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INTEGRAL KERNELS ON COMPLEX SYMMETRIC SPACES AND FOR THE DYSON BROWNIAN MOTION

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In this article, we consider flat and curved Riemannian symmetric spaces in the complex case and we study their basic integral kernels, in potential and spherical analysis: heat, Newton, Poisson kernels and spherical functions, *i.e.* the kernel of the spherical Fourier transform.

We introduce and exploit a simple new method of construction of these W-invariant kernels by alternating sum formulas. We then use the alternating sum representation of these kernels to obtain their asymptotic behavior. We apply our results to the Dyson Brownian Motion on $\mathbf{R}^d$.

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1 Introduction and notations

Analysis on Riemannian symmetric spaces of Euclidean type, also called flat symmetric spaces, continues to develop in recent years ([21, 23, 45, 46]). Its importance is due to its relationship with Dunkl analysis ([9, 12, 42]) together with the correspondence of the complex case with the parameter $k = 1$, in which symmetric spaces of Euclidean type constitute the "geometric case", frequently used as a model case in most challenging open problems of Dunkl theory. The analysis on flat complex symmetric spaces coincides with Weyl group invariant Dunkl analysis associated with multiplicity $k = 1$, see [9]. In particular, the heat kernel $p_t^W(X, Y)$ is a special case of the heat kernel in the Weyl group invariant Dunkl setting. We employ this intimate connection to Dunkl theory in our paper in Section 3 as one of main tools of the proof of main theorems. This connection appears also in Proposition 2.8.

Another important aspect of this paper is to apply analysis on symmetric spaces of Euclidean type to potential theory and to stochastic analysis of Dyson Brownian Motion, one of the most important models of non-colliding particles, see the recent survey [29]. We expect further applications of our results and techniques to other non-intersecting stochastic path problems related to root systems and to multivariate stochastic processes related to Laplace-Beltrami operators on symmetric spaces, to Dunkl Laplacians and to Schrödinger operators, see the discussion in Section 5. We thank an anonymous referee for pointing out to us such further stochastic applications.

The objective of this paper is to study basic integral kernels, in potential theory and spherical analysis: heat, Newton, Poisson kernels, Green function and spherical functions (*i.e.* the kernel of the spherical Fourier transform), in the set-up of flat and curved symmetric spaces of complex type.

Our main results on the exact form and asymptotics of the heat, Poisson and Newton kernels (Theorems 2.2, 3.11, 3.13 and Corollaries 5.6 and 5.7) are crucial for the future development of the potential theory on flat and curved symmetric spaces of complex type, and for the potential theory of the Dyson

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Brownian Motion. These results are a starting point of research and a source of conjectures for the corresponding kernels in the Weyl-invariant Dunkl setting (for the rank one case, refer to [19]).

The main result on asymptotics of the spherical functions contained in Theorem 4.5 is important from the point of view of spherical analysis on symmetric spaces, because it generalizes significantly the results of Helgason in [23], of Narayanan, Pasquale and Pusti in [36] and of Schapira in [44], for the flat and curved symmetric spaces in the complex case, cf. Remark 4.9.

We recall now some basic terminology and facts about symmetric spaces associated to Cartan motion groups.

Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of G. We recall the definition of the Cartan motion group and the flat symmetric space associated with the semisimple Lie group G with maximal compact subgroup K. The Cartan motion group is the semi-direct product 5

$G_0 = K \rtimes \mathfrak{p}$ where the multiplication is defined by $(k_1, X_1) \cdot (k_2, X_2) = (k_1\, k_2, \mathrm{Ad}(k_1)(X_2) + X_1)$. The associated flat symmetric space is then $M = \mathfrak{p} \simeq G_0/K$ (the action of G_0 on $\mathfrak{p}$ is given by $(k, X) \cdot Y = \mathrm{Ad}(k)(Y) + X$).

We tacitly identify K-invariant measures, functions, differential operators on M with W-invariant measures etc. on $\mathfrak{a}$.

The spherical functions for the symmetric space M are then given by

$$\psi_\lambda(X) = \int_K e^{\lambda(\mathrm{Ad}(k)(X))}\, dk \tag{1}$$

where λ is a complex linear functional on $\mathfrak{a} \subset \mathfrak{p}$, a Cartan subalgebra of the Lie algebra of G. To extend λ to $X \in \mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}$, one uses $\lambda(X) = \lambda(\pi_{\mathfrak{a}}(X))$ where $\pi_{\mathfrak{a}}$ is the orthogonal projection with respect to the Killing form (denoted throughout this paper by $\langle \cdot, \cdot \rangle$). Note also that the spherical function for the symmetric space G/K is given by

$$\phi_\lambda(g) = \int_K e^{(\lambda - \rho)(H(g\,k))}\, dk \tag{2}$$

where λ is a complex linear functional on $\mathfrak{a}$ and the map H is defined via the Iwasawa decomposition of G, namely $g = k\, e^{H(g)}\, n \in K\, A\, N$ and $\rho = (1/2) \sum_{\alpha > 0} m_\alpha\, \alpha$. Note that in [23, 24, 25], λ is replaced by $i\,\lambda$.

Throughout this paper, we suppose that G is a semisimple complex Lie group. The complex root systems are respectively A_{n-1} for $n \geq 2$ (where $\mathfrak{p}$ consists of the $n \times n$ hermitian matrices with trace 0), B_n for $n \geq 2$ (where $\mathfrak{p} = i\,\mathfrak{so}(2\,n+1)$), C_n for $n \geq 3$ (where $\mathfrak{p} = i\,\mathfrak{sp}(n)$) and D_n for $n \geq 4$ (where $\mathfrak{p} = i\,\mathfrak{so}(2\,n)$) for the classical cases and the exceptional root systems E_6, E_7, E_8, F_4 and G_2.

Let Δ be the Laplace-Beltrami operator on M and Δ^W its restriction to W-invariant functions on $\mathfrak{a}$ where W is the corresponding Weyl group. Recall the formula

$$\Delta^W f = \pi^{-1}\, \Delta^{\mathbf{R}^d}(\pi\, f), \tag{3}$$

where $\pi(X) = \prod_{\alpha > 0} \alpha(X)$ (see [25, Chap. II, Theorem 5.37]) in the Euclidean case.

In Section 2, we introduce and exploit a simple new method of construction of important K-invariant kernels on the space M.

We show in Theorem 2.2 that if $\mathcal{K}(X, Y)$ is an Euclidean kernel (heat kernel, potential kernel, Poisson kernel, ...) for the Laplacian $\Delta^{\mathbf{R}^d}$, then the corresponding kernel acting on W-invariant functions on M is given by the alternating sum

$$\mathcal{K}^W(X, Y) = \frac{1}{|W|\, \pi(X)\, \pi(Y)} \sum_{w \in W} \epsilon(w) \mathcal{K}(X, w \cdot Y). \tag{4}$$

Here and in Theorem 2.2 below, $\mathcal{K}(X, Y)$ is an Euclidean kernel on the Cartan subalgebra $\mathfrak{a}$ which is isomorphic to $\mathbf{R}^d$ where d is the dimension of $\mathfrak{a}$ and with the underlying scalar product being the Killing form on $\mathfrak{a}$.

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The proof of Theorem 2.2 is short and easy and uses the simple form of the operator Δ^W given in (3).

It is well-known that the spherical functions of the space can be written explicitly as such alternating sums ([25, Chap. IV, Proposition 4.10]).

The alternating sum formulas (4) also include determinantal formulas for transition probabilities $p_t^W(x,y)$ (equivalently, for heat kernels) of Karlin-McGregor type, proven for Dyson Brownian Motions in Weyl chambers [18] and exploited in stochastic analysis (refer to [31, 32]).

The fact that alternating sums formulas (4) are true for many further analytic and stochastic kernels beyond spherical functions and heat kernels, was surprisingly not published or exploited (we asked experts of the field for an existing reference).

The approach with formulas (4) will allow us to provide asymptotics for kernels $\mathcal{K}^W$, using our knowledge of the kernels $\mathcal{K}(X,Y)$ on $\mathbf{R}^d$ as given in Table 1.

In Section 3, we discuss the asymptotic behaviour of the Poisson kernel especially when one or both arguments are singular. These results translate well to the Newton kernel.

In Section 4, we compute asymptotics for the spherical functions $\psi_\lambda(Y)$ which can prove challenging when either λ or Y are singular (*i.e.* such that at least one of the nonzero root vanish on X or Y). Our results depend on a property we call "Killing-max" namely the property that for X, $Y \in \overline{\mathfrak{a}^+}$, $\langle X, w\cdot Y\rangle = \langle X, Y\rangle$ if and only if $w \in W_X\, W_Y$ where $W_X = \{w \in W : w\cdot X = X\}$. It is known that this property is verified when either X or Y is non singular [23]. We prove in Appendix A, using the classification of Lie algebras, that the Killing-max holds in almost all cases (only in the cases related to the root systems E_6, E_7 and E_8 is the question left unanswered).

We conclude with Section 5 where we apply the previous results to the heat kernel and Poisson and Newton kernels for the Dyson Brownian Motion.

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2 Kernels on flat symmetric spaces in the complex case

2.1 Definitions

We first recall the classical integral kernels on $\mathbf{R}^d$ in Table 1.

The integral kernels on the flat symmetric space M are considered with respect to the invariant measure $\mu(dY) = \pi^2(Y)\,dY$ on M. Their definition is analogous to the classical $\mathbf{R}^d$ and Riemannian manifold case, with the W-invariance imposed on the operator, boundary problem and solutions. The Dunkl-Poisson, Newton and Green kernels and their W-invariant versions were introduced and studied in [16] and [19].

Definition 2.1 We define a kernel $\mathcal{K}^W(X,Y)$ for the operator Δ^W and a boundary problem $\mathcal{P}$ as the fundamental solution of this PDE problem, which is W-invariant in X-variable, for each Y. Equivalently, $\mathcal{K}^W(X,Y)$ is an integral reproducing kernel for the W-invariant solutions of the problem $\mathcal{P}$ and this kernel is W-invariant in X.

The uniqueness of $\mathcal{K}^W(X,Y)$ may be deduced, as in the classical case, from the uniqueness of the spherical Fourier transform. Another approach for the existence of Poisson, Newton and Green kernels is available from the point of view of stochastic diffusion processes [7]. Note that W-invariant Dunkl processes are diffusions.

2.2 The method of alternating sums for constructing kernels on M

This method will be introduced and used in the proof of Theorem 2.2 below.

Theorem 2.2 *Let M be a symmetric space of Euclidean type with G a complex simple Lie group of rank d. Then the following formulas hold for X, $Y \in \mathfrak{a}$, a Cartan subalgebra associated with M.*

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PDE	Kernel	Solution
$\begin{cases} \Delta^{\mathbf{R}^d} u(X,t) = \frac{\partial}{\partial t} u(X,t) \\ \lim_{t\to 0^+} u(X,t) = f(X) \end{cases}$ $X \in \mathbf{R}^d,\ t > 0$	$p_t(X,Y) = \dfrac{e^{-\frac{\lvert X-Y\rvert^2}{4\,t}}}{(4\,\pi\,t)^{d/2}}$	$u(X,t) = \displaystyle\int_{\mathbf{R}^d} f(Y)\, p_t(X,Y)\, dY$
$\begin{cases} \Delta^{\mathbf{R}^d} u(X) = f(X) \quad \text{on } \mathbf{R}^d, \\ \lvert u(X)\rvert \to 0 \text{ as } X \to \infty \end{cases}$ $f \in C_c(\mathbf{R}^d)$	$N(X,Y) = \Phi(X-Y),$	$u(X) = \displaystyle\int_{\mathbf{R}^d} f(Y)\, N(X,Y)\, dY$
$\begin{cases} \Delta^{\mathbf{R}^d} u(X) = 0 \quad \text{on } B(X_0,r) \\ u(X) = f(X) \quad \text{on } \partial B(X_0,r) \end{cases}$	$P(X,Y) = \dfrac{r^2 - \lvert X-X_0\rvert^2}{w_d\, r\, \lvert X-Y\rvert^d}$	$u(X) = \displaystyle\int_{\partial B(X_0,r)} f(Y)\, P(X,Y)\, dY$
$\begin{cases} \Delta^{\mathbf{R}^d} u(X) = f(X) \quad \text{on } B = B(0,1) \\ u(X) = 0 \quad\quad\quad\ \text{on } \partial B \end{cases}$	$G_B(X,Y) = \Phi(X-Y)$ $-\Phi(\lvert X\rvert\,(Y - X/\lvert X\rvert^2))$	$u(X) = \displaystyle\int_B f(Y)\, G_B(X,Y)\, dY$

where $w_d = 2\,\pi^{d/2}/\Gamma(d/2)$ (the surface area of a sphere of radius 1 in $\mathbf{R}^d$) and $\Phi(X) = \begin{cases} \frac{1}{2\,\pi}\,\ln\lvert X\rvert & \text{if } d = 2 \\ \frac{1}{(2-d)\,w_d}\,\lvert X\rvert^{2-d} & \text{if } d \geq 3 \end{cases}$.

Table 1 The heat kernel p_t, the Newton kernel N, the Poisson kernel P and the Green kernel G_B for $\mathbf{R}^d$

1. *The heat kernel on M is given by*

$$p_t^W(X,Y) = \frac{1}{\lvert W\rvert\,(4\,\pi\,t)^{d/2}\,\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w) e^{-\frac{\lvert X-w\cdot Y\rvert^2}{4\,t}} \tag{5}$$

2. *The Newton kernel on M is given by*

$$N^W(X,Y) = \frac{1}{2\,\pi\,\lvert W\rvert\,\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w)\,\ln\lvert X - w\cdot Y\rvert \text{ when } d = 2, \tag{6}$$

$$N^W(X,Y) = \frac{1}{\lvert W\rvert\,(2-d)\,w_d\,\pi(X)\,\pi(Y)} \sum_{w\in W} \frac{\epsilon(w)}{\lvert X - w\cdot Y\rvert^{d-2}} \text{ when } d \geq 3.$$

3. *The Poisson kernel of the open unit ball B is given for $X \in B$ and $Y \in \partial B$ by*

$$P^W(X,Y) = \frac{1 - \lvert X\rvert^2}{\lvert W\rvert\,w_d\,\pi(X)\,\pi(Y)} \sum_{w\in W} \frac{\epsilon(w)}{\lvert X - w\cdot Y\rvert^d} \tag{7}$$

4. *The Green function of the unit ball is given by*

$$G_B^W(X,Y) = \frac{1}{\lvert W\rvert\,\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w) G_B(X,Y), \tag{8}$$

where $G_B(X,Y)$ is the classical Green function of the unit ball B in $\mathbf{R}^d$ (refer to Table 1).

P r o o f. It is based on the following steps:

1. Write a kernel on $\mathbf{R}^d$ where d is the rank of M;
2. Exploit formula (3);
3. Apply the W-invariance (the kernels on M must be W-invariant).

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We give the proof in the Poisson kernel case; the other proofs are similar. The Poisson kernel of $B(0,1)$ in the Euclidean case is

$$P(X,Y) = \frac{1-|X|^2}{w_d\,|X-Y|^d}.$$

If u is harmonic with respect to Δ^W then $\pi\,u$ is harmonic with respect to $\Delta^{\mathbf{R}^d}$. Hence

$$\pi(X)\,u(X) = \int_{\partial B} \frac{1-|X|^2}{w_d\,|X-Y|^d}\,\pi(Y)\,f(Y)\,dY.$$

This is equivalent to

$$u(X) = \int_{\partial B} \frac{1-|X|^2}{w_d\,\pi(X)\,\pi(Y)}\,\frac{1}{|X-Y|^d}\,f(Y)\,\pi(Y)^2\,dY.$$

The reproducing kernel $\frac{1-|X|^2}{w_d\,\pi(X)\,\pi(Y)}\,\frac{1}{|X-Y|^d}$ is not W-invariant. We write the last equation $|W|$ times, replacing X by $w\cdot X$

$$\begin{aligned} u(X) = u(w\cdot X) &= \int_{\partial B} \frac{1-|X|^2}{w_d\,\pi(w\cdot X)\,\pi(Y)}\,\frac{1}{|w\cdot X-Y|^d}\,f(Y)\,\pi(Y)^2\,dY \\ &= \int_{\partial B} \frac{1-|X|^2}{w_d\,\pi(X)\,\pi(Y)}\,\frac{\epsilon(w)}{|X-w\cdot Y|^d}\,f(Y)\,\pi(Y)^2\,dY \end{aligned}$$

and we sum up the $|W|$ equations. We obtain

$$u(X) = \frac{1}{|W|\,w_d}\int_{\partial B} \frac{1-|X|^2}{\pi(X)\,\pi(Y)}\sum_{w\in W}\frac{\epsilon(w)}{|X-w\cdot Y|^d}\,f(Y)\,\pi(Y)^2\,dY.$$

The formula for the Newton kernel requires more care. Let $\tilde{u}$ be the solution of the inhomogeneous Laplace equation on $\mathbf{R}^d$, then

$$u(X) = \frac{\sum_{w\in W}\,\epsilon(w)\,\tilde{u}(wX)}{\pi(X)}$$

solves the corresponding problem for Δ^W. We need however to show that $\lim_{X\to\infty}|u(X)| = 0$. It is useful to note that the function $\hat{u}(X) = \sum_{w\in W}\,\epsilon(w)\,\tilde{u}(wX)$ is skew-symmetric.

For $J\subseteq\{1,2,\dots,n\}$, les $A_J = \{x\in\mathbf{R}^n\colon |x_i|>1/2 \text{ for } i\in J,\ |x_i|<1 \text{ for } i\in J^c\}$. Note that $\mathbf{R}^d$ is the union of the open sets A_J. Now, on A_J with $|J|\geq 1$ (so that $X\to\infty$),

$$\begin{aligned} &\lim_{X\to\infty}\left|\frac{\hat{u}(X)}{\pi(x_1,\dots,x_d)}\right| = \lim_{(x_i)_{i\in J}\to\infty}\left|\frac{\hat{u}(X)}{\pi(x_1,\dots,x_d)}\right| = \lim_{\exists i\in J, x_i\to 0}\left|\frac{\hat{u}((1/x_i)_{i\in J},(x_i)_{i\in J^c})}{\pi((1/x_i)_{i\in J},(x_i)_{i\in J^c})}\right| \\ &= \lim_{|x_i|<2, j\in J, |x_i|>1, i\in J^c, \exists i\in J, x_i\to 0}\ \prod_{i\in J}|x_i|^{d-1}\left|\frac{\hat{u}((1/x_i)_{i\in J},(x_i)_{i\in J^c})}{\pi((x_i)_{i\in J})\,\pi((x_i)_{i\in J^c})\,\prod_{i\in J, j\in J^c}(1-x_i\,x_j)}\right|. \end{aligned}$$

Observe that $\hat{u}((1/x_i)_{i\in J},(x_i)_{i\in J^c})$ is continuous since $\lim_{X\to\infty}\tilde{u}(X) = 0$ and skew-symmetric in $(x_i)_{i\in J}$ and in $(x_i)_{i\in J^c}$. We remark also that it is zero when $1-x_i\,x_j = 0$, $i\in J$, $j\in J^c$. Using Lemma 2.3 below, we can conclude that the term

$$\left|\frac{\hat{u}((1/x_i)_{i\in J},(x_i)_{i\in J^c})}{\pi((x_i)_{i\in J})\,\pi((x_i)_{i\in J^c})\,\prod_{i\in J, j\in J^c}(1-x_i\,x_j)}\right|$$

is an analytic function so that it remains bounded and that the limit is 0. We are grateful to the anonymous referee for pointing out the need for additional justification in the Newton kernel case. □

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Lemma 2.3 *Let $f : \mathbf{R}^2 \to \mathbf{R}$, $h : \mathbf{R}^2 \to \mathbf{R}$ two analytical functions such that $h^{-1}(\{0\}) \subset f^{-1}(\{0\})$. Suppose that for each $z_0 = (x_0, y_0) \in h^{-1}(\{0\})$, the order of y_0 as a zero of $h(x_0, \cdot)$ is one (i.e. $h(x_0, y) = (y - y_0)\tilde{h}(y)$, $\tilde{h}$ analytical, $\tilde{h}(y_0) \neq 0$).*

Then there exists an analytical function $g : \mathbf{R}^2 \to \mathbf{R}$ such that $f = hg$.

P r o o f. We apply the Weierstrass division theorem ([8, Th. 0.43(2)], [33, Th.6.1.3(1)]). For each $z_0 = (x_0, y_0) \in h^{-1}(\{0\})$ there exists a neighbourhood V_{z_0} and analytical functions $v_{z_0}(x, y)$ and $b_1(x)$ such that

$$f(x, y) = h(x, y)v_{z_0}(x, y) + b_1(x).$$

For all $(x, y) \in h^{-1}(0) \cap V_{z_0}$ the last equality gives $0 = 0 + b_1(x)$, so that

$$f(x, y) = h(x, y)v_{z_0}(x, y), \qquad (x, y) \in V_{z_0}.$$

An application of the principle of identity ends the proof. □

Remark 2.4 The properties of factorization of analytical functions of several real variables are not as straightforward as one might hope. For example, consider $f(x, y) = y^3$ which is zero whenever $x^2 + y^2 = 0$. However, it is not true that f divided by $x^2 + y^2$ is analytic or even defined.

For the root systems of type A, we obtain the following determinantal formula for the heat kernel on M. This formula may be also deduced from the formula for the transition function of the Dyson Brownian Motion, based on the Doob transform and Karlin-MacGregor formula, see Section 5.

Corollary 2.5 *Consider the flat complex symmetric space M with the root system $\Sigma = A_{d-1}$. Let $g_t(u, v) = \frac{1}{\sqrt{4\,\pi\,t}}\, e^{-|u-v|^2/4t}$ be the 1-dimensional classical heat kernel. The heat kernel on M is given by*

$$p_t^W(X, Y) = \frac{1}{|W|\,\pi(X)\,\pi(Y)}\, \det\left(g_t(x_i, y_j)\right) \tag{9}$$

where $x_1, \ldots, x_d$ are the coordinates of X and $y_1, \ldots, y_d$ are the coordinates of Y.

P r o o f. Formula (9) follows from Theorem 2.2 (1) and the definition of determinant. □

Remark 2.6 In [18], Grabiner computes determinant formulas for the transition probabilities of the Dyson Brownian motion in the Weyl chambers of A_{n-1}, B_n, C_n and D_n.

Note that the alternating sum formula (4) reduces to a determinant if and only if the kernel $\mathcal{K}(X, Y)$ has a multiplicative form

$$\mathcal{K}(X, Y) = \prod_{i=1}^{d} k(x_i, y_i).$$

This holds true for the transition probabilities of the Brownian Motion on $\mathbf{R}^d$ or, more generally, of any multidimensional stochastic process $\underline{X}(t)$ with independent identically distributed components $X_i(t)$.

Let us resume the method of alternating sums, applied in the proof of Theorem 2.2. An Euclidean kernel $\mathcal{K}(X, Y)$ (heat kernel, potential kernel, Poisson kernel, ...) for the Laplacian $\Delta^{\mathbf{R}^d}$ is transformed in the following way into the kernel $\mathcal{K}^W$ acting on W-invariant functions on M:

$$\mathcal{K}^W(X, Y) = \frac{1}{|W|\,\pi(X)\,\pi(Y)} \sum_{w \in W} \epsilon(w)\mathcal{K}(X, w \cdot Y). \tag{10}$$

Formula (5) is immediate from the explicit form of the heat kernel in Dunkl theory (refer to [40]) together with Proposition 2.7 below. The formulas (6)-(9) are new.

However, in the harmonic analysis of flat symmetric spaces of complex type, the alternating sum formula (11) for a spherical function on M given below is well known (see [25, Chap. IV, Proposition 4.8 and Chap. II, Theorem 5.35]). Dunkl had provided a proof for the root system A_{n-1} in [10] using a similar approach as ours.

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Proposition 2.7 *Given $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ (the dual of the complexification of $\mathfrak{a}$), the spherical function $\psi_\lambda(X)$ on M is given by the formula*

$$\psi_\lambda(X) = \frac{\pi(\rho)}{2^\gamma \pi(\lambda)\,\pi(X)} \sum_{w\in W} \epsilon(w) e^{\langle \lambda, w\cdot X\rangle}, \tag{11}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha = \sum_{\alpha\in\Sigma^+} \alpha$ and $\gamma = |\Sigma^+|$ is the number of positive roots.

We finish this section with a relation between the heat kernel and spherical functions which will be useful in stochastic applications of our results, see Proposition 4.10 and Corollary 5.8. Proposition 2.8 is an immediate consequence of well-known results by Rösler in Dunkl theory (see for instance [40, Lemma 4.5]) and of [9], ensuring that heat kernel and spherical functions on flat complex symmetric spaces coincide with their Weyl group invariant analogues in Dunkl analysis when the multiplicity $k = 1$.

Proposition 2.8 *Let M be a flat symmetric space of complex type. The following formula holds*

$$p_t^W(X,Y) = \frac{1}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{4t}}\, \psi_X\left(\frac{Y}{2t}\right). \tag{12}$$

Remark 2.9 We provide here a simple explanation for the constant occurring in (12). From (5) and (11),

$$\begin{aligned} p_t^W(X,Y) &= \frac{1}{|W|\,(4\,\pi\,t)^{d/2}\,\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w) e^{-\frac{|X-w\cdot Y|^2}{4\,t}} \\ &= \frac{1}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\, t^{-\frac{d}{2}-\gamma}\, \frac{\pi(\rho)}{2^\gamma\,\pi(X)\,\pi\left(\frac{Y}{2t}\right)} \sum_{w\in W} \epsilon(w)\, e^{\frac{\langle X, w\cdot Y\rangle}{2t}} \\ &= \frac{1}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{4t}}\, \psi_X\left(\frac{Y}{2t}\right). \end{aligned}$$

Note that the constants in [40] lead to the same constant as in (12) even though the functional ρ is not used in the context of Dunkl theory. The same phenomenon will appear for the constant for the Poisson kernel, see Remark 3.5.

3 Asymptotic behavior of the kernels

To simplify the notation, we will write $f \overset{Y_0}{\sim} g$ if $\lim_{X\to Y_0} \frac{f(X)}{g(X)} = 1$.

The main results of this Section are Theorems 3.11 and 3.13 which give asymptotics of the Poisson and Newton kernels of the flat complex symmetric space M. In their proofs, we need some knowledge of Dunkl analysis on $\mathbf{R}^d$.

Consider $\mathbf{R}^d$ with a root system Σ. The basic information on the Dunkl analysis in this context can be found in [42]. Denote the Dunkl Laplacian by Δ_k and the intertwining operator by V_k.

Recall now the formula of Dunkl ([11, 12]) for the Dunkl-Poisson kernel of the unit open ball $B = B(0,1)$.

$$P_k(X,Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}\, V_k\left[\frac{1-|X|^2}{(1-2\langle X,\cdot\rangle + |X|^2)^{\gamma+d/2}}\right](Y), \quad X\in B,\ Y\in\partial B,\ \gamma = \sum_{\alpha\in\Sigma} k_\alpha. \tag{13}$$

The constant in (13) is different from the one given in [11, 12]. Our constant is explained below in Remark 3.5.

The flat complex symmetric spaces M correspond to the formula (13) in the W-invariant case and with $k_\alpha = 1$. Then $\gamma = |\Sigma_+|$ expresses the number of positive roots.

A formula for the Dunkl-Newton kernel $N_k(X,Y)$, analogous to (13), was proven in [16].

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3.1 Poisson kernel of the flat complex symmetric space

The following technical results will prove useful further on.

Lemma 3.1

$$\partial(\pi)\,|X|^{-d} = 2^\gamma \prod_{k=0}^{\gamma-1} (-d/2-k)\,\pi(X)\,|X|^{-d-2\,\gamma}$$
$$\partial(\pi)\,\log|X| = (-2)^{\gamma-1}\,(\gamma-1)!\,\pi(X)\,|X|^{-2\,\gamma}.$$

P r o o f. We see easily that $|X|^{d+2\,\gamma}\,\partial(\pi)\,|X|^{-d}$ is a skew polynomial of degree at most γ. It must therefore be a constant multiple of $\pi(X)$. Note from the definition of $\partial(\pi)$ that

$$\partial(\pi)\,f(X) = \prod_{\alpha>0} \frac{\partial}{\partial t_\alpha}\Big|_{t_\alpha=0} f(X+\sum_{\alpha>0} t_\alpha\,H_\alpha)$$

where H_α is defined by the relation $\langle X, H_\alpha\rangle = \alpha(X)$ for $X \in \mathfrak{a}$. Hence,

$$\partial(\pi)\,|X|^{-d} = \prod_{\alpha>0} \frac{\partial}{\partial t_\alpha}\Big|_{t_\alpha=0} \langle X+\sum_{\alpha>0} t_\alpha\,H_\alpha, X+\sum_{\alpha>0} t_\alpha\,H_\alpha\rangle^{-d/2}.$$

After applying the operators $\frac{\partial}{\partial t_\alpha}\Big|_{t_\alpha=0}$, we will be left with the term

$$(-d/2)\,(-d/2-1)\,\cdots\,(d/2-(\gamma-1))\,\langle X,X\rangle^{-d/2-\gamma} \prod_{\alpha>0} (2\,\alpha(X))$$

and other terms which do not have the right form. This tells us that desired constant is $2^\gamma \prod_{k=0}^{\gamma-1}(-d/2-k)$.

A similar reasoning applies for the computation of $\partial(\pi)\,\log|X|$. □

Proposition 3.2 *Let* $T(X,Y) = \frac{1}{\pi(X)\,\pi(Y)} \sum_{w\in W} \frac{\epsilon(w)}{|X-w\cdot Y|^d}$. *Then* $T(0,Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)}\,|Y|^{-d-2\,\gamma}$.

P r o o f. Note first that $\partial(\pi)_X\,|X-Y|^{-d} = 2^\gamma \prod_{k=0}^{\gamma-1}(-d/2-k)\,\pi(X-Y)\,|X-Y|^{-d-2\,\gamma}$. Consider $B(X,Y) = \pi(X)\,T(X,Y) = \frac{1}{\pi(Y)} \sum_{w\in W} \epsilon(w)\,|X-w\cdot Y|^{-d}$. We apply the differential operator $\partial(\pi)|_{X=0}$ to B. We find

$$\begin{aligned}\partial(\pi)(\pi)\,T(0,Y) &= 2^\gamma \prod_{k=0}^{\gamma-1}(-d/2-k)\,\frac{1}{\pi(Y)} \sum_{w\in W} \epsilon(w)\,\pi(X-w\cdot Y)\,|X-w\cdot Y|^{-d-2\,\gamma}\Big|_{X=0}\\ &= (-1)^\gamma\,2^\gamma \prod_{k=0}^{\gamma-1}(-d/2-k)\,|W|\,|Y|^{-d-2\,\gamma}.\end{aligned}$$

Finally,

$$\begin{aligned}T(0,Y) &= \frac{(-1)^\gamma\,2^\gamma \prod_{k=0}^{\gamma-1}(-d/2-k)\,|W|}{\partial(\pi)(\pi)}\,|Y|^{-d-2\,\gamma} = \frac{(-1)^\gamma\,2^\gamma \prod_{k=0}^{\gamma-1}(-d/2-k)\,|W|}{\pi(\rho)\,|W|/2^\gamma}\,|Y|^{-d-2\,\gamma}\\ &= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)}\,|Y|^{-d-2\,\gamma}.\end{aligned}$$

□

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Corollary 3.3 *We have*

$$P^W(0,Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}$$
$$N^W(0,Y) = \frac{-2^{2\,\gamma-1}\,(\gamma-1)!}{2\,\pi\,|W|\,\pi(\rho)}\,|Y|^{-2\,\gamma} \text{ if } d=2$$
$$N^W(0,Y) = \frac{2^{2\,\gamma}\,((d-2)/2)_\gamma}{|W|\,(2-d)\,w_d\,\pi(\rho)}\,|Y|^{2-d-2\,\gamma} \text{ if } d\geq 3$$

Proposition 3.4 *The Poisson kernel of the unit ball on the flat complex symmetric space M is given by*

$$P^W(X,Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}\,\mathcal{A}^*\left(\frac{1-|X|^2}{(1-2\,\langle X,\cdot\rangle+|X|^2)^{\gamma+d/2}}\right)(Y), \tag{14}$$

where $\mathcal{A}^*$ denotes the dual Abel transform on M.

Proposition 3.4 will be essential to establish (17) in Theorem 3.11. Recall that the dual of the Abel transform can be defined by the equation

$$\mathcal{A}^*(f)(X) = \int_K f(\pi_{\mathfrak{a}}(\mathrm{Ad}(k)\,X))\,dk$$

where, as before, $\pi_{\mathfrak{a}}$ is the orthogonal projection from $\mathfrak{p}$ to $\mathfrak{a}$ with respect to the Killing form. Note in particular that $\mathcal{A}^*(e^{\lambda(\cdot)})(X) = \psi_\lambda(X)$. Note also (see [25, Ch. IV, Theorem 10.11]) that unless $C(X)$ reduces to $\{X\}$, there exists a density $K(H,X)$ such that

$$\mathcal{A}^*(f)(X) = \int_{C(X)} f(H)\,K(H,X)\,dH.$$

Proof of Proposition 3.4. It should be noted that for Weyl-invariant f, $\mathcal{A}^*(f) = V_k(f)$ (refer to [9]). Since the argument of $\mathcal{A}^*$ in (14) is not Weyl-invariant, some proof is needed. Let $K(Z,Y)$ be the kernel of the dual Abel transform. Using (13), we have

$$\begin{aligned}
P^W(X,Y) &= \frac{\sum_{w,w_0\in W} P_k(w\cdot X, w_0\cdot Y)}{|W|^2} \qquad (\text{with } k=1)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\sum_{w,w_0\in W}\int_{C(w_0\cdot Y)}\frac{1-|w\cdot X|^2}{(1-2\,\langle w\cdot X,Z\rangle+|w\cdot X|^2)^{\gamma+d/2}}\,d\mu_{w_0\cdot Y}(Z)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle w\cdot X,Z\rangle+|X|^2)^{\gamma+d/2}}\,d\mu_Y(w_0^{-1}\cdot Z)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\overbrace{\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle w\cdot X,w_0\cdot Z\rangle+|X|^2)^{\gamma+d/2}}}^{\text{Weyl-invariant}}\,d\mu_Y(Z)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle w\cdot X,w_0\cdot Z\rangle+|X|^2)^{\gamma+d/2}}\,K(Z,Y)\,dZ\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle X,w^{-1}w_0\cdot Z\rangle+|X|^2)^{\gamma+d/2}}\,K(Z,Y)\,dZ\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle X,Z\rangle+|X|^2)^{\gamma+d/2}}\,K(w_0^{-1}\,w\,Z,Y)\,dZ\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}\,(1-|X|^2)\int_{C(Y)}\frac{1}{(1-2\,\langle X,Z\rangle+|X|^2)^{\gamma+d/2}}\,K(Z,Y)\,dZ.
\end{aligned}$$

□

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Remark 3.5 Note that our normalizing constant is different from what is found in [11, 12]. We explain here how they correspond in the complex case. In [11], the Poisson kernel $P^W(X,Y)$ is normalized in the following manner:

$$u(X) = c'_d \int_{\partial B(0,1)} f(y)\, P^W(X,Y)\, \pi(Y)^2\, \frac{dY}{w_d}$$

where c'_d is such that

$$1 = c'_d \int_{\partial B(0,1)} \pi(Y)^2\, \frac{dY}{w_d}$$

where, reading through [11, Page 1215],

$$c'_d = 2^\gamma\, (d/2)_\gamma \prod_{\alpha>0} \left(\frac{|\alpha|^2}{2}\, (\langle \alpha, \rho\rangle / |\alpha|^2 + 1) \right)^{-1}.$$

Our different normalizations come down to the equality

$$\frac{2^{2\,\gamma}\, (d/2)_\gamma}{\pi(\rho)\, |W|\, w_d} = \frac{c'_d}{w_d}$$

which gives the interesting equality

$$\frac{\pi(\rho)\, |W|}{2^\gamma} = \prod_{\alpha>0} \left(\frac{|\alpha|^2}{2}\, (\langle \alpha, \rho\rangle / |\alpha|^2 + 1) \right).$$

This equality is easily verified directly for the classical Lie algebras and for $\mathfrak{g}_2$ (the other exceptional Lie algebras require more work). It should be noted that in [11], Dunkl used the notation ν_h instead of ρ but refers to the fact that Opdam uses ρ in [38].

Corollary 3.6 *The Newton kernel of the flat complex symmetric space M is given by*

$$N^W(X,Y) = \frac{2^{2\,\gamma}\, ((d-2)/2)_\gamma}{|W|\, (2-d)\, w_d\, \pi(\rho)}\, \mathcal{A}^* \left((|Y|^2 - 2\, \langle X, \cdot\rangle + |X|^2)^{(2-d-2\,\gamma)/2} \right)(Y).$$

P r o o f. We apply the same computations as for the Poisson kernel to formula [16, (6.1)] (the constant has been adjusted to follow our conventions as per Remark 3.5). □

We now start to study the asymptotic behavior of the Poisson kernel $P^W(X,Y)$. Let us introduce some notations. We define

$$R(X,Y) = \sum_{w\in W} \frac{\epsilon(w)}{|X - w\cdot Y|^d} \quad \text{and} \quad T(X,Y) = \frac{R(X,Y)}{\pi(X)\pi(Y)}$$

and therefore,

$$P^W(X,Y) = \frac{1-|X|^2}{|W|\, w_d}\, T(X,Y).$$

The function $R(X,Y)$ is defined for $X, Y \in \mathfrak{a}$ such that $X \notin W\cdot Y = \{w\cdot Y |\ w\in W\}$. We will denote this domain by

$$D := \{(X,Y) \in \mathfrak{a}^2 |\ X \notin W\cdot Y\}$$

The function $T(X,Y)$ is, for now, defined for non-singular X, $Y \in \mathfrak{a}$ (*i.e.* such no nonzero root vanish on X or on Y) such that $X \notin WY$. We will see in Proposition 3.10 that the function $T(X,Y)$ extends by continuity to an analytic function on the domain D.

Studying the properties of $P^W(X,Y)$ is equivalent to studying the properties of $T(X,Y)$ and $R(X,Y)$. We will give some of them in Proposition 3.10. We start by introducing two auxiliary results.

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Lemma 3.7 *Assume $a_1, \ldots, a_d$ are not all 0 and let U be an open set. Let q be an analytic function on U which is 0 whenever $\sum_{k=1}^d a_k x_k = 0$. Then $q(X) = \left(\sum_{k=1}^d a_k x_k\right) r(X)$ where r is a analytic function on U.*

P r o o f. The Lemma follows from Lemma 2.3. We give here an elementary proof.

Using a change of variable, we can assume that $a_1 = 1$ and $a_i = 0$ for $i > 1$. It is also enough to show that for every $X_0 = (b_1, \ldots, b_d) \in U$, there exists $\epsilon > 0$ such that the result holds in the ball $B(X_0, \epsilon)$. If $X_0 \neq 0$, then pick $\epsilon > 0$ small enough so that $(x_1, \ldots, x_d) \in B(X_0, \epsilon)$ implies $x_1 \neq 0$. Then we can pick $r(X) = q(X)/x_1$.

Suppose now that $b_1 = 0$. We then have $q(x_1, \ldots, x_d) = x_1 \overbrace{\int_0^1 \frac{\partial}{\partial x_1} q(t\, x_1, x_2, \ldots, x_d)\, dt}^{r(X)}$ for $(x_1, \ldots, x_d) \in B(X_0, \epsilon) \subset U$. □

Proposition 3.8 *Let $p(X) = \prod_{i=1}^d \langle \alpha_i, X\rangle$ where no α_i's is a multiple of another α_j and let U be an open set. If q is an analytic function on U which is 0 whenever $\alpha_i(X) = 0$ for some i then $q(X) = p(X)\, r(X)$ where r is an analytic function on U.*

P r o o f. We use induction on n. Lemma 3.7 shows that the result is true for $n = 1$. Assume it is true for $n-1$, $n \geq 2$ and write $q(X) = \prod_{i=1}^{n-1} \langle \alpha_i, X\rangle\, r(X)$. Since $q(X) = 0$ when $\langle \alpha_n, X\rangle = 0$, we conclude that $r(X) = 0$ on the set $\{X | \langle \alpha_n, X\rangle = 0$ and $\langle \alpha_i, X\rangle \neq 0,\ i < n\}$. By continuity, we deduce that $r(X) = 0$ when $\langle \alpha_n, X\rangle = 0$ and, using Lemma 3.7 once more, we can conclude. □

Remark 3.9 We thank the referee for pointing out that Lemma 3.7 and Proposition 3.8 can be also be proven with the help of the Weierstrass division theorem, via Lemma 2.3.

Proposition 3.10

1. *(Symmetry in X and Y) $R(X,Y) = R(Y,X)$ and $T(X,Y) = T(Y,X)$.*
2. *(Skew-symmetry) $R(w_0 X, Y) = \epsilon(w_0)\, R(X,Y)$ and $R(X, w_0 Y) = \epsilon(w_0)\, R(X,Y)$.*
3. *(Nullity of R on singular arguments) $R(X,Y)$ is zero whenever at least one of X or Y is singular.*
4. *(analytic factorization of R, analytic extension of T to D.) There exists a function f analytic on D such that $R(X,Y) = \pi(X)\pi(Y) f(X,Y)$ on D. Equivalently, the function T extends to an analytic function on D.*
5. *(Non-nullity of T and P^W) When $X \in B$ and $Y \in \partial B$ then $T(X,Y) > 0$ and $P^W(X,Y) > 0$.*

P r o o f. The proof of (1) and (2) is straightforward.

(3) Suppose $\alpha(Y) = 0$. We use Property 2 and $\epsilon(\sigma_\alpha) = -1$ where σ_α is the reflection with respect to the hyperplane $\{\alpha = 0\}$. Since $R(X,Y)/\pi(Y)$ is analytic, the statement follows.

(4) This follows from Proposition 3.8.

(5) This follows from Proposition 3.4. The dual Abel integral transform of a strictly positive function is strictly positive.

□

Theorem 3.11 *Let $Y_0 \in \partial B$, $\Sigma' = \{\alpha \in \Sigma |\ \alpha(Y_0) = 0\}$ and $\Sigma'_+ = \Sigma' \cap \Sigma^+$. Then*

$$P^W(X, Y_0) \overset{Y_0}{\sim} \frac{2^{2\gamma'}\, (d/2)_{\gamma'}}{|W|\, w_d\, \pi'(\rho')\, (\pi''(Y_0))^2}\, \frac{1 - |X|^2}{|X - Y_0|^{2\gamma' + d}} \tag{15}$$

where $\gamma' = |\Sigma'_+|$ is the number of positive roots annihilating Y_0, $\pi'(Y) = \prod_{\alpha \in \Sigma'_+} \langle \alpha, Y\rangle$ and $\pi''(Y) = \prod_{\alpha \in \Sigma_+ \setminus \Sigma'_+} \langle \alpha, Y\rangle$.

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P r o o f. Let $W' = \{w \in W | w \cdot Y = Y\}$. In this proof, we consider $X \in V = B(Y_0, \epsilon)$ with $\epsilon > 0$ fixed and chosen in such a way that

$$\alpha(\bar{V}) \subset (0, \infty) \text{ for } \alpha \in \Sigma_+ \setminus \Sigma'_+ \text{ and } w\,V \cap V = \emptyset \text{ for every } w \in W \setminus W'.$$

Using Theorem 2.2, we have

$$P^W(X,Y) = \frac{1}{|W|\, w_d} \frac{1-|X|^2}{\pi(X)\,\pi(Y)} \sum_{w \in W} \frac{\epsilon(w)}{|X - w \cdot Y|^d}.$$

We consider $X \in V \setminus \{Y_0\}$ and we deal with

$$T(X,Y_0) = \frac{|W|\, w_d}{1-|X|^2} P^W(X,Y_0) = \frac{1}{\pi(X)\,\pi(Y_0)} \sum_{w \in W} \frac{\epsilon(w)}{|X - w \cdot Y_0|^d} = \frac{R(X,Y_0)}{\pi(X)\,\pi(Y_0)}. \tag{16}$$

By Proposition 3.10 applied to the root systems Σ and Σ', all the expressions in (16) are well defined for $X \in V \setminus \{Y_0\}$, if needed in the limit sense.

We decompose the sum $\sum_{w \in W}$ into two terms, the first being the sum over the subgroup $W' = \{w \in W | w \cdot Y_0 = Y_0\}$ which is the Weyl group of the root subsystem Σ'. We obtain

$$T(X,Y_0) = \frac{\sum_{w \in W} \epsilon(w)|X - w \cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)} = \frac{\sum_{w \in W'} \epsilon(w)|X - w \cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)} + \frac{\sum_{w \in W \setminus W'} \epsilon(w)|X - w \cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}.$$

By Proposition 3.10, all the expressions in the last formula are well defined for $X \in V \setminus \{Y_0\}$, if needed in the limit sense. Denote

$$T_1(X,Y_0) = \frac{\sum_{w \in W'} \epsilon(w)|X - w \cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)} \text{ and } T_2(X,Y_0) = \frac{\sum_{w \in W \setminus W'} \epsilon(w)|X - w \cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}.$$

Let $\pi'(X) = \prod_{\alpha \in \Sigma'_+} \alpha(X)$ and $\pi''(X) = \prod_{\alpha \in \Sigma_+ \setminus \Sigma'_+} \alpha(X)$. Observe that by Theorem 2.2,

$$\pi''(X)\,\pi''(Y_0)\,T_1(X,Y_0) = \frac{\sum_{w \in W'} \epsilon(w)|X - w \cdot Y_0|^{-d}}{\pi'(X)\,\pi'(Y_0)} = \frac{|W'|\, w_d}{1-|X|^2} P^{W'}(X,Y_0)$$

where $P^{W'}(X,Y)$ is the Poisson kernel for the flat symmetric space $(\mathbf{R}^d, \Sigma')$ corresponding to the complex root system Σ'. The convex hull $C'(Y_0) = \mathrm{conv}(W'\,Y_0) = \{Y_0\}$, so by Proposition 3.4 and the properties of $\mathcal{A}^*$,

$$\begin{aligned} \frac{1}{1-|X|^2} P^{W'}(X,Y_0) &= \frac{2^{2\gamma'}\,(d/2)_{\gamma'}}{\pi(\rho')\,|W'|\, w_d} \int_{C(Y_0)} \frac{1}{(1 - 2\,\langle X, Z\rangle + |X|^2)^{\gamma' + d/2}} \delta_{\{Y_0\}}(dZ) \\ &= \frac{2^{2\gamma'}\,(d/2)_{\gamma'}}{\pi'(\rho')\,|W'|\, w_d} \frac{1}{|X - Y_0|^{2\gamma' + d}} \end{aligned} \tag{17}$$

where $X \in B \cap V$.

We now prove that the function $X \mapsto T_2(X,Y_0)$ is bounded on V, which, together with (17), will conclude the proof. We denote by

$$N(X,Y) = \sum_{w \in W \setminus W'} \epsilon(w)|X - w \cdot Y|^{-d}$$

the numerator of T_2. Observe that $N(X,Y)$ is an analytic function on $V \times V$. The function

$$T_2(X,Y) = \frac{\sum_{w \in W \setminus W'} \epsilon(w)|X - w \cdot Y|^{-d}}{\pi(X)\,\pi(Y)}$$

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is well defined and analytic for $(X,Y)\in V\times V\setminus D$ with $D=\{(X,Y)\in\mathfrak{a}\times\mathfrak{a}\colon X=Y\}$, since $T(X,Y)$ and $T_1(X,Y)$ have these properties by Proposition 3.10 and $T_2=T-T_1$.

This implies that if $X'\in V$ or $Y'\in V$ are singular (*i.e.* $\alpha(X')=0$ or $\alpha(Y')=0$ for some $\alpha\in\Sigma'_+$) and $X'\neq Y'$ then the numerator $N(X',Y')=0$ since otherwise the limit $N(X,Y)/\pi(X)\pi(Y)$ could not exist when $(X,Y)\to(X',Y')$.

We deduce that if $X'\in V$ or $Y'\in V$ and $\alpha(X')=0$ or $\alpha(Y')=0$ for some $\alpha\in\Sigma'_+$) then $N(X',Y')=0$. This is also true for $X'=Y'$ since such points are limits when t tends to 1 of $(t\,X',Y')$ with singular $t\,X'\neq Y'$ and $N(t\,X',Y')$ converges to $N(X',Y')$.

By Proposition 3.8, there exists a function $F(X,Y)$ analytic on $V\times V$ such that

$$N(X,Y)=\pi'(X)\pi'(Y)F(X,Y),\qquad X,Y\in V$$

and, finally,

$$T_2(X,Y)=\frac{F(X,Y)}{\pi''(X)\pi''(Y)},\qquad X,Y\in V$$

(we have $\min_{X\in\bar V}\pi''(X)>0$ since $\pi''(\bar V)\subset(0,\infty)$). In particular, the function $X\mapsto T_2(X,Y_0)$ is bounded on V. □

Remark 3.12 For the asymptotic properties of P^W, besides the alternating sum formula, the approach via the Dunkl formula (13) and dual Abel transform, *i.e.* Proposition 3.4 is needed. We use it to compute the leading term $T_1(X,Y)$ in $T(X,Y)$.

3.2 Asymptotic behavior of the Newton kernel on flat complex symmetric spaces

Using the same approach as in the proof of Theorem 3.11 together with Corollary 3.6, we conclude that

Theorem 3.13 *Let $Y_0\in\overline{\mathfrak{a}^+}$. If $d=2$ and α, β are the simple roots then*

$$N^W(X,0)=\frac{-2^{2\,\gamma-1}\,(\gamma-1)!}{2\,\pi\,|W|\,\pi(\rho)}\,|X|^{-2\,\gamma}\ \textit{(case } Y_0=0\textit{)},$$
$$N^W(X,Y_0)\overset{Y_0}{\sim}\frac{-2^{2\,\gamma'-1}\,(\gamma'-1)!}{2\,\pi\,|W|\,\pi''(Y_0)^2\,\langle\alpha,\alpha\rangle}\,|X-Y_0|^{-2}\ \textit{where } Y_0\neq 0,\ \alpha(Y_0)\neq 0\ \textit{and}\ \beta(Y_0)=0.$$

If $d\geq 3$

$$N^W(X,Y_0)\overset{Y_0}{\sim}\frac{2^{2\,\gamma'}\,((d-2)/2)_{\gamma'}}{|W|\,(2-d)\,w_d\,\pi'(\rho')\,(\pi''(Y_0))^2}\,\frac{1}{|X-Y_0|^{2\gamma'+d-2}}. \tag{18}$$

Here $\gamma'=|\Sigma'_+|$ is the number of positive roots annihilating Y_0 and $\pi''(Y)=\prod_{\alpha\in\Sigma_+\setminus\Sigma'_+}\langle\alpha,Y\rangle$.

Remark 3.14 In the paper [20] exact estimates of the Poisson and Newton kernels P^W and N^W were proven complementing the results of Theorem 3.11 and Theorem 3.13. For the Poisson kernel it is proven that

$$P^W(X,Y)\asymp\frac{P^{\mathbf{R}^d}(X,Y)}{\prod_{\alpha\in\Sigma^+}|X-\sigma_\alpha Y|^2}$$

where σ_α is the symmetry with respect to the hyperplane perpendicular to α.

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4 Asymptotic behavior of spherical functions on flat complex symmetric spaces

In this section we consider spherical functions on M, satisfying the formula

$$\psi_\lambda(Y) = \frac{\pi(\rho)}{2^\gamma \pi(\lambda)\,\pi(Y)} \sum_{w\in W} \epsilon(w) e^{\langle \lambda, w\cdot Y\rangle}, \quad \lambda \in \mathfrak{a}^{\mathbf{C}},\ Y \in \mathfrak{a}^{\mathbf{R}}. \tag{19}$$

Note that our notation is different from that of Helgason (in his notation the function given by (19) is denoted $\psi_{-i\,\lambda}$).

The following technical lemma will prove useful later in this section.

Lemma 4.1 *Suppose G_1 and G_2 are subgroups of the finite group G. Then $|G_1\,G_2|\,|G_1\cap G_2| = |G_1|\,|G_2|$.*

P r o o f. The group $G_1 \times G_2$ acts on the set $G_1\,G_2 \subset G$ via $(g_1, g_2)(g) = g_1\,g\,g_2^{-1}$. Clearly the action is transitive. The stabilizer of $e \in G_1\,G_2$ (e being the identity) is easily seen to be isomorphic to $G_1 \cap G_2$. The orbit-stabilizer theorem ([43, Theorem 5.8]) implies then that $|G_1\,G_2|\,|G_1\cap G_2| = |G_1|\,|G_2|$. □

We introduce here some notation. If $X \in \mathfrak{a}$, we denote by Σ_X^+ the positive root system $\Sigma_X^+ = \{\alpha \in \Sigma^+ \colon \alpha(X) = 0\}$ and by W_X the Weyl group generated by the symmetries s_α with $\alpha \in \Sigma_X^+$ (consequently, $W_X = \{w \in W \colon w\cdot X = X\}$). We also write $\pi_X(Y) = \prod_{\alpha\in\Sigma_X^+} \alpha(Y)$ and $c_X = \partial(\pi_X)(\pi_X)$ (this derivative is constant on $\mathfrak{a}$).

For $X \in \mathfrak{a}$ we define the polynomial $\pi'_X(Y)$ by $\pi(Y) = \pi_X(Y)\pi'_X(Y)$. Denote

$$W(\lambda_0, Y_0) = \{w \in W \colon \langle \lambda_0, w\cdot Y_0\rangle = \langle \lambda_0, Y_0\rangle\}.$$

Remark 4.2 We conjecture that the property $W(\lambda_0, Y_0) = W_{\lambda_0}\,W_{Y_0}$ is valid for all root systems. In Appendix A, we provide a series of proofs that cover all cases except for the exceptional root systems of type E. We also point out that if one of λ_0 or Y_0 is regular then this property is also verified, see [23].

Denote the Weyl subgroup $W_{\lambda_0,Y_0} = W_{\lambda_0} \cap W_{Y_0} = \{w \in W \colon w\cdot\lambda_0 = \lambda_0 \text{ and } w\cdot Y = Y\}$. The group W_{λ_0,Y_0} corresponds to the root system $\Sigma^+_{\lambda_0,Y_0} = \Sigma^+_{\lambda_0} \cap \Sigma^+_{Y_0}$. We write $\pi_0(Y) = \pi_{\lambda_0,Y_0}(Y) = \prod_{\alpha\in\Sigma^+_{\lambda_0,Y_0}} \alpha(Y)$ and $c_{\lambda_0,Y_0} = \partial(\pi_{\lambda_0,Y_0})(\pi_{\lambda_0,Y_0})$. Denote by $\mathcal{M}$ the set of positive roots that are neither in $\Sigma^+_{\lambda_0}$ nor in $\Sigma^+_{Y_0}$, *i.e.* $\mathcal{M} = \Sigma^+ \setminus (\Sigma^+_{\lambda_0} \cup \Sigma^+_{Y_0})$. We also write $\pi_{\mathcal{M}}(X) = \prod_{\alpha\in\mathcal{M}} \alpha(X)$.

Proposition 4.3

(i) *If $w \in W_Y$ then $\pi_Y(w\cdot X) = \epsilon(w)\,\pi_Y(X)$.*

(ii) *If $w \in W_Y$ then $\pi_Y(\partial)[f(w\cdot Y)] = \epsilon(w)(\pi_Y(\partial)f)(w\cdot Y)$.*

P r o o f. The property (i) is well known [24]. The property (ii) is straightforward for $f(X) = e^{\langle Z,X\rangle}$ and extends by linear density. □

Proposition 4.4 *Let λ_0, Y_0 be singular. The asymptotics of $\psi_{\lambda_0}(t\,Y_0)$ when $t \to \infty$ is given by the following formula:*

$$\psi_{\lambda_0}(t\,Y_0) \sim C(\lambda_0, Y_0) t^{|\Sigma^+_{Y_0}|-|\Sigma^+|} \sum_{w\in W(\lambda_0,Y_0)} \epsilon(w)\pi_{Y_0}(\partial^Y)\,\left(\pi_{\lambda_0}(w\cdot Y)e^{\langle\lambda_0, w\cdot Y\rangle}\right)\big|_{Y=t\,Y_0} \tag{20}$$

where $C(\lambda_0, Y_0) = (c_{\lambda_0}\,c_{Y_0}\,\pi'_{\lambda_0}(\lambda_0)\,\pi'_{Y_0}(Y_0))^{-1}$.

When $W(\lambda_0, Y_0) = W_{\lambda_0}\,W_{Y_0}$, the last formula simplifies to

$$\psi_{\lambda_0}(t\,Y_0) \sim C_1(\lambda_0, Y_0) t^{|\Sigma^+_{Y_0}|-|\Sigma^+|} \pi_{Y_0}(\partial^Y)\,\left(\pi_{\lambda_0}(Y)e^{\langle\lambda_0, Y\rangle}\right)\big|_{Y=t\,Y_0} \tag{21}$$

where $C_1(\lambda_0, Y_0) = C(\lambda_0, Y_0)|W_{\lambda_0}|\,|W_{Y_0}|/|W_{\lambda_0,Y_0}|$.

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P r o o f. We start with the alternating sum formula for the spherical function ψ_λ, written in the following way

$$\pi(\lambda)\pi(Y)\psi_\lambda(Y) = \sum_{w\in W} \epsilon(w) e^{\langle \lambda, w\cdot Y\rangle} \tag{22}$$

We write $\pi(\lambda) = \pi_{\lambda_0}(\lambda)\pi'_{\lambda_0}(\lambda)$ and $\pi(Y) = \pi_{Y_0}(Y)\pi'_{Y_0}(Y)$. We apply the operator $L = \pi_{Y_0}(\partial^Y)\pi_{\lambda_0}(\partial^\lambda)$ to both sides of (22). Using the fact that $\pi_{\lambda_0}(\partial^\lambda)\, e^{\langle \lambda, w\cdot Y\rangle} = \pi_{\lambda_0}(w\cdot Y) e^{\langle \lambda, w\cdot Y\rangle}$, we obtain

$$c_{\lambda_0}\, c_{Y_0}\, \pi'_{\lambda_0}(\lambda_0)\, \pi'_{Y_0}(t\, Y_0)\, \psi_{\lambda_0}(t\, Y_0) = \sum_{w\in W} \epsilon(w)\, \pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(w\cdot Y)\, e^{\langle \lambda_0, w\cdot Y\rangle}\right)\big|_{Y=t\, Y_0}.$$

In order to get the exact asymptotics of $\psi_{\lambda_0}(t\, Y_0)$, we only need to deal with $w \in W$ such that $\langle \lambda_0, w\cdot Y_0\rangle = \langle \lambda_0, Y_0\rangle$. This gives the asymptotics (20).

We now assume that $W(\lambda_0, Y_0) = W_{\lambda_0}\, W_{Y_0}$. The asymptotics (20) simplify, since by Proposition 4.3, we obtain for $w = w_1\, w_2$ with $w_1 \in W_{\lambda_0}$ and $w_2 \in W_{Y_0}$

$$\begin{aligned}
\pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(w\cdot Y)\, e^{\langle \lambda_0, w\cdot Y\rangle}\right) &= \epsilon(w_1)\pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(w_2 Y)\, e^{\langle \lambda_0, w_2 Y\rangle}\right) \\
&= \epsilon(w_1)\, \epsilon(w_2)\pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(Y)\, e^{\langle \lambda_0, Y\rangle}\right) \\
&= \epsilon(w)\, \pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(Y)\, e^{\langle \lambda_0, Y\rangle}\right).
\end{aligned}$$

Using Lemma 4.1, we have $|W_{\lambda_0} W_{Y_0}| = |W_{\lambda_0}||W_{Y_0}|/|W_{\lambda_0, Y_0}|$. We obtain the formula (21). □

Theorem 4.5 *Let λ_0, Y_0 be singular. Assume that $W(\lambda_0, Y_0) = W_{\lambda_0}\, W_{Y_0}$. Then the asymptotics of $\psi_{\lambda_0}(t\, Y_0)$ when $t \to \infty$ are given by the following formula:*

$$\psi_{\lambda_0}(t\, Y_0) \sim D(\lambda_0, Y_0)\, t^{-m}\, e^{t\, \langle \lambda_0, Y_0\rangle} \tag{23}$$

where m is the number of positive roots that are neither in $\Sigma^+_{\lambda_0}$ nor in $\Sigma^+_{Y_0}$ i.e.

$$m = card\mathcal{M} = |\Sigma|^+ - (|\Sigma^+_{\lambda_0}| + |\Sigma^+_{Y_0}| - |\Sigma^+_{\lambda_0} \cap \Sigma^+_{Y_0}|)$$

and

$$D(\lambda_0, Y_0) = \frac{c_{\lambda_0, Y_0}}{c_{\lambda_0}\, c_{Y_0}}\, \frac{|W_{\lambda_0}|\, |W_{Y_0}|}{|W_{Y_0} \cap W_{\lambda_0}|}\, \frac{1}{\pi_{\mathcal{M}}(\lambda_0)\, \pi_{\mathcal{M}}(Y_0)}.$$

Remark 4.6 When Y_0 is regular, the method of proof used in Theorem 3.11 for the asymptotics of the Poisson kernel could have been used here. When both λ_0 and Y_0 are singular, that approach fails to apply.

P r o o f. Using Leibniz formula, we have

$$\begin{aligned}
&\pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(Y)\, e^{\langle \lambda_0, Y\rangle}\right)\big|_{Y=t\, Y_0} \\
&= \pi_0(\partial^Y) \prod_{\alpha\in\Sigma^+_{Y_0}\setminus\Sigma^+_{\lambda_0}} \partial^Y(A_\alpha)\, \left(\pi_{\lambda_0}(Y) e^{\langle \lambda_0, Y\rangle}\right)\big|_{Y=t\, Y_0} \\
&= \pi_0(\partial^Y) \left(\prod_{\alpha\in\Sigma^+_{Y_0}\setminus\Sigma^+_{\lambda_0}} \langle \lambda_0, \alpha\rangle\, \pi_{\lambda_0}(Y)\, e^{\langle \lambda_0, Y\rangle} + P(Y)\, e^{\langle \lambda_0, Y\rangle} \right)\big|_{Y=t\, Y_0}
\end{aligned}$$

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The number of factors in each term $P(Y)$ of the form $\langle \eta, Y\rangle$ where η is a root, is strictly less than the number of factors in π_{λ_0} *i.e.* less than $|\Sigma^+_{\lambda_0}|$.

In the expression in the last line, all derivatives involving the term $e^{\langle \lambda_0, Y\rangle}$ give 0 since $\beta(\lambda_0) = 0$ for $\beta \in \Sigma^+_{Y_0} \cap \Sigma^+_{\lambda_0}$.

In the derivatives of $\pi_{\lambda_0}(Y)$, any term that contains $\langle \beta, Y\rangle$ with $\beta \in \Sigma^+_{Y_0} \cap \Sigma^+_{\lambda_0}$ will be zero when Y is replaced by $t\,Y_0$. Thus, for a nonzero result, the operator $\pi_0(\partial^Y)$ must be applied to $\pi_0(Y)$, what gives $c_{\lambda_0,Y_0} > 0$. We obtain

$$\begin{aligned}
&\pi_{Y_0}(\partial^Y)\,\left(\pi_{\lambda_0}(Y)\,e^{\langle \lambda_0, Y\rangle}\right)\big|_{Y=t\,Y_0}\\
&= \prod_{\alpha\in\Sigma^+_{Y_0}\setminus\Sigma^+_{\lambda_0}} \langle \lambda_0, \alpha\rangle \prod_{\gamma\in\Sigma^+_{\lambda_0}\setminus\Sigma^+_{Y_0}} \langle \gamma, t\,Y_0\rangle\, c_{\lambda_0,Y_0}\, e^{t\,\langle \lambda_0, Y_0\rangle} + \pi_0(\partial^Y)P(t\,Y_0)e^{t\langle \lambda_0, Y_0\rangle}\\
&= c_{\lambda_0,Y_0}\, t^{|\Sigma^+_{\lambda_0}|-|\Sigma^+_{\lambda_0}\cap\Sigma^+_{Y_0}|} \prod_{\alpha\in\Sigma^+_{Y_0}\setminus\Sigma^+_{\lambda_0}} \langle \lambda_0, \alpha\rangle \prod_{\gamma\in\Sigma^+_{\lambda_0}\setminus\Sigma^+_{Y_0}} \langle \gamma, Y_0\rangle\; e^{t\,\langle \lambda_0, Y_0\rangle} + \text{negligible terms.}
\end{aligned}$$

We labeled as "negligible terms" the terms with the derivatives involving $P(Y)$. They have the number of factors of the form $\langle \eta, t\,Y_0\rangle$ strictly less then $|\Sigma^+_{\lambda_0}| - |\Sigma^+_{\lambda_0,Y_0}|$, so strictly less than the term $\prod_{\gamma\in\Sigma^+_{\lambda_0}\setminus\Sigma^+_{Y_0}} \langle \gamma, t\,Y_0\rangle$. The rest follows from the definition of $C(\lambda_0, Y_0)$. □

Remark 4.7 We can give a more explicit expression for the constant D, using the formula

$$\partial(\pi)\,\pi = \frac{|W|\,\pi(\rho)}{2^\gamma},$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\alpha = \sum_{\alpha\in\Sigma^+}\alpha$ and $\gamma = |\Sigma^+|$ is the number of positive roots.

For $X \in \mathfrak{a}$, denote $p_X = \pi_X(\rho_X)$. Analogously, we define p_{X_1,X_2} for the root system annihilating both elements $X_1, X_2 \in \mathfrak{a}$. We have

$$D(\lambda_0, Y_0) = \frac{2^{\gamma_{\lambda_0,Y_0}-\gamma_{\lambda_0}-\gamma_{Y_0}}}{\pi_{\mathcal{M}}(\lambda_0)\,\pi_{\mathcal{M}}(Y_0)}\,\frac{p_{\lambda_0,Y_0}}{p_{\lambda_0}\,p_{Y_0}},$$

and therefore

$$\lim_{t\to\infty}\frac{\psi_{\lambda_0}(t\,Y_0)}{t^{-m}\,e^{t\langle \lambda_0, Y_0\rangle}} = \frac{2^{\gamma_{\lambda_0,Y_0}-\gamma_{\lambda_0}-\gamma_{Y_0}}}{\pi_{\mathcal{M}}(\lambda_0)\,\pi_{\mathcal{M}}(Y_0)}\,\frac{p_{\lambda_0,Y_0}}{p_{\lambda_0}\,p_{Y_0}}$$

Remark 4.8 As a quick application of Theorem 4.5, we find, in the flat complex case, a simple proof of a general result of Vogel and Voit: for symmetric spaces with subexponential (here polynomial) growth, the set of bounded spherical functions coincides with the support of the Plancherel measure of the associated Gelfand pair (G_0, K), G_0 the Cartan motion group. See for instance Sections 3.2 and 3.3. of [41] for details. A proof in the flat complex case was also proposed by Helgason in [23].

Remark 4.9 Taking into account the relationship between the spherical functions in the flat case and those in the curved case for the complex Lie groups, the estimates of spherical functions in [36, 44] extend to the flat case.

In this case, Theorem 4.5 completes the estimates of [36, 44] providing the exact asymptotics. We conjecture that asymptotics with appropriate constants and not only estimates hold in the results of Narayanan, Pasquale and Pusti [36] and Schapira [44].

The asymptotic expansion given in [2, Proposition 3.8] for regular λ_0 and Y_0 implies asymptotics of spherical functions in this case. Theorem 4.5 strengthens this result to singular λ_0 and Y_0.

Proposition 4.10 *Let X and Y be singular and $m' = |\Sigma^+_X \cup \Sigma^+_Y|$. With the same notation as in Theorem 4.5, we have*

$$p^W_t(X,Y) \sim \frac{D(X,Y)\,2^{m-d}}{|W|\,\pi^{d/2}\,\pi(\rho)}\,t^{-\frac{d}{2}-m'}\,e^{\frac{-|X-Y|^2}{4t}}$$

as $t \to 0^+$.

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P r o o f. From Theorem 4.5, we have for $t > 0$ close to 0,

$$\psi_X(Y/(2\,t)) \sim D(X,Y)\,(2\,t)^m\,e^{\langle X,Y/(2\,t)\rangle}.$$

Combined with (12), this leads us to

$$\begin{aligned} p_t^W(X,Y) &\sim \frac{2^m}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\,t^{-\frac{d}{2}-\gamma}\,e^{\frac{-|X|^2-|Y|^2}{4t}}\,D(X,Y)\,t^m\,e^{\langle X,Y/(2\,t)\rangle} \\ &= \frac{D(X,Y)\,2^m}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\,t^{-\frac{d}{2}-(\gamma-m)}\,e^{\frac{-|X-Y|^2}{4t}}. \end{aligned}$$

□

5 Applications to the Dyson Brownian motion and stochastic analysis

5.1 Definition and transition density of the Dyson Brownian motion

When a probabilist looks at formula (3), he or she sees in it the generator of the Doob h-transform (refer to [39]) of the Brownian Motion on $\mathbf{R}^d$ with the excessive function $h(X) = \pi(X)$. For the root system A_d on $\mathbf{R}^d$, the operator Δ^W restricted to functions on $\mathfrak{a}^+$, is the generator of the Dyson Brownian Motion on $\mathfrak{a}^+ \subset \mathbf{R}^d$ ([14]), *i.e.* the d Brownian independent particles $B_t^{(1)}, \ldots, B_t^{(d)}$ conditioned not to collide. More generally, for any root system Σ on $\mathbf{R}^d$, the construction of a Dyson Brownian Motion as a Brownian Motion conditioned not to touch the walls of the positive Weyl chamber, can be done ith a starting point $X \in \overline{\mathfrak{a}^+}$ ([18]).

Let us recall basic facts about the Doob h-transform and the Dyson Brownian Motion. Let Σ be a root system on $\mathbf{R}^d$ and $\pi(X) = \prod_{\alpha>0}\langle \alpha, X\rangle$. It is known that π is $\Delta_{\mathbf{R}^d}$-harmonic on $\mathbf{R}^d$ ([18]), so in particular π is excessive.

Definition 5.1 Let Σ be a root system on $\mathbf{R}^d$ and $\pi(X) = \prod_{\alpha>0}\langle \alpha, X\rangle$. The Dyson Brownian Motion D_t^Σ on the positive Weyl chamber $\mathfrak{a}^+$ is defined as the h-Doob transform of the Brownian Motion on $\mathbf{R}^d$, with $h = \pi$, *i.e.* its transition density is equal to

$$p_t^{\mathrm{D}}(X,Y) = \frac{\pi(Y)}{\pi(X)} p_t^{\mathrm{killed}}(X,Y), \qquad X \in \overline{\mathfrak{a}^+}, Y \in \mathfrak{a}^+, \tag{24}$$

where $p_t^{\mathrm{killed}}(X,Y)$ is the transition density of the Brownian Motion killed at the first strictly positive time of touching $\partial\mathfrak{a}^+$.

The infinitesimal generator of D_t^Σ is given by the formula ([39])

$$\Delta^D f = \pi^{-1}\Delta^{\mathbf{R}^d}(\pi f), \qquad \operatorname{supp} f \subset \mathfrak{a}^+, \tag{25}$$

which coincides on $\mathfrak{a}^+$ with formula (3) for Δ^W. The only differences with the symmetric flat complex case are that the domain of kernels $\mathcal{K}^D(X,Y)$ is restrained to $X \in \overline{\mathfrak{a}^+}, Y \in \mathfrak{a}^+$, and that no invariant measure $\pi^2(Y)\,dY$ appears for the integral kernels in the Dyson Brownian Motion case. Consequently, we obtain

Corollary 5.2 *The transition density and the heat kernel of the Dyson Brownian Motion D_t^Σ on $\mathfrak{a}^+ \subset \mathbf{R}^d$ is given by the formula*

$$p_t^{\mathrm{D}}(X,Y) = \frac{\pi(Y)}{\pi(X)} \sum_{w\in W} \epsilon(w)\,h_t(X - w\cdot Y), \qquad X \in \overline{\mathfrak{a}^+}, Y \in \mathfrak{a}^+,$$

where $h_t(X-Y) = \frac{1}{(4\,\pi\,t)^{d/2}} e^{-\frac{|X-Y|^2}{4\,t}}$ is the Euclidean heat kernel on $\mathbf{R}^d$.

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In the case $\Sigma = A_p$ *we have*

$$p_t^{\mathrm{D}}(X,Y) = \frac{\pi(Y)}{\pi(X)} \det\left(g_t(x_i,y_j)\right), \qquad X \in \overline{\mathfrak{a}^+}, Y \in \mathfrak{a}^+,$$

where $g_t(u,v) = \frac{1}{\sqrt{4\,\pi t}}\, e^{-|u-v|^2/4t}$ *is the 1-dimensional classical heat kernel.*

P r o o f. We use Theorem 2.2 (1) and Corollary 2.5. □

Comparing the formulas from Corollary 5.2 with formula (24), we obtain the following formulas for the heat kernel of the Brownian Motion killed at the first strictly positive time of touching a wall of the positive Weyl chamber.

Corollary 5.3 *The transition density for the Brownian Motion killed when exiting the positive Weyl chamber is given by the formula*

$$p_t^{\mathrm{killed}}(X,Y) = \sum_{w\in W} \epsilon(w)\, h_t(X - w\cdot Y). \tag{26}$$

In the case $\Sigma = A_{d-1}$ *we have*

$$p_t^{\mathrm{killed}}(X,Y) = \det\left(g_t(x_i,y_j)\right), \tag{27}$$

Remark 5.4 Karlin and McGregor [28] showed formula (27) by different methods. In [18], formulas for $p_t^{\mathrm{killed}}(X,Y)$ for the root systems B_d, C_d and D_d are proven. Our method of alternating sums provides a simple proof of formula (26) valid for any root system Σ.

5.2 Poisson and Newton kernels for the Dyson Brownian Motion

The Poisson and Newton kernels $P^D(X,Y)$ and $N^D(X,Y)$ are central objects of the potential theory of the Dyson Brownian Motion D_t^Σ and this is a first reason of studying them. However, these kernels have stochastic interpretation and, consequently, are useful in stochastic analysis of the Dyson Brownian Motion.

Denote by $D_t^{\Sigma,X}$ the Dyson Brownian Motion starting from X. Let $X \in B(0,1)$ and

$$T(X) = \inf\{t>0 \,|\, D_t^{\Sigma,X} \notin B(0,1)\}.$$

By the mean-value theorem for harmonic functions of general strong Markov processes (see [13, 26]), called sometimes Kakutani's Theorem ([7]), the Poisson kernel $P^D(X,Y)$ is the density of the random vector

$$D_{T(X)}^{\Sigma,X}$$

on the sphere. This is the Dyson Brownian Motion starting from X inside the unit ball and stopped at the first time $T(X)$ of exiting the ball. If dY denotes the Lebesgue measure on the unit sphere, then $P^D(X,Y)dY$ is called the harmonic measure of the Dyson Brownian Motion on the unit sphere.

The Newton kernel $N^D(X,Y)$ is related to the transition probability $p_t^D(X,Y)$ of the Dyson Brownian Motion by the formula ([7])

$$N^D(X,Y) = \int_0^\infty p_t^D(X,Y)dt.$$

The alternating sum formulas for the integral Poisson and Newton kernels P^D and N^D of the Dyson Brownian Motion D_t^Σ can be easily deduced from their counterparts (see Theorem 2.2) for the flat complex symmetric spaces M, just by multiplying P^W and N^W by $\pi(Y)^2$.

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Remark 5.5 We have

$$P^D(X,Y) = |W|\,\pi(Y)^2\,P^W(X,Y),$$
$$N^D(X,Y) = |W|\,\pi(Y)^2\,N^W(X,Y) \text{ and}$$
$$p_t^D(X,Y) = |W|\,\pi(Y)^2\,p_t^W(X,Y).$$

The Poisson kernel of the Dyson Brownian Motion extends continuously to $X \in \overline{\mathfrak{a}^+}$, $Y \in \overline{\mathfrak{a}^+}$. In particular, $P^D(X,Y) = 0$ when Y is singular. The same remarks apply to the Newton kernel $N^D(X,Y)$ and to the heat kernel.

These observations allow us to consider the ratios $P^D(X,Y)/\pi(Y)^2$, $N^D(X,Y)/\pi(Y)^2$ and $p_t^D(X,Y)/\pi(Y)^2$ even when $Y \in \partial\mathfrak{a}^+$.

Theorems 3.11 and 3.13 imply asymptotics for the Poisson and Newton kernels for the Dyson Brownian Motion. For completeness and for their applications in the potential theory and in the stochastic analysis of the process D_t^Σ, we state these results here.

Corollary 5.6 *The following formulas hold for $X, Y \in \overline{\mathfrak{a}^+}$*

$$P^{\mathrm{D}}(X,Y) = \frac{(1-|X|^2)\,\pi(Y)}{w_d\,\pi(X)} \sum_{w\in W} \frac{\epsilon(w)}{|X - w\cdot Y|^d}$$
$$N^{\mathrm{D}}(X,Y) = \frac{\pi(Y)}{2\,\pi\,\pi(X)} \sum_{w\in W} \epsilon(w)\,\ln|X - w\cdot Y| \text{ when } d = 2,$$
$$N^{\mathrm{D}}(X,Y) = \frac{\pi(Y)}{(2-d)\,w_d\,\pi(X)} \sum_{w\in W} \frac{\epsilon(w)}{|X - w\cdot Y|^{d-2}} \text{ when } d \geq 3.$$

Keeping in mind Remark 5.5, equations (15) and (18) lead us to the following result.

Corollary 5.7 *Let $Y_0 \in \overline{\mathfrak{a}^+}$, $\Sigma' = \{\alpha \in \Sigma |\ \alpha(Y_0) = 0\}$, $\Sigma'_+ = \Sigma' \cap \Sigma^+$, $\gamma' = |\Sigma'_+|$ and $\pi'(X) = \prod_{\alpha\in\Sigma'_+} \langle \alpha, X\rangle$.*

(i) *Let $Y_0 \in \partial B$. Then*

$$\frac{P^{\mathrm{D}}(X,Y_0)}{\pi'(Y_0)^2} \underset{Y_0}{\sim} \frac{2^{2\gamma'}\,(d/2)_{\gamma'}}{w_d\,\pi'(\rho')} \frac{1-|X|^2}{|X-Y_0|^{2\gamma'+d}}.$$

(ii) *If $d = 2$, α, β are the simple roots, $\alpha(Y_0) \neq 0$, $\beta(Y_0) = 0$, then*

$$\frac{N^{\mathrm{D}}(X,Y_0)}{\pi'(Y_0)^2} \underset{Y_0}{\sim} \frac{-2^{2\gamma'-1}\,(\gamma'-1)!}{2\,\pi\,\langle\alpha,\alpha\rangle}\,|X-Y_0|^{-2}.$$

(iii) *If $d \geq 3$, then*

$$\frac{N^{\mathrm{D}}(X,Y_0)}{\pi'(Y_0)^2} \underset{Y_0}{\sim} \frac{2^{2\gamma'}\,((d-2)/2)_{\gamma'}}{(2-d)\,w_d\,\pi(\rho')} \frac{1}{|X-Y_0|^{2\gamma'+d-2}}.$$

5.3 On the transition probability of the Dyson Brownian Motion

The heat kernel $p_t^D(X,Y)$ of the Dyson Brownian Motion is nonzero for $X \in \overline{\mathfrak{a}^+}$ and $Y \in \mathfrak{a}^+$ and zero if $Y \in \partial\mathfrak{a}^+$ as per Remark 5.5. By Proposition 4.10 we then have the following asymptotic result:

Corollary 5.8 *Let X and Y be singular and $m' = |\Sigma_X^+ \cup \Sigma_Y^+|$. With the same notation as in Theorem 4.5, we have*

$$\frac{p_t^D(X,Y)}{\pi(Y)^2} \sim \frac{D(X,Y)\,2^{m-d}}{|W|\,\pi^{d/2}\,\pi(\rho)}\,t^{-\frac{d}{2}-m'}\,e^{\frac{-|X-Y|^2}{4t}}$$

as $t \to 0^+$.

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Remark 5.9 In [31, 32] an asymptotic formula in terms of Schur functions was used to analyze the heat kernel of Dyson Brownian Motion.

We are grateful to one of the anonymous referees for suggesting that an asymptotic result for Dyson heat kernel was a natural extension of our results. It lead us to Corollary 5.8.

5.4 Remarks on relations to stochastic analysis

At the beginning of Section 5.2 and in the asymptotic formulas for the Poisson and Newton kernels in Corollary 5.6, the factor $\pi(Y)^2$ appears as a common feature. In the context of random matrix theory and non-colliding diffusive particle problems (the original Brownian motion models), this factor is very important as follows.

(i) This factor $\pi(Y)^2$ appearing in the probability density becomes zero if $x_j = x_i$. Then the system has some "repulsive" interaction and it will be regarded as a determinantal (Fermion) point process.

(ii) The squared Vandermonde determinant $\pi(Y)^2$ can be written as the determinant of a matrix whose entries are given by orthonormal polynomials. This opens the way to applications to reproducing kernels of Hilbert spaces spanned by these orthonormal functions.

(iii) The factor $\pi(Y)^2$ is a special case with $\beta = 2$ in the general setting $\prod_{j=1}^{N} (x_j - x_i)^\beta$ important in the theory of random matrices.

It is natural to ask whether it is possible to discuss the O'Connell and Macdonald stochastic processes studied in [5, 30, 37] from the viewpoint of the present paper. The multivariate processes studied there are related to the representation theory (e.g. Gelfand-Zetlin patterns), the symmetric functions and special functions (e.g. Whittaker functions, Macdonald polynomials), and integrable systems (e.g. quantum Toda lattice). This question is best left to another paper.

5.5 Curved case and relations to Schrödinger operators

The alternating sum formulas given in Section 2.2 have analogs in the curved complex case, considered in this section. To underline the difference with the flat case, we denote the spherical and potential analysis objects on M with a tilde ($\tilde{}$). The kernels in this section are with respect to the invariant measure $\delta(Y)\,dY$ where

$$\delta(Y) = \prod_{\alpha>0} \sinh^2 \alpha(Y).$$

The following method of construction of kernels is similar to the one presented in Section 2.2.

1. Exploit the formula for the Laplace-Beltrami operator on M ([25, Chap. II, Theorem 5.37]):

$$\tilde{\Delta}^W f = \delta^{-1/2} \, (\Delta^{\mathbf{R}^d} - |\rho|^2)(\delta^{1/2} \, f)$$

2. Apply the W-invariance.

In this way, the Euclidean kernel $\mathcal{K}^{\Delta^{\mathbf{R}^d} - |\rho|^2}(X, Y)$ (heat, potential, Poisson, ...) for the operator $\Delta^{\mathbf{R}^d} - |\rho|^2$ is transformed into the kernels $\tilde{\mathcal{K}}$ for G/K:

$$\tilde{\mathcal{K}}(X, Y) = \frac{1}{\delta^{1/2}(X)\,\delta^{1/2}(Y)} \sum_{w \in W} \epsilon(w) \mathcal{K}^{\Delta^{\mathbf{R}^d} - |\rho|^2}(X, w \cdot Y). \tag{28}$$

The estimates of the Newton kernel $\tilde{N}(X, Y)$ for all curved Riemannian symmetric spaces G/K were obtained in [1]. In the case when G is complex, it would be possible to apply our methods based on formula (28), using the knowledge of the Newton kernel $N(X, Y)$ of the Schrödinger operator $\Delta^{\mathbf{R}^d} - |\rho|^2$,

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i.e. the $|\rho|^2$-potential ($|\rho|^2$-resolvent) of $\Delta^{\mathbf{R}^d}$. The Newton kernel $N(X,Y)$ may be expressed with the Bessel function of third type $K_{d/2}$.

For the Poisson kernel for Δ^W, we need to know the Poisson kernel for the Schrödinger operator $\Delta^{\mathbf{R}^d} - |\rho|^2$. This kernel is not known explicitly. However much intensive work was and is presently being done in the analytic and stochastic theory of heat and other kernels for Schrödinger operators, see e.g. [3, 4, 7]. In a further work, we plan to study thoroughly these results and apply them to the estimates of the Poisson kernel on curved complex symmetric spaces.

A The Killing-max property

The aim of this appendix is to find precise conditions on $w \in W$ under which

$$\langle \lambda, w \cdot Y \rangle = \langle \lambda, Y \rangle. \tag{29}$$

Definition A.1 Let $W_\lambda = \{w \in W \colon w \cdot \lambda = \lambda\}$ (similarly for W_Y). We will say that the property Killing-max is satisfied if (29) is verified if and only if $w \in W_\lambda W_Y$.

Remark A.2 It is clear that the condition $w \in W_\lambda W_Y$ is sufficient. Property Killing-max is also satisfied whenever at least one of λ or Y is regular (refer to [23]). We observe also that this property only depends on the action of the Weyl group on the Cartan subalgebra $\mathfrak{a}$. Given that $\langle \lambda, w \cdot Y \rangle = \langle w^{-1}\lambda, Y \rangle$, this problem is symmetric in λ and Y.

In Table 2, we describe the action of the Weyl group on the Cartan subalgebra in the case of the noncompact and complex simple Lie algebras. Note that in the case of $(\mathfrak{f}_{4(-26)}, \mathfrak{so}(9))$, which is not in the table, the Killing-max property is trivially true since the rank of the space is 1.

A.1 Type A_n $(\mathfrak{sl}(n+1,\mathbf{F}))$

Lemma A.3 ("max principle" for permutations) *Let λ, $Y \in \mathbf{R}^n$ with their entries in decreasing order and let $w \in S_n$ be a permutation. Suppose that the block of λ_1 in λ has length $j_0 \geq 1$ and that the block of Y_1 in Y has length $i_0 \geq 1$. If $\min w^{-1}(\{1,\dots,i_0\}) > j_0$ then $\langle \lambda, w \cdot Y \rangle < \langle \lambda, Y \rangle$.*

Remark A.4 The lemma states that if $\langle \lambda, w \cdot Y \rangle = \langle \lambda, Y \rangle$ then the permutation w is such that "$\max Y$ meets $\max \lambda$", *i.e.* there exists $i \leq j_0$ such that $(w \cdot Y)_i = y_1$.

Proof. Without loss of generality, we may assume that $\lambda \neq \lambda_1 1^n$ and $Y \neq y_1 1^n$. Let $i = \min w^{-1}(\{1,\dots,i_0\})$. By assumption, the first y_1 appears in $w \cdot Y$ at the i-th position with $i > j_0$. Let $w(1) = k$, *i.e.* $w \cdot Y$ begins with y_k. We have $y_k < y_1$ and $\lambda_i < \lambda_1$. Consider $w_0 = (1i\,)\,w$; we then have

$$\langle \lambda, w_0 \cdot Y \rangle - \langle \lambda, w \cdot Y \rangle = (\lambda_1 - \lambda_i)\,(y_1 - y_k) > 0.$$

By the standard property of the Weyl group, $\langle \lambda, w_0 \cdot Y \rangle \leq \langle \lambda, Y \rangle$. Hence, $\langle \lambda, w \cdot Y \rangle < \langle \lambda, Y \rangle$. □

Corollary A.5 *Property Killing-max is verified in the case of the root system A_n.*

Proof. We use the same notation as in Lemma A.3 and in its proof. Suppose $\langle \lambda, Y \rangle = \langle \lambda, w \cdot Y \rangle$. We use induction on n. The result is clear for $n = 1$. By Lemma A.3, there exists $i \leq j_0$ such that $w(i) \leq i_0$.

We now apply the induction hypothesis to $\lambda = (\lambda_2,\dots,\lambda_n)$ and to $Y = (y_2,\dots,y_n)$. Let $\lambda' = (\lambda_2,\dots,\lambda_n)$, $Y' = (y_2,\dots,y_n)$ and note that $[y_{w(1)},\dots,\widehat{y_{w(i)}},\dots,y_{w(n)}]$ is a permutation of Y' (say $w' \cdot Y'$). We have $\langle \lambda, w \cdot Y \rangle = \lambda_i\, y_1 + \sum_{k=2}^n \lambda_k\, y_{w'(k)}$ where w' is a permutation of $\{2,\dots,n\}$. We then have $\langle \lambda', Y' \rangle = \langle \lambda, w' \cdot Y' \rangle$ since $\lambda_i = \lambda_1$. By the induction hypothesis $w' = w_{\lambda'}\, w_{Y'} \in W_{\lambda'}\, W_{Y'}$. We extend $w_{\lambda'}$ and $w_{Y'}$ to $w_\lambda \in W_\lambda$ and $w_Y \in W_Y$ by having them fix 1 in both cases. With the permutation $w_0 = (1\ i) \in W_\lambda$, we have $w = (1\ i)\, w_\lambda\, w_Y \in W_\lambda\, W_Y$. □

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Symmetric space	Description of $X \in \mathfrak{a}^+$	Action of $w \in W$, the Weyl group	Underlying root system
$\mathbf{SL}(n,\mathbf{F})/\mathbf{SU}(n,\mathbf{F})$, $\mathbf{F}=\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, $n\geq 2$, $\mathbf{F}=\mathbf{O}$, $n=3$ (*i.e.* $\mathbf{E}_6/\mathbf{F}_4$)	$X=\mathrm{diag}[x_1,\dots,x_n]$, $\sum_{i=1}^n x_i=0$, $x_1>\cdots>x_n$	$w\in S_n$ permutes the entries x_i	A_{n-1}
$\mathbf{SO}(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, $1\leq p<q$, $\mathbf{SU}(p,q)/\mathbf{SU}(p)\times\mathbf{SU}(q)$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$, $1\leq p\leq q$,	$X=\begin{bmatrix}0&\mathcal{D}_X&0\\ \mathcal{D}_X&0&0\\ 0&0&0\end{bmatrix}$, $\mathcal{D}_X=\mathrm{diag}[x_1,\dots,x_p]$, $x_1>\cdots>x_p>0$	w permutes the x_i's and changes any number of signs	B_n
$\mathbf{SO}(p,p)/\mathbf{SO}(p)\times\mathbf{SO}(p)$, $p\geq 2$	$X=\begin{bmatrix}0&\mathcal{D}_X\\ \mathcal{D}_X&0\end{bmatrix}$, $\mathcal{D}_X=\mathrm{diag}[x_1,\dots,x_p]$, $x_1>\cdots>x_{p-1}>\lvert x_p\rvert$	w permutes the x_i's and changes any **even** number of signs	D_n
$\mathbf{SO}^*(2\,n)/\mathbf{U}(n)$, $n\geq 3$	$X=\left[\begin{array}{c\|c}0_{n\times n}&\mathcal{E}_X\\ \hline -\mathcal{E}_X&0_{n\times n}\end{array}\right]$, $\mathcal{E}_X=\sum_{k=1}^{[n/2]} x_k\,F_{2\,k,2\,k+1}$, $x_1>\cdots>x_{n/2}>0$	w permutes the x_i's and changes any number of signs	B_n
$\mathbf{Sp}(n,\mathbf{R})/\mathbf{U}(n)$ and $\mathbf{Sp}(n,\mathbf{C})/\mathbf{Sp}(n)$, $n\geq 1$	$\begin{bmatrix}0&i\,\mathcal{D}_X\\ -i\,\mathcal{D}_X&0\end{bmatrix}$, $\mathcal{D}_X=\mathrm{diag}[x_1,\dots,x_p]$, $x_1>\cdots>x_{p-1}>x_p>0$	w permutes the x_i's and changes any number of signs	C_n
$\mathbf{SO}(2\,n,\mathbf{C})/\mathbf{SO}(2\,n)$, $n\geq 3$	$X=i\sum_{k=1}^n x_k\,F_{2\,k-1,2\,k}$, $x_1>\cdots>x_{p-1}>\lvert x_p\rvert$	w permutes the x_i's and changes any **even** number of signs	D_n
$\mathbf{SO}(2\,n+1,\mathbf{C})/\mathbf{SO}(2\,n+1)$, $n\geq 2$	$X=i\sum_{k=1}^n x_k\,F_{2\,k-1,2\,k}$, $x_1>\cdots>x_{p-1}>x_p>0$	w permutes the x_i's and changes any number of signs	B_n
$\mathbf{F}_4^{\mathbf{C}}/\mathbf{F}_4$, $(\mathfrak{f}_{4(4)},\mathfrak{sp}(3)+\mathfrak{su}(2))$	$X=[x_1,x_2,x_3,x_4]$, $x_2>x_3>x_4>0$, $x_1>x_2+x_3+x_4$	Refer to [6]	F_4
$\mathbf{G}_2^{\mathbf{C}}/\mathbf{G}_2$, $(\mathfrak{g}_{2(2)},\mathfrak{su}(3)+\mathfrak{su}(2))$	$X=\mathrm{diag}[x_1,x_2,x_1-x_2,0,x_2-x_1,-x_2,-x_1]$, $x_1>x_2>x_1/2$	Refer to [35]	G_2

Table 2 Action of the Weyl group (except for E_6, E_7 and E_8)

A.2 Type B_n ($\mathfrak{so}(2\,n+1,\mathbf{C})$) and C_n ($\mathfrak{sp}(n,\mathbf{C})$)

Proposition A.6 *Property Killing-max is verified in the case of the root systems B_n and C_n.*

P r o o f. Recall that B_n is the root system of $\mathfrak{so}(2n+1,\mathbf{C})$. The positive Weyl chamber is defined by the condition

$$\lambda_1>\lambda_2>\cdots>\lambda_n>0$$

The Weyl group is $W=S_n\rtimes\{\pm1\}^n$; its elements are called "signed permutations". It is straightforward to see that sign changes in $w\cdot Y$ strictly decrease $\langle\lambda,Y\rangle$ unless the negative terms in $w\cdot Y$ are in front of $\lambda_i=0$.

More precisely, if $w\cdot Y$ has strictly negative terms in positions where $\lambda_i>0$, then $\langle\lambda,w\cdot Y\rangle<\langle\lambda,w_0\,w\cdot Y\rangle\leq\langle\lambda,Y\rangle$ where w_0 changes the negative signs in $w\cdot Y$ into positive ones.

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Thus, if (29) holds, all negative terms in $w \cdot Y$ are in front of $\lambda_i = 0$. Then $w_0 \in W_\lambda$ and $\langle \lambda, w \cdot Y \rangle = \langle \lambda, w_0\, w \cdot Y \rangle$. All the terms of $w_0 w \cdot Y$ are non-negative and the result for A_n applies.

To conclude, it suffices to recall that C_n is the root system for $\mathfrak{sp}(n, \mathbf{C})$. We have $W(C_n) = W(B_n)$, the only difference is in the relative length of roots ([15, p. 227]). □

A.3 Type D_n ($\mathfrak{so}(2\,n, \mathbf{C})$)

The Weyl group W is composed by permutations and the signs change **by pairs**, *i.e.* of two terms simultaneously. The positive Weyl chamber $\mathfrak{a}^+$ is given by the condition

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > |\lambda_n|.$$

Lemma A.7 (The "max principle" for $W(D_n)$) *Suppose that λ, $Y \in \overline{\mathfrak{a}^+}$ and that $\lambda \neq a\,(1, \ldots, 1, -1)$. Suppose that the block of λ_1 in λ has length $1 \leq j_0 < n$. Suppose also that $\min\{k : (w \cdot Y)_k = y_1\} > j_0$ or that $\{k : (w \cdot Y)_k = y_1\} = \emptyset$. Then $\langle \lambda, w \cdot Y \rangle \ < \ \langle \lambda, Y \rangle$.*

P r o o f. Suppose λ and Y are as in the statement of the lemma. If y_1 appears in $w \cdot Y$ then $\langle \lambda, w \cdot Y \rangle < \langle \lambda, Y \rangle$ by Lemma A.3 so we can assume that only $-y_1$ appears.

Using the standard property of the Weyl group over A_n, $\langle \lambda, w \cdot Y \rangle \ \leq \ \langle \lambda, w_0\, w \cdot Y \rangle$ where $w_0 \in S_n$ re-orders the entries of $w \cdot Y$ in decreasing order. The last entry of $w_0\, w \cdot Y$ has to be $-y_1$.

We first assume $n = 2$, or $n \geq 3$ and $j_0 \leq n - 2$. As $\pm y_n \geq -y_i$ for all $i < n$, we can suppose that the $(n-1)$-entry is $-y_i$ for some i. Using the element w_1 of the Weyl group which changes signs and permutes the last two entries, we have $\langle \lambda, w_1\, w_0\, w \cdot Y \rangle - \langle \lambda, w_0\, w \cdot Y \rangle = (\lambda_{n-1} + \lambda_n)\,(y_1 + y_i) \geq 0$. It is easy to check that the last inequality is strict if $n = 2$. Finally, by another application of Lemma A.3, $\langle \lambda, w_1\, w_0\, w \cdot Y \rangle < \langle \lambda, Y \rangle$ and the result follows.

We next handle the case $j_0 \geq n - 1$, with $n \geq 3$. Let $\lambda = (a, \ldots, a, b)$ with $b \in (-a, a]$. and $n \geq 3$. We will show that $\Delta = \langle \lambda, Y \rangle - \langle \lambda, w_0\, w \cdot Y \rangle > 0$. If $-y_n$ appears in $w_0 w \cdot Y$, we have, using $\sum_{i \neq 1,n} a\, y_i \geq \sum_{i \neq 1,n} a\,(\pm y_i)$,

$$\Delta = \langle \lambda, Y \rangle - \langle \lambda, w_0 w \cdot Y \rangle \geq a\, y_1 + b\, y_n - [a\,(-y_n) + b\,(-y_1)] = (a+b)(y_1 + y_n) > 0$$

where we used the hypothesis $b \neq -a$ and the fact that $y_1 + y_n > 0$ (otherwise $-y_n = y_1$ appears in $w_0 w \cdot Y$). If $-y_n$ does not appear in $w_0 w \cdot Y$, another $-y_k$ appears among the $n - 1$ first entries of $w_0 w \cdot Y$. This time, we obtain $\Delta \geq (a + b)\, y_1 + a\,(y_k - y_n) + a\, y_k + b\, y_n > 0$, where we used $y_1 > 0$ (as $Y \neq 0$), the hypothesis $a + b > 0$, and the inequalities $y_k \geq y_n$, $a\, y_k \geq |b\, y_n|$. □

Lemma A.8

P r o o f. Note that $\langle \lambda, Y \rangle = (n-1)\, a\, b + a\, b = n\, a\, b$. The only way that $\langle \lambda, w \cdot Y \rangle = n\, a\, b$ is if $w \cdot Y = Y$ *i.e.* $w \in W_Y = W_\lambda$. □

Proposition A.9 *Property Killing-max is verified in the case of the root system D_n.*

P r o o f. We proceed by induction on $n \geq 2$. Given Lemma A.8, if both λ and $Y \in \mathbf{R}\,(1, \ldots, 1, -1)$ then there is nothing to prove. Given the symmetry of the problem, if $\lambda \in \mathbf{R}\,(1, \ldots, 1, -1)$ and $Y \notin \mathbf{R}\,(1, \ldots, 1, -1)$, we can switch their roles and suppose that $\lambda \notin \mathbf{R}\,(1, \ldots, 1, -1)$.

The base case $n = 2$, in which, by Lemma A.8, we can assume that $\lambda = (\lambda_1, \lambda_2) \notin \mathbf{R}(1, -1)$, is clear by inspection.

Assume the result true for $n-1$, $n \geq 3$. As explained above, we may assume that $\lambda \notin \mathbf{R}(1, \ldots, 1, -1)$. By Lemma A.7, the equality (29) implies that "$\max \lambda$ meets $\max Y$". As in the case A_n, it follows that there exist permutations $\sigma \in W_\lambda$ and $\gamma \in W_Y$ such that $(\sigma w \gamma \cdot Y)_1 = y_1$. We consider $\tilde{\lambda}_1 = (\lambda_2, \lambda_3, \ldots, \lambda_n)$, $\tilde{Y}_1 = (Y_2, \ldots, Y_n)$ and $\tilde{w}_1 = \sigma w \gamma|_{\tilde{\mathfrak{a}}}$ where $\tilde{\mathfrak{a}} = \{(x_2, \ldots, x_n) |\ X = (x_i)_{i \geq 1} \in \mathfrak{a}\}$ and we use the induction hypothesis or Lemma A.8 depending on the situation. □

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A.4 Type F_4

We use Helgason [24] and some simple facts about the Weyl group $W = W(F_4)$ from [6]. We consider the simple roots $\alpha_1 = \mathbf{e}_2 - \mathbf{e}_3$, $\alpha_2 = \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_3 = \mathbf{e}_4$ and $\alpha_4 = (\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)/2$ and the corresponding reflections $s_{\alpha_i} = s_i$. It follows that $\mathfrak{a}^+ = \{(x_1, x_2, x_3, x_4) \colon x_1 > x_2 + x_3 + x_4,\ x_2 > x_3 > x_4 > 0\}$.

Denote $\alpha_{12} = \mathbf{e}_1 - \mathbf{e}_2$ and $s_{12} = s_{\alpha_{12}}$. Note that $\alpha_{12} = \alpha_2 + 2\,\alpha_3 + \alpha_4$ is a positive root. It is easy to check that

$$s_3\, s_4\, s_{12} = s_2\, s_3\, s_4 \tag{30}$$

by inspection or using [6, Table 1] on the basis $(\mathbf{e}_i)$.

Let $X = (x_1, x_2, x_3, x_4)$ with $x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0$, *i.e.* $X \in \overline{\mathfrak{a}^+(B_4)}$. We define $W_X^{B_4} \subset W(B_4)$ as the subgroup generated by a subset of the symmetries $s \in \{s_{12}, s_1, s_2, s_3\}$ such that $s(X) = X$.

Lemma A.10 *Let $\lambda \in \overline{\mathfrak{a}^+(B_4)}$. Then $W_\lambda^{B_4} \subset W_\lambda$.*

Proof. Clear from the definition of $W_\lambda^{B_4}$. □

Let α, β, γ denote the three sets of roots of F_4 defined in [6, p. 85], with $\alpha = (\pm\mathbf{e}_i)_{i=1}^4$. Let δ, $\eta \in \{\alpha, \beta, \gamma\}$ and $W_{\delta\eta} = \{w \in W \colon w(\delta) = \eta\}$. By [6], we have $W = W_{\alpha\alpha} \cup W_{\alpha\beta} \cup W_{\alpha\gamma}$. In order to describe the action of $w \in W$, we define $w_0^\alpha = id, w_0^\beta = s_3\, s_4$ and $w_0^\gamma = s_4$. Then, by [6, Table 1], we have $w_0^\delta(\alpha) = \delta$ with $\delta \in \{\alpha, \beta, \gamma\}$.

The following result is proven in [6]. Recall that $W(B_4)$ is the group of signed permutations of 4 elements.

Lemma A.11 *Let $\delta \in \{\alpha, \beta, \gamma\}$ and $w \in W_{\alpha\delta}$. There exists $\sigma \in W(B_4)$ such that if $Y = \sum_{i=1}^4 y_i\, \mathbf{e}_i$, then $w \cdot Y = \sum_{i=1}^4 y_{\sigma(i)}\, w_0^\delta(\mathbf{e}_i)$. Equivalently, $(w_0^\delta)^{-1}\, w$ is a signed permutation with respect to the basis $(\mathbf{e}_i)$.*

Proposition A.12 *Property Killing-max is verified in the case of the root system F_4.*

Proof. Suppose that $\lambda = \sum_{i=1}^4 \lambda_i\, \mathbf{e}_i$, $Y = \sum_{i=1}^4 y_i\, \mathbf{e}_i \in \overline{\mathfrak{a}^+(F_4)}$ are singular. Our objective is to solve the equation (29). We will assume from now on that (29) holds. We consider the three cases $w \in W_{\alpha\delta}$, where $\delta = \alpha, \beta, \gamma$.

If $w \in W_{\alpha\alpha}$, we note that $\mathfrak{a}^+(F_4) \subset \mathfrak{a}^+(B_4)$. Lemma A.11, Proposition A.6 and Lemma A.10 imply that $w \in W_\lambda^{B_4}\, W_Y^{B_4} \subset W_\lambda\, W_Y$.

In the case $w \in W_{\alpha\beta}$, we use $w_0 = w_0^\beta = s_3\, s_4$. If $\lambda = \sum_{i=1}^4 \lambda_i\, \mathbf{e}_i \in \overline{\mathfrak{a}^+(F_4)}$ then $\lambda' = w_0^{-1} \cdot \lambda = \sum_{i=1}^4 \lambda_i'\, \mathbf{e}_i$ with $\lambda_1' \geq \lambda_2' \geq \lambda_3' \geq \lambda_4' \geq 0$ since

$$\lambda' = \frac{1}{2}[(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)\, \mathbf{e}_1 + (\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4)\, \mathbf{e_2} + (\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)\, \mathbf{e}_3 + (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)\, \mathbf{e}_4]. \tag{31}$$

Using (29), Lemma A.11 and the standard property of the Killing form for B_4, we have

$$\langle \lambda, Y \rangle = \langle \lambda, w \cdot Y \rangle = \langle w_0^{-1} \cdot \lambda, w_0^{-1}\, w \cdot Y \rangle \leq \langle w_0^{-1} \cdot \lambda, Y \rangle = \langle \lambda, w_0 \cdot Y \rangle \leq \langle \lambda, Y \rangle.$$

This means that $\langle \lambda', w_0^{-1}\, w \cdot Y \rangle = \langle \lambda', Y \rangle$ and therefore that $w \in w_0\, W_{\lambda'}^{B_4}\, W_Y^{B_4}$ by Proposition A.6.

We reason similarly if $w \in W_{\alpha\gamma}$, with $w_0 = w_0^\gamma = s_4$ and

$$\begin{aligned}\lambda' = s_4(\lambda) = \frac{1}{2}[&(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\, \mathbf{e}_1 + (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\, \mathbf{e}_2 + (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)\, \mathbf{e}_3 \\ &+ (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)\, \mathbf{e}_4].\end{aligned}$$

It therefore follows that $w \in w_0\, W_{\lambda'}^{B_4}\, W_Y^{B_4}$ with $\lambda' = w_0^{-1} \cdot \lambda$.

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It is important to note that a feature of both cases $w \in W_{\alpha\beta}$ and $w \in W_{\alpha\gamma}$ implies that the respective w_0 satisfy $\langle \lambda, Y\rangle = \langle \lambda, w_0 \cdot Y\rangle$. It follows that these cases do not occur if $\alpha_4 \notin \Sigma_\lambda \cup \Sigma_Y$. Indeed, using the formula $s_i(X) = X - 2\,\frac{\alpha_i(X)}{\|\alpha_i\|^2}\alpha_i$, we have for $w_0^\beta = s_3\, s_4$ and for $w_0^\gamma = s_4$,

$$\begin{aligned} \langle \lambda, Y\rangle - \langle \lambda, s_3\, s_4\, Y\rangle &= 2\,\alpha_4(\lambda)\,\alpha_4(Y) + 2\,\alpha_3(\lambda)\,\alpha_3(Y) + 2\,\alpha_3(\lambda)\,\alpha_4(Y) \\ \langle \lambda, Y\rangle - \langle \lambda, s_4\, Y\rangle &= 2\,\alpha_4(\lambda)\,\alpha_4(Y) \end{aligned} \tag{32}$$

Thus $\langle \lambda, Y\rangle \neq \langle \lambda, w_0 \cdot Y\rangle$ if $\alpha_4 \notin \Sigma_\lambda \cup \Sigma_Y$ and $w \in W_{\alpha\beta}$ or $w \in W_{\alpha\gamma}$. We showed above that in the case $w \in W_{\alpha\alpha}$, formula (29) implies that $w \in W_\lambda\, W_Y$. The Proposition is thus proven for $\alpha_4 \notin \Sigma_\lambda \cup \Sigma_Y$.

It remains to treat the cases $\alpha_4 \in \Sigma_\lambda$ or $\alpha_4 \in \Sigma_Y$. By symmetry of the problem (29), it is sufficient to treat the case $\alpha_4 \in \Sigma_\lambda$, for any singular Y. We assume henceforth that $\alpha_4 \in \Sigma_\lambda$.

We showed above that in the case $w \in W_{\alpha\alpha}$, formula (29) implies that $w \in W_\lambda\, W_Y$.

If $w \in W_{\alpha\gamma}$, we have $w_0 = w_0^\gamma = s_4$ and therefore $\lambda' = s_4 \cdot \lambda = \lambda$ since $\alpha_4 \in \Sigma_\lambda$. Since $s_4 \in W_\lambda$, we have $w \in s_4 W_{\lambda'}^{B_4} W_Y^{B_4} = s_4 W_\lambda^{B_4}\, W_Y^{B_4} \subset W_\lambda\, W_Y$.

Suppose that $w \in W_{\alpha\beta}$ and recall that $w_0 = w_0^\beta = s_3\, s_4$. By (32), we have the following two cases:

(A) $\alpha_3(\lambda) = 0$ or (B) $\alpha_3(\lambda) \neq 0$, $\alpha_3(Y) = 0$ and $\alpha_4(Y) = 0$.

In the case (A), we have $w_0^{-1} \cdot \lambda = \lambda$ *i.e.* $\lambda' = \lambda$ and $s_3\, s_4 \in W_\lambda$. Therefore, we have $w \in s_3\, s_4 W_{\lambda'}^{B_4}\, W_Y^{B_4} = s_3\, s_4\, W_\lambda^{B_4}\, W_Y^{B_4} \subset W_\lambda\, W_Y$.

In the case (B), we compute using (31), $\lambda' = (\lambda_2 + \lambda_3, \lambda_2 + \lambda_4, \lambda_3 + \lambda_4, 0)$, where $\lambda_4 > 0$. We will be using s_3 defined by $s_3(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$. Note that $s_3 \cdot Y = Y$ since $y_4 = \alpha_3(Y) = 0$, and that s_3 commutes with s_1 and s_{12}. We consider the following mutually exclusive cases (B1)–(B4):

(B1) $\Sigma_\lambda = \{\alpha_4\}$: in that case, $W_{\lambda'}^{B_4} = \{id, s_3\}$ and $w \in s_3\, s_4 W_{\lambda'}^{B_4} W_Y^{B_4} \subset W_Y$.

(B2) $\Sigma_\lambda = \{\alpha_1, \alpha_4\}$, *i.e.* $\lambda_2 = \lambda_3 > \lambda_4 > 0$: in that case, $W_{\lambda'}^{B_4} = \{id, s_1\}\,\{id, s_3\}$.

Since s_1 commutes with s_3 and s_4, we have $w \in \{id, s_1\}\, s_3\, s_4\, \{id, s_3\}\, W_Y^{B_4} \subset W_\lambda\, W_Y$.

(B3) $\Sigma_\lambda = \{\alpha_2, \alpha_4\}$, *i.e.* $\lambda_3 = \lambda_4 > 0$: in that case, $W_{\lambda'}^{B_4} = \{id, s_{12}\}\,\{id, s_3\}$. Using (30), we find that $w \in \{id, s_2\} s_3\, s_4 \{id, s_3\}\, W_Y^{B_4} \subset W_\lambda\, W_Y$.

(B4) $\Sigma_\lambda = \{\alpha_1, \alpha_2, \alpha_4\}$, *i.e.* $\lambda_2 = \lambda_3 = \lambda_4 > 0$: in that case, $W_{\lambda'}^{B_4} = \{id, s_{12}, s_1, s_{12}s_1, s_1 s_{12}, s_1 s_{12} s_1\}\,\{id, s_3\}$.

Similarly as in (B2) and (B3), we verify that $s_3\, s_4 W_{\lambda'}^{B_4} \subset W_\lambda\, W_Y$. For example, $s_3\, s_4 (s_1 s_{12} s_1) = s_1\, s_3\, s_4 s_{12} s_1 = s_1\, s_2\, s_3\, s_4\, s_1 = s_1\, s_2\, s_1\, s_3\, s_4 \in W_\lambda\, W_Y$. Thus (29) implies that $w \in s_3\, s_4\, W_{\lambda'}^{B_4}\, W_Y^{B_4} \subset W_\lambda\, W_Y$. □

A.5 Type G_2

The Cartan space is given by $\mathfrak{a}(G_2) = \{H_{A,B} = (A, B, A-B, 0, B-A, -B, -A) \mid A, B \in \mathbf{R}\}$ and two simple positive roots are $\alpha(H_{A,B}) = A - B$ and $\beta(H_{A,B}) = B - (A - B) = 2B - A$. Consequently, the positive Weyl chamber is given by $\mathfrak{a}^+ = \{H_{A,B} \mid A > B > A - B > 0\}$.

Note that it is sufficient to work on the space $\mathfrak{a} = \{h_{A,B} = (A, B, A-B) \colon A, B \in \mathbf{R}\}$ which is isomorphic to $\mathfrak{a}(G_2)$. We will work on this space $\mathfrak{a}$ from now on. Observe also that the Weyl group W is generated by s_α which interchanges the first two entries and changes the sign of the third and $s_\beta = (2, 3)$, so it is included in $S_3 \rtimes \{1, -1\}^3$. This inclusion is strict: the group W has 12 elements and $S_3 \rtimes \{1, -1\}^3$ has $6 \times 2^3 = 48$ elements.

Proposition A.13 *Property Killing-max is verified in the case of the root system G_2.*

Proof. Given that the root system is of rank 2, we only need to consider three cases of singular λ and Y:

(C1) $\alpha(\lambda) = \alpha(Y) = 0$: We have $\lambda = (l, l, 0)$, $Y = (y, y, 0)$, l, $y > 0$ and $\langle \lambda, w \cdot Y\rangle = \langle \lambda, Y\rangle = 2\,l\,y$. It follows that 0 in Y cannot change position in $w \cdot Y$ and no y can become $-y$, so $w \cdot Y = Y$ and $w \in W_Y$.

(C2) $\alpha(\lambda) = \beta(Y) = 0$: We have $\lambda = (l, l, 0)$, $Y = (2\,y, y, y)$, l, $y > 0$ and $\langle \lambda, w \cdot Y\rangle = \langle \lambda, Y\rangle = 3\,l\,y$. Then no minus sign is possible in the first two terms of $w \cdot Y$ and $2\,y$ cannot go to the third position. Consequently, using the fact that $(h_{A,B})_3 = (h_{A,B})_1 - (h_{A,B})_2$, we find that $w \cdot Y = (2\,y, y, y) = Y$ (so $w \in W_Y$) or $w \cdot Y = (y, 2\,y, -y) = s_\alpha Y$, which implies that $s_\alpha w \in W_Y$ and $w \in s_\alpha W_Y \subset W_\lambda W_Y$.

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(C3) $\beta(\lambda) = \beta(Y) = 0$: We have $\lambda = (2\,l, l, l)$, $Y = (2\,y, y, y)$, l, $y > 0$. Then $2\,y$ must remain in the first position in $w \cdot Y$ and no sign change can happen, thus $w \cdot Y = Y$ and $w \in W_Y$.

□

References

[1] J.-P. Anker and L. Ji. *Heat Kernel and Green Function Estimates on Noncompact Symmetric Spaces*, Geometric and Functional Analysis, 1999, v. 9, n. 6, 1035–1091.

[2] D. Barlet and J. L. Clerc. *Le comportement à l'infini des fonctions de Bessel généralisées, I*, Adv. in Math., 61 (1986), 165–183.

[3] K. Bogdan, J. Dziubański, K. Szczypkowski, *Sharp Gaussian estimates for heat kernels of Schrödinger operators,* Integral Equations Operator Theory 91 (2019), no. 1, Paper No. 3, 20 pp.

[4] K. Bogdan, W. Hansen, T. Jakubowski, *Localization and Schrödinger perturbations of kernels*, Potential Anal. 39 (2013), no. 1, 13-28

[5] A. Borodin, I. Corwin, *Macdonald processes*, Probab. Theory Relat. Fields 158, 2014, 225–400.

[6] P. Cahn, R. Haas, A. G. Helminck, J. Li and J. Schwartz. *Permutation notations for the exceptional Weyl group F_4*, INVOLVE 5:1 (2012), 81–89.

[7] K. L. Chung and Z. Zhao. *From Brownian Motion to Schrödinger's equation*, Grundlehren der mathematischen Wissenschaften, Springer, Volume 312, 1995.

[8] Z. Denkowska, M. Denkowski, J. Stasica, *Ensembles sous-analytiques la polonaise: avec une introduction aux fonctions et ensembles analytiques.* Paris : Editions Hermann, 2008. 144 p.

[9] M. de Jeu. *Paley-Wiener theorems for the Dunkl transform*, Trans. Amer. Math. Soc. 358, 2006, 4225–4250.

[10] C. F. Dunkl, *Intertwining Operators Associated to the Group S_3*, Transactions of the American Mathematical Society, Vol. 347, No. 9 (Sep., 1995), 3347–3374.

[11] C. F. Dunkl, *Integral kernels with reflection group invariance*, Canad. J. Math. 43 (1991), 1213–1227.

[12] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, 81. Cambridge University Press, Cambridge, 2001.

[13] E. B. Dynkin, Markov Processes. vols. 1 and 2. Academic Press, New York; Springer, Berlin, 1965.

[14] F. Dyson: A Brownian Motion Model for the Eigenvalues of a Random Matrix. *J. Math. Phys.* 3, 1191–198, 1962.

[15] K. Erdmann and M. J. Wildon. *Introduction to Lie Algebras*, Springer 2006

[16] L. Gallardo and C. Rejeb. *Newtonian potentials and subharmonic functions associated to root systems*, Potential Anal. 47(2017), 369–400.

[17] R. Gangolli. *Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces*, Acta Math. 121, 1968, 151–192

[18] D. J. Grabiner. *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices*, Annales de l'I.H.P. Probabilités et statistiques, 1999 vol. 35, no. 2, 177–204 .

[19] P. Graczyk, T. Luks and M. Rösler. *On the Green function and Poisson integrals of the Dunkl Laplacian*, Potential Anal 48(2018), 337–360.

[20] P. Graczyk, T. Luks and P. Sawyer. *Potential kernels for radial Dunkl Laplacians*, arXiv:1910.03105, 1–31, 2019.

[21] P. Graczyk and P. Sawyer. *The Convolution of orbital measures on symmetric spaces: a survey*, Proceedings of the Conference Probability on Algebraic and Geometric Structures, Contemporary Mathematics, Vol. 668, 81-110, 2016.

[22] Harish-Chandra. *Differential operators on a semisimple Lie algebra*, Amer. J. Math. 79 (1957), 241–310.

[23] S. Helgason, *The bounded spherical functions on the Cartan motion group*, arXiv:1503.07598, 1–7, 2015.

[24] S. Helgason, *Differential Geometry, Lie Groups and Symmetric spaces*, Graduate Studies in Mathematics, 34, American Mathematical Society, Providence, RI, 2001.

[25] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, Mathematical Surveys and Monographs, 83, American Mathematical Society, Providence, RI, 2000.

[26] O. Kallenberg, Foundations of Modern Probability Springer Series in Statistics, Probability and Its Applications. Springer 1997.

[27] J. El Kamel and Ch. Yacoub. *Poisson integrals and Kelvin transform associated to Dunkl-Laplacian operator*, GJPAM (Global Journal of Pure and Applied Mathematics) Vol. 3, nr. 3, 251-261 2007.

[28] K. S. Karlin and J. McGregor. *Coincidence probabilities*, Pacific J. Math. 9 (1959), 1141–1164.

[29] M. Katori. *Bessel Processes, Schramm–Loewner Evolution, and the Dyson Model*, Springer Briefs in Mathematical Physics, Vol.11, 2016.

Copyright line will be provided by the publisher

[30] M. Katori. *Survival probability of mutually killing Brownian motion and the OConnell process*, J. Stat. Phys.147, 2012, 206–223.
[31] M. Katori, H. Tanemura, *Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems*, J. Math. Phys. 45, 2004, 3058–3085.
[32] W. König and N. O'Connell, *Eigenvalues of the Laguerre process as non-colliding squared Bessel process*, Elec. Comm. Probab. 6, 2001, 107–114.
[33] S. G. Krantz and H. R. Parks. *A Primer of Real Analytic Functions*, Birkhäuser Adavnced Texts, Basler Lehrbúcher, Second Edition, 2002.
[34] I. G. Macdonald. *Some conjecture for root systems*, SIAM J. MATH. ANAL. Vol. 13, No. 6, 1982.
[35] A. Moy. *Minimal K-types for G_2 over a p-adic field*, Trans. Amer. Math. Soc. 305 (1988), no. 2, 517–529.
[36] E. K. Narayanan, A. Pasquale and S. Pusti. *Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications*, Advances in Mathematics, 252 (2014), 227–259.
[37] N. OConnell. *Directed polymers and the quantum Toda lattice*, Ann. Probab. 40, 2012, 437–458.
[38] E. M. Opdam. *Some applications of hypergeometric shift operators*, Invent. math. 98, 1989, 1–18.
[39] D. Revuz and M. Yor. *Continuous martingales and Browmian Motion* Third Edition, Springer, 2005.
[40] M. Rösler. *Generalized Hermite Polynomials and the Heat Equation for Dunkl Operators*, Commun. Math. Phys. 192, 519–541, 1998.
[41] M. Rösler and M. Voit. *Dunkl theory, convolutions algebras, and related Markov processes*, in: Harmonic and stochastic analysis of Dunkl processes (eds. P. Graczyk et al.). Travaux en cours 71, 2008.
[42] M. Rösler and M. Voit. *Positivity of Dunkl's intertwining operator via the trigonometric setting*, Int. Math. Res. Not. 2004, no. 63, 3379–3389.
[43] H. E. Rose. *A Course on Finite Groups*, Universitext, Springer, 2010.
[44] B. Schapira. *Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel*, Geom. Funct. Anal. 18 (1) (2008) 222–250.
[45] J. A. Wolf. *Spherical functions on Euclidean space*, J. Funct. Anal. 239 (2006) 127-136.
[46] J. Xu. *The bounded spherical functions on the Cartan Motion group and Generalizations for the eigenspaces of the Laplacian on* $\mathbf{R}^n$ arXiv:1608.05500v1 (2016)

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