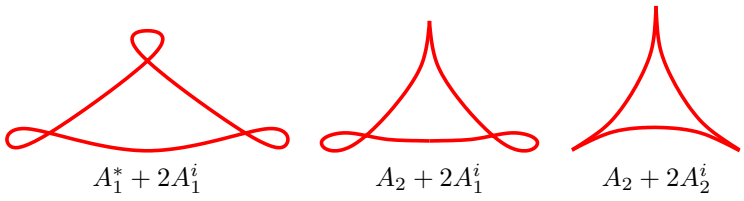
1) Three singular points:

a) $A_1^* + 2A_1^i$ where the real singular point of type A_1^* belongs $C \cap L_\infty$ and L_∞ also intersects C with multiplicity two at a smooth real point of C.

b) $A_2 + 2A_1^i$ or $A_2 + 2A_2^i$. In each case the real singular point of type A_2 belongs to $C \cap L_\infty$, and the tangent line to C at this point is different from L_∞.

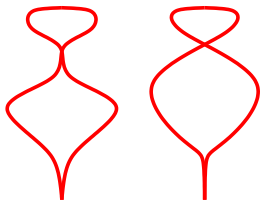
2) Two singular points: $A_2 + A_3^*$ or $A_4 + A_1^*$. In each case, both singular points are real points of $C \cap L_\infty$, and the tangent line to C at each of them is different from L_∞.

3) A unique singular point: A_5^*, A_6, D_4^* or E_6. In each case, the unique singular point is a real point of $C \cap L_\infty$. In the cases D_4^* and E_6, L_∞ is a tangent line to C at the singular point whereas in the case A_5^*, the tangent line to C at the singular point is different from L_∞.



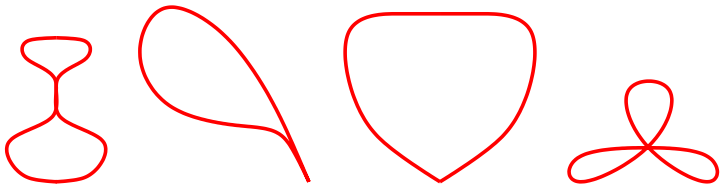
$A_1^* + 2A_1^i$ $A_2 + 2A_1^i$ $A_2 + 2A_2^i$

Three singular points



$A_2 + A_3^*$ $A_4 + A_1^*$

Two singular points



A_5^* A_6 E_6 D_4^*

One singular point

TABLE 2. Real projective quartics associated with rational smooth embeddings of $\mathbb{A}^1_{\mathbb{R}}$.

Proof. 1) Assume that C has three singular points. The genus formula implies they all have multiplicity sequence equal to [2]. At least one of these points, say p_∞ is real and hence necessarily contained in $C \cap L_\infty$. If the other two were real, then since they must also be contained in $C \cap L_\infty$, we would have $(C \cdot L_\infty) \geq 6$ which is absurd. The other two singular points are thus a pair $\{p, \overline{p}\}$ of non-real complex conjugate points of type $2A_1^i$ or $2A_2^i$ of C. Furthermore, neither p nor $\overline{p}$ belongs to $C \cap L_\infty$. We now discuss the possibilities according to the type of the point p_∞.

a) p_∞ is an ordinary double point. Then p_∞ is necessarily of type A_1^* for otherwise its inverse image in the normalization of C would consists of two real point, in contradiction with Lemma 14. This implies in particular that L_∞ is not tangent to C at p_∞, hence that the intersection multiplicity of C and L_∞ at p_∞ is equal to 2. Since the inverse image of $C \cap L_\infty$ in the normalization of C contains a unique real point and $C \cdot L_\infty = 4$, we conclude that $C \cap (L_\infty \setminus p_\infty)$ consists of a unique real point p_s and that L_∞ is the tangent to C at p_s. This yields the possible types $A_1^* + 2A_1^i$ and $A_1^* + 2A_2^i$. To exclude the type $A_1^* + 2A_2^i$, we observe that if it existed then the composition of the normalization morphism $\nu : \mathbb{P}^1_{\mathbb{R}} \to C$ with the morphism $C \to \mathbb{P}^1_{\mathbb{R}}$ induced by the linear projection from the point p_∞ would be a double cover $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$ ramified at the inverse images of p_s and the pair of non-real complex conjugate points p and $\overline{p}$, which is impossible. We thus obtain case a).

b) p_∞ is an ordinary cusp A_2. Then the tangent line to C at p_∞ is distinct from L_∞. Indeed, otherwise we would have $(L_\infty \cdot C)_{p_\infty} = 3$ and then $L_\infty \cap C$ would consists of p_∞ and another smooth real point of C at which C and L_∞ intersects transversally in contradiction with Lemma 14. So $(L_\infty \cdot C)_{p_\infty} = 2$ and since the inverse image of $C \cap L_\infty$ already contains the inverse image of p_s as a real point, it follows again from Lemma 14 that $(C \cap L_\infty) \setminus \{p_\infty\}$ consists of a pair $\{q, \overline{q}\}$ of non-real complex conjugate smooth points q and $\overline{q}$ of C. This yields the possible types $A_2 + 2A_2^i$ and $A_2 + 2A_1^i$.

2) Assume that C has two singular points, say p_0 and p_1. Then the genus formula implies that these points have respective multiplicity sequences $[2]$ and $[2, 2]$. Since these are different, it follows that p_0 and p_1 are real points of C, which are therefore supported on L_∞. Furthermore, since $C \cdot L_\infty = 4$, we have $C \cap L_\infty = \{p_0, p_1\}$ and L_∞ is not tangent to C at p_0 or p_1. Since the inverse image of $C \cap L_\infty$ in the normalization of C contains a unique real point, the union of the branches of C at p_0 and p_1 contains a unique real branch. It follows that either p_0 is of type A_2 and then p_1 must be of type A_3^* or p_0 is of type A_1^* and then p_1 must be of type A_4.

3) Assume that C has a unique singular point p_∞. Then p_∞ is a real point of C, which must then be contained $C \cap L_\infty$. By the genus formula, the possible multiplicity sequences for p_∞ are $[3]$ or $[2, 2, 2]$.

If p_∞ is a triple point, then it follows from Lemma 14 that p_∞ is either a cusp of multiplicity three E_6 or an ordinary triple point of type D_4^*. Indeed, in the other two cases D_5 and D_4, the inverse image of p_∞ in the normalization of C would consist of two and three real points respectively. Since $(C \cdot L_\infty)_{p_\infty} \geq 3$, we conclude that L_∞ is a tangent line to C at p_∞. Indeed otherwise, if $(C \cdot L_\infty)_{p_\infty} = 3$ then L_∞ would also intersect C transversally in a smooth real point, which is again impossible by Lemma 14.

Otherwise, if p_∞ has multiplicity sequence $[2, 2, 2]$ then p_∞ is either a ramphoid cusp A_6 or an oscnode, which is then necessarily of type A_5^* by Lemma 14 again. In the second case, the tangent line to C at p_∞ intersects C with multiplicity four at p_∞, so it cannot be equal to L_∞. Indeed, otherwise we would have $C \cap L_\infty = \{p_\infty\}$ and then the inverse image of $C \cap L_\infty$ in the normalization of C would not contain any real point, in contradiction to Lemma 14. □

Proposition 33. *Every rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated projective curve C is a quartic is biddable. Furthermore, it is rectifiable except possibly in the cases where C has either a singular point of type A_5^* or a singular point of type A_6 whose tangent is different from L_∞.*

To prove Proposition 33, we consider below the different possible cases for C separately, according to the number of its singular points.

4.2.1. *Three singular points.*

Lemma 34. *A rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a quartic with singularities $A_1^* + 2A_1^i$ is rectifiable.*

Proof. The curve C has a real singular point $p_\infty \in C \cap L_\infty$ of type A_1^* with pair of non-real complex conjugate tangents T_∞ and $\overline{T}_\infty$, and $L_\infty \cap C = \{p_\infty, p_0\}$ where p_0 is a smooth real point of C, at which C intersects L_∞ with multiplicity 2. We denote by p and $\overline{p}$ the pair of non-real complex conjugate singular points of C. Since T_∞ intersects C at p_∞ with multiplicity 3, it also intersects C transversally at a point q different from p and $\overline{p}$, and $\overline{T}_\infty$ then intersects C transversally at $\overline{q}$. We let $F = [p_\infty, p]$ and $\overline{F} = [p_\infty, \overline{p}]$ be the pair of non-real complex conjugate lines through p_∞ and p and p_∞ and $\overline{p}$ respectively. Note that the pairs of lines $\{T_\infty, \overline{T}_\infty\}$ and $\{F, \overline{F}\}$ are different. Let $\tau_1 : V_1 = \mathbb{F}_1 \to \mathbb{P}^2_{\mathbb{R}}$ be the blow-up the point p_∞ with exceptional divisor E_∞, and let $\rho_1 : V_1 = \mathbb{F}_1 \to \mathbb{P}^1$ be the induced $\mathbb{P}^1$-bundle structure with exceptional section E_∞. The proper transform of C intersects E_∞ transversally in a pair of non-real complex conjugate points $p_{\infty,1}$ and $\overline{p}_{\infty,1}$. The proper transforms of T_∞ and $\overline{T}_\infty$ are non-real complex conjugate fibers of ρ_1 which intersect E_∞ transversally in $p_{\infty,1}$ and $\overline{p}_{\infty,1}$, respectively, and also intersect the proper transform of C at the points q and $\overline{q}$ respectively. Let $\tau_2 : V_1 \dashrightarrow V_2 = \mathbb{F}_1$ be the birational map defined over $\mathbb{R}$ consisting of the blow-up of $p_{\infty,1}$, $\overline{p}_{\infty,1}$, p and $\overline{p}$ followed by the contraction of the proper transforms of T_∞, $\overline{T}_\infty$, F and $\overline{F}$ (See Figure 4.1).

The proper transform of E_∞ is a section of the $\mathbb{P}^1$-bundle structure $\rho_1' : \mathbb{F}_1 \to \mathbb{P}^1$ with self-intersection -1 while the proper transform of C is a smooth 2-section of ρ_1' with self-intersection 4 disjoint from E_∞. The images of C and L_∞ by the contraction $\tau_3 : V_2 \to \mathbb{P}^2_{\mathbb{R}}$ of E_∞ are then a smooth

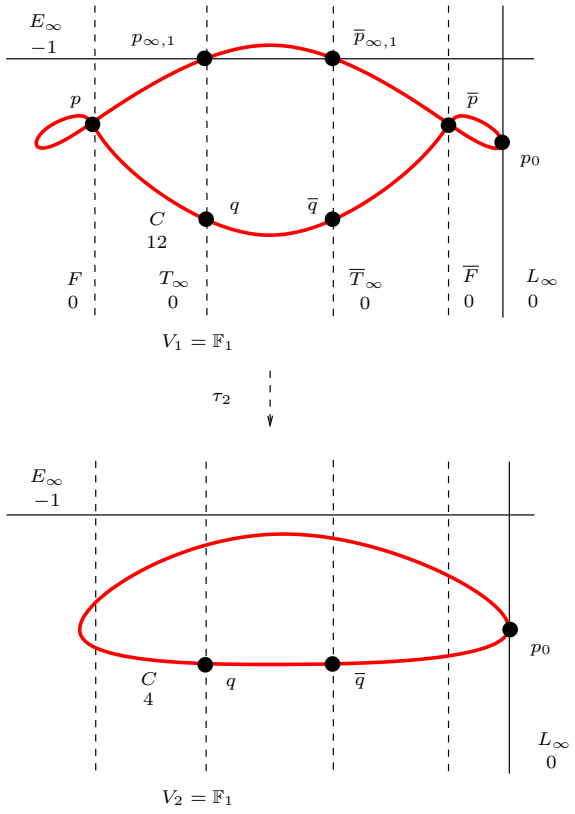


FIGURE 4.1. Three singular points $A_1^* + 2A_1^i$.

conic C' and its tangent line L'_∞ at the real point p_0. The composition $\tau_3 \circ \tau_2 \circ \tau_1^{-1} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^2_{\mathbb{R}}$ is a birational diffeomorphism which maps $C \cup L_\infty$ onto $C' \cup L'_\infty$ and restricts to a birational diffeomorphism $\alpha : \mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \dashrightarrow \mathbb{P}^2_{\mathbb{R}} \setminus L'_\infty = \mathbb{A}^2_{\mathbb{R}}$. The composition $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is then rectifiable by Proposition 31 and hence, f is rectifiable. □

Lemma 35. *A rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a quartic with singularities $A_2 + 2A_1^i$ or $A_2 + 2A_2^i$ is rectifiable.*

Proof. In both cases, C has a unique real ordinary cusp $p_\infty \in C \cap L_\infty$ whose tangent line is different from L_∞. So $(C \cdot L_\infty)_{p_\infty} = 2$ and then, by Lemma 14, $(C \cap L_\infty) \setminus \{p_\infty\}$ consists of a pair $\{q, \overline{q}\}$ of non-real complex conjugate smooth points q and $\overline{q}$ of C. We denote by $\{p, \overline{p}\}$ the pair of non-real complex conjugate singular points of C. To show that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is rectifiable, we proceed as follows.

Step 1) (See Figure 4.2) Let $\tau_1 : \mathbb{F}_1 \to \mathbb{P}^2_{\mathbb{R}}$ be the blow-up the point p_∞ with exceptional divisor E_∞, and let $\rho_1 : V_1 = \mathbb{F}_1 \to \mathbb{P}^1$ be the induced $\mathbb{P}^1$-bundle structure with exceptional section E_∞. The proper transforms of the non-real conjugate lines $L = [p_\infty, p]$ and $\overline{L} = [p_\infty, \overline{p}]$ are fibers of ρ_1 which intersect the proper transform of C in the double points p and $\overline{p}$ respectively. On the other hand C intersects the proper transform of L_∞ transversally in the two conjugate points q and $\overline{q}$, and it intersects E_∞ with multiplicity 2 at a smooth real point $p_{\infty,1}$ supported on $E_\infty \setminus \{L_\infty, L, \overline{L}\}$. Let $\tau_2 : V_1 \dashrightarrow V_2 = \mathbb{F}_1$ be the birational map defined over $\mathbb{R}$ consisting of the blow-up of the pair of points p and $\overline{p}$ followed by the contraction of the proper transforms of L and $\overline{L}$. The proper transform of E_∞ is a section of the $\mathbb{P}^1$-bundle structure $\rho'_1 : \mathbb{F}_1 \to \mathbb{P}^1$ with self-intersection 1 while the proper transform of C is a smooth 2-section of ρ'_1 with self-intersection 4, still intersecting E_∞ with multiplicity 2 in $p_{\infty,1}$.

Step 2) The proper transforms Q and $\overline{Q}$ in V_2 of the pair of non-real complex conjugate conics in $\mathbb{P}^2_{\mathbb{R}}$ passing through $p_\infty, p_{\infty,1}, q, p, \overline{p}$ and $p_\infty, p_{\infty,1}, \overline{q}, p, \overline{p}$ respectively are sections of ρ'_1 passing through

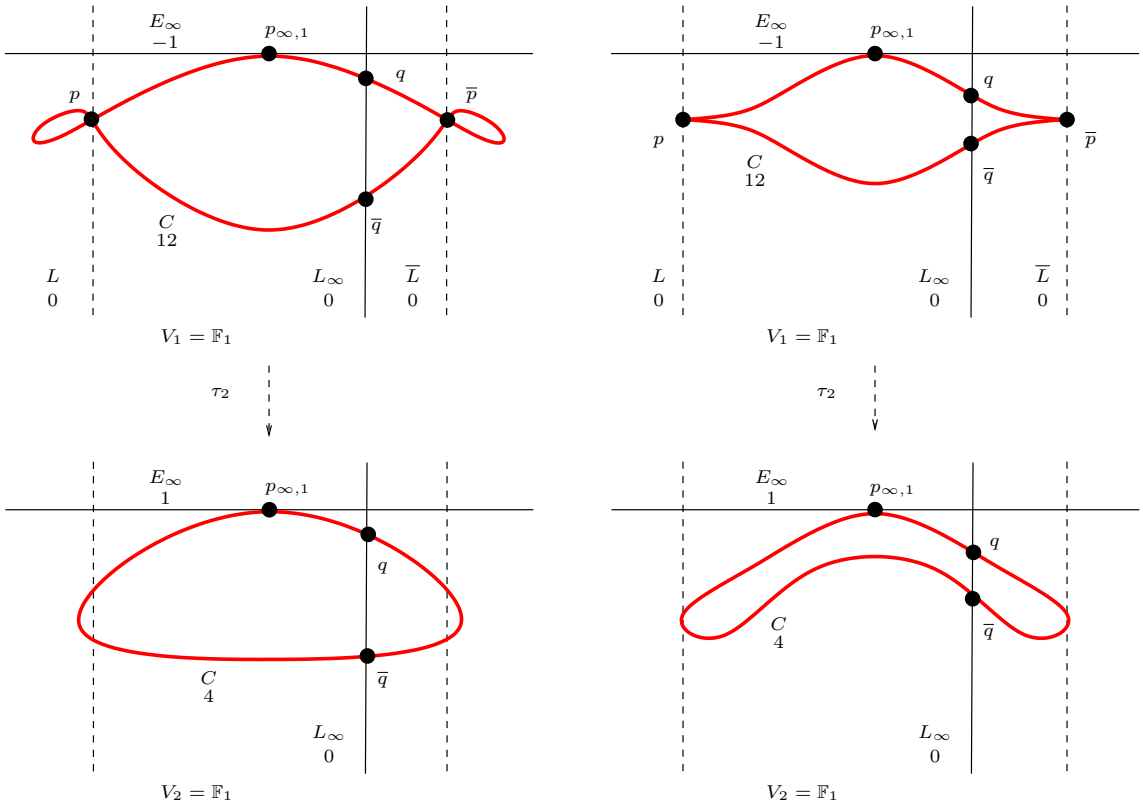


FIGURE 4.2. Three singular points, step 1: case $A_2 + 2A_1^i$ on the left and $A_2 + 2A_2^i$ on the right.

q and $p_{\infty,1}$ and $\overline{q}$ and $p_{\infty,1}$ respectively. Let $\tau_3 : V_2 \dashrightarrow V_3$ be the birational map defined over $\mathbb{R}$ consisting of the blow-up of q and $\overline{q}$, the blow-up of the $\mathbb{R}$-rational point $p_{\infty,1}$ with exceptional divisor $E_{\infty,1}$, followed by the contraction of the proper transforms of Q and $\overline{Q}$. The proper transforms in V_3 of L_∞, E_∞, $E_{\infty,1}$ have self-intersections -2, 0 and 1 respectively, the proper transform of C has self-intersection 1, it intersects E_∞ transversally at the real point $p_{\infty,2} = E_\infty \cap E_{\infty,1}$ and it does no longer intersect L_∞.

Step 3) (See Figure 4.3) Finally, let $\tau_4 : V_3 \dashrightarrow V_4$ be the birational map defined over $\mathbb{R}$ obtained by blowing up $p_{\infty,2}$ with exceptional divisor $E_{\infty,2}$ and then contracting successively the proper transforms of E_∞ and L_∞. The proper transforms of $E_{\infty,1}$ and C have self-intersection 0 while the proper transform of $E_{\infty,2}$ has self-intersection 1. Counting the number of points blown-up and curves contracted, we see that $\mathrm{Pic}(V_4) \simeq \mathbb{Z}^2$. It follows that $V_4 \simeq \mathbb{F}_1$ on which C and $E_{\infty,1}$ and $E_{\infty,2}$ are respectively fibers and a section of the $\mathbb{P}^1$-bundle structure $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$. By construction, the composition $\tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1^{-1} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{F}_1$ restricts to a birational diffeomorphism

$$\alpha : \mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \dashrightarrow \mathbb{F}_1 \setminus (E_{\infty,1} \cup E_{\infty,2}) \simeq \mathbb{A}^2_{\mathbb{R}}$$

such that $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ coincides with the closed immersion of $\mathbb{A}^1_{\mathbb{R}}$ as a real fiber of the projection $\mathrm{pr}_2 = \rho_1|_{\mathbb{A}^2_{\mathbb{R}}} : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$. This shows that f is rectifiable. □

4.2.2. *Two singular points.*

Lemma 36. *A rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ whose associated curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a quartic with singularities $A_2 + A_3^*$ or $A_4 + A_1^*$ is rectifiable.*

Proof. We deal with each case separately (see Figure 4.4).

1) C has a real ordinary cusp p_0 and a real tacnode p_1 with real tangent, whose inverse image in the normalization of C consists of a pair $(q, \overline{q})$ of non-real complex conjugate points. Furthermore the tangent to C at p_0 and p_1 is different from L_∞. By Lemma 14, we have $C \cap L_\infty = \{p_0, p_1\}$. A minimal

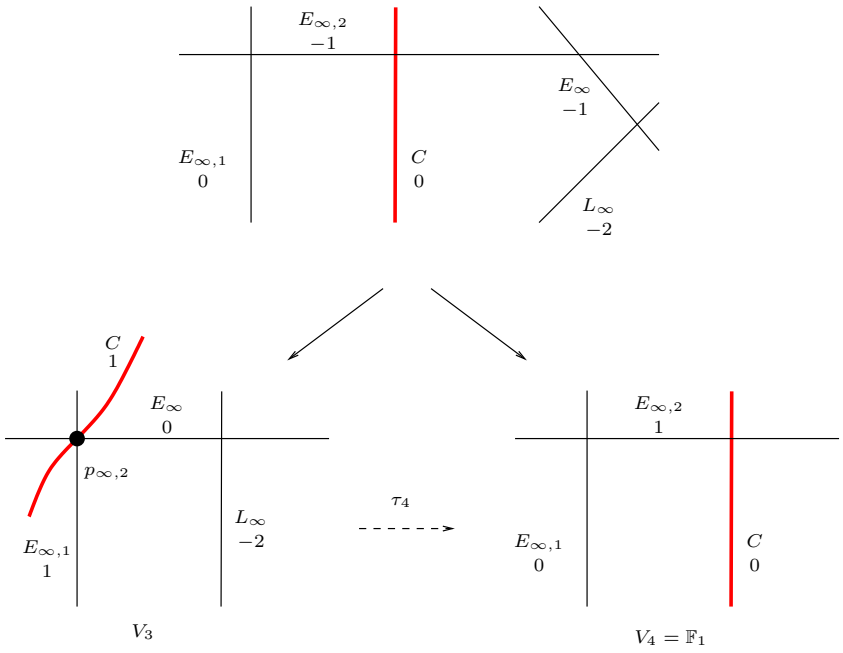


FIGURE 4.3. Three singular points, step 3: case $A_2 + 2A_1^i$ or $A_2 + 2A_2^i$.

log-resolution $\tau : (V, B) \to (\mathbb{P}^2_{\mathbb{R}}, C \cup L_\infty)$ of the pair $(\mathbb{P}^2_{\mathbb{R}}, C \cup L_\infty)$ for which B has no imaginary loop is obtained by blowing-up on the one hand the real point p_0 with exceptional divisor $E_{0,1}$, then the real intersection point $p_{0,1} \neq L_\infty \cap E_{0,1}$ of the proper transform of C with $E_{0,1}$, with exceptional divisor $E_{0,2}$ and then the real intersection point $p_{0,2}$ of the proper transform of C with $E_{0,2}$, with exceptional divisor $E_{0,3}$, and on the other hand the real point p_1 with exceptional divisor $E_{1,1}$, the real intersection point $p_{1,1} \neq L_\infty \cap E_{1,1}$ of the proper transform of C with $E_{1,1}$, with exceptional divisor $E_{1,2}$ and then the pair of non-real complex conjugate intersection points $(q, \overline{q})$ of the proper transform of C with $E_{1,2}$, with respective exceptional divisors F and $\overline{F}$. The proper transform of C in V is a smooth rational curve with self-intersection 0, which intersects $E_{0,3}$, F and $\overline{F}$ transversally. The proper transforms in V of the non-real complex conjugate smooth conics Q and $\overline{Q}$ passing through $p_0, p_{0,1}, p_1, p_{1,1}, q$ and $p_0, p_{0,1}, p_1, p_{1,1}, \overline{q}$ respectively are disjoint (-1)-curves which do not intersect the proper transform of C. The contraction $\tau_1 : V \to V_1$ of $Q \cup \overline{Q} \cup L_\infty \cup E_{1,1} \cup E_{0,1} \cup E_{1,2}$ is a birational morphism defined over $\mathbb{R}$. The images in V_1 of C and $E_{0,2}$ have self-intersection 0 while the image of $E_{0,3}$ has self-intersection 1. We conclude in the same way as in the proof of Lemma 35 that $V_1 \simeq \mathbb{F}_1$ on which C, $E_{0,2}$ and $E_{0,3}$ are respectively fibers and a section of $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$, that $\tau_1 \circ \tau^{-1} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{F}_1$ restricts to a birational diffeomorphism

$$\alpha : \mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \dashrightarrow \mathbb{F}_1 \setminus (E_{0,3} \cup E_{0,2}) \simeq \mathbb{A}^2_{\mathbb{R}}$$

for which $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ as a real fiber of the projection $\mathrm{pr}_2 : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$.

2) C has real double cusp p_0 and a real ordinary double point p_1 with non-real complex conjugate tangents. Furthermore, the tangent line to C at each of them is different from L_∞. As in the previous case, we have $C \cap L_\infty = \{p_0, p_1\}$. A minimal log-resolution $\tau : (V, B) \to (\mathbb{P}^2_{\mathbb{R}}, C \cup L_\infty)$ of the pair $(\mathbb{P}^2_{\mathbb{R}}, C \cup L_\infty)$ for which B has no imaginary loop is obtained by blowing-up on the one hand the real point p_0 with exceptional divisor $E_{0,1}$, then the real intersection point $p_{0,1} \neq E_{0,1} \cap L_\infty$ of the proper transform of C with $E_{0,1}$, with exceptional divisor $E_{0,2}$, then the real intersection point $p_{0,2}$ of the proper transform of C with $E_{0,2}$ with exceptional divisor $E_{0,3}$ and then the real intersection point $p_{0,3}$ of the proper transform of C with $E_{0,3}$ with exceptional divisor $E_{0,4}$, and on the other the real point p_1 with exceptional divisor $E_{1,1}$ and then the pair of non-real complex conjugate intersection points $(q, \overline{q})$ of the proper transform of C with $E_{1,1}$, with respective exceptional divisors F and $\overline{F}$. The proper transform of C in V is a smooth rational curve with self-intersection 0, which intersects $E_{0,4}$, F and $\overline{F}$ transversally. The proper transforms in V of the non-real complex conjugate smooth conics Q and $\overline{Q}$

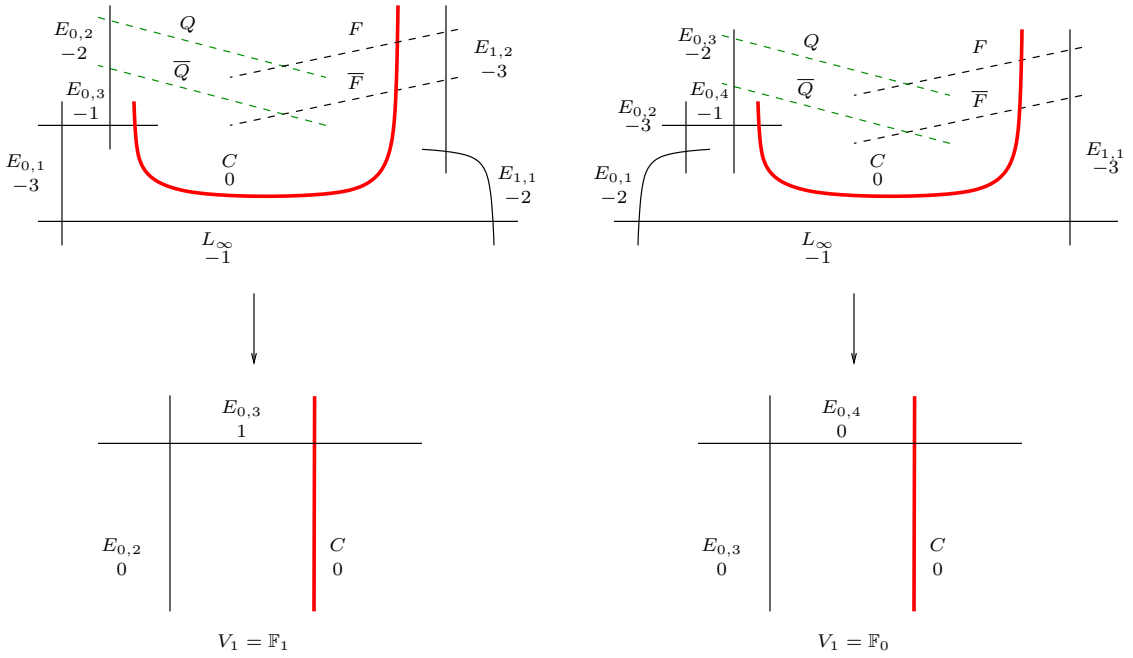


FIGURE 4.4. Two singular points: case $A_2 + A_3^*$ on the left and $A_4 + A_1^*$ on the right.

passing through $p_0, p_{0,1}, p_{0,2}, p_1, q$ and $p_0, p_{0,1}, p_{0,2}, p_1, \overline{q}$ respectively are disjoint (-1)-curves which do not intersect the proper transform of C. The contraction $\tau_1 : V \to V_1$ of $Q \cup \overline{Q} \cup L_\infty \cup E_{0,1} \cup E_{1,1} \cup E_{0,2}$ is a birational morphism defined over $\mathbb{R}$. The images in V_1 of C, $E_{0,3}$ and $E_{0,4}$ all have self-intersection 0. Counting the number of points blown-up and curves contracted, we conclude that $V_1 \simeq \mathbb{F}_0 = \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ in which the images of C, $E_{0,3}$ and $E_{0,4}$ are respectively fibers and a section of $\mathrm{pr}_2 : \mathbb{F}_0 \to \mathbb{P}^1_{\mathbb{R}}$, and that

$$\alpha = \tau_1 \circ \tau^{-1}|_{\mathbb{P}^2_{\mathbb{R}} \setminus L_\infty} : \mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \dashrightarrow \mathbb{F}_0 \setminus (E_{0,4} \cup E_{0,3}) \simeq \mathbb{A}^2_{\mathbb{R}}$$

is a birational diffeomorphism for which $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ as a real fiber of the projection $\mathrm{pr}_2 : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$. □

4.2.3. *A unique singular point.*

Lemma 37. *Let $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ be a rational smooth embedding whose associated curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a quartic with a unique singular point p_∞. Then the following holds:*

a) If p_∞ is an oscnode A_5^* or a ramphoid cusp A_6 such that $T_{p_\infty} C \neq L_\infty$ then f is biddable.

b) If p_∞ is either triple point D_4^* or E_6 or a ramphoid cusp A_6 such that $T_{p_\infty} C = L_\infty$ then f is rectifiable.

Proof. We consider each of the possible singularities E_6, D_4^*, A_6 and A_5^* listed in Lemma 32 separately. We begin with the curves listed in case b).

1) If p_∞ is a multiplicity 3 cusp E_6 or a ramphoid cusp A_6 with tangent $T_{p_\infty} C = L_\infty$, then $C \cap L_\infty = p_\infty$ and hence $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is an everywhere defined closed embedding, which is thus rectifiable by the Abhyankar-Moh Theorem over $\mathbb{R}$.

2) If p_∞ is a real triple point of type D_4^* then the tangent cone to C at p_∞ is the union of L_∞ and a pair L and $\overline{L}$ of non-real complex conjugate tangents to C. Since every line in $\mathbb{P}^2_{\mathbb{R}}$ passing through p_∞ intersects $C \setminus \{p_\infty\}$ transversally in at most one real point, the proper transform of C in the blow-up $\tau_1 : \mathbb{F}_1 \to \mathbb{P}^2_{\mathbb{R}}$ of p_∞ with exceptional divisor $E \simeq \mathbb{P}^1_{\mathbb{R}}$ is a section of the $\mathbb{P}^1$-bundle structure $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1_{\mathbb{R}}$. It intersects E transversally at the pair of non-real complex conjugate intersection points q and $\overline{q}$ of the proper transforms of L and $\overline{L}$ with E respectively and at the real intersection point of the proper transform of L_∞ with E. Let $\tau_2 : \mathbb{F}_1 \dashrightarrow \mathbb{F}_3$ be the birational map defined over $\mathbb{R}$

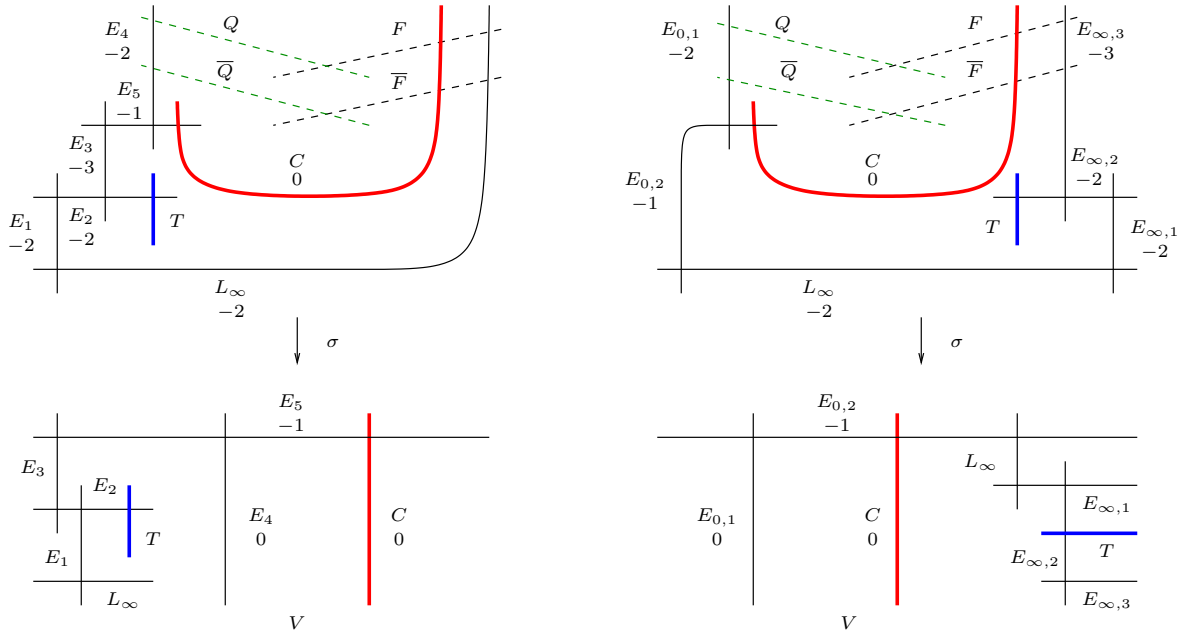


FIGURE 4.5. One singular point: case A_6 on the left and A_5^* on the right.

consisting of the blow-up of q and $\overline{q}$ followed by the contraction of the proper transform of L and $\overline{L}$. The proper transforms of L_∞ and E are respectively a fiber and the negative section of the $\mathbb{P}^1$-bundle structure $\rho_3 : \mathbb{F}_3 \to \mathbb{P}^1_{\mathbb{R}}$ while the proper transform of C in $\mathbb{F}_3$ is a section of ρ_3 intersecting E transversally at the real point $E \cap L_\infty$. The birational map $\tau_2 \circ \tau_1^{-1}$ restricts to a birational diffeomorphism $\alpha : \mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \xrightarrow{\sim} \mathbb{F}_3 \setminus (E \cup L_\infty) \simeq \mathbb{A}^2_{\mathbb{R}}$ for which the composition $\alpha \circ f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is an everywhere defined closed embedding. By the Abhyankar-Moh Theorem over $\mathbb{R}$, $\alpha \circ f$ is rectifiable, and so f is rectifiable.

3) If p_∞ is a ramphoid cusp A_6 such that $T_{p_\infty} C \neq L_\infty$ then $C \cap (L_\infty \setminus \{p_\infty\})$ consists of a pair $\{q, \overline{q}\}$ of non-real complex conjugate points while the tangent T to C at p_∞ intersects C with multiplicity 4 at p_∞. Let $\overline{\rho} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ be the rational map defined by the pencil $\mathcal{M} \subset |\mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(4)|$ generated by C and $3T + L_\infty$. A minimal resolution $\tau : V_1 \to \mathbb{P}^2_{\mathbb{R}}$ of the indeterminacy of $\overline{\rho}$ is obtained by blowing-up q and $\overline{q}$ with respective non-real complex conjugate exceptional divisor F and $\overline{F}$, the point p_∞ with exceptional divisor $E_1 \simeq \mathbb{P}^1_{\mathbb{R}}$ and four times successively the intersection point $p_{\infty,i}$ of the proper transform of C and E_i, with exceptional divisor $E_{i+1} \simeq \mathbb{P}^1_{\mathbb{R}}$. The reduced total transform B_1 of L_∞ is a tree of smooth geometrically rational curves. The proper transform of C in V is a smooth rational curve with self-intersection 0, and so $\overline{\pi}_1 = \overline{\rho} \circ \tau : V_1 \to \mathbb{P}^1_{\mathbb{R}}$ is a $\mathbb{P}^1$-fibration, having the proper transform of C as a smooth real fiber, and the exceptional divisors F, $\overline{F}$ and E_5 as sections (See left hand side of Figure 4.5).

The fiber of $\overline{\pi}_1$ over the real point $\overline{\pi}_1(E_4)$ consists of the union of E_4 and the proper transforms of the non-real complex conjugate smooth conics Q and $\overline{Q}$ passing respectively through $p_\infty, p_{\infty,1}, p_{\infty,2}, p_{\infty,3}, q$ and $p_\infty, p_{\infty,1}, p_{\infty,2}, p_{\infty,3}, \overline{q}$. These are disjoint (-1)-curves in $V_{1,\mathbb{C}}$ which can thus be contracted by a birational morphism $\sigma : V_1 \to V$ defined over $\mathbb{R}$. We let $B = L_\infty \cup E_1 \cup \cdots \cup E_5$ be the proper transform of $B_{1,\mathbb{R}}$. The proper transform of E_4 is then a full fiber of the induced $\mathbb{P}^1$-fibration $\overline{\pi} = \overline{\pi}_1 \circ \sigma^{-1} : V \to \mathbb{P}^1_{\mathbb{R}}$ and the latter restricts on $S = V \setminus B$ to an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ with a unique singular fiber equal to $T_{p_\infty} \cap S \simeq \mathbb{A}^1_{\mathbb{R}}$ with multiplicity 3. By Theorem 2, S is an algebraic model of $\mathbb{R}^2$ and by construction, $\alpha = \sigma \circ \tau^{-1} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow V$ induces a birational diffeomorphism $\mathbb{P}^2_{\mathbb{R}} \setminus L_\infty \dashrightarrow S$ that maps C to an irreducible fiber of $\overline{\pi}$. This shows that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is biddable. Note that since π has $T \cap S \simeq \mathbb{A}^1_{\mathbb{R}}$ as a non-reduced fiber, none of the sufficient conditions for rectifiablity given Proposition 24 is satisfied.

4) If p_∞ is an oscnode whose inverse image in the normalization of C consists of a pair $\{q, \overline{q}\}$ of non-real complex conjugate points and whose tangent is different from L_∞. It follows from Lemma 14 that $C \cap (L_\infty \setminus \{p_\infty\})$ consists of a unique other smooth real point p_0 at which C and L_∞ intersect with multiplicity 2. As in the previous case, the tangent T to C at p_∞ intersects C with multiplicity 4 at p_∞, and we consider the rational map $\overline{\rho} : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^1_{\mathbb{R}}$ defined by the pencil $\mathcal{M} \subset |\mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(4)|$ generated by C and $3T + L_\infty$. A minimal resolution $\tau : V_1 \to \mathbb{P}^2_{\mathbb{R}}$ of the indeterminacy of $\overline{\rho}$ for which B has no imaginary loop is obtained by blowing-up on the one hand p_0 with exceptional divisors $E_{0,1}$ and then the intersection point $p_{0,1}$ of the proper transform of C and $E_{0,1}$ with exceptional divisor $E_{0,2}$, and on the other hand blowing-up p_∞ with exceptional divisor $E_{\infty,1}$, then two times successively the intersection point $p_{\infty,i}$ of the proper transform of C and $E_{\infty,i}$ with exceptional divisor $E_{\infty,i+1}$ and finally blowing-up the pair $\{q, \overline{q}\}$ of non-real complex conjugate points of intersection of $E_{\infty,3}$ with the proper transform of C, with respective exceptional divisors F and $\overline{F}$ (See right hand side of Figure 4.5). The reduced total transform B_1 of L_∞ is a tree of smooth geometrically rational curves. The proper transform of C in V is a smooth rational curve with self-intersection 0. So $\overline{\pi}_1 = \overline{\rho} \circ \tau_1 : V_1 \to \mathbb{P}^1_{\mathbb{R}}$ is a $\mathbb{P}^1$-fibration having the proper transform of C as a smooth real fiber and the curves $E_{0,2}$, F and $\overline{F}$ as disjoint sections. The fiber of $\overline{\pi}_1$ over the real point $\overline{\pi}_1(E_{0,1})$ consists of the union of $E_{0,1}$ and the proper transforms of the non-real complex conjugate smooth conics Q and $\overline{Q}$ passing respectively through $p_0, p_\infty, p_{\infty,1}, p_{\infty,2}, q$ and $p_0, p_\infty, p_{\infty,1}, p_{\infty,2}, \overline{q}$. These are disjoint (-1)-curves in $V_{1,\mathbb{C}}$ which can thus be contracted by a birational morphism $\sigma : V_1 \to V$ defined over $\mathbb{R}$. We let $B = L_\infty \cup E_{\infty,1} \cup E_{\infty,2} \cup E_{\infty,3} \cup E_{0,1} \cup E_{0,2}$ be the proper transform of $B_{1,\mathbb{R}}$. The proper transform of $E_{0,1}$ is then a full fiber of the induced $\mathbb{P}^1$-fibration $\overline{\pi} = \overline{\pi}_1 \circ \sigma^{-1} : V \to \mathbb{P}^1_{\mathbb{R}}$, and the latter restricts on $S = V \setminus B$ to an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ with a unique singular fiber equal to $T \cap S \simeq \mathbb{A}^1_{\mathbb{R}}$ with multiplicity 3. We conclude as in the previous case that $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ is biddable but that none of the sufficient conditions for rectifiablity given Proposition 24 is satisfied. □

Remark 38. Let $f_i : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$, $i = 1, 2$, be rational smooth embeddings whose associated curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a quartic whose unique singular point p_∞ is an oscnode A_5^* or a ramphoid cusp A_6 such that $T_{p_\infty} C \neq L_\infty$ respectively. In each case, the $\mathbb{A}^1$-fibered model $\pi_i : S_i \to \mathbb{A}^1_{\mathbb{R}}$ of $\mathbb{R}^2$ for which $\alpha_i \circ f_i : \mathbb{A}^1_{\mathbb{R}} \to S_i$ is a closed embedding as a smooth fiber of π_i constructed in the proof of Lemma 37 has a unique multiple fiber, of multiplicity 3, say $\pi_i^{-1}(0)$. On the other hand, one sees on Figure 4.5 that the $\mathbb{P}^1$-fibrations $\overline{\pi}_i : V_i \to \mathbb{P}^1_{\mathbb{R}}$ extending π_i have non-isomorphic real fibers $\overline{\pi}_1^{-1}(0)$ and $\overline{\pi}_2^{-1}(0)$: in the A_5^* case, the irreducible component of $\overline{\pi}_1^{-1}(0)$ intersecting the section of $\overline{\pi}_1$ contained in B_1 has self-intersection -3, while the ireducible component of $\overline{\pi}_2^{-1}(0)$ intersecting the section of $\overline{\pi}_2$ contained in B_2 has self-intersection -2 in the A_6 case. This implies that there cannot exist any birational map $\beta : V_1 \dashrightarrow V_2$ which restricts to a birational diffeomorphism between S_1 and S_2 and such that $\overline{\pi}_2 \circ \beta = \overline{\pi}_1$.

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