NUMERICAL DYNAMICS OF GRANULAR MATERIALS

Jean Jacques MOREAU
Laboratoire de Mécanique et Génie Civil
Université Montpellier II/CNRS, Montpellier, France.

Abstract

Contact Dynamics is a numerical method, suitable for computing the dynamical motion of large collections of rigid bodies, with Coulomb friction taken into account in the event of contact. The principles of the method are sketched, in particular the way possible collisions or other nonsmooth features of the evolution are handled. As applications to granular dynamics, the construction of dry deposits and banks is simulated, in order to investigate their microstructure: force chains, geometrical anisotropy, Cauchy stress and some unexpected features of force transmission.

1 Introduction

1.1 Addressed problems

In diverse domains, computation methods are needed for the statics or the dynamics of collections of rigid bodies subject to the constraints of non-interpenetrability, with friction taken into account in the event of contact. This includes the dynamics of machines, in particular robots, the dynamics of masonry works submitted to transient actions (earthquakes, gusts of wind or impacts), animated computer graphics and numerical simulation in granular mechanics. Possibly, some variables are added in order to also account for a certain deformability of the bodies [13] without essentially changing the computational strategies. Of course, for the handling of non-interpenetrability and friction, much may be learned from the rich literature devoted to the treatment of contact between deformable media, discretized through finite elements; see e.g. [2][5][6][7][32].

The main part of the lecture at the Symposium consisted of projected animations, intended to show what sort of knowledge may be gained from the numerical simulation of granular processes. The present paper begins with an overview of the 'Contact Dynamics' time-stepping technique used in these simulations. More detailed accounts of the method, originated in [20], may be found in [25][26]. Finally, some drawings created in the course of computations are commented.

1.2 Analytical setting

The analytical description of a multibody system begins with the choice of a parametrization – at least a local one – of the set of possible configurations through an element $q := (q^1, \ldots, q^n)$ of $\mathbf{R}^n$. In terms of this parametrization, the geometric effect of the non-interpenetration of the system members, or of their confinement by external obstacles with prescribed motion, is assumed expressed by a finite set of inequalities

$$g_\alpha(t, q) \geq 0, \quad \alpha \in \{1, \ldots, \kappa\}, \tag{1}$$

where the functions $g_\alpha(t,q)$ are measures of the *gaps* between bodies, counted as negative if the configuration q at time t involves overlap. The event of equality in (1) for some value of α, corresponds to *contact* between a member $\mathcal{B}$ of the system and a body $\mathcal{B}'$ which may be another member of the system or an external obstacle.

Non-interpenetrability is a *unilateral constraint*; as always in Mechanics, its description cannot reduce to merely specifying the corresponding set of feasible states (such, in contrast, is the meaning of the word 'constraint' in Optimization and similar contexts). Some information must be added about the mechanical process through which the geometrical restriction is enforced. For instance the use of servomechanisms to secure (1) should result in other motions than those investigated in what follows.

Generically, let us conceive a *contact law* as a relationship involving the contact force $\mathcal{R}^\alpha$ experienced by $\mathcal{B}$ from $\mathcal{B}'$ at the contact point M_α and the local velocity $\mathcal{U}_\alpha$ of $\mathcal{B}$ relative to $\mathcal{B}'$, relationship which a priori depends on time and on the configuration attained by the system, say

$$\mathrm{law}_\alpha(t, q, \mathcal{U}_\alpha, \mathcal{R}^\alpha) = \mathrm{true}. \tag{2}$$

On numerical and analytical purposes, the definitions of the elements $\mathcal{U}_\alpha$ and $\mathcal{R}^\alpha$ of $\mathbf{E}^3$ are extended, at least locally, to configurations with nonzero $g_\alpha(t,q)$. Saying that (2) models a contact phenomenon involves that this law yields $\mathcal{R}^\alpha = 0$ whenever $g_\alpha(t,q) > 0$. Of course, more complex situations may be addressed, for instance with contact actions not reducing to single forces.

The system motion on a time-interval I, with origin t_0, is described by a function $t \mapsto q(t)$ of I into $\mathbf{R}^n$. Classical Kinematics requires of this function to be locally absolutely continuous, so that its derivative q' makes an element of $\mathcal{L}^1_{\mathrm{loc}}(I, \mathbf{R}^n)$ from which q may be retrieved through Lebesgue integration. The components of $q'(t)$ are commonly referred to as the velocity components of the system with regard to the parametrization in use.

When dealing with stereodynamical problems one may also characterize the system velocity by some otherwise defined set of components, making an element u of $\mathbf{R}^n$ whose dependence on t may generically be called the *velocity function.* Commonly, one attaches to each rigid body an orthonormal frame emanating from its mass center and principal with regard to body inertia. The components relative to these axes of the *spin vector* of the body are entered as elements of the column $u(t)$. Retrieving from the function u the evolution of q then rests on the integration of elementary kinematical relationships. The moments, about the same axes, of the forces experienced by the body are concomitantly entered at the corresponding places in the row of the covariant components of the forces acting on the system.

The classical framework of 'smooth' dynamics requires in turn of u to be locally absolutely continuous on the interval I. By Lagrange's method or any stereodynamical practice, one writes the dynamical equations as the following equality of elements of $\mathbf{R}^n$

$$A(t,q)\frac{du}{dt} = F(t,q,u) + \sum_\alpha r^\alpha, \tag{3}$$

to hold for almost every t. Here A denotes in general the $n \times n$ *inertia matrix*, diagonal and constant in (t,q) if u has been constructed according to the technique of mass centers and principal axes mentioned above. Expression F comprises certain standard terms, sometimes referred to as 'centrifugal' and 'gyroscopic', and also the covariant components of possible applied forces, supposed given as functions of time, the position of the system and its velocity. The elements $r^\alpha, \alpha \in \{1, 2, \ldots, \kappa\}$, are made of the covariant components of the respective contact forces.

In order to take the contact laws (2) into account, one has to connect the elements r^α and u of $\mathbf{R}^n$ with the vectors $\mathcal{R}^\alpha$ and $\mathcal{U}_\alpha$ of $\mathbf{E}^3$. By the kinematical analysis of the way the parametrization (q) has been constructed, one obtains an expression of the form

$$\mathcal{U}_\alpha = G_\alpha u + \mathcal{W}_\alpha, \tag{4}$$

where $G_\alpha : \mathbf{R}^n \to \mathbf{E}^3$ denotes a linear mapping, depending on t and q. In case the contact labelled α involves an external obstacle, the term $\mathcal{W}_\alpha \in \mathbf{E}^3$, a known function of t and q, accounts for the given motion of this obstacle. No attention is paid at this stage to the imagined motion preserving contact or not.

By applying the standard *virtual power* machinery, one correlatively obtains the following expression for the covariant components of the contact forces acting on the system at M_α (a pair of opposite forces if $\mathcal{B}'$ is a member of the system, a single force if it is an external obstacle)

$$r^\alpha = G^*_\alpha \, \mathcal{R}^\alpha, \tag{5}$$

with $G^*_\alpha : \mathbf{E}^3 \to \mathbf{R}^n$ denoting the *transpose* of G_α.

As said before, the elements associated with each α, such as G_α here or a common normal unit $\mathbf{n}^\alpha$, directed toward $\mathcal{B}$, invoked in the sequel, are conventionally extended in a smooth way to neighbouring configurations without requiring $g_\alpha = 0$.

1.3 Approximation

Relationships (1) to (5) convey all the retained information about the considered mechanical system. For the numerical treatment of evolution problems, it is easy to conceive time-stepping procedures of approximation to the differential equation (3), under the kinematical relationship connecting q an u. Through (4) and (5), the contact laws (2) are entered into the management of each time-step. The difficulty comes from the necessity of taking the non-interpenetration inequalities (1) also into account.

In the majority of the techniques described in literature and applied in commercial software packages, this is achieved through the classical trick of *penalizing* inequalities. Mechanically, this means that the strict non-interpenetrability of a pair of bodies is approximately replaced by elastic repulsion forces which become effective when the bodies come close to each other. Such a procedure amounts to approximate the joint conditions (1) and (2) by relationships connecting the same variables, but smoother. In principle, the resulting system of differential equations should be sufficiently regular for standard time-stepping procedures to apply. Similar interaction laws are used in the computer simulation of molecular motions, hence the name of Molecular Dynamics (abbr.: MD) commonly given to this approach.

The drawback is that the need of precision requires of the artificial repulsion laws to be very stiff. Numerical stability in integration then calls for the use of very short time-steps and frequently also for the introduction of some artificial damping or artificial increase of inertia. When treating dynamical applications, the effect of such alterations of the mechanical data may blur the picture. Significant simulations of loose, collisional, flows of granular materials have been obtained in that way, but when dense collections of bodies are concerned (masonry works or compact granulates) the method is mainly applied to *quasi-static evolutions* in which only a succession of equilibrium states is looked for.

For these reasons, other numerical strategies may be preferred in unilateral multibody dynamics.

In the line of the traditional stereodynamic practice are approaches qualified as *event-driven* (abbr: ED). Starting from a state in which certain contacts are in effect, one attempts to calculate the subsequent motion under the provisional assumption that these contacts persist and that their status – sticking or sliding – in regard to Coulomb law is invariant. Calculation is then the same as with bilateral constraints, but requires to watch the evolution of some indicators. If, for instance, the normal components of some contact forces take directions incompatible with unilaterality, one concludes that the subsequent motion has to be calculated otherwise. Finding the further status of contacts without having to try all combinations is a nontrivial matter. It is usually reduced to *Linear Complementarity Problems* [19][29][32][1].

The ED strategy becomes unpractical if the number of contacts in presence is too large. Some time-stepping integration procedures with preselected time-intervals have been developed instead [28][30]. The following one belongs to this class.

2 The 'Contact Dynamics' approach

2.1 Handling non-interpenetration in terms of velocity

For every label α, it has been agreed to extend, at least locally, the definition of contact descriptors such as the normal unit $\mathbf{n}^\alpha$, to configurations with nonzero g_α. Then put

$$\mathcal{K}_\alpha(t,q) := \begin{cases} \{\mathcal{V} \in \mathbf{E}^3 \mid \mathcal{V}.\mathbf{n}^\alpha \geq 0\} & \text{if} \quad g_\alpha(t,q) \leq 0 \\ \mathbf{E}^3 \quad \text{otherwise.} \end{cases}$$

If $g_\alpha(t,q) \geq 0$, this is the set of the values of $\mathcal{U}_\alpha^+$, the contact *right-velocity*, which are kinematically compatible with non-interpenetration.

The elements for the proof of the following may be found in [24]:

Lemma. *Let a motion be defined by a locally integrable velocity function $u : I \to \mathbf{R}^n$. If the corresponding expression of $\mathcal{U}_\alpha : I \to \mathbf{E}^3$, as it results from* (4), *satisfies $\mathcal{U}_\alpha \in \mathcal{K}_\alpha(t,q)$ for almost every t and if the non-interpenetration inequality* (1) *holds at the initial instant t_0, then it holds throughout I.*

The decisive move of the Contact Dynamics strategy consists in complementing the contact laws (2) so as to obtain relationships in the same variables, containing the same stipulations as before and additionally the following ones: • *in all cases* $\mathcal{U}_\alpha \in \mathcal{K}_\alpha$, • *if* $\mathcal{U}_\alpha \in$ interior $\mathcal{K}_\alpha$, *then* $\mathcal{R}^\alpha = 0$. In other words, these relationships secure the implications

$$g_\alpha(t,q) \leq 0 \;\Rightarrow\; \mathbf{n}^\alpha . \mathcal{U}_\alpha \geq 0 \tag{6}$$

$$\mathbf{n}^\alpha . \mathcal{U}_\alpha > 0 \;\Rightarrow\; \mathcal{R}^\alpha = 0. \tag{7}$$

We propose to say that a package of information concerning the possible contact labelled α, if it possesses these properties, is a *contact law of prospective type*. The underlying idea is that such a law does not properly govern the values of $\mathcal{U}_\alpha$ and $\mathcal{R}^\alpha$ at the actual instant, but their *limits on the right* of this instant, assumed to exist. In fact, if $\mathbf{n}^\alpha . \mathcal{U}_\alpha . > 0$, the concerned instant is followed by a contactless time-interval. Since $\mathcal{R}^\alpha$ must vanish over this interval, the same should be true for its right-limit.

Once all the contact laws (2) have been complemented in that way, the Lemma shows that condition (1) becomes redundant, provided it is satisfied at the initial instant. It then becomes easy to imagine time-stepping procedures for the approximation of the evolution problem.

The very structure of the Coulomb model of friction induces to organize the step computation as an *implicit* scheme, at least with regard to the velocity function u, i.e. it is the unknown value of this function *at the step end* which, as an approximation, is entered into the law. A contact law of the prospective type is adequate here, since step end values mimick right-side limits. The possibility of contacts to get loose at every instant is then managed automatically, without needing any analysis of complementarity conditions.

Each time-step makes a nonlinear problem whose computational cost grows rapidly with the number of contacts in presence. As a compensation for this cost, the method accepts much larger step-lengths than those required by the regularization techniques referred to in 1.3. Usually an iteration procedure à la Gauss-Seidel is applied. It amounts to treat cyclically a sequence of dynamical problems, each of them involving a single contact. In dense collections of bodies with slowly evolving contact list, computation efficiency may be greatly enhanced by taking as starting guess in iterations the contact forces calculated at the antecedent time step, whenever the same contact was already in effect.

The Contact Dynamics strategy may also be applied with other solvers used at each time step [15].

A single computation step is needed to check whether a given position of the system is that of a possible equilibrium. One just have to launch iterations with zero initial velocity; if the step-end velocity is found zero, the values obtained for the contact forces agree with equilibrium.

2.2 Nonsmoothness in time and collisions

The sudden occurrence at some instant t_c of a new contact, i.e. a *collision*, is expected to generate a velocity jump, the dynamics of which cannot be governed by the differential equation (3). Even in the absence of events of this sort, dry friction at a contact point has long been known, in some cases, to forbid the existence of any smooth solution beyond a catastrophic instant, on the left of which the contact forces, as well as the derivative u', may become unbounded [10]. This is a dynamic analogue to the *locking* situations familiarly met in the statics of frictional systems. Around year 1900, such a lack of solution for an apparently well set problem seemed inadmissible to P. Painlevé and induced him to question the very concept of a contact force. Hence the misleading denomination of 'Painlevé's paradox' which has been given to the observation. Today, one is accustomed to see models reaching the limits of their validity domain. Such is the case for smooth dynamics at a catastrophic instant of any sort, leading to enlarge the function space where u is looked for [20][30].

A readily available mathematically framework is that of $\mathbf{R}^n$-valued functions of *locally bounded variation* on the time-interval I; notation $\mathrm{lbv}(I, \mathbf{R}^n)$ [21]. With every u in this space, an $\mathbf{R}^n$-valued measure on I is classically associated that we shall denote by du: this is the *differential measure* (or Stieltjes measure) of u. In the smooth case, this measure admits the derivative u' as density function relative to dt (the Lebesgue measure on I). A natural generalization of (3) therefore is

$$A(t,q)\,du = F(t,q,u)\,dt + \sum_{\alpha} ds^{\alpha}, \tag{8}$$

where the $\mathbf{R}^n$-valued measures ds^α constitute the 'components', relative to the parametrization in use, of the respective contact impulses $d\mathcal{S}^\alpha$ which are $\mathbf{E}^3$-valued measures. This is a *measure differential equation.* Concerning the theoretical foundation of such an extension of Classical Dynamics, one may refer to [22].

On a time interval of smooth motion $d\mathcal{S}^\alpha = \mathcal{R}^\alpha\,dt$, while at the instant t_c of a collision, each measure $d\mathcal{S}^\alpha$ is expected to present an *atom* whose value is nothing else than the *percussion* vector of the traditional theory of shocks, say $\mathcal{P}^\alpha$. The latter is a priori unknown and the question arises whether it may be involved in some phenomenological relationship playing at instant t_c a role similar to that of (2) on intervals of smooth motion. The *laws of restitution* of Newton, Poisson or others amount to pragmatic tricks of this sort, but it is now clear [31][12] that the empirical coefficients involved in such laws cannot in general be identified a priori. Apart the simple case of the collision of two, otherwise free, nearly spherical objects, the outcome of a collision can be predicted with some precision only through a detailed analysis of the contact interaction which actually involves a certain amount of deformation of the concerned bodies. When so studied, the process takes place on a nonzero time interval, contact forces and accelerations exist in the proper sense, leaving smooth mechanics valid. Another classical approach consists of a multiple scaling technique, using some microtime variable in the analysis of the process.

A critical issue, when restitution laws are used, is the energy balance of the instant process. A theoretical framework securing the dissipativity required by the Second Law of Thermodynamics may be found in [9].

2.3 The CD handling of nonsmooth evolutions

Formally, at each time-step of a CD computation scheme, say $[t_\mathrm{i}, t_\mathrm{f}]$ ('i' as in *initial*, 'f' as in *final*), $t_\mathrm{f} = t_\mathrm{i} + h$, the algorithm is ready to face collisions. Throughout this interval, the elements A, G_α, $\mathcal{W}_\alpha$ are approximated by constants, namely the value they take at the 'midpoint' ($t_\mathrm{m} = t_\mathrm{i} + h/2, q_\mathrm{m} = q_\mathrm{i} + h\,u_\mathrm{i}/2$) and the element F by the value it takes at $(t_\mathrm{m}, q_\mathrm{m}, u_\mathrm{i})$. It is also at $(t_\mathrm{m}, q_\mathrm{m})$ that inequalities (1) are checked in order to determine the set J_m of the labels α to be treated as effective. By integrating both members of (3) over the interval, *or both members of* (8) *as well*, one obtains

$$u_{\rm f} = u_{\rm i} + h\, A^{-1} F + A^{-1} \sum_{\alpha} p^{\alpha}, \tag{9}$$

where the quantities p^{α} are the covariant components of the contact *impulses*, i.e. the integrals of the ds^{α}. The relationship that (2), (4), (5) establish between r^{α} and u, does not commute in general with time integration because contact laws are nonlinear. When the motion one intends to approximate is smooth, the variation of r^{α} and u over the time-step may be small enough for the decision of connecting p^{α} with the *final* velocity $u_{\rm f}$ by the said relationship to merely reflect the choice made in 2.1 of a time-stepping strategy of the *implicit* type.

But if some collisions occur in $[t_{\rm i}, t_{\rm f}]$, i.e. $J_{\rm m}$ includes labels it did not contain at the antecedent step, large variations of u are expected. The designation of any value of u to be connected with p^{α}, constitutes an assertion about the collisional process, the phenomenological quality of which has to be checked in regard to physical circumstances.

To $u_{\rm f}$ corresponds the final value $\mathcal{U}_{\alpha {\rm f}}$ of the local velocity. The above choice amounts to connect with it, through the admitted contact law, the total local impulse on $[t_{\rm i}, t_{\rm f}]$. If this interval contains a single collisional instant $t_{\rm c}$, the initial and final values of $\mathcal{U}_{\alpha}$ may be viewed as approximants of the left and right limits $\mathcal{U}_{\alpha}^{-}(t_{\rm c})$ and $\mathcal{U}_{\alpha}^{+}(t_{\rm c})$.

Therefore, the collisions met in the course of the above computation are treated according to the following rule: *for each contact α present at instant $t_{\rm c}$, the contact percussion $\mathcal{P}^{\alpha}$ is related, through the admitted contact law, to the right limit $\mathcal{U}_{\alpha}^{+}(t_{\rm c})$.*

Contact laws here are assumed of the *prospective type.* In view of (6) and (7), this yields the implication $\mathcal{P}^{\alpha} \neq 0 \;\Rightarrow\; \mathbf{n}^{\alpha}.\,\mathcal{U}_{\alpha} = 0$, expressing that the collision is *soft*, i.e. *completely inelastic.* But the possibility $\mathbf{n}^{\alpha}.\,\mathcal{U}_{\alpha} > 0$ with $\mathcal{P}^{\alpha} = 0$ is left open. This model of soft collision in multicontact systems thus improves on the mere assumption of zero Newton restitution coefficient, because all the contacts present at instant $t_{\rm c}$ are involved together in the dynamics.

An extension of the preceding enables CD algorithms to produce bouncy collisions: it consists in connecting, through the admitted contact law, the total local impulse with some *weighted mean* of $\mathcal{U}_{\alpha {\rm i}}$ and $\mathcal{U}_{\alpha {\rm f}}$ [20][25][26]. The choice of the weighting coefficients (possibly distinct for the normal and tangential components of velocities) directly determines some restitution rules, improving as before on those of Newton.

Anyway, such laws of restitution do not elude the criticism made in 2.2. Their advantage is only that their implementation in a CD algorithm adds nothing to the computation cost.

3 Numerical simulation in Granular Mechanics

3.1 What can be expected from computation

Granular mechanics has long been a topic in Civil Engineering as a part of Soil Mechanics. For a certain number of years, it has also attracted the attention of another public, coming from the Condensed Matter and Disordered Systems community. It is expected that the forms of statistical reasoning used in the latter domain could provide some insight into the many intriguing features of the granular behaviour. On this purpose, the model consisting of rigid grains which interact only through contact and Coulomb friction, without any adhesive effect, makes a sufficiently rich concept for investigating some fundamental questions.

In contrast dry sand, to which the above model is relevant, is not the most frequently met soil in Civil Engineering. In the numerical treatment of most problems, the soil material is viewed as a continuous medium, for which some empirical constitutive law should be available. Finite Element softwares are then applied to the corresponding boundary value problem. The difficulty lies in the determination of the constitutive law which anyway has to be tested against all available experimental information about the considered soil. Engineering needs may be met hrough this approach, but no insight into the intimate granular behaviour can be gained.

While the abbreviation FEM (for *Finite Element Method*) is used to refer to the above strategy, the representation of a granular sample as a collection of solids is called a *Distinct Element Method* (abbr. DEM, sometimes also read as *Discrete Element Method*). Numerical simulation may be conducted thanks to the methods referred to in the foregoing. Clearly the boundary problems of Civil Engineering cannot directly be handled that way since the number of grains involved in real soils exceeds the possibility of computers. The availability of scaling rules which would allow one to approximate these boundary problems through DEM with oversized grains is still an object of investigation [27].

3.2 Force chains

It is known from experiments with assemblies of photoelastic grains that the transmission of forces in the bulk of a granular material is concentrated in chains of grains more loaded than the surrounding ones.

A two-dimensional model of railway track is shown on Figure 1. In view of the elastic deformation of the rail, the force exerted by a passing wheel is distributed between the closest supporting concrete blocks, in relation with their distances. The blocks, as well as the ballast grains, are drawn with levels of gray corresponding to their respective *loads*, i.e. the sum of the normal components of the experienced contact forces. This makes visible that forces are transmitted to the ground across the ballast layer along some preferred chains of grains.

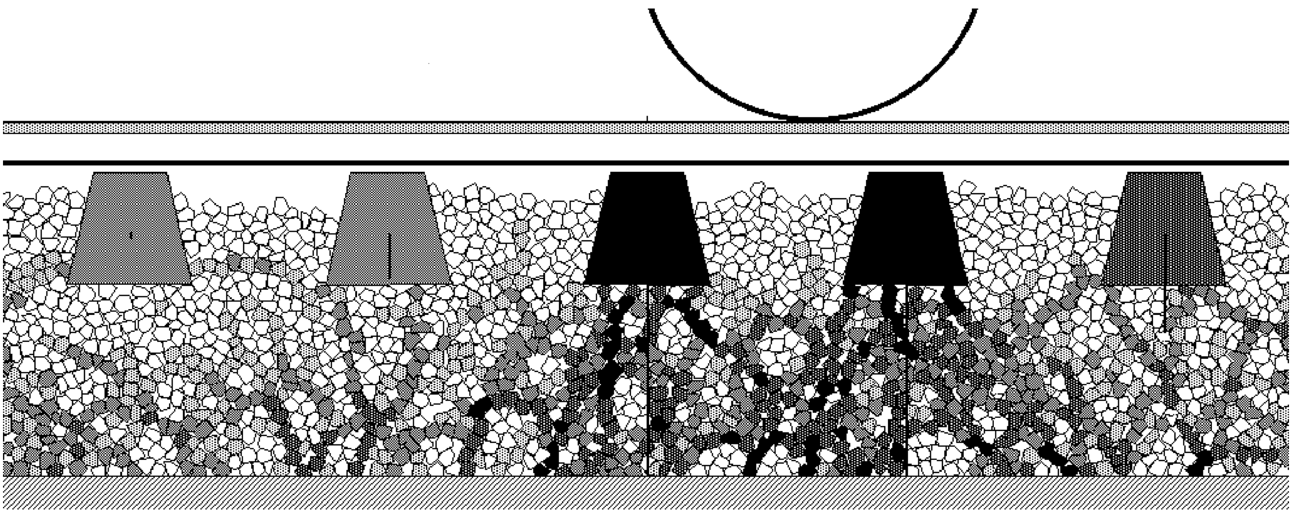

Figure 1: A two-dimensional model of railway ballast.

3.3 Deposition anisotropy

The response of a grain collection to some mechanical action strongly depends on the way it has been prepared or, more generally, on its past evolution. Some message from history is therefore written in the current state of the granulate. As a part of this message the statistical distribution of *the directions of the normals* at the grain to grain contacts is commonly viewed as a descriptor of a possible microstructural anisotropy. Here is an intriguing example.

A usual way of preparing granulates consists of producing a random rain of grains under gravity. If this rain is received on a fixed horizontal ground, a certain statistical anisotropy of the collected granular mass is expected, since the vertical direction plays a special role in the deposition process. The surprise was to discover experimentally that the most frequent directions of contact normals are not vertical nor horizontal but at some angle on both sides of the vertical. The physical experiment was performed with the model of two-dimensional granulate called a *Schneebeli material*, i.e. a collection of cylindrical rods of equal lengths stacked parallel and observed laterally; it was well reproduced by CD numerical simulations [8].

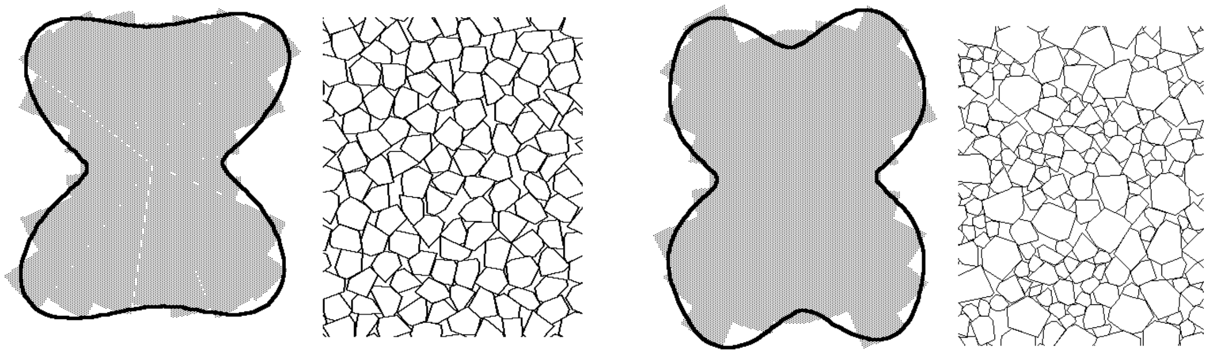

Figure 2: Histograms of contact directions.

Instead of circular objects, the two examples presented on Figure 2 concern convex irregular polygonal grains of random shapes, with two different distributions of sizes. In each sample, for all contact points detected in the rain-deposited layer, the normal directions – in an extended sense, since contact may affect a vertex – are recorded. Histograms of the distributions are constructed by dividing the 360° range about the origin into 24 equal sectors. Each sector is filled in gray up to a radius proportional to the number of contact points whose normals have the corresponding directions. For legibility or for quantitative treatment, one may choose to fit a continuous curve to the distribution, the graph of a probability density expressed as a function of the angle θ by a linear combination of cosine and sine of multiples of 2θ. The same preference for two oblique directions as with circular grains is found.

Three-dimensional simulations yield similar results: in the granular layer created by raining spherical grains of dispersed sizes, contact directions of largest frequency make an angle with the vertical.

3.4 The stress tensor

In the Civil Engineering situations referred to in 3.1, the granular material is treated in the framework of classical Continuum Mechanics, with the Cauchy stress field as central concept. This macroscopic approach does not prevent one, at the stage of assessing the constitutive laws wich govern the medium behaviour, to refer to smaller scale features. Connecting with micromechanical quantities some tensor which could be identified as the stress is not trivial [11]. Since the very time of Cauchy, this question has mainly been addressed for assemblies of points subject to some interaction potentials. Granular media offer a very different sort of microstructure.

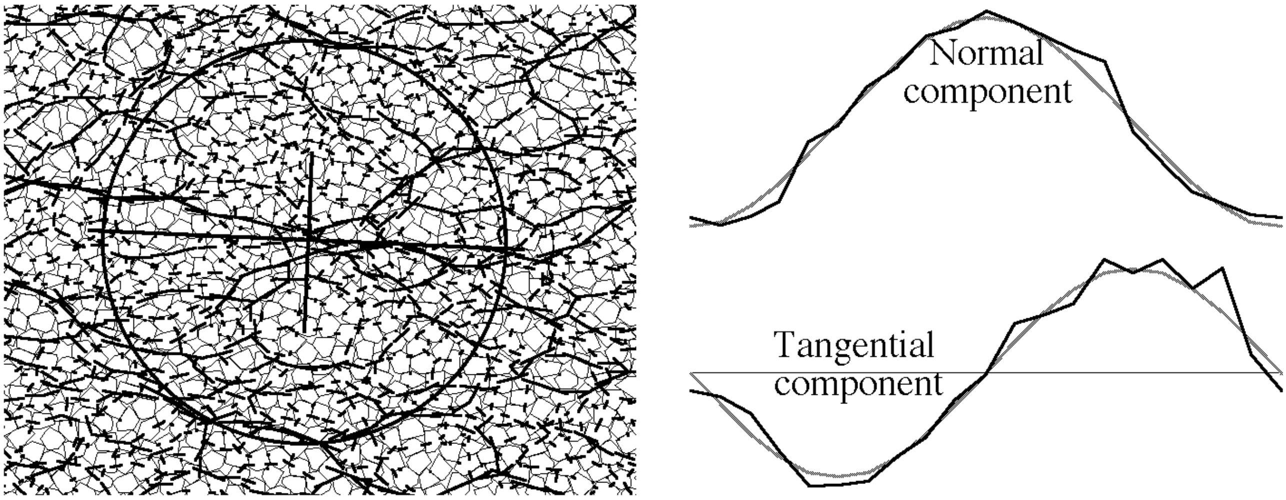

Figure 3: Quality of the stress tensor estimate.

On Figure 3 there is shown a part of a two-dimensional collection of irregular polygonal grains (grain to grain friction: 0.4) submitted to a 'biaxial test', i.e. the granulate is squeezed between frictionless rectilinear vertical boundaries in imposed motion while some containing pressure is exerted by horizontal ones. The representation of contact forces as line segments makes the force chains visible. The programme allows one to superimpose a circular probing area over which the *average stress* is computed according to the definition introduced in [23], practically equivalent to some other ones found in literature. If this tensor is used in the standard way to express the normal and tangential components of the tension upon a cut with direction ϑ, the results are sinusoidal functions of ϑ whose graphs are drawn in gray. On the other hand, 31 diameters of the probing area have been selected. By computing, for each of them, the resultant of the contact forces exerted by the grains on one side upon the grains on the other side, one estimates the corresponding tensions. When the plots of the normal and tangential components of these vectors are compared with the preceding sinusoids, one concludes that the computed tensor, in spite of the small size of the sample and the inhomogenity due to force chains, plays with satisfactory precision the role expected from the Cauchy stress in continuous media. The cross drawn in the circular area represents the principal stresses.

3.5 Banks created by avalanches of dry grains

Figure 4 has been created in the course of the following two-dimensional simulation. Random polygonal grains are deposited at the top of a rigid incline, made of a collection of fixed grains with similar characteristics, so as to simulate roughness. Thanks to a landing plate placed at the bottom of the incline, a layer of some thickness accumulates, sporadically perturbed by avalanches. The angle that the free surface makes with horizontal fluctuates between two values: the *angle of repose* θ_{rep} (approx. 25°), observed after an overall avalanche has remodelled the layer, and some larger value θ_{lim}(approx. 29°), the *limit angle*, attained as the result of progressive accretion. At θ_{lim}, the layer becomes instable.

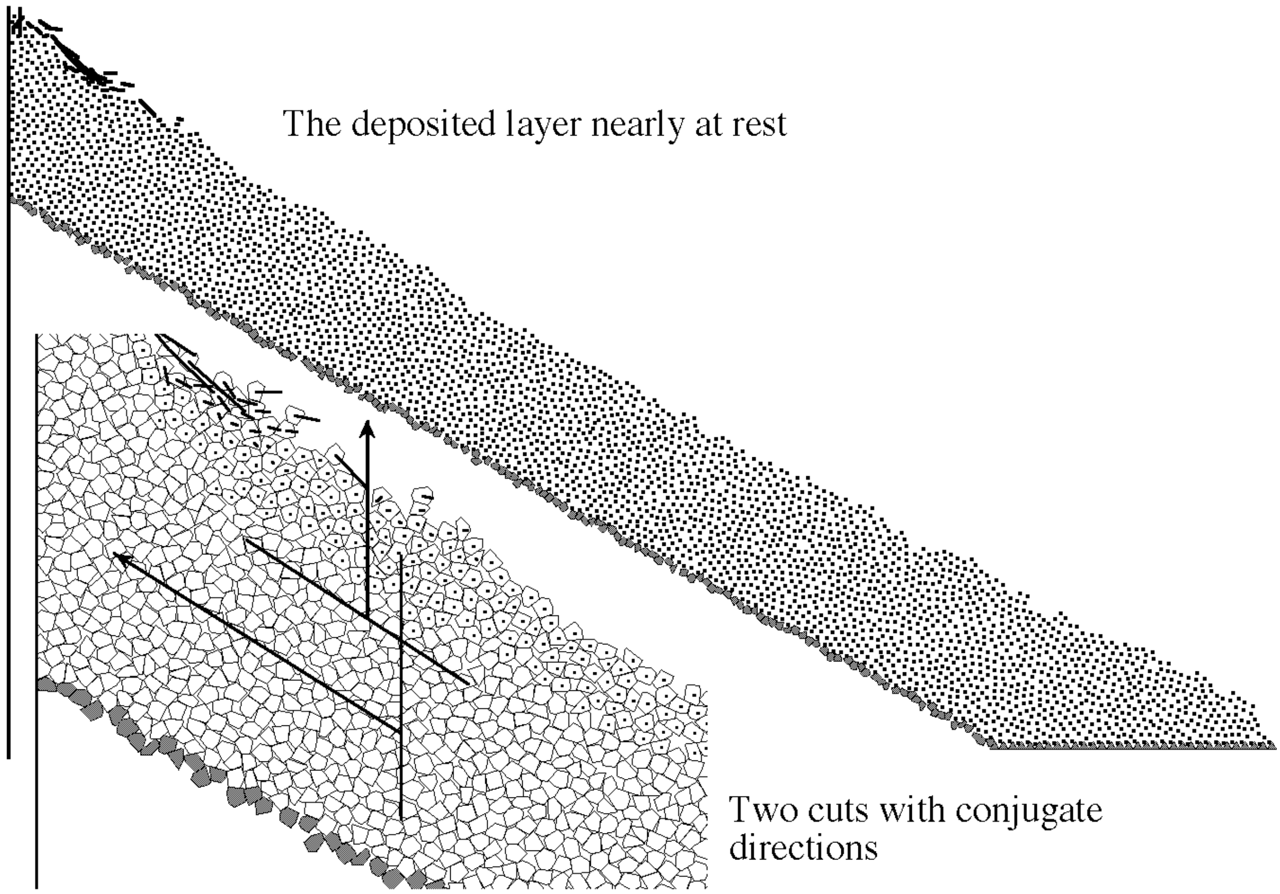

Figure 4: Two cuts in a bank.

Similarly to that was done in 3.4, the programme allows one to choose a line segment and to calculate the resultant contact force transmitted across such an imagined cut.

For a line segment *parallel to the free surface* (the difference between θ_{rep} and θ_{lim} is neglected here), one finds a *vertical force*. This feature looks natural since the bank is made of layers successively deposited by avalanches. When such a layer stops, the supporting forces it experiences from the material beneath should equilibrate its weight, and this verticality of transmitted forces is expected to persist after other layers possessing the same property have been superimposed. This confirms the assumption commonly made in Civil Engineering that a freshly deposited bank is everywhere in a state of *incipient failure*.

For a *vertical line segment*, one finds a *force parallel to the free surface*. This observation, which seems novel, is connected in an essential way with the preceding one through the *symmetry of the Cauchy stress tensor*. It provides a convincing insight into the following subject which, in recent years, started a lot of speculation and controversy.

Let a conical pile be created by pouring grains from a source onto a rough rigid horizontal ground. Some experimentalists have been surprised to observe that the distribution of the pressure exerted upon the ground was not proportional to the height of the material above and even that a local minimum of ground pressure may be present at the vertical of the apex.

Actually, the fact that the free surface is not horizontal makes clear that the statics of the granular assembly has nothing to do with hydrostatics. In a three-dimensional CD numerical experiment [26] not shown here, we have reproduced the construction of a conical pile of 14.000 spherical grains with dispersed sizes. Computed results reveal first that a test surface portion placed inside the peripheral bank of the pile and parallel to the free surface transmits a vertical force, indicative of a state of incipient failure. Secondly, a test surface in the shape of a cylinder coaxial with the conical pile is found to transmit average forces parallel to the free surface. As before, this couple of observations reflects the symmetry of the stress tensor. If such a cylinder is used to delimit some 'central core' of the pile, this core consequently has part of its weight supported by an *arching* action from the surrounding bank, which explains the experimental findings.

References

[1] Abadie, M. (2000) Dynamic simulation of rigid bodies : modelling of frictional contact, in *Impacts in Mechanical Systems. Analysis and Modelling*, edited by B. Brogliato, Springer-Verlag, Berlin Heidelberg, 61–144.

[2] Alart, P., Curnier A. (1991) A mixed formulation for frictional contact problems prone to Newton like methods, *Comput. Meth. in Appl. Mech. Engng.* **92**, 353-375.

[3] Brogliato, B. (1999) *Nonsmooth Mechanics*, 2d. edition. Springer-Verlag, London.

[4] Brogliato, B., ten Dam, A. A., Paoli, L., Génot, F. and Abadie, M. (2001) Numerical simulation of finite dimensional multibody nonsmooth mechanical systems, *ASME Applied Mechanics Reviews*, to appear.

[5] Chabrand, P., Dubois, F., and Raous, M. (1998) Various numerical methods for solving unilateral contact problems with friction, *Mathl. Comput. Modelling* **28**, 97–108.

[6] Christensen, P. W., Klarbring, A., Pang J. S., and Strömberg N. (1998) Formulation and comparison of algorithms for frictional contact problems, *Int. J. Num. Meth. Engng.* **42**, 145–173.

[7] Curnier, A. (1984) A theory of friction, *Int. J. Solids Struct.* **20**, 637–647.

[8] Daudon, D., Lanier, J., and Jean, M. (1997) A micromechanical comparison between experimental results and numerical simulation of a biaxial 2D granular material, in *Powders and Grains 97*, edited by R. P. Behringer and J. T. Jenkins, Balkema, Rotterdam, 219–222.

[9] Frémond, M. (1995) Rigid body collisions, *Physics Letters A* **204**, 33–41.

[10] Génot, F., and Brogliato, B. (1999) New results on Painlevé paradoxes, *European Journal of Mechanics*, A/Solids, **18**, 653–677.

[11] Goddard, J. D. (1998) Continuum modelling of granular assemblies, in *Physics of Dry Granular Materials*, edited by H. J. Herrmann et al., Kluwer, Dordrecht Boston London, 1–24.

[12] Ivanov, A. P. (1997) The problem of constrained impact, *J. Appl. Math. Mech.* **61**, 341–353.

[13] Jean, M. (1999) The Non Smooth Contact Dynamics method, in *Computational Modeling of Contact and Friction*, edited by J. A. C. Martins and A. Klarbring, special issue of *Computer Meth. in Appl. Mech. and Engng.* **177**, 235–257.

[14] Jean, M. (2001) Simulation numérique discrète de matériaux granulaires, in *Micromécanique des matériaux granulaires*, edited by B. Cambou and M. Jean, Hermes, Paris.

[15] Johansson, L., and Klarbring, A. (2000) Study of frictional impact using a nonsmooth equation solver, *ASME J. Appl. Mech.* **67**, 267–273.

[16] Jourdan, F., Alart, P., and Jean, M. (1998) A Gauss-Seidel-like algorithm to solve frictional contact problems. *Computer Meth. Appl. Mech. Engng.* **155**, 31–47.

[17] Kunze, M., and Monteiro Marques, M. D. P. (2000) An introduction to Moreau's sweeping process, in *Impacts in Mechanical Systems. Analysis and Modelling*, edited by B. Brogliato, Springer-Verlag, Berlin Heidelberg, 1–60.

[18] Monteiro Marques, M. D. P. (1993) *Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction*. Birkhäuser, Basel, Boston, Berlin.

[19] Moreau, J. J. (1966) Quadratic programming in mechanics: dynamics of one-sided constraints, *SIAM J. Control* **4**, 153–158.

[20] Moreau, J. J. (1988) Unilateral contact and dry friction in finite freedom dynamics, in *Nonsmooth Mechanics and Applications*, edited by J. J. Moreau and P. D. Panagiotopoulos, CISM Courses and Lectures, Vol. **302**. Springer-Verlag, Wien New York, 1–82.

[21] Moreau, J. J. (1988) Bounded variation in time, in *Topics in Nonsmooth Mechanics*, edited by J. J. Moreau, P. D. Panagiotopoulos, and G. Strang, Birkhäuser, Basel Boston Berlin, 1–74.

[22] Moreau, J. J. (1989) An expression of classical dynamics, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **6** (suppl.), 1-48. Volume also available as *Analyse Non Linéaire*, edited by H. Attouch, J.-P. Aubin, F. Clarke, and I. Ekeland, Gauthier-Villars, Paris.

[23] Moreau J. J., (1997) Numerical investigation of shear zones in granular materials, in *Proc. HLRZ-Workshop on Friction, Arching, Contact Dynamics*, edited by : P. Grassberger P., and D. Wolf , World Scientific, Singapore, 233–247.

[24] Moreau, J. J. (1999) Some basics of unilateral dynamics, in *Unilateral Multibody Contacts*, edited by F. Pfeiffer and Ch. Glocker, Kluwer, Dordrecht/Boston/London, 1–14.

[25] Moreau J. J. (2000) Contact et frottement en dynamique des systèmes de corps rigides, *Rev. Europ. des Eléments Finis* **9**, 9–28.

[26] Moreau J. J. (2001) An introduction to unilateral dynamics, in *Novel approaches in Civil Engineering*, edited by M. Frémond and F. Maceri, Springer-Verlag, to appear.

[27] Nouguier, C., Bohatier, C., Moreau, J. J., and Radjai, F. (2000) Force fluctuations in a pushed granular material, *Granular matter* **2**, 171–178.

[28] Pang, J. S., and Stewart, D. E. (1999) A unified approach to discrete frictional contact problems, *Int. J. Engng. Sci.*, **37**, 1747–1768.

[29] Pfeiffer F., Glocker Ch. (1996) *Multibody Dynamics with Unilateral Contacts*, John Wiley and Sons, New York.

[30] Stewart, D. E. (1998) Convergence of a time-stepping scheme for rigid body dynamics and resolution of Painlevé's problem, *Arch. Rational Mech. Anal.* **145**, 215–260.

[31] Stoianovici, S. P., and Hurmuzlu, Y. (1996) A critical study of the concepts of rigid body collision theory, *J. Appl. Mech.* **63**, 307–316.

[32] Vola, D., Pratt, E., Jean, M., and Raous, M. (1998) Consistent time discretization for a dynamical frictional contact problem and complementarity techniques, *Rev. Europ. des Eléments Finis* **7**, 149–162.