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ON THE NON-DETECTABILITY OF SPIKED LARGE RANDOM TENSORS

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ABSTRACT
This paper addresses the detection of a low rank high-dimensional tensor corrupted by an additive complex Gaussian noise. In the asymptotic regime where all the dimensions of the tensor converge towards $+\infty$ at the same rate, existing results devoted to rank 1 tensors are extended. It is proved that if a certain parameter depending explicitly on the low rank tensor is below a threshold, then the null hypothesis and the presence of the low rank tensor are undistinguishable hypotheses in the sense that no test performs better than a random choice.

1. INTRODUCTION

The problem of testing whether an observed $n_1 \times n_2 \times \ldots \times n_d$ matrix $\mathbf{Y}$ is either a zero-mean independent identically distributed Gaussian random matrix $\mathbf{Z}$ with variance $\frac{1}{n_d}$, or $\mathbf{X}_0 + \mathbf{Z}$ (where $\mathbf{X}_0$ is a low rank matrix: a useful signal, called also spike) is a fundamental problem arising in numerous applications such as the detection of low-rank multivariate signals or the Gaussian hidden clique problem. When the two dimensions $n_1$, $n_2$ converge towards $+\infty$ at the same rate, the rank of $\mathbf{X}_0$ remaining fixed, the context is this of the so-called additive spiked large random matrix models. Various results on the singular values of $\mathbf{X}_0 + \mathbf{Z}$ have been established; in particular it is possible to show that the Generalized Likelihood Ratio Test (GLRT) is consistent (i.e. the probability of false alarm and the probability of missed detection both converge towards 0 when $n_1$, $n_2$ converge towards $+\infty$ in such a way that $n_1/n_2 \rightarrow c > 0$) if and only if the largest singular value of $\mathbf{X}_0$ is above the threshold $\sqrt{\frac{d}{n_1 n_2}}$ (see e.g. [12], [3], [2]).

In a number of real life problems, the observation is not a matrix, but a tensor $\mathbf{Y}$ of order $d \geq 3$, i.e. a $d$-dimensional array $\mathbf{Y} = Y_{i_1,i_2,\ldots,i_d}$ where for each $k = 1, \ldots, d$, $i_k \in [1, \ldots, n_k]$. In this context, the generalization of the above matrix hypothesis testing problem becomes: test that the observed order $d \geq 3$ tensor is either a zero-mean independent identically distributed Gaussian random tensor $\mathbf{Z}$, or the sum of $\mathbf{Z}$ and a low rank deterministic tensor $\mathbf{X}_0 = \sum_{i=1}^r \lambda_i \mathbf{x}_0^{(1)} \otimes \mathbf{x}_0^{(2)} \otimes \ldots \otimes \mathbf{x}_0^{(d)}$ (1)

where $r$ is the rank of $\mathbf{X}_0$. Here $(\lambda_i)_{i=1,\ldots,r}$ are strictly positive real numbers, and for each $i = 1, \ldots, r$ and $k = 1, \ldots, d$, $\mathbf{x}_0^{(k)}$ is a $n_k \times 1$ unit norm vector. Recent works (see e.g. [8], [11], [10], [14]) addressed the detection/estimation of $\mathbf{X}_0$ when $r$ is reduced to 1 and when the dimensions $n_1, \ldots, n_d$ converge towards $+\infty$ at the same rate. We also mention that [8] and [14] only considered the case where the rank 1 tensor $\mathbf{X}_0$ is symmetric, that is: $n_1 = n_2 = \ldots = n_d$ and the $d$ vectors $(\mathbf{x}_0^{(k,i)})_{i=1,\ldots,n_d}$ are equal. Since the concept of singular value decomposition cannot be extended to tensors, ad hoc statistical strategies have been considered to prove the (non-)existence of consistent tests: [11] and [14] ($r = 1$) established that if $\lambda_1$ is larger than a certain upper bound, then consistent detection of $\mathbf{X}_0$ is possible. In the other direction, [10] [14] (again, $r = 1$) proved that if $\lambda_1$ is less than a certain lower bound (which is strictly less than the upper bound), then $\mathbf{X}_0$ is non-detectable in the sense that any test behaves as a random choice between the two hypotheses. This is a remarkable phenomenon because such a behaviour is not observed in the matrix case ($d = 2$); indeed, in this context, if the largest eigenvalue of $\mathbf{X}_0$ is below $\sqrt{d/2}$, it is proved in [13] ($r = 1$) that there exist statistical tests having a better performance than a random choice - a result that [10,14] obtained a different way.

The replica method has been successfully considered [4,9]. In these contributions, the model does not match exactly ours since 1) the spike is assumed symmetric, i.e. $\mathbf{X}_0 = \sum_{i=1}^r \lambda_i \mathbf{x}_0^{(i)} \otimes \mathbf{x}_0^{(i)}$ and 2) the rows of the matrix $(\mathbf{x}_0^{(1)}, \ldots, \mathbf{x}_0^{(r)})$ are random i.i.d. with a known distribution (the prior). When $r = 1$, and the prior is of the Rademacher type, the observed tensor follows the pure p-spin model [15]: in an illuminating contribution [4], a tight threshold when $d \geq 3$ is provided (above which consistent detection is possible and under which any detector performs as a random guess). The case $r \geq 1$ with a general prior is addressed in [9]: there, the estimation of the spike is considered rather than the detection; specifically, the asymptotic performance of the MMSE estimator is computed and an estimation threshold is deduced. This latter is rigorously proved when $r = 1$. The threshold is not explicit and intrinsically depends on the prior.

In the present contribution, we follow the methodology of [11] [10] [14] and extend it to the general case $r \geq 1$, though suboptimal (the thresholds provided are not tight in general), the machinery is much lighter than this of the replica method, it provides explicit bounds for the non-detectability and lastly allows one to deal with deterministic spikes. Precisely, we find out a simple sufficient condition on the spike $\mathbf{X}_0$ under which $\mathbf{X}_0$ is non-detectable. The problem of finding conditions under which the existence of a consistent detection is guaranteed is not addressed here.

2. MODEL, NOTATION, AND BACKGROUND

The order-$d$ tensors are complex-valued, and it is assumed that $n_1 = n_2 = \ldots = n = n$ in order to simplify the notations. The set $\otimes^d \mathbb{C}^n$ is a complex vector-space endowed with the standard scalar product

$$\forall \mathbf{X}, \mathbf{Y} \in \otimes^d \mathbb{C}^n \quad (\mathbf{X}, \mathbf{Y}) = \sum_{i_1,\ldots,i_d} \mathbf{X}_{i_1,\ldots,i_d} \mathbf{Y}_{i_1,\ldots,i_d}$$

and the Frobenius norm $\|\mathbf{X}\|_F = \sqrt{(\mathbf{X}, \mathbf{X})}$.
The spike ("the signal") is assumed to be a tensor of fixed rank \( r \) following (1). Along this contribution, \( n \) is large or, mathematically, \( n \to \infty \). We hence have for each \( n \) a set of \( n \times 1 \) vectors \( (x_{0,i}^{(k)})_{k=1,\ldots,d,i=1,\ldots,r} \). For each \( k = 1, \ldots, d \), we denote by \( X^{(k)} \) the \( n \times r \) matrix \( X^{(k)} = (x_{0,1}^{(k)}, \ldots, x_{0,n}^{(k)}) \). We impose a non-erratic asymptotic behavior of the spike, and specifically, as all the vectors \( x_{0,i}^{(k)} \in \mathbb{C}^{n \times 1} \) have unit norm, we suppose that for all \( i, j \), \( \langle x_{0,i}^{(k)}, x_{0,j}^{(k)} \rangle = \langle x_{0,i}^{(k)}, x_{0,j}^{(k)} \rangle \), converges as \( n \to \infty \). The rate of convergence is a technical aspect that is out of the scope of this contribution: we will simply assume that the matrices \( (x_{0,i}^{(k)}, x_{0,j}^{(k)})_{k=1,\ldots,d,i=1,\ldots,r} \) do not depend on \( n \). We define the SVD of \( X^{(k)} \) as \( U_k = (\mathbb{S}_k^T 0) V_k^T \) for \( U_k \) and \( V_k \) unitary matrices respectively of size \( n \times n \) and \( r \times r \) and \( \mathbb{S}_k \) diagonal matrix with non-negative entries on the diagonal, \( V_k \) and \( \mathbb{S}_k \) do not depend on \( n \) because \( x_{0,i}^{(k)} = V_k \Sigma_k \Sigma_k V_k \).

We denote by \( Z \) the noise tensor, and assume that its entries are \( \mathcal{N}(0,1/n) \) independent identically distributed complex Gaussian random variables.

In the following, we consider the alternative \( \mathcal{H}_0 : Y = Z \) versus \( \mathcal{H}_1 : Y = X_0 + Z \). We denote by \( p_{1,n}(y) \) the probability probability density of \( Y \) under \( \mathcal{H}_0 \) and \( p_{1,n}(y) \) the density of \( Y \) under \( \mathcal{H}_0 \). \( \Lambda(Y) = \frac{p_{1,n}(y)}{p_{0,n}(y)} \) is the likelihood ratio and we denote by \( E_r \) the expectation under \( \mathcal{H}_0 \). We now recall the fundamental information geometry results used in [10] in order to address the detection problem. The following properties are well known (see also [1] section 3):

- (i) If \( E_r [\Lambda(Y)^2] \) is bounded, then no consistent detection test exists.
- (ii) If moreover \( E_0 [\Lambda(Y)^2] = 1 + o(1) \), then the total variation distance between \( p_{0,n} \) and \( p_{1,n} \) converges towards 0, and no test performs better than a decision at random.

Therefore, the computation of the second-order moment of \( \Lambda(Y) \) under \( p_{1,n} \) may provide insights on the detection. We however notice that conditions (i) and (ii) are only sufficient. In particular, if \( \limsup_n E_r [\Lambda(Y)^2] = +\infty \), nothing can be inferred on the behavior of the detection problem when \( n \to +\infty \).

3. EXPRESSION OF THE SECOND-ORDER MOMENT.

The density of \( Z \), seen as a collection of \( n^d \) complex-valued random variables, is obviously \( p_{0,n}(z) = \kappa_n \exp(-n \|z\|^2_2) \) where \( \kappa_n = \frac{1}{\pi^{\frac{d}{2}}} \). On the one hand, we notice that the second-order moment approach is not suited to the deterministic model of the spike as presented previously. Indeed, in this case \( E_0 [\Lambda(Y)^2] \) has the simple expression \( \exp(2n \|X_0\|^2_2) \) and always diverges. On the other hand, the noise tensor shows an invariance property: if \( \Theta_1, \ldots, \Theta_d \) are unitary \( n \times n \) matrices, then the density of the mode products \( \Theta_1 \Theta_2 \cdots \Theta_d \cdot Z \) equals this of \( Z \). For \( d = 2 \), the notation \( (\Theta_1 \Theta_2 \cdots \Theta_d) \cdot Z \) simply means \( \Theta_1 Z \Theta_2 \) and for a general \( d \), \( (\Theta_1 \Theta_2 \cdots \Theta_d) \cdot Z \) is

\[
\sum_{l_1, l_2, \ldots, l_d} \left( \Theta_1 \right)_{l_1, l_1} \left( \Theta_2 \right)_{l_2, l_2} \cdots \left( \Theta_d \right)_{l_d, l_d} Z_{l_1, \ldots, l_d}.
\]

We hence modify the data according to the procedure: we pick i.i.d. complex Haar samples \( \Theta_1, \ldots, \Theta_d \) and consider the "new" data tensor defined as \( (\Theta_1 \Theta_2 \cdots \Theta_d) \cdot Y \). This does not affect the distribution of the noise, but this amounts to assume an artificial prior on the spike. Indeed, the vectors \( x_{0,i}^{(k)} \) are replaced by \( \Theta_1 x_{0,i}^{(k)} \). They are all uniformly distributed on the unit sphere of \( \mathbb{C}^n \) and for \( k \neq l \), vectors \( \Theta_1 x_{0,i}^{(k)} \) and \( \Theta_1 x_{0,j}^{(k)} \) are independent for each \( i, j \). However, vectors \( \Theta_1 x_{0,i}^{(k)} \) are not independent. In the following, the data and the noise tensors after this procedure are still denoted respectively by \( Y \) and \( Z \). This transformation of the spike is an extension of a trick used in Section III.C of [10].

We are now in position to give a closed-form expression of the second-order moment of \( \Lambda(Y) \). We have \( p_{1,n}(Y) = \mathbb{E}_X [p_{0,n}(Y - X)] \), where \( \mathbb{E}_X \) is the mathematical expectation over the distribution of the spike, or equivalently over the Haar matrices \( (\Theta_k)_{k=1,\ldots,d} \). It holds that

\[
\mathbb{E}_X [\Lambda(Y)^2] = \mathbb{E}_{X,X'} \left[ \exp \left( 2n \mathcal{R} \langle X, X' \rangle \right) \right] = \mathbb{E}_{X,X'} \left[ \exp \left( 2n \mathcal{R} \sum_{i,j=1}^r \lambda_i \lambda_j \prod_{k=1}^d \left( \langle \Theta_k, x_{0,i}^{(k)} \rangle \langle \Theta_k, x_{0,j}^{(k)} \rangle \right) \right) \right],
\]

where \( \mathbb{E}_{X,X'} \) is over independent copies \( X, X' \) of the spike associated respectively with \( (\Theta_k)_{k=1,\ldots,d} \) and \( (\Theta_k)_{k=1,\ldots,d} \). It stands for the real part. As \( \Theta_k \) and \( \Theta_k \) are Haar and independent, then \( (\Theta_k, \Theta_k) \) is also Haar distributed and \( \mathbb{E}_0 [\Lambda(Y)^2] = \mathbb{E}[\mathcal{R} \langle X, X \rangle] \), where the expectation is over the i.i.d. Haar matrices \( \Theta_1, \Theta_2, \ldots, \Theta_d \). and

\[
\eta = \limsup_{n \to \infty} E_r [\Lambda(Y)^2] = \limsup_{n \to \infty} E_r [\mathcal{R} \langle X, X \rangle] \lambda.
\]

4. EXTENDING KNOWN RESULTS

When \( r = 1 \), Montanari et al. [10] found a bound on the parameter \( \lambda_1 \) ensuring that \( E_0 [\Lambda(Y)^2] \) is bounded. In this case, \( \eta \) has a simple expression since \( \gamma = \lambda_2^2 \mathcal{R} \prod_{k=1}^d \lambda_k \) where the \( (\lambda_k)_{k=1,\ldots,d} \) are i.i.d. distributed as the first component of a uniform vector of the unit sphere of \( \mathbb{C}^n \). As in [10], we introduce

\[
\beta_2^{\text{bound}} = \min_{u \in (-1, 1)} \frac{1}{u^2} \log(1 - u^2).
\]

Adapting the result of the aforementioned article to the complex-circular context is straightforward:

**Theorem 1** (case \( r = 1 \) (Montanari et al.), Let \( \xi_1, \ldots, \xi_d \) be i.i.d. distributed as the first component of a vector uniformly distributed on the unit sphere of \( \mathbb{C}^n \)). If

\[
\sum_{k=1}^d \frac{1}{\beta_2^{\text{bound}}} \text{ then } E_r [\mathcal{R} \prod_{k=1}^d \lambda_k] \text{ is bounded; moreover, if } d > 2, \text{ the above expectation is } 1 + o(1).
\]
This non-obvious result may be used in order to derive a condition ensuring that hypotheses \(H_0\) and \(H_1\) are indistinguishable when \(r > 1\). In this respect, recall the expansion (2). Thanks to the Hölder inequality, \(E_0[\Lambda(Y)^2]\) is upper bounded by (see (2) for the definition of \(\xi(k,i,j)\))

\[
\prod_{i,j=1}^r \exp\left(2n\rho_{i,j}\lambda_i\lambda_j S_{k}^{d}(i,j)\right)
\]

for any non-negative numbers \(\rho_{i,j}\) such that \(\sum_{i,j} \frac{1}{\rho_{i,j}} = 1\). For fixed \(i,j\), we notice that the random variables \(\xi(k,i,j)\) verify the condition of Theorem 1. Any of the expectations in (5) are upper-bounded when \(n \to \infty\) provided that, for all \(i,j\), \(\rho_{i,j}\lambda_i < \frac{d}{2}(\xi(k,i,j)^2)\). Choosing eventually \(\rho_{i,j} = \frac{(\xi(k,i,j)^2)}{\lambda_i}\), we deduce

Theorem 2 (case \(r \geq 1\) extension of Theorem 1). \(E_0[\Lambda(Y)^2]\) is bounded. If moreover \(d > 2\), we have \(E_0[\Lambda(Y)^2] = 1 + o(1)\) and the hypotheses \(H_0\) and \(H_1\) are indistinguishable.

Remark 3. Due to the use of the Hölder inequality, Theorem 2 is suboptimum in general. The inequality is patently an equality when \(\forall k, i, j, x_{(k,i,j)} = x_{(k,i)}\). I.e. the spike has rank \(r = 1\) and amplitude \(\sum_{r=1}^\infty \lambda_i\).

5. A TIGHTER BOUND

The main result of our contribution is the following

Theorem 4 (case \(r \geq 1\)). We define \(\eta_{\text{max}}\) as

\[
\eta_{\text{max}} = \lambda \left(\bigotimes_{k=1}^d \langle x_{(k,i)} \rangle \right) \lambda.
\]

If \(\sqrt{\eta_{\text{max}}} > \sqrt{\frac{2}{d}}\) then, for \(d > 2\), \(E_0[\Lambda(Y)^2] = 1 + o(1)\).

Before providing elements of the proof of the above result, we may briefly justify why the bound in Theorem 4 is tighter than that of Theorem 2, whatever the choice of \(\lambda\). On the one hand, indeed, \(\bigotimes_{k=1}^d \langle x_{(k,i)} \rangle \lambda\) equals \(1\) and for any \(i \neq j\), \(\langle x_{(k,i)} \rangle \lambda\) equals \(1\). This proves that \(\bigotimes_{k=1}^d \langle x_{(k,i)} \rangle \lambda\) equals \(1\) and for any \(i \neq j\), \(\langle x_{(k,i)} \rangle \lambda\) equals \(1\). This proves that \(\bigotimes_{k=1}^d \langle x_{(k,i)} \rangle \lambda\) equals \(1\).

We provide the key elements of the proof of Theorem 4. Re- mind that we are looking for a condition on the spike under which \(E[\exp(2n\eta)]\) is bounded. Evidently, the divergence may occur only when \(\eta > 0\). We hence consider \(E_1 = E[\exp(2n\eta)]\) and \(E_2 = E[\exp(2n\eta)]\), and prove that under the condition \(\sqrt{\eta_{\text{max}}} < \sqrt{\frac{2}{d}}\), for a certain small enough \(\epsilon\), \(E_1 = o(1)(d \geq 2)\) and that \(E_2 = 1 + o(1)(d \geq 2)\).

The \(E_1\) term. It is clear that the boundedness of the integral \(E_1\) is achieved when \(\eta\) rarely deviates from \(0\). As remarked in [10], the natural machinery to consider to understand \(E_1\) is this of the Large Deviation Principle (LDP). In essence, if \(\eta\) follows the LDP with rate \(1\), there can be found a certain non-negative function called Good Rate Function (GRF) \(I_\eta\) such that for any Borel set \(A\) of \(R^n\),

\[
\frac{1}{n} \log E[\exp(n\eta)] \to \sup_{\xi \in \mathcal{M}_n} \left(\eta_\text{max} + \frac{d}{2}u^2 \log(1 - u^2)\right) + \delta
\]

for \(n\) large enough, where \(\xi = (\xi/\max)^{1/d}\). Recalling (4) and choosing \(\delta\) small enough, we deduce that the condition \(\eta_{\text{max}} < \frac{d}{2}(\xi(k,i,j)^2)\) implies that \(E_1 \to 0\). This holds for any order \(d \geq 2\).
The $E_2$ term. The Varadhan lemma may be invoked: but its conclusion, namely $\frac{1}{2} \log E_2 \to 0$, says nothing on the boundedness of $E_2$. We have, however

$$E_2 = \int_0^\infty \Pr(\exp(2nu) \leq t \text{ and } \eta \leq \epsilon) \, dt$$

$$= \int_0^\infty \Pr(\eta \geq u \text{ and } \eta \leq \epsilon) \, 2n \exp(2nu) \, du + \int_0^\infty \Pr(\eta \geq u \text{ and } \eta \leq \epsilon) \, 2n \exp(2nu) \, du$$

$$\leq \Pr(\eta \leq \epsilon) + \int_0^\infty \Pr(\eta \geq u) \, 2n \exp(2nu) \, du.$$

A weak consequence of the LDP on $\eta$ is the concentration of $\eta$ around 0, namely $\Pr(\eta \leq \epsilon) = 1 - \Pr(\eta > \epsilon) = 1 - o(1)$. We recall the expanded expression for $\eta$: see (2). Notice that $\eta \geq u$ implies that at least one of the $r^2$ terms of this expansion is at least equal to $\frac{r^2}{2n \lambda}$. By the union bound, and the fact that $\Pr(\prod_{k=1}^d \xi_{k}^{(i,j)} \leq \prod_{k=1}^d \xi_{k}^{0(i,j)})$ we deduce that $\Pr(\eta \leq u) \leq \sum_{i,j=1}^r \Pr(\prod_{k=1}^d \xi_{k}^{(i,j)} \geq \frac{1}{r^2 \lambda} \xi_{k}^{0(i,j)})$. Invoking again the union bound and noticing that for fixed $i,j$, $(\xi_{k}^{(i,j)})_{k=1,\ldots,d}$ have the same distribution, we deduce that

$$\Pr(\eta \geq u) \leq d \sum_{i,j=1}^r \Pr\left(\left|\xi_{k}^{(i,j)}\right| \geq \left(\frac{u}{r^2 \lambda \lambda_j}\right)^{1/d}\right).$$

Now, the density of $\xi_{k}^{(i,j)}$ is in polar coordinates $\frac{\pi}{2} \frac{1}{r} (1 - r^2)^{n-2}$ hence, choosing $\epsilon$ such that $\epsilon \leq r^2 \max_{i,j} \lambda_j \lambda_j$;

$$\Pr\left(\left|\xi_{k}^{(i,j)}\right| \geq \left(\frac{u}{r^2 \lambda \lambda_j}\right)^{1/d}\right) = \left(1 - \left(\frac{u}{r^2 \lambda \lambda_j}\right)^{2/d}\right)^{n-1}.$$

For any $0 \leq x \leq 1$, $\log(1 - x) \leq -x$, hence

$$E_2 \leq d \sum_{i,j=1}^r \int_0^\epsilon \exp\left(-(n-1) \left(\frac{u}{r^2 \lambda \lambda_j}\right)^{2/d} + 2nu\right) \, du.$$  

When $d > 2$, it is always possible to determine $\epsilon$ sufficiently small such that $-\epsilon \leq \epsilon^{2/d} + 2nu \leq -\frac{1}{2} \left(\frac{u}{r^2 \lambda \lambda_j}\right)^{2/d}$. This implies that, for such an $\epsilon$, we have

$$E_2 \leq d^2 r^2 n \left(\frac{2}{n-1}\right)^{d/2} \sum_{i,j=1}^r \lambda_j \lambda_j \int_0^\epsilon \exp(-\nu) d\nu.$$  

The r.h.s. of course $o(1)$ since $d > 2$.

Remark 7. The bound $\sqrt{\eta_{\text{max}}} < \sqrt{2/\beta^2}$ guarantees the non-detectability but it is not tight in general because, in order to study the asymptotics of $E_1$, we replaced the true GRF $I_0$ by the lower bound (8). Based on the loose inequality $\log \det(I_0 - \psi_k^* \psi_k) \leq \log \det(1 - \|\psi_k\|^2)$, (8) may not be very accurate. It is easy to check that the equality is reached in (8) when all the matrices $(\tilde{X}_{0,k})_{k=1,\ldots,d}$ are rank 1, i.e. if the rank of $X_0$ is equal to 1. Therefore, the lower bound (8) is all the better as the matrices $(\tilde{X}_{0,k})_{k=1,\ldots,d}$ are close to being rank 1 matrices. This suggests that, conversely, the bound (8) is likely to be loose when matrices $(\tilde{X}_{0,k})_{k=1,\ldots,d}$ are close to being orthogonal. As an illustration, we would like to consider experimental results. For a given configuration of the spike, we have chosen at random the matrices $\psi_k$ with $\|\psi_k\| \leq 1$. For each trial, we plot the points of coordinates $x = \psi_1, \ldots, \psi_k$ and $y = \sum_{k=1}^r \log \det(I_0 - \psi_k^* \psi_k)$ and we obtain a cloud the upper envelope of which is a representation of the true GRF of $\eta$; for comparison, we have plotted the graph of the function defined by the lower bound (8). We have chosen $r = 2$, $d = 3$, and two configurations of the spike: in the first one, all the matrices $\tilde{X}_k$ have orthogonal columns (top graph of 1), in the second one, the eigenvalues of $\tilde{X}_k^* \tilde{X}_k$ are the same for $k = 1, 2$ equal to 1.8 and 0.2 (bottom graph of 1).
7. REFERENCES


