# ONE DIMENSIONAL CRITICAL KINETIC FOKKER-PLANCK EQUATIONS, BESSEL AND STABLE PROCESSES

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Abstract. We consider a particle moving in one dimension, its velocity being a reversible diffusion process, with constant diffusion coefficient, of which the invariant measure behaves like $(1+|v|)^{-\beta}$ for some $\beta>0$. We prove that, under a suitable rescaling, the position process resembles a Brownian motion if $\beta\geq 5$, a stable process if $\beta\in[1,5)$ and an integrated symmetric Bessel process if $\beta\in(0,1)$. The critical cases $\beta=1$ and $\beta=5$ require special rescalings. We recover some results of [24, 10, 19] and [1] with an alternative approach.

## 1. Introduction and results

We consider the following one-dimensional stochastic kinetic model:

$$V_t=V_0+B_t-\frac{\beta}{2}\int_0^t F(V_s)\mathrm{d}s \quad \text{and} \quad X_t=X_0+\int_0^t V_s\mathrm{d}s. \tag{1}$$

Here $(B_t)_{t\geq 0}$ is a Brownian motion independent of the initial condition $(X_0,V_0)$. We assume that $\beta>0$ and that the force is of the form

$$F=-\frac{\Theta'}{\Theta}, \text{ for some even } \Theta:\mathbb{R}\mapsto(0,\infty) \text{ of class } C^2 \text{ satisfying } \lim_{|v|\to\infty}|v|\Theta(v)=1. \tag{2}$$

The typical example we have in mind is $F(v)=v/(1+v^2)$, which corresponds to $\Theta(v)=(1+v^2)^{-1/2}$.

The drift $F$ being $C^1$, (1) classically has a unique (possibly local) strong solution, and we will see that it is global. An invariant measure $\mu_\beta$ of the velocity process $(V_t)_{t\geq0}$ must solve $\frac{1}{2}\mu_\beta''+\frac{\beta}{2}(F\mu_\beta)'=0$ in the sense of distributions. The unique (up to constants) solution is given by

$$\mu_\beta(\mathrm{d}v)=c_\beta[\Theta(v)]^\beta\mathrm{d}v,$$

and we choose $c_\beta^{-1}=\int_{\mathbb{R}}[\Theta(v)]^\beta\mathrm{d}v<\infty$ if $\beta>1$ and $c_\beta=1$ if $\beta\in(0,1]$.

1.1. **Main result.** Our goal is to describe the large time behavior of the position process $(X_t)_{t\geq0}$. For each $\beta\geq1$, we define the constant $\sigma_\beta>0$ as follows:

- $\sigma_\beta^2=8c_\beta\int_0^\infty\Theta^{-\beta}(v)[\int_v^\infty u\Theta^\beta(u)\mathrm{d}u]^2\mathrm{d}v$ if $\beta>5$,
- $\sigma_5^2=4c_5/27$,
- $\sigma_\beta^\alpha=3^{1-2\alpha}2^{\alpha-1}c_\beta\pi/[(\Gamma(\alpha))^2\sin(\pi\alpha/2)]$, where $\alpha=(\beta+1)/3$, if $\beta\in(1,5)$,
- $\sigma_1^{2/3}=2^{2/3}3^{-5/6}\pi/[\Gamma(2/3)]^2$.

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Consider a family $((Z_t^\epsilon)_{t\geq 0})_{\epsilon\geq 0}$ of processes. We write $(Z_t^\epsilon)_{t\geq 0} \stackrel{f.d.}{\longrightarrow} (Z_t^0)_{t\geq 0}$ if for all finite subset $S \subset [0,\infty)$ the vector $(Z_t^\epsilon)_{t\in S}$ goes in law to $(Z_t^0)_{t\in S}$ as $\epsilon \to 0$. We write $(Z_t^\epsilon)_{t\geq 0} \stackrel{d}{\longrightarrow} (Z_t)_{t\geq 0}$ if the convergence in law holds in the usual sense of continuous processes. Here is our main result.

**Theorem 1.** *Fix $\beta > 0$ and consider the solution $(X_t, V_t)_{t\geq 0}$ to (1). Let $(B_t)_{t\geq 0}$ be a Brownian motion, let $(S_t^{(\alpha)})_{t\geq 0}$ be a symmetric stable process with index $\alpha \in (0,2)$ such that $\mathbb{E}[\exp(i\xi S_t^{(\alpha)})] = \exp(-t|\xi|^\alpha)$ and let $(U_t^{(\delta)})_{t\geq 0}$ be a symmetric Bessel process of dimension $\delta \in (0,1)$, see Definition 4.*

*(a) If $\beta > 5$, $(\epsilon^{1/2} X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\longrightarrow} (\sigma_\beta B_t)_{t\geq 0}$.*

*(b) If $\beta = 5$, $(\epsilon^{1/2}|\log\epsilon|^{-1/2} X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\longrightarrow} (\sigma_5 B_t)_{t\geq 0}$.*

*(c) If $\beta \in (1,5)$, $(\epsilon^{1/\alpha} X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\longrightarrow} (\sigma_\beta S_t^{(\alpha)})_{t\geq 0}$, where $\alpha = (\beta+1)/3$.*

*(d) If $\beta = 1$, $(|\epsilon\log\epsilon|^{3/2} X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\longrightarrow} (\sigma_1 S_t^{(2/3)})_{t\geq 0}$.*

*(e) If $\beta \in (0,1)$, $(\epsilon^{1/2} V_{t/\epsilon}, \epsilon^{3/2} X_{t/\epsilon})_{t\geq 0} \stackrel{d}{\longrightarrow} (U_t^{(1-\beta)}, \int_0^t U_s^{(1-\beta)} \mathrm{d}s)_{t\geq 0}$.*

We will deduce the following decoupling between the position and the velocity.

**Corollary 2.** *Fix $\beta > 1$, adopt the same notation as in Theorem 1 and consider a $\mu_\beta$-distributed random variable $\bar V$ independent of everything else.*

*(a) If $\beta > 5$, for each $t > 0$, $(\epsilon^{1/2} X_{t/\epsilon}, V_{t/\epsilon}) \stackrel{d}{\longrightarrow} (\sigma_\beta B_t, \bar V)$.*

*(b) If $\beta = 5$, for each $t > 0$, $(\epsilon^{1/2}|\log\epsilon|^{-1/2} X_{t/\epsilon}, V_{t/\epsilon}) \stackrel{d}{\longrightarrow} (\sigma_5 B_t, \bar V)$.*

*(c) If $\beta \in (1,5)$, for each $t > 0$, $(\epsilon^{1/\alpha} X_{t/\epsilon}, V_{t/\epsilon}) \stackrel{d}{\longrightarrow} (\sigma_\beta S_t^{(\alpha)}, \bar V)$, where $\alpha = (\beta+1)/3$.*

1.2. **The Fokker-Planck equation.** Consider a solution $(X_t, V_t)_{t\geq 0}$ to (1) and denote, for each $t \geq 0$, by $f_t \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$ the law of $(X_t, V_t)$. We write the paragraph doing as if $f_t$ had a density, that we still denote by $f_t$, for each $t > 0$. This probably always holds true, but this is not the subject of the paper. Of course, everything can be written, with heavier notation, when $f_t$ is (possibly) singular. By the Itô formula, $(f_t)_{t\geq 0}$ is a weak solution to the Fokker-Planck equation

$$\partial_t f_t(x,v) + v\partial_x f_t(x,v) = \frac{1}{2}\big(\partial_{vv} f_t(x,v) + \beta\partial_v[F(v) f_t(x,v)]\big). \tag{3}$$

Corollary 2-(a) tells us that, when $\beta > 5$, in a weak sense, $\epsilon^{-1/2} f_{\epsilon^{-1}t}(\epsilon^{-1/2}x, v)$ goes, as $\epsilon \to 0$, to $g_t(x)\mu_\beta(v)$, where $g_t(x)$ is the centered Gaussian density with variance $\sigma_\beta^2 t$.

If $\beta = 5$, by Corollary 2-(b), still in a weak sense, $\epsilon^{-1/2}|\log\epsilon|^{1/2} f_{\epsilon^{-1}t}(\epsilon^{-1/2}|\log\epsilon|^{1/2}x, v)$ goes, as $\epsilon \to 0$, to $g_t(x)\mu_\beta(v)$, where $g_t(x)$ is the centered Gaussian density with variance $\sigma_5^2 t$.

If $\beta \in (1,5)$, by Corollary 2-(c), setting $\alpha = (\beta+1)/3$, $\epsilon^{-1/\alpha} f_{\epsilon^{-1}t}(\epsilon^{-1/\alpha}x, v)$ goes, as $\epsilon \to 0$, to $g_t(x)\mu_\beta(v)$, where $g_t(x)$ is the symmetric stable law characterized by its Fourier transform $\int_\mathbb{R} g_t(x) e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_\beta\xi|^\alpha)$.

If $\beta = 1$, by Theorem 1-(d), setting $\rho_t(x) = \int_\mathbb{R} f_t(x,v)\mathrm{d}v$, $|\epsilon\log\epsilon|^{-3/2}\rho_{\epsilon^{-1}t}(|\epsilon\log\epsilon|^{-3/2}x)$ goes, in a weak sense, to $g_t(x)$, where $g_t(x)$ is characterized by $\int_\mathbb{R} g_t(x)e^{i\xi x}\mathrm{d}x = \exp(-t|\sigma_1\xi|^{2/3})$.

Finally, when $\beta \in (0,1)$, Theorem 1-(e) tells us that $\epsilon^{-2} f_{\epsilon^{-1}t}(\epsilon^{-3/2}x, \epsilon^{-1/2}v)$ converges to $h_t(x,v)$, the density of the law of $(U_t^{(1-\beta)}, \int_0^t U_s^{(1-\beta)}\mathrm{d}s)$. It should be that $(h_t)_{t\geq 0}$ is a symmetric solution, in some very weak sense, of (3), with $h_0 = \delta_{(0,0)}$ and $F(v) = v^{-1}\mathbf{1}_{\{v\neq 0\}}$.

1.3. **Motivation and references.** Performing a space-time rescaling in Boltzmann-like equations to get some diffusion limit for the position process is an old and classical subject, see for example Larsen and Keller [18], Bensoussans, Lions and Papanicolaou [4]. More recently, Bodineau, Gallagher and Saint-Raymond [6] obtained some Brownian motion, starting from a system of hard spheres.

However, anomalous diffusion often arises in physics, and many works show how to modify the collision kernel in some Boltzmann-like linear equations to get some fractional diffusion limit (i.e. a symmetric stable jumping position process). One can e.g. linearize the Boltzmann equation around a fat tail equilibrium or consider some *ad hoc* cross section. This was initiated by Mischler, Mouhot and Mellet [22], with close links to the earlier work of Milton, Komorowski and Olla [23] on Markov chains. This was continued by Mellet [21], Ben Abdallah, Mellet and Puel [2, 3] and others.

The Fokker-Planck equation is often used in physics to approximate Boltzmann-like equations, because it is generally more simple to tackle. When the repealing/friction force $F$ is strong enough, the velocities have an invariant measure with fast decay. When $F(v) = v$, the diffusion limit for the position process was then predicted by Langevin in [17]. For generalizations and recent analysis see e.g. Cattiaux, Chafaï and Guillin [8].

The only way to hope for some anomalous diffusion limit, for a Fokker-Planck toy model like (1), is to choose the repealing force in such a way that the invariant measure of the velocity process has a fat tail. Then one realizes that we have to choose $F$ behaving like $F(v) \sim 1/v$ as $|v| \to \infty$, and the most natural choice is $F(v) = v/(1 + v^2)$. This case has been studied by Nasreddine and Puel [24] (normal diffusion, $\beta > 5$, in any dimension), Cattiaux, Nasreddine and Puel [10] (critical case $\beta = 5$, in any dimension) and Lebeau and Puel [19] (anomalous diffusion, $\beta \in (1, 5)$, in one dimension), the case $\beta \in (0, 1]$ being left open.

For this model, there is no (weighted) Poincaré inequality for the velocity process, so that this process converges quite slowly to equilibrium. This seems to be an issue when $\beta \leq 5$. Some more complicated inequalities, see Cattiaux, Gozlan, Guillin and Roberto [9], are used in [10, 19].

The anomalous diffusion case $\beta \in (1, 5)$ for (1) is rather difficult to treat, in comparison to above cited works [22, 21, 2, 3] on Boltzmann-like equations. In particular, while the stable index $\alpha$ is more or less prescribed from the beginning in [22, 21, 2, 3], it is a rather mysterious function of $\beta$ in the present case. The paper [19] relies on a deep spectral analysis, making a wide use of special functions. From the probabilistic point of view, the models studied in [22, 21, 2, 3] correspond to jumping velocity processes with fat tail Lévy measures, so that stable Lévy limits are quite natural.

In a somewhat different perspective, physicists discovered that atoms, when cold by a laser, diffuse anomalously, like *Lévy walks.* See Castin, Dalibard and Cohen-Tannoudji [7], Sagi, Brook, Almog and Davidson [26] and Marksteiner, Ellinger and Zoller [20]. A theoretical study has been proposed by Barkai, Aghion and Kessler [1] (see also and Hirschberg, Mukamel and Schütz [12]). They precisely model the motion of atoms by (1) with the force $F(v) = v/(1 + v^2)$ induced by the laser field. They prove, with a quite high level of rigor, the results of Theorem 1, excluding the critical cases and stating point (e), that they call Obukhov-Richardson phase, differently.

Let us say a word of the proof of [1]. One easily gets convinced that the velocity process behaves like a (symmetrized) Bessel process with dimension $\delta = 1 - \beta$ when far away from 0, simply because $F(v) \simeq v^{-1}$. But, even when $\delta \leq 0$, this velocity process is not stuck when it reaches 0. Consequently, the position process can be approximated by a sum of i.i.d. (signed) areas of excursions of Bessel processes with dimension $\delta$, with a sense to be precised when $\delta \leq 0$ (consider the area under the Bessel process until it reaches 0, when starting from $v > 0$, and let $v \to 0$ with in a clever way). Using some explicit computations relying on modified Bessel functions, they show that this area has a fat tail distribution, with a density decaying at infinity like $x^{-1-\alpha}$, where $\alpha = (\beta + 1)/3$. Hence if $\beta > 5$, then $\alpha > 2$ and this area has a second order moment, whence the classical central limit theorem applies, and normal diffusion occurs. If now $\alpha < 2$, one has to use a *stable* limit theorem, and anomalous diffusion arises.

Actually, when $\beta < 1$, the *length* of the Bessel excursion is no more integrable and the above proof breaks down. Barkai, Aghion and Kessler [1] introduce some Bessel bridges and handle some tedious explicit computations. But the situation is actually much easier since, at least at the informal level, the Bessel process with dimension $\delta = 1-\beta > 0$ is not stuck when it reaches 0, so that one can simply approximate the velocity process by a *true* (symmetrized) Bessel process with dimension $\delta$.

1.4. **Goal and strategy.** Our initial goal was to completely formalize the arguments of [1], in order to provide a probabilistic proof, including the critical cases, of the results of [24, 10, 19]. We found another way, which is more qualitative and even more probabilistic, making use of the connections (or similarities) between Bessel and stable processes, see Section 2. We provide a concise proof, that moreover allows us to deal with general forces of the form (2).

The core of the paper (when $\beta \leq 5$, which is the most interesting case) consists in making precise the following *informal* arguments. For $(W_t)_{t\geq 0}$ a Brownian motion and for $\tau_t$ the inverse of the time change $A_t = (\beta+1)^{-2}\int_0^t |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s$, the process $Y_t = W_{\tau_t}$ should classically solve, see e.g. Revuz-Yor [25, Proposition 1.13 page 373], $Y_t = (\beta+1)\int_0^t |Y_s|^{\beta/(\beta+1)}\mathrm{d}B_s$, for some other Brownian motion $(B_t)_{t\geq 0}$. Hence, still informally, $V_t = \mathrm{sg}(Y_t)|Y_t|^{1/(\beta+1)}$ should solve, by the Itô formula,

$$V_t = B_t - (\beta/2)\int_0^t \mathrm{sg}(V_s)|V_s|^{-1}\mathrm{d}s, \tag{4}$$

This is a rough version of (1) with $F(v) = \mathrm{sg}(v)|v|^{-1}$ and we admit that, after rescaling, this describes some large time behavior of the true solution to (1).

We recognize in (4) a symmetric version of the SDE for a Bessel process of dimension $\delta = 1-\beta$.

If $\beta \in (0,1)$, i.e. $\delta > 0$, such a (symmetric) Bessel process is well-defined and non-trivial, see also Definition 4 below. Thus Theorem 1-(e) is not surprising.

If $\beta \in (1,5)$, i.e. $\delta \leq 0$, it is well-known that $V_t$ will remain stuck at 0. But it actually appears that $A_t$ is infinite and, in some sense to be precised, proportional to the local time $L_t^0$ of $(W_t)_{t\geq 0}$. Hence, up to correct rescaling, $X_t = \int_0^t V_s\mathrm{d}s = \int_0^t \mathrm{sg}(W_{\tau_s})|W_{\tau_s}|^{1/(\beta+1)}\mathrm{d}s = \int_0^{\tau_t}\mathrm{sg}(W_s)|W_s|^{1/(\beta+1)}\mathrm{d}A_s = (\beta+1)^{-2}\int_0^{\tau_t}\mathrm{sg}(W_s)|W_s|^{(1-2\beta)/(\beta+1)}\mathrm{d}s$. Since $(\tau_t)_{t\geq 0}$ is proportional to the inverse of the local time of $(W_t)_{t\geq 0}$, we know from Biane-Yor [5] that $(X_t)_{t\geq 0}$ is an $\alpha$-stable Lévy process, with $\alpha = (\beta+1)/3$. See Theorem 3 below for a precise statement and a few explanations. All this is completely informal, in particular observe that we always have $W_{\tau_t} = 0$, so that the equality $\int_0^t \mathrm{sg}(W_{\tau_s})|W_{\tau_s}|^{1/(\beta+1)}\mathrm{d}s = \int_0^{\tau_t}\mathrm{sg}(W_s)|W_s|^{1/(\beta+1)}\mathrm{d}A_s$ is far from being fully justified.

The proof of Theorem 3 does not require deep computations involving specials functions, unless one wants to know the value of the diffusion constant. This is why we say that our proof is *qualitative*.

## 2. Stable and Bessel processes

In the whole paper, we denote by sg the sign function with the convention that $\mathrm{sg}(0) = 0$.

2.1. **Stable processes.** We will use the following theorem that we found in Biane-Yor [5]. Very similar results were already present in Itô-McKean [13, page 226] and Jeulin-Yor [15] when $\alpha \in (0,1)$.

**Theorem 3.** *Fix $\alpha \in (0,2)$. Consider a Brownian motion $(W_t)_{t\geq 0}$, its local time $(L_t^0)_{t\geq 0}$ at 0 and its right-continuous generalized inverse $\tau_t = \inf\{u \geq 0 : L_u^0 > t\}$. For $\eta > 0$, let $K_t^\eta = \int_0^t \mathrm{sg}(W_s)|W_s|^{1/\alpha-2}\mathbf{1}_{\{|W_s|\geq\eta\}}\mathrm{d}s$. Then $(K_t^\eta)_{t\geq 0}$ a.s. converges to some process $(K_t)_{t\geq 0}$ as $\eta \to 0$, and $(K_{\tau_t})_{t\geq 0}$ is a symmetric $\alpha$-stable process such that*

$$\mathbb{E}[\exp(i\xi K_{\tau_t})] = \exp(-\kappa_\alpha t|\xi|^\alpha), \quad \textit{where} \quad \kappa_\alpha = \frac{2^\alpha \pi \alpha^{2\alpha}}{2\alpha[\Gamma(\alpha)]^2\sin(\pi\alpha/2)}.$$

When $\alpha \in (0,1)$, we simply have $K_t = \int_0^t \mathrm{sg}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s$, since this integral is a.s. absolutely convergent. Theorem 3 is very natural and easy to verify, as far as we are not interested in the exact value of $\kappa_\alpha$. Indeed, $\tau_t$ is a stopping-time, with $W_{\tau_t} = 0$, for each $t \geq 0$. Hence the strong Markov property implies that the process $Z_t^\varphi = \int_0^{\tau_t} \varphi(W_s)\mathrm{d}s$ is Lévy, for any reasonable function $\varphi : \mathbb{R} \mapsto \mathbb{R}$. It is furthermore of course symmetric if $\varphi$ is odd. If finally one wants $Z_t^\varphi$ to satisfy the scaling property of $\alpha$-stable processes, i.e. $Z_t^\varphi \stackrel{d}{=} c^{-1/\alpha} Z_{ct}^\varphi$, there is no choice for $\varphi$: it has to be $\varphi(z) = \mathrm{sg}(z)|z|^{1/\alpha-2}$, recall that $((\tau_t)_{t\geq 0}, (W_t)_{t\geq 0}) \stackrel{d}{=} (c^{-2}\tau_{ct}, c^{-1}W_{c^2t})$. All this perfectly holds true when $\alpha \in (0,1)$, but there are some small difficulties when $\alpha \in [1,2)$ because the integral $\int_0^t \mathrm{sg}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s$ is not absolutely convergent. The computation of $\kappa_\alpha$ is rather tedious and involves special functions.

2.2. **Bessel processes.** Bessel processes are studied in details in Revuz-Yor [25, Chapter XI]. These are nonnegative processes, and we need a signed symmetric version. Roughly, we would like to take a Bessel process and to change the sign of each excursion, independently, with probability 1/2. Inspired by Donati-Roynette-Vallois-Yor [11], we will rather use the following (equivalent) definition.

**Definition 4.** *Let $\delta \in (0,2)$. Consider a Brownian motion $(W_t)_{t\geq 0}$, introduce the time-change $\bar{A}_t = (2-\delta)^{-2} \int_0^t |W_s|^{-2(1-\delta)/(2-\delta)}\mathrm{d}s$ and its inverse $(\bar{\tau}_t)_{t\geq 0}$. We set $U_t^{(\delta)} = \mathrm{sg}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(2-\delta)}$ and say that $(U_t^{(\delta)})_{t\geq 0}$ is a symmetric Bessel process with dimension $\delta$.*

Since $2(1-\delta)/(2-\delta) < 1$, $\mathbb{E}[\bar{A}_t] < \infty$ for all $t \geq 0$. The map $t \mapsto \bar{A}_t$ is a.s. continuous, strictly increasing and $\bar{A}_\infty = \infty$ by recurrence of $(W_t)_{t\geq 0}$, so that $(\bar{\tau}_t)_{t\geq 0}$ is well-defined and continuous. Also, $U_t^{(1)} = W_t$: the Brownian motion is the symmetric version of the Bessel process of dimension 1.

To justify the terminology, let us mention that $(|U_t^{(\delta)}|)_{t\geq 0}$ is a Bessel process with dimension $\delta$. Indeed, [11, Corollary 2.2] tells us that, for $(R_t)_{t\geq 0}$ a Bessel process with dimension $\delta$, $R_t = |W_{C_t}|^{1/(2-\delta)}$ for some Brownian motion $(W_t)_{t\geq 0}$ and for $C_t = (2-\delta)^2 \int_0^t R_s^{2(1-\delta)}\mathrm{d}s$. But $C_t = \bar{\tau}_t$, whence $R_t = |U_t^{(\delta)}|$, because $\bar{A}_{C_t} = (2-\delta)^{-2}\int_0^{C_t} |W_s|^{-2(1-\delta)/(2-\delta)}\mathrm{d}s = \int_0^t |W_{C_u}|^{-2(1-\delta)/(2-\delta)} R_u^{2(1-\delta)}\mathrm{d}u = t$.

## 3. Proofs

Here is the strategy of the proof. We first verify quickly that the velocity process is well-defined for all times, regular and recurrent, and we explain why it suffices to prove Theorem 1 when $X_0 = V_0 = 0$.

In a second subsection, we (classically) check Theorem 1 in the normal diffusive case $\beta > 5$.

In Subsection 3.3, we introduce some functions $\Psi_\epsilon : \mathbb{R} \mapsto \mathbb{R}$ and $\sigma_\epsilon : \mathbb{R} \mapsto (0,\infty)$ such that, for $(W_t)_{t\geq 0}$ a Brownian motion and for $(\tau_t^\epsilon)_{t\geq 0}$ the inverse of $(A_t^\epsilon)_{t\geq 0}$ defined by $A_t^\epsilon = \int_0^t [\sigma_\epsilon(W_s)]^{-2}\mathrm{d}s$, the processes $(V_{t/\epsilon})_{t\geq 0}$ and $(\Psi_\epsilon(W_{\tau_t^\epsilon}))_{t\geq 0}$ have the same law.

In Subsection 3.5, we prove our main result when $\beta \in (0,1)$. We first write $(\epsilon^{1/2}V_{t/\epsilon}, \epsilon^{3/2}X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (\epsilon^{1/2}\Psi_\epsilon(W_{\tau_t^\epsilon}), \int_0^t \epsilon^{1/2}\Psi_\epsilon(W_{\tau_s^\epsilon})\mathrm{d}s)_{t\geq 0}$. And we show that $\epsilon^{1/2}\Psi_\epsilon(z)$ resembles $\mathrm{sg}(z)|z|^{1/(1+\beta)}$ and that $\sigma_\epsilon(z)$ resembles $(\beta+1)|z|^{\beta/(\beta+1)}$. Consequently, recalling Definition 4 (with $\delta = 1-\beta$), $A_t^\epsilon \simeq \bar{A}_t$, whence $\tau_t^\epsilon \simeq \bar{\tau}_t$, and $\epsilon^{1/2}\Psi_\epsilon(W_{\tau_t^\epsilon}) \simeq \mathrm{sg}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)} = U_t^{(1-\beta)}$ as desired.

We study the case where $\beta \in [1,5]$ in Subsection 3.6. Up to logarithmic corrections when $\beta = 1$ or $5$, we write $(\epsilon^{1/\alpha}X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (\epsilon^{1/\alpha-1}\int_0^t \Psi_\epsilon(W_{\tau_s^\epsilon})\mathrm{d}s)_{t\geq 0} = (\epsilon^{1/\alpha-1}\int_0^{\tau_t^\epsilon} \Psi_\epsilon(W_s)[\sigma_\epsilon(W_s)]^{-2}\mathrm{d}s)_{t\geq 0}$. We then recall Theorem 3 with $\alpha = (\beta+1)/3$, and we check that $A_t^\epsilon \simeq L_t^0$, whence $\tau_t^\epsilon \simeq \tau_t$, and that $\epsilon^{1/\alpha-1}\Psi_\epsilon(z)[\sigma_\epsilon(z)]^{-2} \simeq \mathrm{sg}(z)|z|^{(1-2\beta)/(1+\beta)} = \mathrm{sg}(z)|z|^{1/\alpha-2}$. All in all, we deduce that $(\epsilon^{1/\alpha}X_{t/\epsilon})_{t\geq 0} \stackrel{d}{\simeq} (\int_0^{\tau_t} \mathrm{sg}(W_s)|W_s|^{1/\alpha-2}\mathrm{d}s)_{t\geq 0}$, which is a symmetric $\alpha$-stable process.

In this last case $\beta \in [1,5]$, the situation is actually more complicated, there are some constants appearing everywhere. Let us also mention that the case $\beta = 5$ requires a special study, using both the standard method (as when $\beta > 5$) and some local time arguments.

Finally, the last subsection is devoted to the proof of Corollary 2.

3.1. **Preliminaries.** Let us first prove the following.

**Lemma 5.** *(a) The solution $(V_t)_{t\geq 0}$ to (1) is global, regular and recurrent.*

*(b) There is $C>0$ such that, if $V_0=0$, $\mathbb{E}[V_t^2+|V_t|^{\beta+1}]\leq C(1+t)$ for all $t\geq 0$.*

*(c) If Theorem 1-(a)-(b)-(c)-(d) holds when $X_0=V_0=0$ a.s., then it holds for any initial condition.*

*(d) To prove Theorem 1-(e), it suffices to prove that $(\epsilon^{1/2}V_{t/\epsilon})_{t\geq 0}\stackrel{d}{\longrightarrow}(U_t^{(1-\beta)})_{t\geq 0}$ when $V_0=0$.*

*Proof.* We first verify (a). As we will see in Lemma 7 with $\epsilon=a_\epsilon=1$, the solution $(V_t)_{t\geq 0}$ to (1) with $V_0=0$ has the same law as $(h^{-1}(W_{\tau_t}))_{t\geq 0}$, for some Brownian motion $(W_t)_{t\geq 0}$, some (random) continuous bijective time change $\tau_t:[0,\infty)\mapsto[0,\infty)$ and some continuous bijective function $h:\mathbb{R}\mapsto\mathbb{R}$. Hence $(V_t)_{t\geq 0}$ is non-exploding and thus global, and it is regular and recurrent (when starting from any initial condition).

We next prove (b). The even function $\ell(v)=2\int_0^v\Theta^{-\beta}(x)\int_0^x\Theta^\beta(u)\mathrm{d}u\mathrm{d}x$ solves the Poisson equation $\ell''(v)-\beta F(v)\ell'(v)=2$, whence $\mathbb{E}[\ell(V_t)]=t$, by the Itô formula and since $\ell(V_0)=\ell(0)=0$. Using (2), we see that there is a constant $c>0$ such that, as $|v|\to\infty$, $\ell(v)\sim c|v|^{\beta+1}$ if $\beta>1$, $\ell(v)\sim c|v|^2\log|v|$ if $\beta=1$, and $\ell(v)\sim cv^2$ if $\beta\in(0,1)$. Thus in any case, we can find a constant $C$ such that $v^2+|v|^{\beta+1}\leq C(\ell(v)+1)$ for all $v\in\mathbb{R}$, whence $\mathbb{E}[V_t^2+|V_t|^{\beta+1}]\leq C(1+\mathbb{E}[\ell(V_t)])=C(1+t)$.

We now check (c). Assume that for $\beta\geq 1$, Theorem 1 holds when starting from $(0,0)$ and consider the solution $(V_t,X_t)_{t\geq 0}$ to (1) starting from some $(V_0,X_0)$. We introduce $\tau=\inf\{t\geq 0:V_t=0\}$, which is a.s. finite by recurrence. Then $(\hat V_t,\hat X_t)=(V_{\tau+t},X_{\tau+t}-X_\tau)$ solves (1), starts from $(0,0)$, and is independent of $\tau$ by the strong Markov property. We thus know that $(v_\epsilon^{(\beta)}\hat X_{t/\epsilon})_{t\geq 0}\stackrel{f.d.}{\to}(X_t^{(\beta)})_{t\geq 0}$, where $v_\epsilon^{(\beta)}\to 0$ and $(X_t^{(\beta)})_{t\geq 0}$ are the rate and limit process appearing in Theorem 1. We now prove that for each $t\geq 0$, $v_\epsilon^{(\beta)}|X_{t/\epsilon}-\hat X_{t/\epsilon}|\to 0$ in probability, and this will complete the proof. We introduce $D^1=|X_0|+\int_0^{2\tau}|V_s|\mathrm{d}s$ and $D_t^{2,\epsilon}=\mathbf{1}_{\{t/\epsilon\geq\tau\}}\int_{t/\epsilon-\tau}^{t/\epsilon}|\hat V_s|\mathrm{d}s$ and observe that $|X_{t/\epsilon}-\hat X_{t/\epsilon}|\leq D^1+D_t^{2,\epsilon}$. Indeed,

- if $t/\epsilon\leq\tau$, $|X_{t/\epsilon}-\hat X_{t/\epsilon}|\leq|X_{t/\epsilon}|+|X_{\tau+t/\epsilon}-X_\tau|\leq|X_0|+\int_0^{t/\epsilon}|V_s|\mathrm{d}s+\int_\tau^{\tau+t/\epsilon}|V_s|\mathrm{d}s\leq D^1$,
- if $t/\epsilon>\tau$, $|X_{t/\epsilon}-\hat X_{t/\epsilon}|=|X_\tau+\hat X_{t/\epsilon-\tau}-\hat X_{t/\epsilon}|\leq|X_0|+\int_0^\tau|V_s|\mathrm{d}s+\int_{t/\epsilon-\tau}^{t/\epsilon}|\hat V_s|\mathrm{d}s\leq D^1+D_t^{2,\epsilon}$.

But $v_\epsilon^{(\beta)}D^1$ a.s. tends to 0, and using (b),

$$\mathbb{E}[v_\epsilon^{(\beta)}D_t^{2,\epsilon}|\mathcal{F}_\tau]\leq\mathbf{1}_{\{t/\epsilon\geq\tau\}}Cv_\epsilon^{(\beta)}\int_{t/\epsilon-\tau}^{t/\epsilon}(1+s)^{1/(\beta+1)}\mathrm{d}s\leq C\tau v_\epsilon^{(\beta)}(1+t/\epsilon)^{1/(\beta+1)},$$

which a.s. tends to 0 for all values of $\beta\geq 1$. Hence $v_\epsilon^{(\beta)}D_t^{2,\epsilon}$ tends to 0 in probability.

We finally prove (d). First, $(\epsilon^{1/2}V_{t/\epsilon})_{t\geq 0}\stackrel{d}{\longrightarrow}(U_t^{(1-\beta)})_{t\geq 0}$ implies that $(\epsilon^{1/2}V_{t/\epsilon},\epsilon^{3/2}X_{t/\epsilon})_{t\geq 0}\stackrel{d}{\longrightarrow}(U_t^{(1-\beta)},\int_0^tU_s^{(1-\beta)}\mathrm{d}s)_{t\geq 0}$, simply because $\epsilon^{3/2}X_{t/\epsilon}=\epsilon^{3/2}X_0+\int_0^t(\epsilon^{1/2}V_{s/\epsilon})\mathrm{d}s$. We assume that this convergence holds true when $V_0=0$, consider any other solution $(V_t)_{\geq 0}$, introduce $\tau>0$ and $\hat V_t=V_{\tau+t}$ as previously. Our goal is to check that $\Delta_T^\epsilon=\epsilon^{1/2}\sup_{[0,T]}|V_{t/\epsilon}-\hat V_{t/\epsilon}|\to 0$ in probability. We write $\Delta_T^\epsilon\leq\Delta^{1,\epsilon}+\Delta_T^{2,\epsilon}$, where $\Delta^{1,\epsilon}=2\epsilon^{1/2}\sup_{[0,2\tau]}|V_s|$ and $\Delta_T^{2,\epsilon}=\sup_{[0,T]}\epsilon^{1/2}|\hat V_{(t+\epsilon\tau)/\epsilon}-\hat V_{t/\epsilon}|$. Indeed,

- if $t\in[0,T]$ and $t/\epsilon\leq\tau$, $\epsilon^{1/2}|V_{t/\epsilon}-\hat V_{t/\epsilon}|\leq\epsilon^{1/2}|V_{t/\epsilon}|+\epsilon^{1/2}|V_{\tau+t/\epsilon}|\leq\Delta^{1,\epsilon}$,
- if $t\in[0,T]$ and $t/\epsilon>\tau$, $\epsilon^{1/2}|V_{t/\epsilon}-\hat V_{t/\epsilon}|=\epsilon^{1/2}|\hat V_{t/\epsilon-\tau}-\hat V_{t/\epsilon}|\leq\Delta_T^{2,\epsilon}$.

First, $\Delta^{1,\epsilon}$ a.s. tends to 0. Next, it is not hard to check that $\Delta_T^{2,\epsilon}$ goes in probability to 0, using that $(\epsilon^{1/2}\hat V_{t/\epsilon})_{t\geq 0}$ goes in law, in $C([0,\infty),\mathbb{R})$, to the continuous process $(U_t^{(1-\beta)})_{t\geq 0}$. □

3.2. **The normal diffusion regime.** The following proof is standard, see e.g. Jacod-Shiryaev [14, Chapter VIII, Section 3f].

*Proof of Theorem 1-(a).* We assume that $\beta > 5$ and, in view of Lemma 5-(c), that $X_0 = V_0 = 0$. Thanks to Lemma 5-(a) and since $\mu_\beta$ is a probability measure (because $\beta > 1$), we classically deduce, see e.g. Kallenberg [16, Lemma 23.17 page 466 and Thm 23.14 page 464], that

(i) $V_t$ goes in law to $\mu_\beta$ as $t \to \infty$,

(ii) for all $\varphi \in L^1(\mathbb{R}, \mu_\beta)$, $\lim_{t\to\infty} t^{-1} \int_0^t \varphi(V_s)\mathrm{d}s = \int_\mathbb{R} \varphi \mathrm{d}\mu_\beta$ a.s.

The function $g(v) = 2\int_0^v \Theta^{-\beta}(x) \int_x^\infty u\Theta^\beta(u)\mathrm{d}u\mathrm{d}x$ is odd (since $\Theta$ is even and $\int_\mathbb{R} u\Theta^\beta(u)\mathrm{d}u\mathrm{d}x = 0$) and solves the Poisson equation $g''(v) - \beta F(v)g'(v) = -2v$, whence, by the Itô formula,

$$g(V_t) = \int_0^t g'(V_s)\mathrm{d}B_s - \int_0^t V_s\mathrm{d}s, \quad \textit{i.e.} \quad X_t = \int_0^t g'(V_s)\mathrm{d}B_s - g(V_t).$$

Consequently, we have $\epsilon^{1/2}X_{t/\epsilon} = M_t^\epsilon - \epsilon^{1/2}g(V_{t/\epsilon})$, where $M_t^\epsilon = \epsilon^{1/2}\int_0^{t/\epsilon} g'(V_s)\mathrm{d}B_s$.

For each $t \geq 0$, $\epsilon^{1/2}g(V_{t/\epsilon})$ tends to 0 in probability: this follows from point (i) above. Here is why we deal with finite-dimensional distributions: it is not clear that $\sup_{t\in[0,1]} |\epsilon^{1/2}g(V_{t/\epsilon})|$ tends to 0.

We now show that $(M_t^\epsilon)_{t\geq 0}$ goes in law (in the usual sense of continuous processes) to $(\sigma_\beta B_t)_{t\geq 0}$, and this will complete the proof. It suffices, see e.g. Jacod-Shiryaev [14, Theorem VIII-3.11 page 473], to verify that for each $t \geq 0$, $\lim_{\epsilon\to 0}\langle M^\epsilon\rangle_t = \sigma_\beta^2 t$ in probability. But $\langle M^\epsilon\rangle_t = \epsilon\int_0^{t/\epsilon}[g'(V_s)]^2\mathrm{d}s$, which a.s. tends to $\sigma_\beta^2 t$ by point (ii). Indeed, using a symmetry argument,

$$\int_\mathbb{R}[g'(v)]^2\mu_\beta(\mathrm{d}v) = 8\int_0^\infty\Big[\Theta^{-\beta}(v)\int_v^\infty u\Theta^\beta(u)\mathrm{d}u\Big]^2\mu_\beta(\mathrm{d}v) = 8c_\beta\int_0^\infty\Theta^{-\beta}(v)\Big[\int_v^\infty u\Theta^\beta(u)\mathrm{d}u\Big]^2\mathrm{d}v = \sigma_\beta^2,$$

recall Subsection 1.1. This value is finite, since $\Theta^{-\beta}(v)\Big[\int_v^\infty u\Theta^\beta(u)\mathrm{d}u\Big]^2 \sim (\beta-2)^{-2}v^{4-\beta}$ as $v \to \infty$ by (2) and since $\beta > 5$. $\square$

**Remark 6.** *When $\beta = 5$, our goal is to prove that $(\epsilon^{1/2}|\log\epsilon|^{-1/2}X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\longrightarrow} (\sigma_5 B_t)_{t\geq 0}$. We can use exactly the same proof, provided we can show that for each $t \geq 0$, in probability, as $\epsilon \to 0$,*

$$\frac{\epsilon}{|\log\epsilon|}\int_0^{t/\epsilon}[g'(V_s)]^2\mathrm{d}s \longrightarrow \sigma_5^2 t.$$

3.3. **Scale function and speed measure.** We introduce a few notation, closely linked with the scale function and speed measure of the process $(V_t)_{t\geq 0}$. This will allow us to rewrite $(X_t)_{t\geq 0}$ in a way that it resembles the objects appearing in Theorem 3 and Definition 4. All the functions below are defined on $\mathbb{R}$. Recall our conditions on $\Theta$, see (2).

First, $h(v) = (\beta+1)\int_0^v[\Theta(u)]^{-\beta}\mathrm{d}u$ is odd, increasing, bijective, solves $h'' = \beta F h'$, and we have $h(v) \stackrel{|v|\to\infty}{\sim} \mathrm{sg}(v)|v|^{\beta+1}$ and $h^{-1}(z) \stackrel{|z|\to\infty}{\sim} \mathrm{sg}(z)|z|^{1/(\beta+1)}$.

Next, $\sigma(z) = h'(h^{-1}(z))$ is even, bounded below by some $c > 0$ and $\sigma(z) \stackrel{|z|\to\infty}{\sim} (\beta+1)|z|^{\beta/(\beta+1)}$.

The function $\phi(z) = h^{-1}(z)/\sigma^2(z)$ is odd and $\phi(z) \stackrel{|z|\to\infty}{\sim} (\beta+1)^{-2}\mathrm{sg}(z)|z|^{(1-2\beta)/(\beta+1)}$.

When $\beta = 5$, $\psi(z) = [g'(h^{-1}(z))]^2/\sigma^2(z)$ is even, bounded and $\psi(z) \stackrel{|z|\to\infty}{\sim} 1/(81|z|)$. The even function $g'(v) = 2\Theta^{-5}(v)\int_v^\infty u\Theta^5(u)\mathrm{d}u \stackrel{|v|\to\infty}{\sim} 2|v|^2/3$ was introduced in the proof of Theorem 1-(a).

**Lemma 7.** *Fix $\beta > 0$, $\epsilon > 0$ and $a_\epsilon > 0$. Consider a Brownian motion $(W_t)_{t\geq 0}$. Define $A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t[\sigma(W_s/a_\epsilon)]^{-2}\mathrm{d}s$ and its inverse $(\tau_t^\epsilon)_{t\geq 0}$, which is a continuous increasing bijection from $[0,\infty)$ into itself. Set*

$$V_t^\epsilon = h^{-1}(W_{\tau_t^\epsilon}/a_\epsilon) \quad \textit{and} \quad X_t^\epsilon = H_{\tau_t^\epsilon}^\epsilon \quad \textit{where} \quad H_t^\epsilon = a_\epsilon^{-2}\int_0^t \phi(W_s/a_\epsilon)\mathrm{d}s.$$

*For $(V_t, X_t)_{t\geq 0}$ the unique solution of* (1) *starting from* $(0,0)$*, we have* $(V_{t/\epsilon}, X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (V^\epsilon_t, X^\epsilon_t)_{t\geq 0}$.

The result holds for any value of $a_\epsilon > 0$, but in each situation, we will choose it judiciously, in such a way that $(A^\epsilon_t)_{t\geq 0}$ a.s. converges, as $\epsilon \to 0$, to the desired limit time-change.

*Proof.* Since $\sigma$ is bounded below, $t \mapsto A^\epsilon_t$ is a.s. continuous and strictly increasing. By recurrence of the Brownian motion, we also have $A^\epsilon_\infty = \infty$ a.s. Hence $\tau^\epsilon_t$ is well-defined, continuous, bijective from $[0,\infty) \mapsto [0,\infty)$ and $Y^\epsilon_t = W_{\tau^\epsilon_t}$ classically solves, see e.g. Revuz-Yor [25, Proposition 1.13 page 373], $Y^\epsilon_t = \epsilon^{-1/2} a_\epsilon \int_0^t \sigma(Y^\epsilon_s/a_\epsilon)\mathrm{d}B^\epsilon_s$, for some Brownian motion $(B^\epsilon_t)_{t\geq 0}$. We then use the Itô formula to write $V^\epsilon_t = h^{-1}(Y^\epsilon_t/a_\epsilon)$ as

$$V^\epsilon_t = a_\epsilon^{-1}\int_0^t (h^{-1})'(Y^\epsilon_s/a_\epsilon)\epsilon^{-1/2}a_\epsilon\sigma(Y^\epsilon_s/a_\epsilon)\mathrm{d}B^\epsilon_s + \frac{1}{2}a_\epsilon^{-2}\int_0^t (h^{-1})''(Y^\epsilon_s/a_\epsilon)\epsilon^{-1}a_\epsilon^2\sigma^2(Y^\epsilon_s/a_\epsilon)\mathrm{d}s.$$

But $(h^{-1})'(y)\sigma(y) = 1$ and $(h^{-1})''(y)\sigma^2(y) = -\sigma'(y) = -h''(h^{-1}(y))/h'(h^{-1}(y)) = -\beta F(h^{-1}(y))$, whence finally

$$V^\epsilon_t = \epsilon^{-1/2}B^\epsilon_t - \frac{\beta}{2}\epsilon^{-1}\int_0^t F(h^{-1}(Y^\epsilon_s/a_\epsilon))\mathrm{d}s = \epsilon^{-1/2}B^\epsilon_t - \frac{\beta}{2}\epsilon^{-1}\int_0^t F(V^\epsilon_s)\mathrm{d}s.$$

Starting from (1), we find

$$V_{t/\epsilon} = B_{t/\epsilon} - \frac{\beta}{2}\int_0^{t/\epsilon} F(V_s)\mathrm{d}s = \epsilon^{-1/2}(\epsilon^{1/2}B_{t/\epsilon}) - \frac{\beta}{2}\epsilon^{-1}\int_0^t F(V_{s/\epsilon})\mathrm{d}s.$$

Hence $(V^\epsilon_t)_{t\geq 0}$ and $(V_{t/\epsilon})_{t\geq 0}$ are two solutions of the same well-posed SDE, driven by different Brownian motions, namely $(B^\epsilon_t)_{t\geq 0}$ and $(\epsilon^{1/2}B_{t/\epsilon})_{t\geq 0}$. They thus have the same law.

Since $X_{t/\epsilon} = \int_0^{t/\epsilon} V_s\mathrm{d}s = \epsilon^{-1}\int_0^t V_{s/\epsilon}\mathrm{d}s$, we conclude that $(V_{t/\epsilon}, X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (V^\epsilon_t, \epsilon^{-1}\int_0^t V^\epsilon_s\mathrm{d}s)_{t\geq 0}$. But using the substitution $u = \tau^\epsilon_s$, i.e. $s = A^\epsilon_u$, whence $\mathrm{d}s = \epsilon a_\epsilon^{-2}[\sigma(W_u/a_\epsilon)]^{-2}\mathrm{d}u$, we find

$$\epsilon^{-1}\int_0^t V^\epsilon_s\mathrm{d}s = \epsilon^{-1}\int_0^t h^{-1}(W_{\tau^\epsilon_s}/a_\epsilon)\mathrm{d}s = a_\epsilon^{-2}\int_0^{\tau^\epsilon_t}\frac{h^{-1}(W_u/a_\epsilon)}{[\sigma(W_u/a_\epsilon)]^2}\mathrm{d}u = a_\epsilon^{-2}\int_0^{\tau^\epsilon_t}\phi(W_u/a_\epsilon)\mathrm{d}u,$$

which equals $H^\epsilon_{\tau^\epsilon_t}$ as desired. □

3.4. **Inverting time-changes.** We recall the following classical and elementary results.

**Lemma 8.** *Consider, for each $n \geq 1$, a continuous increasing bijective function $(a^n_t)_{t\geq 0}$ from $[0,\infty)$ into itself, as well as its inverse $(r^n_t)_{t\geq 0}$.*

*(a) Assume that $(a^n_t)_{t\geq 0}$ converges pointwise to some function $(a_t)_{t\geq 0}$ such that $\lim_\infty a_t = \infty$, denote by $r_t = \inf\{u \geq 0 : a_u > t\}$ its right-continuous generalized inverse and set $J = \{s \in [0,\infty) : r_{t-} < r_t\}$. For all $t \in [0,\infty) \setminus J$, we have $\lim_{t\to\infty} r^n_t = r_t$.*

*(b) If $(a^n_t)_{t\geq 0}$ converges (locally) uniformly to some strictly increasing function $(a_t)_{t\geq 0}$ such that $\lim_\infty a_t = \infty$, then $(r^n_t)_{t\geq 0}$ converges (locally) uniformly to $(r_t)_{t\geq 0}$, the (classical) inverse of $(a_t)_{t\geq 0}$.*

3.5. **The integrated Bessel regime.** We can now give the

*Proof of Theorem 1-(e).* Let $\beta \in (0,1)$ be fixed. We consider a Brownian motion $(W_t)_{t\geq 0}$ and, as in Definition 4 with $\delta = 1-\beta$, we introduce the continuous strictly increasing bijective time-change $\bar{A}_t = (\beta+1)^{-2}\int_0^t |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s$, its inverse $(\bar{\tau}_t)_{t\geq 0}$ and the process $U^{(1-\beta)}_t = \mathrm{sg}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)}$.

We now apply Lemma 7 with the choice $a_\epsilon = \epsilon^{(\beta+1)/2}$: with the same Brownian motion as above, we consider, for each $\epsilon > 0$, the time-change $A^\epsilon_t = \epsilon^{-\beta}\int_0^t [\sigma(W_s/\epsilon^{(\beta+1)/2})]^{-2}\mathrm{d}s$, its inverse $\tau^\epsilon_t$, and $V^\epsilon_t = h^{-1}(W_{\tau^\epsilon_t}/\epsilon^{(\beta+1)/2})$. Recalling Lemma 5-(d) and that $(\epsilon^{1/2}V_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (\epsilon^{1/2}V^\epsilon_t)_{t\geq 0}$, it suffices to prove that a.s., for all $T \geq 0$, $\lim_{\epsilon\to 0}\sup_{[0,T]}|\epsilon^{1/2}V^\epsilon_t - U^{(1-\beta)}_t| = 0$.

Since $\sigma(z) \geq c > 0$ and $\sigma(z) \overset{|z|\to\infty}{\sim} (\beta+1)|z|^{\beta/(\beta+1)}$, whence $\sigma^{-2}(z) \leq C|z|^{-2\beta/(\beta+1)}$, one has

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |A_t^\epsilon - \bar{A}_t| \leq \lim_{\epsilon\to 0} \int_0^T \Big|\epsilon^{-\beta}[\sigma(W_s/\epsilon^{(\beta+1)/2})]^{-2} - [(\beta+1)|W_s|^{\beta/(\beta+1)}]^{-2}\Big| \mathrm{d}s = 0 \quad \text{a.s.}$$

by dominated convergence. Indeed, we have $\sup_{\epsilon>0} \epsilon^{-\beta}[\sigma(W_s/\epsilon^{(\beta+1)/2})]^{-2} \leq C|W_s|^{-2\beta/(\beta+1)}$, and $\int_0^T |W_s|^{-2\beta/(\beta+1)}\mathrm{d}s < \infty$ a.s. because $2\beta/(\beta+1) < 1$.

By Lemma 8-(b), we deduce that for all $T \geq 0$, $\lim_{\epsilon\to 0} \sup_{[0,T]} |\tau_t^\epsilon - \bar{\tau}_t| = 0$ a.s. whence, by continuity of $(W_t)_{t\geq 0}$,

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |W_{\tau_t^\epsilon} - W_{\bar{\tau}_t}| = 0 \quad \text{a.s. for all } T > 0. \tag{5}$$

We next claim that for all $M > 0$,

$$\kappa_\epsilon(M) = \sup_{|z|\leq M} |\epsilon^{1/2} h^{-1}(z/\epsilon^{(\beta+1)/2}) - \mathrm{sg}(z)|z|^{1/(\beta+1)}| \to 0.$$

Indeed, $h^{-1}$ being $C^1$, with $h^{-1}(0) = 0$ and $h^{-1}(z) \overset{|z|\to\infty}{\sim} \mathrm{sg}(z)|z|^{1/(\beta+1)}$ the function $\gamma$ defined by $\gamma(z) = h^{-1}(z)/[\mathrm{sg}(z)|z|^{1/(\beta+1)}] - 1$ (and $\gamma(0) = -1$) is continuous, and $\lim_{|z|\to\infty} \gamma(z) = 0$. Hence,

$$\kappa_\epsilon(M) = \sup_{|z|\leq M} |z|^{1/(\beta+1)}|\gamma(z/\epsilon^{(\beta+1)/2})| \leq \epsilon^{1/4}||\gamma||_\infty + M^{1/(\beta+1)} \sup_{|z|\geq \epsilon^{(\beta+1)/4}} |\gamma(z/\epsilon^{(\beta+1)/2})|,$$

which equals $\epsilon^{1/4}||\gamma||_\infty + M^{1/(\beta+1)} \sup_{|z|\geq\epsilon^{-(\beta+1)/4}} |\gamma(z)| \to 0$.

All in all, denoting by $M_T = \sup_{[0,T]} \sup_{\epsilon\in(0,1)} |W_{\tau_t^\epsilon}|$, which is a.s. finite by (5),

$$\begin{aligned}\sup_{[0,T]} |\epsilon^{1/2} V_t^\epsilon - U_t^{(1-\beta)}| &= \sup_{[0,T]} |\epsilon^{1/2} h(W_{\tau_t^\epsilon}/\epsilon^{(\beta+1)/2}) - \mathrm{sg}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)}| \\ &\leq \kappa_\epsilon(M_T) + \sup_{[0,T]} \Big|\mathrm{sg}(W_{\tau_t^\epsilon})|W_{\tau_t^\epsilon}|^{1/(\beta+1)} - \mathrm{sg}(W_{\bar{\tau}_t})|W_{\bar{\tau}_t}|^{1/(\beta+1)}\Big| \to 0\end{aligned}$$

a.s., by (5) again. The proof is complete. □

3.6. **The Lévy regime and the critical cases.** We start with the following crucial lemma.

**Lemma 9.** *Fix $\beta \in [1,5]$ and a Brownian motion $(W_t)_{t\geq 0}$, denote by $(L_t^0)_{t\geq 0}$ its local time at 0 and by $(K_t)_{t\geq 0}$ the process defined in Theorem 3 with $\alpha = (\beta+1)/3$. For each $\epsilon > 0$, consider the processes $(A_t^\epsilon)_{t\geq 0}$ and $(H_t^\epsilon)_{t\geq 0}$ built in Lemma 7 with the choice $a_\epsilon = \epsilon/[(\beta+1)c_\beta]$ if $\beta \in (1,5]$ and $a_\epsilon = \epsilon|\log\epsilon|/2$ if $\beta = 1$, and with the same Brownian motion $(W_t)_{t\geq 0}$ as above.*

*(a) We always have $\lim_{\epsilon\to 0} \sup_{[0,T]} |A_t^\epsilon - L_t^0| = 0$ a.s. for all $T > 0$.*

*(b) If $\beta \in (1,5)$, $\lim_{\epsilon\to 0} \sup_{[0,T]} |\epsilon^{1/\alpha} H_t^\epsilon - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} K_t| = 0$ a.s. for all $T > 0$.*

*(c) If $\beta = 1$, $\lim_{\epsilon\to 0} \sup_{[0,T]} ||\epsilon\log\epsilon|^{3/2} H_t^\epsilon - K_t/\sqrt{2}| = 0$ a.s. for all $T > 0$.*

*(d) If $\beta = 5$, $\lim_{\epsilon\to 0} \sup_{[0,T]} |T_t^\epsilon - \sigma_5^2 L_t^0| = 0$ a.s. for all $T > 0$, with $T_t^\epsilon = \frac{\epsilon}{a_\epsilon^2|\log\epsilon|}\int_0^t \psi(W_s/a_\epsilon)\mathrm{d}s$.*

We recall that $c_\beta = 1/[\int_{\mathbb{R}}[\Theta(v)]^\beta \mathrm{d}v]$ (when $\beta > 1$), that $\sigma_5^2 = 4c_5/27$ that the functions $h$, $\sigma$, $\phi$ and $\psi$ were introduced at the beginning of Subsection 3.3.

*Proof.* We start with (a) when $\beta > 1$. We set $\gamma = (\beta+1)c_\beta$ and recall that $a_\epsilon = \epsilon/\gamma$, whence $A_t^\epsilon = \gamma^2\epsilon^{-1}\int_0^t [\sigma(\gamma W_s/\epsilon)]^{-2}\mathrm{d}s$. Using the occupation times formula, see Revuz-Yor [25, Corollary 1.6 page 224], we may write

$$A_t^\epsilon = \int_{\mathbb{R}} \frac{\gamma^2 L_t^x \mathrm{d}x}{\epsilon\sigma^2(\gamma x/\epsilon)} = \int_{\mathbb{R}} \frac{\gamma L_t^{\epsilon y/\gamma}\mathrm{d}y}{\sigma^2(y)},$$

where $(L_t^x)_{t\geq 0}$ is the local time of $(W_t)_{t\geq 0}$ at $x$. Observe now that

$$\int_{\mathbb{R}} \frac{\gamma \mathrm{d}y}{\sigma^2(y)} = \int_{\mathbb{R}} \frac{\gamma \mathrm{d}y}{[h'(h^{-1}(y))]^2} = \int_{\mathbb{R}} \frac{\gamma \mathrm{d}v}{h'(v)} = \int_{\mathbb{R}} \frac{\gamma \Theta^\beta(v)\mathrm{d}v}{(\beta+1)} = 1.$$

Consequently,

$$\sup_{[0,T]} |A_t^\epsilon - L_t^0| \leq \gamma \int_{\mathbb{R}} \frac{\sup_{[0,T]} |L_t^{\epsilon y/\gamma} - L_t^0| \mathrm{d}y}{\sigma^2(y)},$$

which a.s. tends to 0 as $\epsilon \to 0$ by dominated convergence, since $\sup_{[0,T]} |L_t^{\epsilon y/\gamma} - L_t^0|$ a.s. tends to 0 for each fixed $y$ by [25, Corollary 1.8 page 226] and since $\sup_{[0,T]\times\mathbb{R}} L_t^x$ is a.s. finite.

We now turn to point (a) when $\beta = 1$, so that $a_\epsilon = \epsilon|\log \epsilon|/2$. Also, $\sigma(z)$ is bounded below and $\sigma(z) \overset{|z|\to\infty}{\sim} 2|z|^{1/2}$, from which

$$\int_{-x}^{x} \frac{\mathrm{d}z}{\sigma^2(z)} \overset{x\to\infty}{\sim} \frac{\log x}{2}. \tag{6}$$

We now fix $\delta > 0$ and write $A_t^\epsilon = I_t^{\epsilon,\delta} + J_t^{\epsilon,\delta}$, where

$$I_t^{\epsilon,\delta} = \int_0^t \frac{\epsilon \mathrm{d}s}{a_\epsilon^2 \sigma^2(W_s/a_\epsilon)} \mathbf{1}_{\{|W_s|\leq\delta\}} \quad \text{and} \quad J_t^{\epsilon,\delta} = \int_0^t \frac{\epsilon \mathrm{d}s}{a_\epsilon^2 \sigma^2(W_s/a_\epsilon)} \mathbf{1}_{\{|W_s|>\delta\}}.$$

There is $c > 0$ such that $\sigma^2(z) \geq c(1+|z|)$, from which one verifies, using only that $|W_s| > \delta$ implies $\sigma^2(W_s/a_\epsilon) \geq c(1+\delta/a_\epsilon) \geq c\delta/a_\epsilon$, that $\sup_{[0,T]} |J_t^{\epsilon,\delta}| \leq T\epsilon/(ca_\epsilon\delta)$, which tends to 0 as $\epsilon \to 0$. We next use the occupation times formula to write

$$I_t^{\epsilon,\delta} = \int_{-\delta}^{\delta} \frac{\epsilon L_t^x \mathrm{d}x}{a_\epsilon^2 \sigma^2(x/a_\epsilon)} = \Big(\int_{-\delta}^{\delta} \frac{\epsilon \mathrm{d}x}{a_\epsilon^2 \sigma^2(x/a_\epsilon)}\Big) L_t^0 + \int_{-\delta}^{\delta} \frac{\epsilon (L_t^x - L_t^0)\mathrm{d}x}{a_\epsilon^2 \sigma^2(x/a_\epsilon)} = r_{\epsilon,\delta} L_t^0 + R_t^{\epsilon,\delta},$$

the last identity standing for a definition. But a substitution and (6) allow us to write

$$r_{\epsilon,\delta} = \int_{-\delta/a_\epsilon}^{\delta/a_\epsilon} \frac{\epsilon \mathrm{d}y}{a_\epsilon \sigma^2(y)} \overset{\epsilon\to 0}{\sim} \frac{\epsilon \log(\delta/a_\epsilon)}{2a_\epsilon} \longrightarrow 1$$

as $\epsilon \to 0$ since $a_\epsilon = \epsilon|\log\epsilon|/2$. Recalling that $A_t^\epsilon = r_{\epsilon,\delta} L_t^0 + R_t^{\epsilon,\delta} + J_t^{\epsilon,\delta}$, we have proved that a.s.,

$$\text{for all } \delta > 0, \quad \limsup_{\epsilon\to 0} \sup_{[0,T]} |A_t^\epsilon - L_t^0| \leq \limsup_{\epsilon\to 0} \sup_{[0,T]} |R_t^{\epsilon,\delta}|.$$

But $\sup_{[0,T]} |R_t^{\epsilon,\delta}| \leq r_{\epsilon,\delta} \times \sup_{[0,T]\times[-\delta,\delta]} |L_t^x - L_t^0|$, which implies that $\limsup_{\epsilon\to 0} \sup_{[0,T]} |A_t^\epsilon - L_t^0| \leq \sup_{[0,T]\times[-\delta,\delta]} |L_t^x - L_t^0|$ a.s. Letting $\delta \to 0$, using [25, Corollary 1.8 page 226], completes the proof.

Point (d), where $\beta = 5$, is very similar to point (a) when $\beta = 1$ and we only sketch the proof. We set $\gamma = 6c_5$ and recall that $a_\epsilon = \epsilon/\gamma$. Since $\psi$ is bounded on $\mathbb{R}$ and satisfies $\psi(z) \overset{|z|\to\infty}{\sim} |81z|^{-1}$,

$$\int_{-x}^{x} \psi(z)\mathrm{d}z \overset{x\to\infty}{\sim} \frac{2\log x}{81}. \tag{7}$$

Proceeding as previously, we can show rigorously that, for any $\delta > 0$, uniformly in $t \in [0,T]$,

$$T_t^\epsilon = \int_0^t \frac{\gamma^2 \psi(\gamma W_s/\epsilon)\mathrm{d}s}{\epsilon|\log\epsilon|} \simeq \int_0^t \frac{\gamma^2 \psi(\gamma W_s/\epsilon)\mathrm{d}s}{\epsilon|\log\epsilon|} \mathbf{1}_{\{|W_s|\leq\delta\}} = \int_{-\delta}^{\delta} \frac{\gamma^2 \psi(\gamma x/\epsilon) L_t^x \mathrm{d}x}{\epsilon|\log\epsilon|},$$

whence

$$T_t^\epsilon \simeq \Big(\frac{\gamma}{|\log\epsilon|} \int_{-\delta\gamma/\epsilon}^{\delta\gamma/\epsilon} \psi(x)\mathrm{d}x\Big)\Big(L_t^0 \pm \sup_{[0,T]\times[-\delta,\delta]} |L_t^x - L_t^0|\Big) \simeq \frac{2\gamma}{81}\Big(L_t^0 \pm \sup_{[0,T]\times[-\delta,\delta]} |L_t^x - L_t^0|\Big)$$

by (7). We conclude by letting $\delta$ tend to 0, since $2\gamma/81 = 4c_5/27 = \sigma_5^2$.

We now check (b), where $\beta \in (1,5)$ and $a_\epsilon = \epsilon/\gamma$ with $\gamma = (\beta+1)c_\beta$. Recall that $\alpha = (\beta+1)/3$, whence $1/\alpha - 2 = (1-2\beta)/(\beta+1)$. First, recalling Theorem 3 and the occupation times formula,

$$K_t^\eta = \int_{\mathbb{R}} \mathrm{sg}(x)|x|^{(1-2\beta)/(\beta+1)} \mathbf{1}_{\{|x|\geq \eta\}} L_t^x \mathrm{d}x = \int_{\mathbb{R}} \mathrm{sg}(x)|x|^{(1-2\beta)/(\beta+1)} \mathbf{1}_{\{|x|\geq \eta\}} (L_t^x - L_t^0 \mathbf{1}_{\{|x|\leq 1\}}) \mathrm{d}x$$

by symmetry. But we know from [25, Corollary 1.8 page 226] and the fact that $\sup_{[0,T]\times\mathbb{R}} L_t^x$ is a.s. finite that for all $\theta \in (0,1/2)$, all $T>0$,

$$M_{\theta,T} = \sup_{[0,T]\times\mathbb{R}} (|x|\wedge 1)^{-\theta} |L_t^x - L_t^0 \mathbf{1}_{\{|x|\leq 1\}}| < \infty \quad \text{a.s.}$$

Since $(1-2\beta)/(\beta+1) > -3/2$ (because $\beta < 5$), we deduce that $(K_t^\eta)_{t\geq 0}$ a.s. converges uniformly on $[0,T]$, as $\eta \to 0$, to

$$K_t = \int_{\mathbb{R}} \mathrm{sg}(x)|x|^{(1-2\beta)/(\beta+1)} (L_t^x - L_t^0 \mathbf{1}_{\{|x|\leq 1\}}) \mathrm{d}x.$$

Similarly, by oddness of $\phi$ (and since $\epsilon^{1/\alpha} a_\epsilon^{-2} = \gamma^2 \epsilon^{1/\alpha-2} = \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)}$),

$$\epsilon^{1/\alpha} H_t^\epsilon = \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \int_0^t \phi(\gamma W_s/\epsilon)\mathrm{d}s = \int_{\mathbb{R}} \gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi(\gamma x/\epsilon)(L_t^x - L_t^0 \mathbf{1}_{\{|x|\leq 1\}})\mathrm{d}x.$$

Hence

$$\begin{aligned}
&\sup_{[0,T]} |\epsilon^{1/\alpha} H_t^\epsilon - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} K_t| \\
\leq & \int_{\mathbb{R}} \Big|\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi(\gamma x/\epsilon) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \mathrm{sg}(x)|x|^{(1-2\beta)/(\beta+1)}\Big| \sup_{[0,T]} |L_t^x - L_t^0 \mathbf{1}_{\{|x|\leq 1\}}| \mathrm{d}x \\
\leq & M_{\theta,T} \int_{\mathbb{R}} \Big|\gamma^2 \epsilon^{(1-2\beta)/(\beta+1)} \phi(\gamma x/\epsilon) - (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha} \mathrm{sg}(x)|x|^{(1-2\beta)/(\beta+1)}\Big| (|x|\wedge 1)^\theta \mathrm{d}x
\end{aligned}$$

for any $\theta \in (0,1/2)$. Using the equivalence $\phi(z) \overset{|z|\to\infty}{\sim} (\beta+1)^{-2} \mathrm{sg}(z)|z|^{(1-2\beta)/(\beta+1)}$, the bound $|\phi(z)| \leq C(1+|z|^{(1-2\beta)/(\beta+1)})$ and that $(1-2\beta)/(\beta+1) > -3/2$, we conclude, by dominated convergence, that $\lim_{\epsilon\to 0} \sup_{[0,T]} |\epsilon^{1/\alpha} H_t^\epsilon - (\beta+1)^{-2} K_t| = 0$ a.s. This uses that

$$\gamma^{2+(1-2\beta)/(\beta+1)} (\beta+1)^{-2} = (\beta+1)^{1/\alpha-2} c_\beta^{1/\alpha}.$$

We finally check (c), where $\beta = 1$ and $a_\epsilon = \epsilon|\log\epsilon|/2$. No principal value is needed here and it actually holds true that $K_t = \lim_{\eta\to 0} \int_0^t \mathrm{sg}(W_s)|W_s|^{-1/2} \mathbf{1}_{\{|W_s|\geq\eta\}} \mathrm{d}s = \int_0^t \mathrm{sg}(W_s)|W_s|^{-1/2}\mathrm{d}s$. Also, we have $|\epsilon\log\epsilon|^{3/2} H_t^\epsilon = 4|\epsilon\log\epsilon|^{-1/2} \int_0^t \phi(2W_s/|\epsilon\log\epsilon|)\mathrm{d}s$. Using that $\phi(z) \overset{|z|\to\infty}{\sim} \mathrm{sg}(z)|z|^{-1/2}/4$, that $|\phi(z)| \leq C|z|^{-1/2}$ and that $\int_0^T |W_s|^{-1/2}\mathrm{d}s < \infty$ a.s., one verifies, by dominated convergence, that

$$\lim_{\epsilon\to 0} \sup_{[0,T]} \Big| |\epsilon\log\epsilon|^{3/2} H_t^\epsilon - K_t/\sqrt{2} \Big| \leq \lim_{\epsilon\to 0} \int_0^T \Big| 4|\epsilon\log\epsilon|^{-1/2} \phi(2W_s/|\epsilon\log\epsilon|) - \mathrm{sg}(W_s)|W_s|^{-1/2}/\sqrt{2} \Big| \mathrm{d}s = 0$$

a.s., as desired. Such a simple proof can also be handled to check (b) when $\alpha < 1$, i.e. $\beta < 2$. □

We now give the

*Proof of Theorem 1-(b)-(c)-(d).* Fix $\beta \in [1,5]$ and a Brownian motion $(W_t)_{t\geq 0}$, denote by $(L_t^0)_{t\geq 0}$ its local time and by $\tau_t = \inf\{u \geq 0 : L_u^0 > t\}$. Consider the process $(K_t)_{t\geq 0}$ defined in Theorem 3 with $\alpha = (\beta+1)/3$. For each $\epsilon > 0$, consider the processes $(A_t^\epsilon)_{t\geq 0}$, $(\tau_t^\epsilon)_{t\geq 0}$, $(V_t^\epsilon)_{t\geq 0}$ and $(H_t^\epsilon)_{t\geq 0}$ built in Lemma 7 with the choice $a_\epsilon = \epsilon/[(\beta+1)c_\beta]$ if $\beta \in (1,5]$ and $a_\epsilon = \epsilon|\log\epsilon|/2$ if $\beta = 1$.

*Point (b):* $\beta = 5$. As seen in Remark 6, we only have to verify that $\epsilon|\log\epsilon|^{-1}\int_0^{t/\epsilon}[g'(V_s)]^2\mathrm{d}s$, which equals $|\log\epsilon|^{-1}\int_0^t[g'(V_{s/\epsilon})]^2\mathrm{d}s$, goes in probability to $\sigma_5^2 t$ for each $t\geq 0$. By Lemma 7, we may equivalently show that $J_t^\epsilon = |\log\epsilon|^{-1}\int_0^t[g'(V_s^\epsilon)]^2\mathrm{d}s \longrightarrow \sigma_5^2 t$. But as usual,

$$J_t^\epsilon = \int_0^t \frac{[g'(h^{-1}(W_{\tau_s^\epsilon}/a_\epsilon))]^2}{|\log\epsilon|}\mathrm{d}s = \int_0^{\tau_t^\epsilon}\frac{\epsilon[g'(h^{-1}(W_u/a_\epsilon))]^2}{a_\epsilon^2|\log\epsilon|[\sigma(W_u/a_\epsilon)]^2}\mathrm{d}u = \frac{\epsilon}{a_\epsilon^2|\log\epsilon|}\int_0^{\tau_t^\epsilon}\psi(W_u/a_\epsilon)\mathrm{d}u = T^\epsilon_{\tau_t^\epsilon}$$

with the notation of Lemma 9-(e). By Lemma 9-(a), $\sup_{[0,T]}|A_t^\epsilon - L_t^0| \to 0$ a.s. Since $(\tau_t)_{t\geq 0}$, the generalized inverse of $(L_t^0)_{t\geq 0}$, has no fixed times of jump, we deduce from Lemma 8-(a) that for each $t\geq 0$, $\tau_t^\epsilon \to \tau_t$ a.s. Using now Lemma 9-(d), $\sup_{[0,T]}|T_t^\epsilon - \sigma_5^2 L_t^0| \to 0$. All in all, for $t\geq 0$ fixed,

$$|J_t^\epsilon - \sigma_5^2 t| \leq |T^\epsilon_{\tau_t^\epsilon} - \sigma_5^2 L^0_{\tau_t^\epsilon}| + |\sigma_5^2 L^0_{\tau_t^\epsilon} - \sigma_5^2 L^0_{\tau_t}| + |\sigma_5^2 L^0_{\tau_t} - \sigma_5^2 t|$$

which a.s. tends to 0: for the first term, we use that $\sup_{[0,T]}|T_s^\epsilon - \sigma_5^2 L_s^0| \to 0$ a.s. and that $\sup_{\epsilon\in(0,1)}\tau_t^\epsilon$ is a.s. finite (since it has a finite limit as $\epsilon\to 0$), for the second one, we use that $(L_t^0)_{t\geq 0}$ is a.s. continuous and that $\tau_t^\epsilon \to \tau_t$ a.s., for the last one, we use that (for $t\geq 0$ fixed) $L^0_{\tau_t} = t$ a.s.

*Point (c):* $\beta \in (1,5)$. By Lemma 5-(c), we may assume that $X_0 = V_0 = 0$, whence, by Lemma 7, $(X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (H^\epsilon_{\tau_t^\epsilon})_{t\geq 0}$. To verify that $(\epsilon^{1/\alpha}X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\to} (\sigma_\beta S_t^{(\alpha)})_{t\geq 0}$, it is thus sufficient to verify that for each $t\geq 0$ fixed, a.s.,

$$\Delta_t(\epsilon) = |\epsilon^{1/\alpha}H^\epsilon_{\tau_t^\epsilon} - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t}| \to 0.$$

Indeed, Theorem 3 tells us that $S_t^{(\alpha)} = \sigma_\beta^{-1}(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t}$ is a symmetric $\alpha$-stable process with $\mathbb{E}[\exp(i\xi S_t^{(\alpha)})] = \exp(-\kappa_\alpha t|\sigma_\beta^{-1}(\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}\xi|^\alpha) = \exp(-t|\xi|^\alpha)$ by definition of $c_\beta$ and $\kappa_\alpha$.

By Lemma 9-(a), $\sup_{[0,T]}|A_t^\epsilon - L_t^0| \to 0$ a.s. Since $(\tau_t)_{t\geq 0}$, the generalized inverse of $(L_t^0)_{t\geq 0}$, has no fixed time of jump, we deduce from Lemma 8-(a) that $|\tau_t^\epsilon - \tau_t| \to 0$ a.s. By Lemma 9-(b), $\lim_{\epsilon\to 0}\sup_{[0,T]}|\epsilon^{1/\alpha}H_t^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_t| = 0$ a.s. for all $T>0$. All in all,

$$\Delta_t(\epsilon) \leq |\epsilon^{1/\alpha}H^\epsilon_{\tau_t^\epsilon} - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_{\tau_t^\epsilon}| + (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}|K_{\tau_t^\epsilon} - K_{\tau_t}|,$$

which a.s. tends to 0: for the first term, we use that $\lim_{\epsilon\to 0}\sup_{[0,T]}|\epsilon^{1/\alpha}H_s^\epsilon - (\beta+1)^{1/\alpha-2}c_\beta^{1/\alpha}K_s| = 0$ a.s. and that $\sup_{\epsilon\in(0,1)}\tau_t^\epsilon$ is a.s. finite (since it has a finite limit as $\epsilon\to 0$), for the second one, we use that $(K_t)_{t\geq 0}$ is a.s. continuous and that $\tau_t^\epsilon \to \tau_t$ a.s.

*Point (d):* $\beta = 1$. By Lemma 5-(c), we may assume that $X_0 = V_0 = 0$, whence, by Lemma 7, $(X_{t/\epsilon})_{t\geq 0} \stackrel{d}{=} (H^\epsilon_{\tau_t^\epsilon})_{t\geq 0}$. To verify that $(|\epsilon\log\epsilon|^{3/2}X_{t/\epsilon})_{t\geq 0} \stackrel{f.d.}{\to} (\sigma_1 S_t^{(2/3)})_{t\geq 0}$, it is thus sufficient to verify that for each $t\geq 0$ fixed, a.s.,

$$\Delta_t'(\epsilon) = \Big||\epsilon\log\epsilon|^{3/2}H^\epsilon_{\tau_t^\epsilon} - K_{\tau_t}/\sqrt{2}\Big| \to 0.$$

Indeed, Theorem 3 tells us that $(S_t^{(2/3)})_{t\geq 0} = (\sqrt{2}\sigma_1)^{-1}K_{\tau_t}$ is a symmetric $\alpha$-stable process with $\mathbb{E}[\exp(i\xi S_t^{(2/3)})] = \exp(-\kappa_{2/3}t|(\sqrt{2}\sigma_1)^{-1}\xi|^{2/3}) = \exp(-t|\xi|^{2/3})$ by definition of $\sigma_1$ and $\kappa_{2/3}$.

By Lemma 9-(a), $\sup_{[0,T]}|A_t^\epsilon - L_t^0| \to 0$ a.s., whence, by Lemma 8-(a), $|\tau_t^\epsilon - \tau_t| \to 0$ a.s. By Lemma 9-(d), $\lim_{\epsilon\to 0}\sup_{[0,T]}||\epsilon\log\epsilon|^{3/2}H_t^\epsilon - K_t/\sqrt{2}| = 0$ a.s. Thus

$$\Delta_t'(\epsilon) \leq \Big||\epsilon\log\epsilon|^{3/2}H^\epsilon_{\tau_t^\epsilon} - K_{\tau_t^\epsilon}/\sqrt{2}\Big| + |K_{\tau_t^\epsilon} - K_{\tau_t}|/\sqrt{2} \to 0$$

by continuity of $(K_t)_{t\geq 0}$. □

3.7. **Decoupling.** We end the paper with the

*Proof of Corollary 2.* Let $\beta > 1$ be fixed, as well as the solution $(V_t, X_t)_{t\geq 0}$ to (1), starting from some given initial condition $(V_0, X_0)$ and driven by some Brownian motion $(B_t)_{t\geq 0}$. We introduce $\mathcal{F}_t = \sigma(X_0, V_0, B_s, s \leq t)$. We know that $(v_\epsilon^{(\beta)} X_{t/\epsilon})_{t\geq 0} \overset{f.d.}{\to} (X_t^{(\beta)})_{t\geq 0}$, where $v_\epsilon^{(\beta)} \to 0$ and $(X_t^{(\beta)})_{t\geq 0}$ are the rate and limit process appearing in Theorem 1. We fix $t > 0$, $\varphi \in C_b^1(\mathbb{R})$ and $\psi \in B_b(\mathbb{R})$, and our goal is to verify that, setting $\mu_\beta(\psi) = \int_{\mathbb{R}} \psi \mathrm{d}\mu_\beta$, as $\epsilon \to 0$,

$$\Delta_\epsilon = \Big|\mathbb{E}[\varphi(v_\epsilon^{(\beta)} X_{t/\epsilon})\psi(V_{t/\epsilon})] - \mathbb{E}[\varphi(X_t^{(\beta)})]\mu_\beta(\psi)\Big| \to 0.$$

*Step 1.* We check here that for any $h \in (0, t)$, $\delta_\epsilon = \mathbb{E}[|\mathbb{E}[\psi(V_{t/\epsilon})|\mathcal{F}_{(t-h)/\epsilon}] - \mu_\beta(\psi)|] \to 0$ as $\epsilon \to 0$. We use the common notation notation $P_t\psi(v) = \mathbb{E}_v[\psi(V_t)]$ (although we still use $\mathbb{E}$ and $\mathbb{P}$ without subscript when working with the initial condition $V_0$). We introduce the total variation norm $|| \cdot ||_{TV}$. By the Markov property,

$$\delta_\epsilon = \mathbb{E}[|P_{h/\epsilon}\psi(V_{(t-h)/\epsilon}) - \mu_\beta(\psi)|] \leq \delta_\epsilon^1 + \delta_\epsilon^2,$$

where, introducing some $\mu_\beta$-distributed $\bar{V}$ such that $\mathbb{P}(\bar{V} \neq V_{(t-h)/\epsilon}) = ||\mathcal{L}(V_{(t-h)/\epsilon}) - \mu_\beta||_{TV}$,

$$\begin{aligned} \delta_\epsilon^1 =& \mathbb{E}[|P_{h/\epsilon}\psi(V_{(t-h)/\epsilon}) - P_{h/\epsilon}\psi(\bar{V})|] \leq 2||\psi||_\infty \mathbb{P}(\bar{V} \neq V_{(t-h)/\epsilon}) = 2||\psi||_\infty ||\mathcal{L}(V_{(t-h)/\epsilon}) - \mu_\beta||_{TV}, \\ \delta_\epsilon^2 =& \mathbb{E}[|P_{h/\epsilon}\psi(\bar{V}) - \mu_\beta(\psi)|]. \end{aligned}$$

But, by Lemma 5-(a) and [16, Lemma 23.17 page 466], $||\mathcal{L}(V_s) - \mu_\beta||_{TV} \to 0$ (when starting from any initial condition $V_0$) as $s \to \infty$. Hence $\delta_\epsilon^1$ tends to 0 as $\epsilon \to 0$. We also have $\lim_{s\to\infty} P_s\psi(v) = \mu_\beta(\psi)$ for all $v \in \mathbb{R}$, so that $\delta_\epsilon^2$ tends to 0 by dominated convergence.

*Step 2.* For any $h \in (0, t)$, we write $\Delta_\epsilon \leq \Delta_{\epsilon,h}^1 + \Delta_{\epsilon,h}^2 + \Delta_{\epsilon,h}^3 + \Delta_{\epsilon,h}^4$, where

$$\begin{aligned} \Delta_{\epsilon,h}^1 =& \Big|\mathbb{E}[\varphi(v_\epsilon^{(\beta)} X_{t/\epsilon})\psi(V_{t/\epsilon})] - \mathbb{E}[\varphi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})\psi(V_{t/\epsilon})]\Big|, \\ \Delta_{\epsilon,h}^2 =& \Big|\mathbb{E}[\varphi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})\psi(V_{t/\epsilon})] - \mathbb{E}[\varphi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})]\mu_\beta(\psi)\Big|, \\ \Delta_{\epsilon,h}^3 =& \Big|\mathbb{E}[\varphi(v_\epsilon^{(\beta)} X_{(t-h)/\epsilon})]\mu_\beta(\psi) - \mathbb{E}[\varphi(X_{t-h}^{(\beta)})]\mu_\beta(\psi)\Big|, \\ \Delta_{\epsilon,h}^4 =& \Big|\mathbb{E}[\varphi(X_{t-h}^{(\beta)})]\mu_\beta(\psi) - \mathbb{E}[\varphi(X_t^{(\beta)})]\mu_\beta(\psi)\Big|. \end{aligned}$$

By Theorem 1, $\lim_{\epsilon\to 0} \Delta_{\epsilon,h}^3 = 0$ and, with $C = ||\psi||_\infty(||\varphi||_\infty + ||\varphi'||_\infty)$

$$\limsup_{\epsilon\to 0} \Delta_{\epsilon,h}^1 \leq C \limsup_{\epsilon\to 0} \mathbb{E}[|v_\epsilon^{(\beta)} X_{t/\epsilon} - v_\epsilon^{(\beta)} X_{(t-h)/\epsilon}| \wedge 1] = C\mathbb{E}[|X_t^{(\beta)} - X_{t-h}^{(\beta)}| \wedge 1].$$

Also, $\Delta_{\epsilon,h}^4 \leq C\mathbb{E}[|X_t^{(\beta)} - X_{t-h}^{(\beta)}| \wedge 1]$ and $\Delta_{\epsilon,h}^2 \leq ||\varphi||_\infty \mathbb{E}[|\mathbb{E}[\psi(V_{t/\epsilon})|\mathcal{F}_{(t-h)/\epsilon}] - \mu_\beta(\psi)|] \to 0$ by Step 1. All in all, $\limsup_{\epsilon\to 0} \Delta_\epsilon \leq 2C\mathbb{E}[|X_t^{(\beta)} - X_{t-h}^{(\beta)}| \wedge 1]$ for any $h \in (0, t)$. Letting $h \downarrow 0$ ends the proof. □

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