Riccati Observer Design for Pose, Linear Velocity and Gravity Direction Estimation using Landmark Position and IMU Measurements

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Abstract— **This paper revisits the problem of estimating the pose (i.e. position and attitude) of a robotic vehicle by combining landmark position measurements provided by a stereo camera with measurements of an Inertial Measurement Unit. The distinguished features with respect to similar works on the topic are two folds: First, the vehicle's linear velocity is not measured neither in the body frame nor in the inertial frame; Second, no prior knowledge on the gravity direction expressed in the inertial frame is required. Instead both the linear velocity and the gravity direction are estimated together with the pose. Another innovative feature of the paper relies on the idea of over-parametrizing the gravity direction vector evolving on the unit 2-sphere S^2 by an element of $SO(3)$ so that the error system in first order approximations can be written in an "elegant" linear time-varying form. The proposed deterministic observer is accompanied with an observability analysis that points out an explicit observability condition under which local exponential stability is granted. Reported simulation results further indicate that the observer's domain of convergence is large.**

I. Introduction

A robust and reliable pose (i.e. position and orientation) estimator is a key requirement for robust and efficient autonomous navigation of robotic vehicles. The problem of full pose estimation of a rigid body moving in a three-dimensional space is highly nonlinear, essentially due to its orientation (i.e. attitude) that lives in the compact Lie group $SO(3)$ encoded in the larger group $SE(3)$ of 3D rigid transformation. But high nonlinearity is not the unique source of complexity. Indeed, topological obstructions associated with all groups containing rotations impede to exactly linearize the equations of motion evolving on these groups and also exclude the possibility of designing continuous estimators on these groups that allow for *global* asymptotic stability of the zero estimation errors [7].

Classical approaches for state estimation are typically based on filtering techniques such as extended Kalman filters (EKF), unscented Kalman filters or particle filters. However, nonlinear observers have increasingly become an alternative to these classical techniques, starting with the work of Salcudean on attitude observer [20] and subsequent contributions by other researchers [1], [5], [6], [8], [10], [15]–[19]. Full pose observer design has recently attracted some particular attention [2]–[4], [9], [11], [13]–[15], [21]–[23]. Depending on the nature of measurements, several research directions have been investigated. For instance, in the case where the complete pose is measured together with the linear and angular velocities, two observers posed on $SE(3)$ have been proposed by Baldwin et al. [3], achieving almost global asymptotic stability of the estimation errors. In the same context an invariant observer on $SO(3) \times \mathbb{R}^3$ has been derived by Barczyk et al. [4] on the basis of a recently elaborated theory on invariant observer [8]. On the other hand, pose estimation from the observations of known source points (or landmarks) is significantly more challenging than the previously mentioned case. As long as bearing measurements in the body frame of known landmarks are concerned, the pose estimation problem is classically referred to as Perspective-n-Point (PnP) problem that is typically associated with the use of a monocular perspective camera for bearing measurements. Even though the PnP problem has been much studied with algebraic approaches [12], it has seldom been addressed by nonlinear observer design techniques [2], [11]. Noteworthily, in the excellent recent work of Hamel and Samson [11] some Riccati observers for this PnP problem have been derived on the basis of a Riccati observer design framework developed in the same paper. A noticeable conclusion of this work is that, by contrast with static PnP solutions, the body motion is not only allowed but also used as a source of information to improve the estimation performance. In particular, in the case where the linear velocity expressed in the inertial frame is measured, the body motion with sufficient persistence of excitation is instrumental for pose estimation from bearing measurements of a solitary landmark.

In parallel with the PnP problem that exploits landmark bearing measurements, the problem of pose observer design using *landmark position measurements* in the body frame has also been increasingly investigated [13], [14], [22], [23], with motivations strongly related to robotic vision applications that involve a stereo camera or a Kinect sensor. For instance, an observer posed on $SO(3) \times \mathbb{R}^3$ that ensures almost global asymptotic stability of the estimation errors has been developed by Vasconcelos et al. [22]. In some prior works [13], [14] of one of the authors of this paper, gradient-like nonlinear observers on $SE(3)$ have been designed directly on output spaces, achieving either local exponential stability or almost global asymptotic stability. In [23] a hybrid technique has been incorporated in some observers designed on $SE(3)$, allowing one to overcome the topological obstruction associated with this group so that global asymptotic stability is provable. It is interesting to note that the availability of linear velocity measurements in either the inertial frame or the body frame is also an important source of the variety of existing pose observers. As a matter of fact, most existing pose observers relying on landmark position measurements

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necessitate not only position measurements of at least three non-aligned landmarks but also (biased) measurements of the linear and angular velocities. Whereas three non-aligned landmarks guarantees that the pose is observable, velocities measurements on the other hand allow one to address the observer design uniquely with kinematic equations. However, this latter technical convenience is often hindered by some practical issues in many robotic applications. Indeed, whereas the angular velocity can be easily measured at high frequency and precision with low-price (or reasonable-price) MEMS gyrometers, linear velocity measurements in either the inertial frame or the body frame are by contrast more difficult (or even impossible) to obtain for various reasons. For instance, the use of a GPS sensor to measure the linear velocity in the inertial frame is simply excluded in indoor or GPS-denied environments. On the other hand, obtaining linear velocity measurements in the body frame is often a "luxury". Indeed, the current technologies have not yet allowed for reliable, small-size and cost-affordable body velocity sensors for mini aerial drone applications. Moreover, Doppler velocity sensors such as Doppler Velocity Log (DVL) commonly used for the navigation of autonomous underwater vehicles are extremely expensive and difficult to implement due to their size and weight. For those reasons, accelerometer measurements have been recently exploited as an alternative to linear velocity measurements for pose observer design as proposed in [9], which relies on the assumption that the gravity direction in the inertial frame is known a priori. The latter assumption is, however, rather restrictive from a practical point of view for some applications such as stereo visual odometry where the reference keyframe is regularly changed (and so are the inertial frame and the gravity direction vector expressed in this frame) when the number of point correspondences is dropped below a certain threshold.

In the present paper the pose observer design from landmark position measurements in combination with gyrometer and accelerometer readings is revisited, but here without relying on the assumption on the gravity direction used in [9]. The design of the proposed observer is inspired by a recent deterministic Riccati observer design framework [11] that relies on the solutions to the Continuous Riccati Equation (CRE). Without using linear velocity measurements and prior knowledge of the gravity direction in the inertial frame increases significantly the complexity of observer design and its associated convergence and stability analysis. Interestingly, the existence of three non-aligned landmarks is here again required to address the structural question of observability that in turn ensures the exponential stability of the proposed observer. Another innovative feature of the proposed solution relies on the idea of over-parametrizing the gravity direction vector evolving on the unit 2-sphere S^2 by an element of $SO(3)$ so that the error system in first order approximations can be written in an "elegant" linear time-varying (LTV) form that allows for direct application of a modified version of the Riccati observer design framework proposed in [11]. Since only local exponential stability is demonstrated, simulation results are also provided as a complement to show the large domain of attraction of the proposed observer.

The paper is organized as follows. Notation, system equations and measurements used for observer design are specified in Section II. In the same section a few basic definitions and conditions about uniform observability are recalled, together with a modified version of the deterministic Riccati observer design framework proposed in [11]. In Section III the observer design is presented and an associated observability and stability analysis is given in Section IV. Simulation results illustrating the observer performance are reported in Section V. A short concluding section then follows.

II. Preliminary material

A. Notation

- $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes the canonical basis of $\mathbb{R}^3$ and $[\cdot]_\times$ denotes the skew-symmetric matrix associated with the cross product, i.e. $[\mathbf{x}]_\times \mathbf{y} = \mathbf{x} \times \mathbf{y}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. The identity matrix of $\mathbb{R}^{n\times n}$ is denoted as $\mathbf{I}_n$ and $\boldsymbol{\pi}_{\mathbf{x}} \triangleq \mathbf{I}_3 - \mathbf{x}\mathbf{x}^\top$, $\forall \mathbf{x} \in S^2$, is the projection operator onto the plane orthogonal to $\mathbf{x}$. Note that $\boldsymbol{\pi}_{\mathbf{x}} = -[\mathbf{x}]_\times^2$, $\forall \mathbf{x} \in S^2$.
- $\{\mathcal{I}\}$ and $\{\mathcal{B}\}$ denote, respectively, an inertial frame attached to the Earth and a body-fixed frame attached to the vehicle.
- $\boldsymbol{\xi} \in \mathbb{R}^3$ denotes the vehicle's position expressed in $\{\mathcal{I}\}$. Its attitude is represented by a rotation matrix $\mathbf{R} \in \mathrm{SO}(3)$ of the frame $\{\mathcal{B}\}$ relative to $\{\mathcal{I}\}$.
- $\mathbf{V} \in \mathbb{R}^3$ and $\boldsymbol{\Omega} \in \mathbb{R}^3$ are the vectors of coordinates of the vehicle's linear and angular velocities expressed in $\{\mathcal{B}\}$. Its linear velocity expressed in $\{\mathcal{I}\}$ is denoted as $\mathbf{v} \in \mathbb{R}^3$ so that $\mathbf{v} = \mathbf{R}\mathbf{V}$.
- $\mathring{\boldsymbol{\gamma}} \in S^2$ and $\boldsymbol{\gamma} \in S^2$ denote the gravity direction vectors expressed in $\{\mathcal{I}\}$ and $\{\mathcal{B}\}$, respectively. One has $\boldsymbol{\gamma} = \mathbf{R}^\top \mathring{\boldsymbol{\gamma}}$.

B. System equations and measurements

The vehicle kinematic equations are

$$\begin{cases} \dot{\boldsymbol{\xi}} = \mathbf{R}\mathbf{V} & \text{(1a)} \\ \dot{\mathbf{R}} = \mathbf{R}[\boldsymbol{\Omega}]_\times & \text{(1b)} \end{cases}$$

Assume that the vehicle is equipped with an Inertial Measurement Unit that consists of a 3-axis gyrometer to measure the angular velocity $\boldsymbol{\Omega} \in \mathbb{R}^3$ and a 3-axis accelerometer to measure the so-called *specific acceleration* $\mathbf{a}_\mathcal{B} \in \mathbb{R}^3$, expressed in $\{\mathcal{B}\}$. Using the flat non-rotating Earth assumption, one has [8]

$$\dot{\mathbf{V}} = -[\boldsymbol{\Omega}]_\times \mathbf{V} + \mathbf{a}_\mathcal{B} + g\boldsymbol{\gamma} \tag{2}$$

where g is the gravity constant. In this work, we consider the case where both $\mathring{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ are not known.

We further assume that the vehicle is equipped with a stereo camera that provides the position measurements $\mathbf{p}_i$, $i \in \{1, \cdots, N\}$, expressed in $\{\mathcal{B}\}$ of N landmarks whose positions $\mathring{\mathbf{p}}_i$ expressed in $\{\mathcal{I}\}$ are also known a priori. One verifies that

$$\mathbf{p}_i = \mathbf{R}^\top(\mathring{\mathbf{p}}_i - \boldsymbol{\xi}) \tag{3}$$

C. Recalls of observability definitions and conditions

The following definitions and conditions are recalled for the sake of completeness. Consider a LTV system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ \mathbf{y} = \mathbf{C}(t)\mathbf{x} \end{cases} \tag{4}$$

with $\mathbf{x} \in \mathbb{R}^n$ the system state vector, $\mathbf{u} \in \mathbb{R}^l$ the system input vector, and $\mathbf{y} \in \mathbb{R}^m$ the system output vector.

Definition 1 *(uniform observability) System* (4) *is uniformly observable if there exist $\delta, \mu > 0$ such that $\forall t \geq 0$*

$$\mathbf{W}(t, t+\delta) \triangleq \frac{1}{\delta}\int_t^{t+\delta} \mathbf{\Phi}^\top(t,s)\mathbf{C}^\top(s)\mathbf{C}(s)\mathbf{\Phi}(t,s)ds \geq \mu \mathbf{I}_n \tag{5}$$

with $\mathbf{\Phi}(t,s)$ the transition matrix associated with $\mathbf{A}(t)$, i.e. such that $\frac{d}{dt}\mathbf{\Phi}(t,s) = \mathbf{A}(t)\mathbf{\Phi}(t,s)$ with $\mathbf{\Phi}(t,t) = \mathbf{I}_n$.

$\mathbf{W}(t, t+\delta)$ is called the observability Gramian of System (4). When (5) is satisfied one also says that the pair $(\mathbf{A}(t), \mathbf{C}(t))$ is uniformly observable. The following useful condition points out a sufficient condition for uniform observability.

Lemma 1 *(see [21]) If there exists a matrix-valued function $\mathbf{M}(\cdot)$ of dimension $(p \times n)$ $(p \geq 1)$ composed of row vectors of $\mathbf{N}_0 = \mathbf{C}$, $\mathbf{N}_k = \mathbf{N}_{k-1}\mathbf{A} + \dot{\mathbf{N}}_{k-1}, k = 1, \cdots$ such that for some positive numbers $\bar{\delta}, \bar{\mu}$ and $\forall t \geq 0$*

$$\frac{1}{\bar{\delta}}\int_t^{t+\bar{\delta}} \mathbf{M}^\top(s)\mathbf{M}(s)ds \geq \bar{\mu}\mathbf{I}_n \tag{6}$$

then the observability Gramian of System (4) *satisfies condition* (5)*.*

D. A modified version of an existing Riccati observer design framework

The proposed observer design is based on the following modified version of the deterministic observer design framework developed in [11]. Although some modifications are made, the proof for this version proceeds analogously to the proof of [11, Theorem 3.1]. Consider the nonlinear system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, t)\mathbf{x} + \mathbf{u} + O(|\mathbf{x}|^2) + O(|\mathbf{x}||\mathbf{u}|) \\ \mathbf{y} = \mathbf{C}(\mathbf{x}, t)\mathbf{x} + O(|\mathbf{x}|^2) \end{cases} \tag{7}$$

with $\mathbf{x} = [\mathbf{x}_1^\top, \mathbf{x}_2^\top]^\top$, $\mathbf{x}_1 \in \mathcal{B}_r^{n_1}$ (the closed ball in $\mathbb{R}^{n_1}$ of radius r), $\mathbf{x}_2 \in \mathbb{R}^{n_2}$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{C}(\mathbf{x}, t)$ a continuous matrix-valued function uniformly bounded with respect to (w.r.t.) t and uniformly continuous w.r.t. $\mathbf{x}$, and $\mathbf{A}(\mathbf{x}, t)$ of the form[1]

$$\mathbf{A}(\mathbf{x}, t) = \begin{bmatrix} \mathbf{A}_{1,1}(t) & \mathbf{0} \\ \mathbf{A}_{2,1}(\mathbf{x}, t) & \mathbf{A}_{2,2}(\mathbf{x}_1, t) \end{bmatrix}$$

with $\mathbf{A}_{1,1}(t)$ and $\mathbf{A}_{2,2}(\mathbf{x}_1, t)$ continuous matrix-valued functions uniformly bounded w.r.t. t and uniformly continuous w.r.t. $\mathbf{x}_1$, and $\mathbf{A}_{2,1}(\mathbf{x}, t)$ verifying

$$\mathbf{A}_{2,1}(\mathbf{x}, t)\mathbf{x}_1 = \bar{\mathbf{A}}_{2,1}(\mathbf{x}_1, t)\mathbf{x}_1 + O(|\mathbf{x}_1||\mathbf{x}_2|)$$

with $\bar{\mathbf{A}}_{2,1}(\mathbf{x}_1, t)$ a continuous matrix-valued function uniformly bounded w.r.t. t and uniformly continuous w.r.t. $\mathbf{x}_1$.

Apply

$$\mathbf{u} = -\mathbf{P}\mathbf{C}^\top\bar{\mathbf{Q}}(t)\mathbf{y} \tag{8}$$

with $\mathbf{P} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ a symmetric positive definite matrix solution to the following CRE:

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top - \mathbf{P}\mathbf{C}^\top\bar{\mathbf{Q}}(t)\mathbf{C}\mathbf{P} + \bar{\mathbf{V}}(t) \tag{9}$$

with $\mathbf{P}(0) \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ a symmetric positive definite matrix, $\bar{\mathbf{Q}}(t) \in \mathbb{R}^{m\times m}$ bounded continuous symmetric positive semidefinite, and $\bar{\mathbf{V}}(t) \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ bounded continuous symmetric positive definite.

Lemma 2 *Consider System* (7) *and apply the input $\mathbf{u}$ calculated according to* (8) *and* (9) *with the matrices $\bar{\mathbf{Q}}(t)$ and $\bar{\mathbf{V}}(t)$ chosen larger than some positive matrices. If the pair $(\mathbf{A}^\star(t), \mathbf{C}^\star(t))$, with $\mathbf{A}^\star(t) \triangleq \mathbf{A}(\mathbf{0}, t)$ and $\mathbf{C}^\star(t) \triangleq \mathbf{C}(\mathbf{0}, t)$, is uniformly observable, then $\mathbf{x} = 0$ is locally exponentially stable.*

The proof of this lemma proceeds identically to the one of Theorem 3.1 and Corollary 3.2 in [11].

III. Observer design

Let $\hat{\boldsymbol{\xi}} \in \mathbb{R}^3$, $\hat{\mathbf{R}} \in \mathrm{SO}(3)$, $\hat{\mathbf{V}} \in \mathbb{R}^3$ and $\hat{\gamma} \in S^2$ denote the estimates of $\boldsymbol{\xi}$, $\mathbf{R}$, $\mathbf{V}$ and $\boldsymbol{\gamma}$, respectively. The form of the proposed observer is given by

$$\begin{cases} \dot{\hat{\boldsymbol{\xi}}} = \hat{\mathbf{R}}\hat{\mathbf{V}} - \boldsymbol{\sigma}_{\boldsymbol{\xi}} & \text{(10a)} \\ \dot{\hat{\mathbf{R}}} = \hat{\mathbf{R}}[\mathbf{\Omega}]_\times - [\boldsymbol{\sigma}_{\mathbf{R}}]_\times\hat{\mathbf{R}} & \text{(10b)} \\ \dot{\hat{\mathbf{V}}} = -[\mathbf{\Omega}]_\times\hat{\mathbf{V}} + \mathbf{a}_{\mathcal{B}} + g\hat{\gamma} - \boldsymbol{\sigma}_{\mathbf{V}} & \text{(10c)} \end{cases}$$

where $\boldsymbol{\sigma}_{\boldsymbol{\xi}}, \boldsymbol{\sigma}_{\mathbf{R}}, \boldsymbol{\sigma}_{\mathbf{V}} \in \mathbb{R}^3$ are innovation terms to be designed and, in particular, the design of $\hat{\gamma}$ will be thoroughly discussed thereafter.

For this application we need to make the following technical (but non-restrictive) assumption.

Assumption 1 *The vehicle's position $\boldsymbol{\xi}$, linear velocity $\mathbf{v}$, angular velocity $\mathbf{\Omega}$ and angular acceleration $\dot{\mathbf{\Omega}}$ are bounded for all time.*

Defining the observer errors

$$\tilde{\boldsymbol{\xi}} \triangleq \boldsymbol{\xi} - \hat{\boldsymbol{\xi}},\ \tilde{\mathbf{R}} \triangleq \mathbf{R}\hat{\mathbf{R}}^\top,\ \tilde{\mathbf{V}} \triangleq \mathbf{V} - \hat{\mathbf{V}},\ \tilde{\gamma} \triangleq \boldsymbol{\gamma} - \hat{\gamma}$$

then the design objective is the exponential stability of $(\tilde{\boldsymbol{\xi}}, \tilde{\mathbf{R}}, \tilde{\mathbf{V}}, \tilde{\gamma}) = (\mathbf{0}, \mathbf{I}_3, \mathbf{0}, \mathbf{0})$. From (1), (2) and (10) one verifies that

$$\begin{cases} \dot{\tilde{\boldsymbol{\xi}}} = (\tilde{\mathbf{R}} - \mathbf{I}_3)\hat{\mathbf{R}}\hat{\mathbf{V}} + \tilde{\mathbf{R}}\hat{\mathbf{R}}\tilde{\mathbf{V}} + \boldsymbol{\sigma}_{\boldsymbol{\xi}} & \text{(11a)} \\ \dot{\tilde{\mathbf{R}}} = \tilde{\mathbf{R}}[\boldsymbol{\sigma}_{\mathbf{R}}]_\times & \text{(11b)} \\ \dot{\tilde{\mathbf{V}}} = -[\mathbf{\Omega}]_\times\tilde{\mathbf{V}} + g(\boldsymbol{\gamma} - \hat{\gamma}) + \boldsymbol{\sigma}_V & \text{(11c)} \end{cases}$$

The next step consists in defining the dynamics of $\hat{\gamma}$ and working out first order approximations of the error system complemented with first order approximations of the measurement equations. The application to these approximations

[1] In fact, 2 modifications w.r.t. the observer design framework reported in [11] are made. First, it is not necessary that $n_1 = n_2$ and $r = 1$. Second, in [11] $\mathbf{A}$ is a matrix-valued function of $\mathbf{x}_1$ and t, i.e. $\mathbf{A}(\mathbf{x}_1, t)$, with the matrices $\mathbf{A}_{2,1}$ and $\mathbf{A}_{2,2}$ of the form $\mathbf{A}_{2,1}(\mathbf{x}_1, t)$ and $\mathbf{A}_{2,2}(t)$.

of the Riccati observer design framework reported in Section II-D will then provide us with the expressions of the innovation terms.

In the case where the gravity direction vector expressed in the inertial frame (i.e. $\mathring{\gamma}$) is known a priori, the estimate $\hat{\gamma}$ of γ may be written as $\hat{\gamma} = \hat{\mathbf{R}}^\top \mathring{\gamma}$ (recalling that $\gamma = \mathbf{R}^\top \mathring{\gamma}$) and, thus, the observer design objective can only focus on the exponential stability of $(\tilde{\boldsymbol{\xi}}, \tilde{\mathbf{R}}, \tilde{\mathbf{V}}) = (\mathbf{0}, \mathbf{I}_3, \mathbf{0})$. However, we address here a *more challenging problem with* $\mathring{\gamma}$ *unknown.*

The first thing one may ask for is a suitable parametrization for γ and $\hat{\gamma}$. Bearing in mind that the Riccati observer design framework [11] here employed involves a stage of first order approximations of the error system, one quickly figures out that all minimal parametrization techniques for elements on S^2 such as *spherical coordinate system* lead to very complex and highly nonlinear equations. For instance, one needs to develop first order approximations of the term $\gamma - \hat{\gamma}$ on the right-hand side of (11c). However, when the spherical coordinate system is made use so that

$$\gamma = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \ \hat{\gamma} = \begin{bmatrix} \sin\hat{\theta}\cos\hat{\phi} \\ \sin\hat{\theta}\sin\hat{\phi} \\ \cos\hat{\theta} \end{bmatrix}$$

the first order approximations of $\gamma - \hat{\gamma}$ (i.e. first order functions of $\tilde{\theta} \triangleq \theta - \hat{\theta}$ and $\tilde{\phi} \triangleq \phi - \hat{\phi}$) are very complex for observer design purposes. Moreover, the derivative equations of $\tilde{\theta}$ and $\tilde{\phi}$ are not simple either.

Motivated by the above discussion, a more "elegant" solution (i.e. parametrization) will be proposed next. The underlying idea is to *over-parametrize an element of* S^2 *(dimension 2) by an element of* $\mathrm{SO}(3)$ *(dimension 3)*. More precisely, we make use of a rotation matrix $\hat{\mathbf{Q}} \in \mathrm{SO}(3)$ in order to over-parametrize $\hat{\gamma} \in S^2$ so that

$$\hat{\gamma} = \hat{\mathbf{Q}}^\top \mathbf{e}_3 \tag{12}$$

Note also that there exist an infinity of matrices $\mathbf{Q} \in \mathrm{SO}(3)$ such that $\mathbf{Q}^\top \mathbf{e}_3 = \gamma$. Then, the convergence of $\hat{\mathbf{Q}}^\top \mathbf{e}_3$ to $\mathbf{Q}^\top \mathbf{e}_3$ is clearly equivalent to the one of $\hat{\gamma}$ to γ.

Now, let $\tilde{\mathbf{Q}} \triangleq \mathbf{Q}\hat{\mathbf{Q}}^\top \in \mathrm{SO}(3)$ denote the group error of $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ and let $\mathbf{q}_{\tilde{\mathbf{Q}}} = (\eta_{\tilde{\mathbf{Q}}}, \boldsymbol{\nu}_{\tilde{\mathbf{Q}}})$, with $\eta_{\tilde{\mathbf{Q}}} \in \mathcal{B}_1^1$ and $\boldsymbol{\nu}_{\tilde{\mathbf{Q}}} \in \mathcal{B}_1^3$ the real and imaginary parts, denote the Rodrigues unit quaternion associated with $\tilde{\mathbf{Q}}$. Rodrigues formula relating $\mathbf{q}_{\tilde{\mathbf{Q}}}$ to $\tilde{\mathbf{Q}}$ is given by

$$\tilde{\mathbf{Q}} = \mathbf{I}_3 + 2[\boldsymbol{\nu}_{\tilde{\mathbf{Q}}}]_\times(\eta_{\tilde{\mathbf{Q}}}\mathbf{I}_3 + [\boldsymbol{\nu}_{\tilde{\mathbf{Q}}}]_\times) = \mathbf{I}_3 + [\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}]_\times + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}|^2)$$

with $\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}} \triangleq 2\mathrm{sign}(\eta_{\tilde{\mathbf{Q}}})\boldsymbol{\nu}_{\tilde{\mathbf{Q}}} \in \mathcal{B}_1^3$. One then deduces

$$\begin{aligned} \gamma - \hat{\gamma} &= \mathbf{Q}^\top \mathbf{e}_3 - \hat{\mathbf{Q}}^\top \mathbf{e}_3 = \hat{\mathbf{Q}}^\top(\tilde{\mathbf{Q}}^\top - \mathbf{I}_3)\mathbf{e}_3 \\ &= \hat{\mathbf{Q}}^\top[\mathbf{e}_3]_\times \boldsymbol{\lambda}_{\tilde{\mathbf{Q}}} + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}|^2) \\ &= \lambda_{\tilde{\mathbf{Q}},1}\hat{\mathbf{Q}}^\top \mathbf{e}_2 - \lambda_{\tilde{\mathbf{Q}},2}\hat{\mathbf{Q}}^\top \mathbf{e}_1 + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}|^2) \end{aligned} \tag{13}$$

with $\lambda_{\tilde{\mathbf{Q}},1}$, $\lambda_{\tilde{\mathbf{Q}},2}$ the first and second components of $\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}$, so that in view of (11c) one obtains

$$\dot{\tilde{\mathbf{V}}} = -[\boldsymbol{\Omega}]_\times \tilde{\mathbf{V}} + \lambda_{\tilde{\mathbf{Q}},1} g \hat{\mathbf{Q}}^\top \mathbf{e}_2 - \lambda_{\tilde{\mathbf{Q}},2} g \hat{\mathbf{Q}}^\top \mathbf{e}_1 + \boldsymbol{\sigma}_V + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}|^2) \tag{14}$$

Remark 1 *In view of* (13)*, the first order approximations of* $\gamma - \hat{\gamma}$ *only involve 2 components of* $\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}$ *(i.e.* $\lambda_{\tilde{\mathbf{Q}},1}$, $\lambda_{\tilde{\mathbf{Q}},2}$*), which is quite remarkable and desirable. Indeed, although we over-parametrize an element of* S^2 *(i.e.* $\hat{\gamma}$*) by a 3 degrees-of-freedom element of* $\mathrm{SO}(3)$ *(i.e.* $\hat{\mathbf{Q}}$*), we end up exploiting only a minimal number of parameters, in first order approximations, for observer design. The remaining degree of freedom related to the third component* $\lambda_{\tilde{\mathbf{Q}},3}$ *of* $\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}$ *is, thus, left uninvolved.*

We continue by developing first order approximations of the dynamics of $\lambda_{\tilde{\mathbf{Q}},1}$ and $\lambda_{\tilde{\mathbf{Q}},2}$. Since $\gamma = \mathbf{R}^\top \mathring{\gamma}$, the dynamics of γ satisfies $\dot{\gamma} = -[\boldsymbol{\Omega}]_\times \gamma$, which allows one to deduce the dynamics of $\mathbf{Q}$ as

$$\dot{\mathbf{Q}} = \mathbf{Q}[\boldsymbol{\Omega}]_\times \tag{15}$$

From here, we propose the following dynamics of $\hat{\mathbf{Q}}$

$$\dot{\hat{\mathbf{Q}}} = \hat{\mathbf{Q}}[\boldsymbol{\Omega}]_\times - [\boldsymbol{\sigma}_{\mathbf{Q}}]_\times \hat{\mathbf{Q}} \tag{16}$$

with $\boldsymbol{\sigma}_Q \in \mathbb{R}^3$ an innovation term to be designed thereafter. One then deduces from (15) and (16) that

$$\dot{\tilde{\mathbf{Q}}} = \tilde{\mathbf{Q}}[\boldsymbol{\sigma}_{\mathbf{Q}}]_\times$$

which in turn allows one to verify (see also [11]) that

$$\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{Q}}} = \boldsymbol{\sigma}_{\mathbf{Q}} + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{Q}}}||\boldsymbol{\sigma}_{\mathbf{Q}}|) \tag{17}$$

Similarly, let $\mathbf{q}_{\tilde{\mathbf{R}}} = (\eta_{\tilde{\mathbf{R}}}, \boldsymbol{\nu}_{\tilde{\mathbf{R}}})$ be the Rodrigues unit quaternion associated with $\tilde{\mathbf{R}}$. One has

$$\tilde{\mathbf{R}} = \mathbf{I}_3 + [\boldsymbol{\lambda}_{\tilde{\mathbf{R}}}]_\times + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{R}}}|^2)$$

with $\boldsymbol{\lambda}_{\tilde{\mathbf{R}}} \triangleq 2\mathrm{sign}(\eta_{\tilde{\mathbf{R}}})\boldsymbol{\nu}_{\tilde{\mathbf{R}}} \in \mathcal{B}_1^3$. Then, in view of the dynamics of $\tilde{\boldsymbol{\xi}}$ and $\tilde{\mathbf{R}}$ in (11a) and (11b) one verifies that

$$\dot{\tilde{\boldsymbol{\xi}}} = -[\hat{\mathbf{R}}\hat{\mathbf{V}}]_\times \boldsymbol{\lambda}_{\tilde{\mathbf{R}}} + \hat{\mathbf{R}}\tilde{\mathbf{V}} + \boldsymbol{\sigma}_{\boldsymbol{\xi}} + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{R}}}|^2) + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{R}}}||\tilde{\mathbf{V}}|) \tag{18}$$

and

$$\dot{\boldsymbol{\lambda}}_{\tilde{\mathbf{R}}} = \boldsymbol{\sigma}_{\mathbf{R}} + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{R}}}||\boldsymbol{\sigma}_{\mathbf{R}}|) \tag{19}$$

Concerning the landmark position measurements, defining

$$\hat{\mathbf{p}}_i \triangleq \hat{\mathbf{R}}^\top(\mathring{\mathbf{p}}_i - \hat{\boldsymbol{\xi}}), \quad i \in \{1, \cdots, N\} \tag{20}$$

one then deduces from (3) and (20) that

$$\begin{aligned} \mathbf{p}_i - \hat{\mathbf{p}}_i &= \mathbf{R}^\top(\mathring{\mathbf{p}}_i - \boldsymbol{\xi}) - \hat{\mathbf{R}}^\top(\mathring{\mathbf{p}}_i - \hat{\boldsymbol{\xi}}) \\ &= \hat{\mathbf{R}}^\top(\tilde{\mathbf{R}}^\top - \mathbf{I}_3)(\mathring{\mathbf{p}}_i - \hat{\boldsymbol{\xi}}) - \hat{\mathbf{R}}^\top \tilde{\boldsymbol{\xi}} \\ &= \hat{\mathbf{R}}^\top[\mathring{\mathbf{p}}_i - \hat{\boldsymbol{\xi}}]_\times \boldsymbol{\lambda}_{\tilde{\mathbf{R}}} - \hat{\mathbf{R}}^\top \tilde{\boldsymbol{\xi}} + O(|\boldsymbol{\lambda}_{\tilde{\mathbf{R}}}|^2) \end{aligned} \tag{21}$$

In view of (14), (17), (18), (19) and (21), by setting the system output vector equal to

$$\mathbf{y} = \begin{bmatrix} \mathbf{p}_1 - \hat{\mathbf{p}}_1 \\ \vdots \\ \mathbf{p}_N - \hat{\mathbf{p}}_N \end{bmatrix}$$

one obtains LTV first order approximations in the form (7) with

$$\begin{cases} \mathbf{x} = \begin{bmatrix} \lambda_{\tilde{\mathbf{Q}},1} \\ \lambda_{\tilde{\mathbf{Q}},2} \\ \boldsymbol{\lambda}_{\tilde{\mathbf{R}}} \\ \tilde{\boldsymbol{\xi}} \\ \tilde{\mathbf{V}} \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} \lambda_{\tilde{\mathbf{Q}},1} \\ \lambda_{\tilde{\mathbf{Q}},2} \\ \boldsymbol{\lambda}_{\tilde{\mathbf{R}}} \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} \tilde{\boldsymbol{\xi}} \\ \tilde{\mathbf{V}} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \sigma_{\mathbf{Q},1} \\ \sigma_{\mathbf{Q},2} \\ \boldsymbol{\sigma}_{\mathbf{R}} \\ \boldsymbol{\sigma}_{\boldsymbol{\xi}} \\ \boldsymbol{\sigma}_{\mathbf{V}} \end{bmatrix} \\ \mathbf{A} = \left[\begin{array}{ccc|cc} \mathbf{0}_{5\times 1} & \mathbf{0}_{5\times 1} & \mathbf{0}_{5\times 3} & \mathbf{0}_{5\times 3} & \mathbf{0}_{5\times 3} \\ \hline \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} & -[\hat{\mathbf{R}}\hat{\mathbf{V}}]_\times & \mathbf{0}_{3\times 3} & \hat{\mathbf{R}} \\ g\hat{\mathbf{Q}}^\top \mathbf{e}_2 & -g\hat{\mathbf{Q}}^\top \mathbf{e}_1 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & -[\boldsymbol{\Omega}]_\times \end{array}\right] \\ \mathbf{C} = \begin{bmatrix} \mathbf{0}_{3\times 2} & \hat{\mathbf{R}}^\top [\mathring{\mathbf{p}}_1 - \hat{\boldsymbol{\xi}}]_\times & -\hat{\mathbf{R}}^\top & \mathbf{0}_{3\times 3} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{3\times 2} & \hat{\mathbf{R}}^\top [\mathring{\mathbf{p}}_N - \hat{\boldsymbol{\xi}}]_\times & -\hat{\mathbf{R}}^\top & \mathbf{0}_{3\times 3} \end{bmatrix} \end{cases} \tag{22}$$

From there the proposed observer is given by (10) and (16) with $\hat{\gamma}$ defined by (12) and with the innovation terms $\boldsymbol{\sigma}_{\boldsymbol{\xi}}$, $\boldsymbol{\sigma}_{\mathbf{R}}$, $\boldsymbol{\sigma}_{\mathbf{V}}$, $\sigma_{\mathbf{Q},1}$ and $\sigma_{\mathbf{Q},2}$ determined from the input $\mathbf{u}$ calculated according to (8) and (9) with the matrices $\bar{\mathbf{Q}}(t)$ and $\bar{\mathbf{V}}(t)$ chosen larger than some positive matrices.

Note that the third component $\sigma_{\mathbf{Q},3}$ of $\boldsymbol{\sigma}_{\mathbf{Q}}$ is uninvolved and is only required to be bounded for all time. For convenience we simply set it equal to zero.

IV. Observability and stability analysis

According to [11, Corollary 3.2] and Lemma 2, good conditioning of the solutions $\mathbf{P}(t)$ to the continuous Riccati equation (9) and exponential stability of the proposed observer rely on the uniform observability of the pair $(\mathbf{A}^\star(t), \mathbf{C}^\star(t))$ obtained by setting $\mathbf{x} = 0$ in the expressions of the matrices $\mathbf{A}$ and $\mathbf{C}$ derived previously. In view of (22) one has

$$\mathbf{A}^\star = \begin{bmatrix} \mathbf{0}_{5\times 1} & \mathbf{0}_{5\times 1} & \mathbf{0}_{5\times 3} & \mathbf{0}_{5\times 3} & \mathbf{0}_{5\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} & -[\mathbf{v}]_\times & \mathbf{0}_{3\times 3} & \mathbf{R} \\ g\mathbf{Q}^{\star\top}\mathbf{e}_2 & -g\mathbf{Q}^{\star\top}\mathbf{e}_1 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & -[\boldsymbol{\Omega}]_\times \end{bmatrix}$$
$$\mathbf{C}^\star = \begin{bmatrix} \mathbf{0}_{3\times 2} & \mathbf{R}^\top [\mathring{\mathbf{p}}_1 - \boldsymbol{\xi}]_\times & -\mathbf{R}^\top & \mathbf{0}_{3\times 3} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{3\times 2} & \mathbf{R}^\top [\mathring{\mathbf{p}}_N - \boldsymbol{\xi}]_\times & -\mathbf{R}^\top & \mathbf{0}_{3\times 3} \end{bmatrix} \tag{23}$$

with $\mathbf{Q}^\star \in \mathrm{SO}(3)$ satisfying $\mathbf{Q}^{\star\top}\mathbf{e}_3 = \boldsymbol{\gamma}$.

According to Lemma 1 the pair $(\mathbf{A}^\star, \mathbf{C}^\star)$ is uniformly observable if $\exists \delta, \mu > 0$ such that

$$\frac{1}{\delta}\int_t^{t+\delta} \mathbf{M}^\top(s)\mathbf{M}(s)ds \geq \mu \mathbf{I}_{11}, \ \forall t > 0 \tag{24}$$

with $\mathbf{M} \triangleq \begin{bmatrix} \mathbf{N}_0^\top & \mathbf{N}_1^\top & \mathbf{N}_2^\top \end{bmatrix}^\top$, $\mathbf{N}_0 \triangleq \mathbf{C}^\star$, $\mathbf{N}_1 \triangleq \mathbf{N}_0\mathbf{A}^\star + \dot{\mathbf{N}}_0$, $\mathbf{N}_2 \triangleq \mathbf{N}_1\mathbf{A}^\star + \dot{\mathbf{N}}_1$. The next step consists in specifying a more explicit condition guaranteeing (24).

For observability reason the following assumption is made.

Assumption 2 *Assume that $N \geq 3$ and that there exist 3 distinct landmarks with indices i_1, i_2, i_3 such that $\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_2}$ and $\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_3}$ are non-collinear.*

From here the main result of the paper is stated next.

Proposition 1 *Assume also that Assumptions 1 and 2 hold. Then, condition (24) is satisfied. Consequently, the pair $(\mathbf{A}^\star, \mathbf{C}^\star)$ given by (23) is uniformly observable and the equilibrium $(\hat{\boldsymbol{\xi}}, \hat{\mathbf{R}}, \hat{\mathbf{V}}, \hat{\mathbf{Q}}^\top\mathbf{e}_3) = (\boldsymbol{\xi}, \mathbf{R}, \mathbf{V}, \boldsymbol{\gamma})$ of the proposed Riccati observer is (locally) exponentially stable.*

Proof: One verifies that

$$\begin{cases} \mathbf{N}_0 = \begin{bmatrix} \mathbf{0}_{3N\times 2} & \mathbf{N}_{01} & \mathbf{0}_{3N\times 3} \end{bmatrix} \\ \mathbf{N}_1 = \begin{bmatrix} \mathbf{0}_{3N\times 2} & \mathbf{N}_{11} & \mathbf{N}_{12} \end{bmatrix} \\ \mathbf{N}_2 = \begin{bmatrix} \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} \end{cases}$$

with

$$\begin{cases} \mathbf{N}_{01} \triangleq \begin{bmatrix} \mathbf{R}^\top [\mathring{\mathbf{p}}_1 - \boldsymbol{\xi}]_\times & -\mathbf{R}^\top \\ \vdots & \vdots \\ \mathbf{R}^\top [\mathring{\mathbf{p}}_N - \boldsymbol{\xi}]_\times & -\mathbf{R}^\top \end{bmatrix} \\ \mathbf{N}_{11} \triangleq \begin{bmatrix} -[\boldsymbol{\Omega}]_\times \mathbf{R}^\top [\mathring{\mathbf{p}}_1 - \boldsymbol{\xi}]_\times & [\boldsymbol{\Omega}]_\times \mathbf{R}^\top \\ \vdots & \vdots \\ -[\boldsymbol{\Omega}]_\times \mathbf{R}^\top [\mathring{\mathbf{p}}_N - \boldsymbol{\xi}]_\times & [\boldsymbol{\Omega}]_\times \mathbf{R}^\top \end{bmatrix} \\ \mathbf{N}_{12} \triangleq \begin{bmatrix} -\mathbf{I}_3 \\ \vdots \\ -\mathbf{I}_3 \end{bmatrix} \\ \mathbf{N}_{21} \triangleq \begin{bmatrix} -g\mathbf{Q}^{\star\top}\mathbf{e}_2 & g\mathbf{Q}^{\star\top}\mathbf{e}_1 \\ \vdots & \vdots \\ -g\mathbf{Q}^{\star\top}\mathbf{e}_2 & g\mathbf{Q}^{\star\top}\mathbf{e}_1 \end{bmatrix} \\ \mathbf{N}_{22} \triangleq \begin{bmatrix} \boldsymbol{\Delta}_{\boldsymbol{\Omega}} \mathbf{R}^\top [\mathring{\mathbf{p}}_1 - \boldsymbol{\xi}]_\times & -\boldsymbol{\Delta}_{\boldsymbol{\Omega}}\mathbf{R}^\top & 2[\boldsymbol{\Omega}]_\times \\ \vdots & \vdots & \vdots \\ \boldsymbol{\Delta}_{\boldsymbol{\Omega}} \mathbf{R}^\top [\mathring{\mathbf{p}}_N - \boldsymbol{\xi}]_\times & -\boldsymbol{\Delta}_{\boldsymbol{\Omega}}\mathbf{R}^\top & 2[\boldsymbol{\Omega}]_\times \end{bmatrix} \end{cases} \tag{25}$$

and $\boldsymbol{\Delta}_{\boldsymbol{\Omega}} \triangleq -[\dot{\boldsymbol{\Omega}}]_\times + [\boldsymbol{\Omega}]_\times^2$. Subsequently, one deduces

$$\begin{cases} \mathbf{N}_0^\top \mathbf{N}_0 = \left[\begin{array}{c|cc} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 6} & \mathbf{0}_{2\times 3} \\ \hline \mathbf{0}_{6\times 2} & \mathbf{N}_{01}^\top \mathbf{N}_{01} & \mathbf{0}_{6\times 3} \\ \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 3} \end{array}\right] \\ \mathbf{N}_1^\top \mathbf{N}_1 = \left[\begin{array}{c|cc} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 6} & \mathbf{0}_{2\times 3} \\ \hline \mathbf{0}_{6\times 2} & \mathbf{N}_{11}^\top \mathbf{N}_{11} & \mathbf{N}_{11}^\top \mathbf{N}_{12} \\ \mathbf{0}_{3\times 2} & \mathbf{N}_{12}^\top \mathbf{N}_{11} & \mathbf{N}_{12}^\top \mathbf{N}_{12} \end{array}\right] \\ \mathbf{N}_2^\top \mathbf{N}_2 = \left[\begin{array}{c|c} \mathbf{N}_{21}^\top \mathbf{N}_{21} & \mathbf{N}_{21}^\top \mathbf{N}_{22} \\ \hline \mathbf{N}_{22}^\top \mathbf{N}_{21} & \mathbf{N}_{22}^\top \mathbf{N}_{22} \end{array}\right] \end{cases}$$

where $\mathbf{N}_{12}^\top \mathbf{N}_{12} = N\mathbf{I}_3$ and $\mathbf{N}_{21}^\top \mathbf{N}_{21} = Ng^2\mathbf{I}_2$, which are positive definite. According to Lemma 3 (see Appendix A), the resulting matrix of $\mathbf{N}_{01}^\top \mathbf{N}_{01}$ is also positive definite (i.e. $\exists \alpha_0 > 0$ such that $\mathbf{N}_{01}^\top \mathbf{N}_{01} \geq \alpha_0 \mathbf{I}_6$) under Assumption 2. From here, the following matrix

$$\boldsymbol{\Delta}_1 \triangleq \begin{bmatrix} \mathbf{N}_{01}^\top \mathbf{N}_{01} & \mathbf{0}_{6\times 3} \\ \mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 3} \end{bmatrix} + \begin{bmatrix} \mathbf{N}_{11}^\top \mathbf{N}_{11} & \mathbf{N}_{11}^\top \mathbf{N}_{12} \\ \mathbf{N}_{12}^\top \mathbf{N}_{11} & \mathbf{N}_{12}^\top \mathbf{N}_{12} \end{bmatrix}$$

is also positive definite (i.e. $\exists \alpha_1 > 0$ such that $\boldsymbol{\Delta}_1 \geq \alpha_1 \mathbf{I}_9$) according to Lemma 4 (see Appendix A). Using again Lemma 4, one deduces that there exists $\alpha_2 > 0$ such that

$$\begin{aligned} \mathbf{M}^\top \mathbf{M} &= \mathbf{N}_0^\top \mathbf{N}_0 + \mathbf{N}_1^\top \mathbf{N}_1 + \mathbf{N}_2^\top \mathbf{N}_2 \\ &= \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{9\times 2} & \boldsymbol{\Delta}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{N}_{21}^\top \mathbf{N}_{21} & \mathbf{N}_{21}^\top \mathbf{N}_{22} \\ \mathbf{N}_{22}^\top \mathbf{N}_{21} & \mathbf{N}_{22}^\top \mathbf{N}_{22} \end{bmatrix} \\ &\geq \alpha_2 \mathbf{I}_{11} \end{aligned}$$

which in turn guarantees the satisfaction of the uniform observability condition (24). The remainder of the proof follows. ■

V. SIMULATION RESULTS

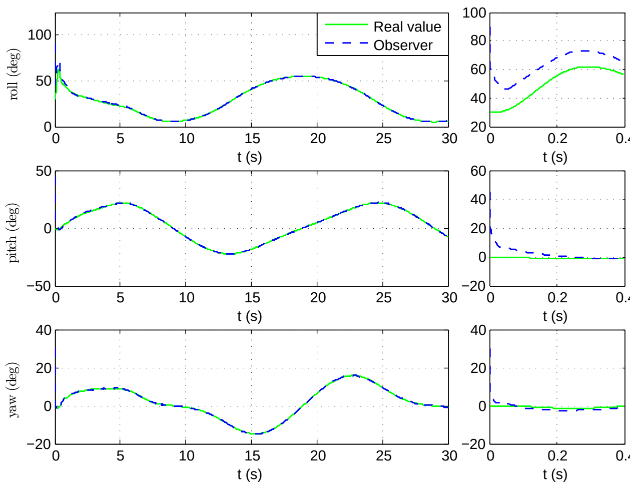

Fig. 1. Estimated and real attitude represented by roll, pitch and yaw Euler angles (deg) versus time (s) (LEFT) and zooms (RIGHT)

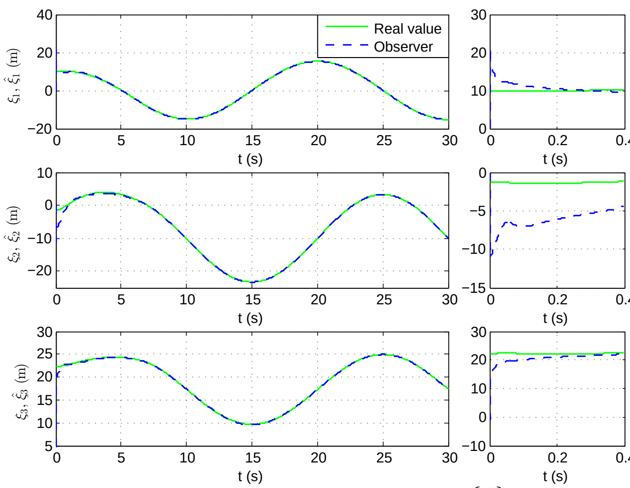

Fig. 2. Estimated and real position expressed in $\{\mathcal{I}\}$ versus time (s) (LEFT) and zooms (RIGHT)

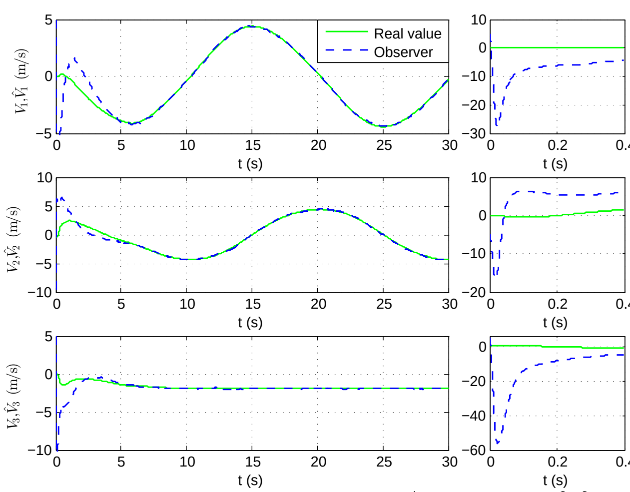

Fig. 3. Estimated and real linear velocity (m/s) expressed in $\{\mathcal{B}\}$ versus time (s) (LEFT) and zooms (RIGHT)

In this section the performance of the proposed observer is illustrated by simulation results. The reported simulations are conducted on a model of a ducted-fan aerial drone, which is controlled to follow a circular reference trajectory

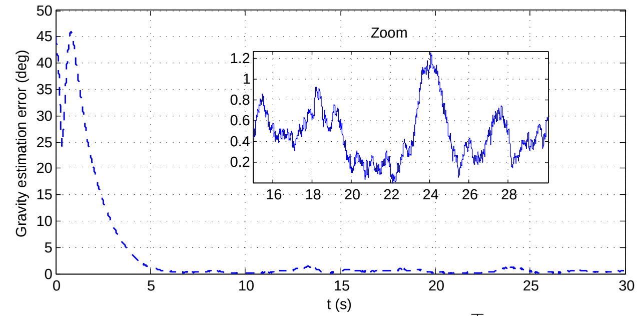

Fig. 4. Gravity estimation error given by $\arccos(\boldsymbol{\gamma}^\top\hat{\boldsymbol{\gamma}})$ versus time (s)

given by $\boldsymbol{\xi}_r = [15\cos(\alpha t), \frac{15\sqrt{3}}{2}\sin(\alpha t) - 10, \frac{15}{2}\sin(\alpha t) + 10\sqrt{3}]^\top(\mathrm{m})$, with $\alpha = \pi/10$. Due to aerodynamic and centripetal forces acting on the vehicle, its attitude and linear acceleration vary quickly and in large proportions. We assume that the drone is equipped with a stereo camera that provides the position measurements expressed in $\{\mathcal{B}\}$ of three known landmarks, whose coordinates in $\{\mathcal{I}\}$ are respectively given by $\mathring{\mathbf{p}}_1 = [2, -\sqrt{3}/2, -1/2]^\top(\mathrm{m})$, $\mathring{\mathbf{p}}_2 = [-3, \sqrt{3}, 1]^\top(\mathrm{m})$, $\mathring{\mathbf{p}}_3 = [-3, -\sqrt{3}/2, -1/2]^\top(\mathrm{m})$. The gravity direction in $\{\mathcal{I}\}$ is $\mathring{\boldsymbol{\gamma}} = [0, -1/2, \sqrt{3}/2]^\top$.

The observer is tuned analogously to Kalman-Bucy filters where the matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{Q}}^{-1}$ are interpreted as covariance matrices of the additive noise on the system state and output respectively. The following parameters are chosen for the presented simulations: $\mathbf{P}(0) = \mathrm{diag}(\mathbf{I}_2, \mathbf{I}_3, 100^2\mathbf{I}_3, 100^2\mathbf{I}_3)$, $\bar{\mathbf{Q}} = 100\mathbf{I}_9$ and $\bar{\mathbf{V}} = \mathrm{diag}(0.01\mathbf{I}_2, 0.01\mathbf{I}_3, 0.01\mathbf{I}_3, \mathbf{I}_3)$. The measurements (i.e. angular velocities, specific acceleration and position measurements $\mathbf{p}_i$ of the landmarks) are corrupted by Gaussian zero-mean additive noises with standard deviations reflecting the above choices of $\bar{\mathbf{Q}}$ $[0.1(\mathrm{m})$ for $\mathbf{p}_i]$ and of $\bar{\mathbf{V}}$ $[0.1(\mathrm{rad/s})$ for $\boldsymbol{\Omega}$ and $1(\mathrm{m/s^2})$ for $\mathbf{a}_{\mathcal{B}}]$.

The following large initial estimation errors are considered: $\tilde{\mathbf{q}}(0) = [0.70, 0.56, 0.43, -0.09]$ (corresponding to errors in roll, pitch, yaw Euler angles of $90(\mathrm{deg})$, $45(\mathrm{deg})$, $30(\mathrm{deg})$, respectively), $\tilde{\boldsymbol{\xi}}(0) = [10, -1.34, 22.3]^\top(\mathrm{m})$, $\tilde{\mathbf{V}}(0) = [-5, 5, -5]^\top(\mathrm{m/s})$, $\arccos(\boldsymbol{\gamma}^\top(0)\hat{\boldsymbol{\gamma}}(0)) = \frac{\pi}{4}(\mathrm{rad})$.

The time evolutions of the estimated and real attitude (represented by Euler angles) along with the estimated and real position and linear velocity are shown in Figs. 1, 2 and 3, respectively. One easily observes that the estimate trajectories converge near to the real ones after a short transition period despite large initial estimation errors and significant measurement noise. Fig. 4 shows that the gravity direction estimation error (represented by the angle between γ and $\hat{\gamma}$) also quickly converges near to zero without much oscillations. In overall we find the performance of the proposed observer quite satisfactory.

VI. CONCLUSIONS

The problem of pose estimation from landmark position measurements in combination with IMU measurements has been addressed by a nonlinear Riccati observer that is derived from a recent Riccati observer design framework. The distinguished feature w.r.t. related works on the topic is the non-requirement of linear velocity measurements and of prior knowledge on the gravity direction in the inertial frame. The proposed observer is supported by rigourous observability and stability analysis together with convincing simulation results. Owning to its simplicity in terms of sensor suite and algorithm, the proposed solution provides a natural plug-and-play capability for many robotic applications.

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Appendix

A. Technical lemmas

Lemma 3 *Under Assumption 2, there exists a positive number $\alpha_0 > 0$ such that $\boldsymbol{\Delta}_0 \triangleq \mathbf{N}_{01}^\top \mathbf{N}_{01} \geq \alpha_0 \mathbf{I}_6$, with $\mathbf{N}_{01}$ defined in* (25).

Proof: One verifies that

$$\boldsymbol{\Delta}_0 = \begin{bmatrix} -\sum_i [\mathring{\mathbf{p}}_i - \boldsymbol{\xi}]_\times^2 & \sum_i [\mathring{\mathbf{p}}_i - \boldsymbol{\xi}]_\times \\ -\sum_i [\mathring{\mathbf{p}}_i - \boldsymbol{\xi}]_\times & N\mathbf{I}_3 \end{bmatrix}$$

Since $\boldsymbol{\Delta}_0 = \mathbf{N}_{01}^\top \mathbf{N}_{01}$, it is symmetric and positive semi-definite (i.e. $\boldsymbol{\Delta}_0 = \boldsymbol{\Delta}_0^\top \geq 0$). Thus, one only needs to prove that its determinant is greater than a positive number. One verifies that

$$\begin{aligned} \det(\boldsymbol{\Delta}_0) &= \det\Big(-N\textstyle\sum_i [\mathring{\mathbf{p}}_i - \boldsymbol{\xi}]_\times^2 + (\sum_i [\mathring{\mathbf{p}}_i - \boldsymbol{\xi}]_\times)^2\Big) \\ &= \det\big(-\textstyle\sum_i \sum_{j<i} [\mathring{\mathbf{p}}_i - \mathring{\mathbf{p}}_j]_\times^2\big) \\ &= \det(\bar{\boldsymbol{\Delta}}_0) \end{aligned}$$

where $\bar{\boldsymbol{\Delta}}_0 \triangleq -\sum_i \sum_{j<i} [\mathring{\mathbf{p}}_i - \mathring{\mathbf{p}}_j]_\times^2$ is a constant symmetric positive semi-definite matrix. From there, one only needs to prove that $\bar{\boldsymbol{\Delta}}_0$ is positive definite. According to Assumption 2 that ensures the existence of 3 distinct landmarks i_1, i_2, i_3 such that $\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_2}$ and $\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_3}$ are non-collinear, one then deduces

$$\bar{\boldsymbol{\Delta}}_0 \geq |\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_2}|^2 \boldsymbol{\pi}_{\frac{\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_2}}{|\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_2}|}} + |\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_3}|^2 \boldsymbol{\pi}_{\frac{\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_3}}{|\mathring{\mathbf{p}}_{i_1} - \mathring{\mathbf{p}}_{i_3}|}} > \mathbf{0}$$

which allows one to conclude the proof. ■

Lemma 4 *Consider the bounded matrices $\mathbf{A} \in \mathbf{R}^{l\times n}$, $\mathbf{B} \in \mathbf{R}^{l\times m}$, $\mathbf{C} \in \mathbf{R}^{n\times n}$. Assume that there exist some positive numbers $\alpha_B, \alpha_C > 0$ such that $\mathbf{B}^\top\mathbf{B} \geq \alpha_B \mathbf{I}_m$, $\mathbf{C} \geq \alpha_C \mathbf{I}_n$. Then, there exists a positive number α such that*

$$\mathbf{D}_1 \triangleq \begin{bmatrix} \mathbf{C} & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^\top\mathbf{A} & \mathbf{A}^\top\mathbf{B} \\ \mathbf{B}^\top\mathbf{A} & \mathbf{B}^\top\mathbf{B} \end{bmatrix} \geq \alpha \mathbf{I}_{n+m} \quad (26)$$

and

$$\mathbf{D}_2 \triangleq \begin{bmatrix} \mathbf{0}_{m\times m} & \mathbf{0}_{m\times n} \\ \mathbf{0}_{n\times m} & \mathbf{C} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^\top\mathbf{B} & \mathbf{B}^\top\mathbf{A} \\ \mathbf{A}^\top\mathbf{B} & \mathbf{A}^\top\mathbf{A} \end{bmatrix} \geq \alpha \mathbf{I}_{n+m} \quad (27)$$

Proof: One only needs to prove (26). The proof for (27) proceeds analogously. Let γ_A denote the upper-bound of $\mathbf{A}$. One rewrites

$$\mathbf{D}_1 = \begin{bmatrix} \mathbf{C} - \beta_1 \mathbf{A}^\top\mathbf{A} & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & \frac{\beta_1}{1+\beta_1}\mathbf{B}^\top\mathbf{B} \end{bmatrix} + \begin{bmatrix} (1+\beta_1)\mathbf{A}^\top\mathbf{A} & \mathbf{A}^\top\mathbf{B} \\ \mathbf{B}^\top\mathbf{A} & \frac{1}{1+\beta_1}\mathbf{B}^\top\mathbf{B} \end{bmatrix} \quad (28)$$

Then, using the fact that the second matrix on the right-hand side of (28) is semi-positive definite and choosing $\beta_1 \triangleq \frac{\alpha_C}{2\gamma_A^2}$, one deduces

$$\begin{aligned} \mathbf{D}_1 &\geq \begin{bmatrix} \mathbf{C} - \beta_1 \mathbf{A}^\top\mathbf{A} & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & \frac{\beta_1}{1+\beta_1}\mathbf{B}^\top\mathbf{B} \end{bmatrix} \geq \begin{bmatrix} \frac{\alpha_C}{2}\mathbf{I}_n & \mathbf{0}_{n\times m} \\ \mathbf{0}_{m\times n} & \frac{\beta_1\alpha_B}{1+\beta_1}\mathbf{I}_m \end{bmatrix} \\ &\geq \alpha \mathbf{I}_{n+m} \end{aligned}$$

with $\alpha \triangleq \min(\frac{\alpha_C}{2}, \frac{\beta_1\alpha_B}{1+\beta_1})$. ■