Resolution of the symmetric algebra of an (n+1)-generated ideal of depth n

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RESOLUTION OF THE SYMMETRIC ALGEBRA OF AN (N+1)-GENERATED IDEAL OF DEPTH N

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ABSTRACT. We provide a locally free resolution of the symmetric algebra of the ideal of an $(n+1)$-generated ideal sheaf of depth n over a regular quasi-projective variety. The resolution is given in terms of the resolution of the ideal itself and of a Buchsbaum-Rim complex associated to an explicit map.

1. INTRODUCTION

Consider a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ with base locus Z. In order to compute some invariants of Φ, for instance its degree, one should resolve the indeterminacies of Φ, which amounts to blow-up Z or equivalently to work with the Rees algebra of the ideal I_Z of Z in $\mathbb{P}^n$. This is not quite easy in general, however a first step is to take the symmetric algebra $\mathrm{Sym}(I_Z)$ of I_Z, this is a larger algebra as the Rees algebra is obtained from it by killing the torsion part [Mic64, Lemme 1]. So a natural question is what is the shape of the resolution of $\mathrm{Sym}(I_Z)$, in particular, is it determined by some process involving the resolution of I_Z?

Focusing on the case where Z is finite, the goal of this paper is to give an affirmative answer to this question by providing a resolution of $\mathrm{Sym}(I_Z)$ as the mapping cone of two complexes, one being a modification of the resolution of I_Z, the other being the first Buchsbaum-Rim complex associated to an explicit map.

Although our main motivation comes from the algebro-geometric question above, finding a resolution of the symmetric algebra also has a purely algebraic interest. This problem was tackled in [BJ03] or [BC05] by investigating the symmetric algebra of a base ideal I of a map $\Phi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$. Moreover, in [CU02, Proposition 3.6(iii)], the two authors established that the resolution of $\mathrm{Sym}(I)$ is in particular sublinear, see Definition 2.9, by controlling spectral sequences related to the resolution of I.

In this paper, as our main result, we give the explicit description of the morphisms of a resolution of $\mathrm{Sym}(I)$ provided some assumptions on I. The sublinearity of the resolution of $\mathrm{Sym}(I)$ can then be recovered via this description, see Corollary 2.10 below.

Let us now state the main result of this note in the algebraic framework and come back later to the geometric problem above. Let R be a noetherian ring, I be an R-ideal generated by $n+1$ elements and

$$(F_\bullet) \qquad \dots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

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be a free resolution of R/I where $F_0 = R$ and $\operatorname{rank}(F_1) = n+1$. Recall that the depth $\operatorname{depth}(I,R)$ is the length of any maximal R-sequence in I [Eis95, 17.2]. Put $\widetilde{R} = \operatorname{Sym}(F_1)$, $\widetilde{(\_)} = (\_) \otimes_R \widetilde{R}$ and $\tau : \widetilde{F_1} = F_1 \otimes \widetilde{R} \to \widetilde{R}$ for the multiplication morphism. Then:

Theorem A. *Assuming that* $\operatorname{depth}(I,R) = n$, *a resolution of* $\operatorname{Sym}(I)$ *by finitely generated free* $\widetilde{R}$*-modules is the mapping cone of two complexes:*

- *one is the complex*
$$\dots \xrightarrow{\widetilde{d_5}} \widetilde{F_4} \xrightarrow{\widetilde{d_4}} \widetilde{F_3} \xrightarrow{\widetilde{d_3}} \widetilde{F_2} \xrightarrow{\tau\circ\widetilde{d_2}} \widetilde{R},$$
- *the second one is the Buschsbaum-Rim complex associated to the morphism*
$$\Psi = \begin{pmatrix} \widetilde{d_1} \\ \tau \end{pmatrix} : \widetilde{F_1} \to \widetilde{G}$$
where $\widetilde{G}$ *is a free* $\widetilde{R}$*-module of rank* 2.

See Theorem 2.4 for a more precise version of this result. Here is an example whose numerical data are extracted from [PRV01, Théorème 3.3.1]).

Example 1.1. Over any field k, let
$$I = (x_1^2 - x_1x_3, x_2^2 - x_2x_3, x_1x_2, x_0x_3)$$
be an ideal of $R = \mathrm{k}[x_0,\dots,x_3]$. In this situation, remark that the scheme $Z = \mathbb{V}(I)$ in $\mathbb{P}^3$ is finite and supported over four points. One can compute a minimal graded free resolution of I, for example using Macaulay2:

$$(F_\bullet) \qquad 0 \longrightarrow \underbrace{R(-3)^2}_{F_3} \xrightarrow{d_3} \underbrace{\begin{matrix} R(-1)^2 \\ \oplus \\ R(-2)^3 \end{matrix}}_{F_2} \xrightarrow{d_2} \underbrace{R^4}_{F_1} \xrightarrow{d_1 = \begin{pmatrix} \phi_0 & \dots & \phi_3 \end{pmatrix}} \underbrace{R(2)}_{F_0}.$$

where $\phi_0 = x_1^2 - x_1x_3, \dots, \phi_3 = x_0x_3$ are the previously mentioned generators of I.

Let $R[y_0,\dots,y_3] = \widetilde{R} = \operatorname{Sym}(R^4)$, then $\operatorname{Sym}(I)$ is generated over $\widetilde{R}$ by the entries in the matrix $\begin{pmatrix} y_0 & \dots & y_3 \end{pmatrix} \widetilde{d_2}$, where $\widetilde{d_2}$ stands for the base change of d_2 to $\widetilde{R}$. The ring $\widetilde{R}$ being bi-graded by the variables $x_0,\dots,x_3$ from one side and by $y_0,\dots,y_3$ for the other side, we write $\widetilde{R}(-i,-j)$ for a shift of degree i in the variables x and a shift of degree j in the variables y. Consider now the first Buchsbaum-Rim complex (cf Subsection 2.1) associated to the map

$$\Psi \ : \qquad \widetilde{F_1} = \widetilde{R}^4 \xrightarrow{\begin{pmatrix} \phi_0 & \dots & \phi_3 \\ y_0 & \dots & y_3 \end{pmatrix}} \widetilde{G} = \begin{matrix} \widetilde{R}(2,0) \\ \oplus \\ \widetilde{R}(0,1) \end{matrix}$$

which provides a minimal free resolution of $\ker(\Psi)$:

$$(B_\bullet) \qquad 0 \longrightarrow \underbrace{\begin{matrix} \widetilde{R}(-2,-3) \\ \oplus \\ \widetilde{R}(-4,-2) \end{matrix}}_{B_2} \xrightarrow{\delta_2} \underbrace{\widetilde{R}(-2,-2)^4}_{B_1} \longrightarrow \ker(\Psi) \longrightarrow 0.$$

Then Theorem A establishes that a bi-graded resolution of $\mathrm{Sym}(I)$ reads:

$$(\mathbb{F}_\bullet)\qquad 0 \longrightarrow \underbrace{\begin{matrix}\widetilde{R}(-2,-3)\\ \oplus \\ \widetilde{R}(-4,-2)\end{matrix}}_{\mathbb{F}_3} \longrightarrow \underbrace{\begin{matrix}\widetilde{R}(-2,-2)^4\\ \oplus \\ \widetilde{R}(-3,-1)^2\end{matrix}}_{\mathbb{F}_2} \longrightarrow \underbrace{\begin{matrix}\widetilde{R}(-1,-1)^2\\ \oplus \\ \widetilde{R}(-2,-1)^3\end{matrix}}_{\widetilde{F_2}=\mathbb{F}_1} \xrightarrow{\begin{pmatrix} y_0 & \dots & y_3\end{pmatrix}\widetilde{d_2}} \underbrace{\widetilde{R}}_{\mathbb{F}_0}$$

which can be verified by a numerical computation.

In the third and last section of this paper, we formulate this result in the geometric setting, that is when I is replaced by the ideal sheaf of a subscheme Z of depth n and generated by $n+1$ elements over a regular quasi-projective variety X. Looking back to the rational map $\Phi : X \dashrightarrow \mathbb{P}^n$ and the canonical map $p : \mathbb{P}^n_X \to X$, our motivation in this case is to study the length of a subscheme obtained as zero locus of a global section of the sheaf $p_*(\mathcal{O}_{\mathbb{P}(I)}(1)^n)$ and relate it to the topological degree of Φ, see [Dol11] for these definitions. Theorem 3.3 ensures that all higher direct image sheaves of p_* vanish.

The explicit computations given in this paper were made using Macaulay2. The corresponding codes are available on request.

2. Free resolution of $\mathrm{Sym}(I)$

In all the section, we let R be a commutative noetherian ring and I be an R-ideal generated by $n+1$ elements with $\mathrm{depth}(I,R) = n$. Let also

$$(F_\bullet)\qquad \dots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

be a free resolution of R/I where $F_0 = R$ and $\mathrm{rank}(F_1) = n+1$.

Let $\widetilde{R} = \mathrm{Sym}(F_1)$, $\widetilde{\ }$ be the functor $-\otimes \widetilde{R}$ and $\tau : \widetilde{F_1} \to \widetilde{R}$ be the multiplication morphism

$$\tau : \qquad \widetilde{F_1} = F_1 \otimes \mathrm{Sym}(F_1) \xrightarrow{\text{multiplication}} \widetilde{R}$$

where, more explicitly, given a basis of F_1, $\widetilde{R} = \mathrm{Sym}(F_1)$ is the polynomial ring $R[y_0,\dots,y_n]$ and $\tau : \widetilde{F_1} \to \widetilde{R}$ is represented by the matrix $\begin{pmatrix} y_0 & \dots & y_n\end{pmatrix}$.

Since the surjection $F_1 \overset{d_1}{\to} I$ gives rise to a surjection $\mathrm{Sym}(F_1) \to \mathrm{Sym}(I)$ whose kernel is $\mathrm{Im}(d_2) \subset F_1 \subset \mathrm{Sym}(F_1)$ [Bou70, A.III.69.4], it is relevant to describe a resolution $\mathbb{F}_\bullet$ of $\mathrm{Sym}(I)$ by free $\mathrm{Sym}(F_1)$-modules, the geometric counterpart of this fact simply being that given a projective scheme $Z = \mathbb{V}(I)$ with I generated by $n+1$ elements, the projectivization $\mathbb{P}(I) = \mathrm{Proj}\big(\mathrm{Sym}(I)\big)$ of I can be embedded into $\mathbb{P}^n_R$.

Consider now the following slight modification of the resolution ($F_\bullet$) tensored by $\widetilde{R}$:

$$(\widetilde{F}_\bullet)\qquad \dots \xrightarrow{\widetilde{d_4}} \widetilde{F_3} \xrightarrow{\widetilde{d_3}} \widetilde{F_2} \xrightarrow{\tau\circ\widetilde{d_2}} \widetilde{R}.$$

Since $\mathrm{Im}(d_2) \subset F_1 \subset \mathrm{Sym}(F_1)$ is the kernel of the surjection $\mathrm{Sym}(F_1) \to \mathrm{Sym}(I)$, it means that the zero-th homology in ($\widetilde{F}_\bullet$) is $\mathrm{Sym}(I)$. Moreover, since ($F_\bullet$) is exact, the homologies $H_\bullet(\widetilde{F_{i+2}} \overset{\widetilde{d_{i+2}}}{\to} \widetilde{F_{i+1}} \overset{\widetilde{d_{i+1}}}{\to} \widetilde{F}_i)$ vanish for $i \geq 2$.

Hence, the point is to handle the homology $H_\bullet(\widetilde{F_3} \overset{\widetilde{d_3}}{\to} \widetilde{F_2} \overset{\tau\circ\widetilde{d_2}}{\to} \widetilde{R})$ in ($\widetilde{F}_\bullet$).

2.1. **Algebraic background about the Buchsbaum-Rim complex.** We describe here the Buchsbaum-Rim complex associated to the morphism

$$\Psi = \begin{pmatrix} \widetilde{d_1} \\ \tau \end{pmatrix} : \widetilde{F_1} \to \widetilde{G}$$

where $\widetilde{G}$ is a free $\widetilde{R}$-module of rank 2. We follow the presentation and notation in [Eis95, A2.6]. This background, and especially the description of the differentials, will be used in Subsection 2.3.

Definition 2.1. The Buchsbaum-Rim complex associated to the morphism

$$\Psi \ : \qquad \widetilde{F_1} \xrightarrow{\begin{pmatrix} \widetilde{d_1} \\ \tau \end{pmatrix}} \widetilde{G} = \widetilde{R}^2$$

is the complex

$$0 \longrightarrow \wedge^{n+1}\widetilde{F_1} \otimes_{\widetilde{R}} D_{n-2}\widetilde{G}^* \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_2} \wedge^3\widetilde{F_1} \otimes_{\widetilde{R}} D_0\widetilde{G}^* \xrightarrow{\epsilon} \widetilde{F_1} \xrightarrow{\Psi} \widetilde{G} \tag{2.1}$$

where $D_i\widetilde{G}^*$ is the ith divided power of $\widetilde{G}^*$, $\widetilde{G}^*$ standing for $\mathrm{Hom}_{\widetilde{R}}(\widetilde{G}, \widetilde{R})$, and $\epsilon : \wedge^3\widetilde{F_1} \otimes_{\widetilde{R}} D_0\widetilde{G}^* \to \widetilde{F_1}$ is the splice morphism in the Buchsbaum-Rim complex which, in this case, is the composition

$$\begin{tikzcd} \wedge^3\widetilde{F_1} \arrow[r, "\text{co-multiplication } c"] \arrow[rr, bend right, "\epsilon"'] & \wedge^1\widetilde{F_1} \otimes_{\widetilde{R}} \wedge^2\widetilde{F_1} \arrow[r, "1 \otimes \wedge^2\Psi"] & \widetilde{F_1} \otimes_{\widetilde{R}} \wedge^2\widetilde{G}. \end{tikzcd}$$

More explicitly, given the choice of an isomorphism $\wedge^2\widetilde{G} \simeq \widetilde{R}$, a basis $(\tilde{\phi}_0, \dots, \tilde{\phi}_n)$ of $\widetilde{F_1}$ and letting ι being a subset $\{i_1, i_2, i_3,\ i_1 < i_2 < i_3\}$ of $\{0, \dots, n\}$ and $\tilde{\phi}_\iota = \tilde{\phi}_{i_1} \wedge \tilde{\phi}_{i_2} \wedge \tilde{\phi}_{i_3}$ for the corresponding product, then

$$\epsilon(\tilde{\phi}_\iota) = \sum_{\kappa \subset \iota, |\kappa| = 2} sgn(\kappa \subset \iota)\big(\det(\Psi_\kappa)\big)\tilde{\phi}_{\iota \setminus \kappa} \tag{2.2}$$

where Ψ_κ is the 2×2-submatrix of Ψ with columns corresponding to the basis elements indexed by κ, $sgn(\kappa \subset \iota)$ is the sign of the permutation of ι that puts the elements of κ into the first 2 positions, and $\tilde{\phi}_{\iota\setminus\kappa}$ is indexed by the element of the set $\iota\setminus\kappa$. Given $2 \leq i \leq n-1$ the morphism $\delta_i : \wedge^{i+2}\widetilde{F_1} \otimes_{\widetilde{R}} D_{i-1}\widetilde{G}^* \to \wedge^{i+1}\widetilde{F_1} \otimes_{\widetilde{R}} D_{i-2}\widetilde{G}^*$ is defined as follows

$$\begin{array}{rccc} \delta_i : & \wedge^{i+2}\widetilde{F_1} \otimes_{\widetilde{R}} D_{i-1}\widetilde{G}^* & \longrightarrow & \wedge^{i+1}\widetilde{F_1} \otimes_{\widetilde{R}} D_{i-2}\widetilde{G}^* \\ & u \otimes v & \longmapsto & \sum\limits_{s,t} [\Psi^*(u'_s)](v'_t) \cdot u''_s \otimes v''_t \end{array} \tag{2.3}$$

with the notation that $u \in \wedge^{i+2}\widetilde{F_1}$ decomposes as $\sum_s u'_s \otimes u''_s \in \widetilde{F_1} \otimes \wedge^{i+1}\widetilde{F_1}$ and $v \in D_{i-1}\widetilde{G}^*$ decomposes as $\sum_l v'_t \otimes v''_t \in \widetilde{G}^* \otimes D_{i-2}\widetilde{G}^*$ (this explicit description of these Eagon-Northcott morphisms δ_i can be found in [Eis05, A2H]).

The fact that (2.1) is a complex is explained for example in [Eis95, A2.6].

Proposition 2.2. *Assume that* $\operatorname{depth}(I,R)=n$ *where* $I=(\phi_0,\dots,\phi_n)$*, then the Buchsbaum-Rim complex* (2.1) *is a free resolution of* $\operatorname{coker}(\Psi)$*. Thus, the complex*

$$(B_\bullet)\qquad 0\longrightarrow \underbrace{\wedge^{n+1}\widetilde{F_1}\otimes_{\widetilde{R}} D_{n-2}\widetilde{G}^*}_{B_{n-1}} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_2} \underbrace{\wedge^{3}\widetilde{F_1}\otimes_{\widetilde{R}} D_0\widetilde{G}^*}_{B_1} \longrightarrow \ker(\Psi)\longrightarrow 0$$

is a free resolution of $\ker(\Psi)$*.*

Proof. By [Eis95, Theorem A.2.10 and the discussion about the Buchsbaum-Rim complex just before], it suffices to show that, since the ideal $I=(\phi_0,\dots,\phi_n)$ verifies $\operatorname{depth}(I,R)=n$, the ideal $I_2(\Psi)$ contains a regular sequence of length n. This follows from the fact that $\widetilde{R}/I_2(\Psi)$ is Cohen-Macaulay because it is determinantal [Eis95, Theorem 18.18] so its depth is equal to its height. Hence it suffices to show that $I_2(\Psi)$ has height n. Now, given basis such that $\widetilde{d_1}$ is represented by the matrix $\begin{pmatrix}\widetilde{\phi_0} & \dots & \widetilde{\phi_n}\end{pmatrix}$ and τ by the matrix $\begin{pmatrix}y_0 & \dots & y_n\end{pmatrix}$, Ψ is represented by the matrix

$$M=\begin{pmatrix}\widetilde{\phi_0} & \dots & \widetilde{\phi_n}\\ y_0 & \dots & y_n\end{pmatrix}.$$

Remark first that $\operatorname{height}\big(I_2(M)\big)\leq n$ by [BV88, Theorem 2.1]. Now, since $\operatorname{depth}(I,R)=n$ over the ring R in $n+1$ variables, we have necessarily $\operatorname{height}(I)=n$ and there are two cases:

- given $p\in\operatorname{Spec}(R)$ such that $I\subset p$ (hence $\operatorname{height}(p)\geq n$), then $I_2(M)\subset\widetilde{p}$ and $\operatorname{height}\big(I_2(\Psi)\big)\leq\operatorname{height}\widetilde{p}=\operatorname{height}p$,
- if $p\in R$ is such that $I\not\subset p$, then the sequence $\phi_0\otimes R_p,\dots,\phi_n\otimes R_p$ is regular in R_p. This implies that $I_2(M)$ has height n by [BV88, Theorem 2.5] when localising at p.

Hence, we have that $\operatorname{height}\big(\widetilde{R}/I_2(\Psi)\big)=n$.

□

2.2. **Mapping cone between $B_\bullet$ and $\widetilde{F}_\bullet$.** Let us describe now a map α between the complex

$$(B_\bullet)\qquad 0\longrightarrow \underbrace{\wedge^{n+1}\widetilde{F_1}\otimes_{\widetilde{R}} D_{n-2}\widetilde{G}^*}_{B_{n-1}} \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_2} \underbrace{\wedge^{3}\widetilde{F_1}\otimes_{\widetilde{R}} D_0\widetilde{G}^*}_{B_1}$$

and the complex

$$(\widetilde{F}_\bullet)\qquad \dots \xrightarrow{\widetilde{d_4}} \widetilde{F_3} \xrightarrow{\widetilde{d_3}} \widetilde{F_2}$$

defined above. First, take any map of complexes $\mu:\wedge^\bullet F_1\to F_\bullet$ from the Koszul complex

$$(\wedge^\bullet F_1)\qquad 0\longrightarrow \wedge^{n+1}F_1 \xrightarrow{k_{n+1}(d_1)} \dots \xrightarrow{k_3(d_1)} \wedge^2 F_1 \xrightarrow{k_2(d_1)} F_1 \xrightarrow{d_1} F_0$$

associated to d_1 to the complex ($F_\bullet$) that extends the identity map in degrees 0 and 1. Such a map μ exists because of the freeness of the considered modules and because $\operatorname{Im}(k_2(d_1))\subset\ker(d_1)$. Now, define the map $\alpha:B_\bullet\to\widetilde{F}_\bullet$ such that given $1\leq i\leq n-1$, the morphisms

$$\alpha_i: B_i=\wedge^{i+2}\widetilde{F_1}\otimes_{\widetilde{R}} D_{i-1}\widetilde{G}^*\to\widetilde{F_{i+1}}$$

are the compositions

$$\wedge^{i+2}\widetilde{F_1}\otimes_{\widetilde{R}} D_{i-1}\widetilde{G}^* \xrightarrow{c\otimes p_1} \wedge^1\widetilde{F_1}\otimes\wedge^{i+1}\widetilde{F_1} \xrightarrow{\tau\otimes\widetilde{\mu_{i+1}}} \widetilde{R}\otimes\widetilde{F_{i+1}} \tag{2.4}$$

where the composite is α_i, $c:\wedge^{i+2}\widetilde{F_1}\to\wedge^1\widetilde{F_1}\otimes\wedge^{i+1}\widetilde{F_1}$ is the co-multiplication map sending $\theta_1\wedge\ldots\wedge\theta_{i+2}$ to $\sum_i \theta_i\otimes\theta_1\wedge\ldots\wedge\hat{\theta}_i\wedge\ldots\wedge\theta_{i+2}$ and

$$\begin{aligned} p_1: \quad & D_{i-1}\widetilde{G}^* \longrightarrow \widetilde{R} \\ & e_1^{*(i-1-j)}\otimes e_2^{*(j)} \longmapsto \begin{cases} 1 \text{ if } j=0 \\ 0 \text{ if } j\geq 1 \end{cases} \end{aligned}$$

for the dual basis (e_1^*, e_2^*) of a basis (e_1, e_2) of $\widetilde{G}$.

Lemma 2.3. *The map $\alpha: B_\bullet \to \widetilde{F}_\bullet$ is a map of complexes.*

Proof. Given $1\leq i\leq n-2$, we have to show that the square:

$$\begin{array}{ccc} B_{i+1}=\wedge^{i+3}\widetilde{F_1}\otimes_{\widetilde{R}} D_i\widetilde{G}^* & \xrightarrow{\delta_{i+1}} & \wedge^{i+2}\widetilde{F_1}\otimes_{\widetilde{R}} D_{i-1}\widetilde{G}^* = B_i \\ \downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \alpha_i} \\ \widetilde{F_{i+2}} & \xrightarrow[\widetilde{d_{i+1}}]{} & \widetilde{F_{i+1}} \end{array} \tag{2.5}$$

is commutative. This commutativity is verified by writing explicitly the composition $\alpha_i\circ\delta_{i+1}$ and the composition $\widetilde{d_{i+1}}\circ\alpha_{i+1}$ and by using the fact that the square

$$\begin{array}{ccc} \wedge^{i+2}F_1 & \xrightarrow{k_{i+2}(d_1)} & \wedge^{i+1}F_1 \\ \downarrow{\scriptstyle \mu_{i+2}} & & \downarrow{\scriptstyle \mu_{i+1}} \\ F_{i+2} & \xrightarrow[d_{i+2}]{} & F_{i+1} \end{array}$$

is commutative so that $\widetilde{d_{i+2}}\circ\widetilde{\mu_{i+2}}=\widetilde{\mu_{i+1}}\circ\widetilde{k_{i+2}(d_1)}$. $\square$

As we are going to explain now, the mapping cone of α provides a resolution of $\mathrm{Sym}(I)$.

2.3. **Description of the resolution of** $\mathrm{Sym}(I)$**.** Let us state our main result. Recall that the morphisms α_i and δ_i are defined respectively in Subsection 2.2 and Subsection 2.1 and, by convention, we complete the resolution $\widetilde{F}_\bullet$ and $B_\bullet$ by zero maps and zero $\widetilde{R}$-modules in order to make the dots consistent in ($\mathbb{F}_\bullet$).

Theorem 2.4. *Let R be a commutative noetherian ring and I be an R-ideal generated by $n+1$ elements such that* $\mathrm{depth}(I,R)=n$*. Following the previous notation,* *a minimal free resolution of* $\mathrm{Sym}(I)$ *by* $\widetilde{R}$*-modules reads:*

$$(\mathbb{F}_\bullet) \qquad \cdots \xrightarrow{\begin{pmatrix}\delta_3 & 0\\ \alpha_3 & \widetilde{d_5}\end{pmatrix}} \underbrace{\begin{matrix}B_2\\ \oplus\\ \widetilde{F_4}\end{matrix}}_{\mathbb{F}_3} \xrightarrow{\begin{pmatrix}\delta_2 & 0\\ \alpha_2 & \widetilde{d_4}\end{pmatrix}} \underbrace{\begin{matrix}B_1\\ \oplus\\ \widetilde{F_3}\end{matrix}}_{\mathbb{F}_2} \xrightarrow{\begin{pmatrix}\alpha_1 & \widetilde{d_3}\end{pmatrix}} \underbrace{\widetilde{F_2}}_{\mathbb{F}_1} \xrightarrow{\tau\circ\widetilde{d_2}} \underbrace{\widetilde{R}}_{\mathbb{F}_0}.$$

The proof of Theorem 2.4 is contained in the following two lemmas.

Lemma 2.5. *Let* S *be a commutative ring,*

$$F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R$$

be an exact sequence of S*-modules and* $\tau : F_1 \to R$ *be an* S*-module homomorphism. Then there is an* S*-module isomorphism*

$$\ker(F_1 \xrightarrow{\begin{pmatrix}d_1\\ \tau\end{pmatrix}} R^2) \simeq H_\bullet(F_3 \overset{d_3}{\to} F_2 \overset{\tau\circ d_2}{\longrightarrow} R)$$

Proof. If $u \in \ker\begin{pmatrix}d_1\\ \tau\end{pmatrix}$, then $u = d_2v$ for some $v \in F_2$ since $\ker(d_1) = \mathrm{Im}(d_2)$. Thus we can define the morphism

$$\begin{aligned} \ker\begin{pmatrix}d_1\\ \tau\end{pmatrix} &\longrightarrow H_\bullet(F_3 \overset{d_3}{\to} F_2 \overset{\tau\circ d_2}{\longrightarrow} R)\\ u &\longmapsto [v] \end{aligned}$$

where $[v]$ stands for the homology class of v. Since $\ker(d_2) = \mathrm{Im}(d_3)$, the image of u depends only on the class of $[v]$ so this morphism is well defined and its inverse is explicitly given by:

$$d_2v \longleftarrow [v]$$

which is also well defined by definition of the homology $H_\bullet(F_3 \overset{d_3}{\to} F_2 \overset{\tau\circ d_2}{\longrightarrow} R)$. □

Lemma 2.6. *We have the following equality:*

$$\widetilde{d}_2 \circ \alpha_1 = \epsilon$$

where ϵ *is the splice map, cf Definition 2.1.*

Proof. Recall that the splice map ϵ is defined as the following composition

$$\begin{tikzcd} \wedge^3\widetilde{F_1} \arrow[r, "\text{co-multiplication } c"] \arrow[rr, bend right, "\epsilon"'] & \wedge^1\widetilde{F_1}\otimes_{\widetilde{R}}\wedge^2\widetilde{F_1} \arrow[r, "1\otimes\wedge^2\Psi"] & \widetilde{F_1}\otimes_{\widetilde{R}}\wedge^2\widetilde{G} \simeq \widetilde{F_1} \end{tikzcd}$$

so, as in the proof of Lemma 2.3, it suffices to write down explicitly the composition $\widetilde{d}_2 \circ \alpha_1$, that is:

$$\begin{tikzcd}
\wedge^3\widetilde{F_1} \arrow[r, "\text{co-multiplication } c"] \arrow[rr, bend right, "\alpha_1"'] & \wedge^1\widetilde{F_1}\otimes_{\widetilde{R}}\wedge^2\widetilde{F_1} \arrow[r, "\tau\otimes\widetilde{\mu_2}"] & \widetilde{R}\otimes_{\widetilde{R}}\wedge^2\widetilde{F_1}\simeq\widetilde{F_2} \arrow[r, "\widetilde{d_2}"] & \widetilde{F_1}
\end{tikzcd}$$

and to use that $\widetilde{d_2}\circ\widetilde{\mu_2}=\widetilde{k_2(d_1)}$ which implies the equality $\widetilde{d_2}\circ\alpha_1=\epsilon$ (recall the explicit description (2.2) of the splice map ϵ). □

Proof of Theorem 2.4. By Lemma 2.5 with $S=\widetilde{R}$, we have that

$$H_\bullet(\widetilde{F_3}\overset{\widetilde{d_3}}{\to}\widetilde{F_2}\overset{\tau\circ\widetilde{d_2}}{\to}\widetilde{R})\simeq\ker(\Psi).$$

Since this isomorphism is the one appearing in $\epsilon=\widetilde{d_2}\circ\widetilde{\mu_2}$ by Lemma 2.6, we have that ($\mathbb{F}_\bullet$) is a resolution of $\operatorname{Sym}(I)$ by $\widetilde{R}$-modules. □

2.4. **Consequences of Theorem 2.4.** We develop here some consequences of Theorem 2.4.

Corollary 2.7. *Let R be a commutative noetherian ring and I be an R-ideal generated by $n+1$ element and such that* $\operatorname{depth}(I,R)=n$. *Assume in addition that R/I has projective dimension n, then $\widetilde{R}/\operatorname{Sym}(I)$ has projective dimension n.*

Proof. Under our assumption, let:

$$(F_\bullet)\qquad 0\longrightarrow F_n\xrightarrow{d_n}\dots\xrightarrow{d_3}F_2\xrightarrow{d_2}F_1\xrightarrow{d_1}F_0$$

be a resolution of R/I. By the explicit description of the resolution of $\operatorname{Sym}(I)$ in Theorem 2.4, $\widetilde{R}/\operatorname{Sym}(I)$ has resolution

$$(\mathbb{F}_\bullet)\qquad 0\longrightarrow\underbrace{B_{n-1}}_{\mathbb{F}_n}\longrightarrow\underbrace{\begin{matrix}B_{n-2}\\ \oplus\\ \widetilde{F_n}\end{matrix}}_{\mathbb{F}_{n-1}}\longrightarrow\dots\longrightarrow\underbrace{\begin{matrix}B_1\\ \oplus\\ \widetilde{F_3}\end{matrix}}_{\mathbb{F}_2}\longrightarrow\underbrace{\widetilde{F_2}}_{\mathbb{F}_1}\longrightarrow\underbrace{\widetilde{R}}_{\mathbb{F}_0}$$

which concludes the proof. □

Given $1\le i\le n-1$, let denote $B_i=\bigoplus_{j=0}^{i-1}\widetilde{R}(-(i-j)\delta,-j-2)^{\binom{n+1}{i+2}}$ where the convention is that $\widetilde{R}(-k,-l)$ denotes a shift k in the variables of R and a shift l in the variables of $\widetilde{R}$.

Corollary 2.8. *Assume now that R is a graded commutative noetherian ring and let I be an R-ideal with* $\operatorname{depth}(I,R)=n$ *and generated by $n+1$ elements all homogeneous of the same degree δ. Let also*

$$(F_\bullet)\qquad\dots\xrightarrow{d_4}\underbrace{\bigoplus_{j\ge2}R(-j)^{\beta_{3j}}}_{F_3}\xrightarrow{d_3}\underbrace{\bigoplus_{j\ge1}R(-j)^{\beta_{2j}}}_{F_2}\xrightarrow{d_2}\underbrace{R^{n+1}}_{F_1}\xrightarrow{d_1}\underbrace{R(\delta)}_{F_0}$$

be a minimal graded free resolution of I, where $\beta_{ji}\ge0$ for all $i\ge2$, $j\ge i-1$.

Then a bi-graded minimal free resolution of $\mathrm{Sym}(I)$ *reads:*

$$(\mathbb{F}_\bullet) \qquad \cdots \xrightarrow{\begin{pmatrix}\delta_2 & 0\\ \alpha_2 & \widetilde{d_4}\end{pmatrix}} \underbrace{\begin{matrix} B_1 \\ \oplus \\ \underbrace{\bigoplus_{j\geq 2} \widetilde{R}(-j,-1)^{\beta_{3j}}}_{\widetilde{F_3}}\end{matrix}}_{\mathbb{F}_2} \xrightarrow{\begin{pmatrix}\alpha_1 & \widetilde{d_3}\end{pmatrix}} \underbrace{\bigoplus_{j\geq 1} \widetilde{R}(-j,-1)^{\beta_{2j}}}_{\mathbb{F}_1} \xrightarrow{\tau\circ\widetilde{d_2}} \underbrace{\widetilde{R}}_{\mathbb{F}_0}.$$

Proof. It suffices to consider all the explicit morphisms in the resolution ($\mathbb{F}_\bullet$) of $\mathrm{Sym}(I)$ from Theorem 2.4 and to keep track of the shifts in the graduation. From the definition of τ and $\widetilde{d_2}$, it is clear that $\mathbb{F}_1 = \bigoplus_{j\geq 1} \widetilde{R}(-j,-1)^{\beta_{2i}}$.

Let focus now on B_1. First $B_1 = \wedge^3 \widetilde{R}^{n+1} \simeq \widetilde{R}^{\binom{n+1}{3}}$ and it remains to identify the shifts. To this end, recall that the composition $\tau \circ \widetilde{d_2} \circ \alpha_1$ is equal to $\tau \circ \epsilon$, see Lemma 2.6. By the description (2.2) of ϵ via the 2-minors of Ψ and then by the composition of ϵ with τ, the shift has thus degree δ in the variables of R and degree 2 in the variables of $\widetilde{R}$. Hence $B_1 = \widetilde{R}(-\delta,-2)^{\binom{n+1}{3}}$. The other shifts in the pieces B_i for $2 \leq i \leq n-1$ are then given by the morphisms δ_i, see (2.3).

The resolution is moreover minimal because no morphism in this resolution has constant entries (constant entries meaning entries belonging to the degree-zero part of R). $\square$

Next we observe that the differentials in our resolution are linear or constant in the y variables.

Definition 2.9. Assuming that R is a graded commutative noetherian ring, let I be an R-ideal generated by $n+1$ elements all homogeneous of the same degree δ and let denote $y_0, \ldots, y_n$ the variables of $\widetilde{R} = \mathrm{Sym}(R^{n+1})$. A complex $\mathbb{F}_\bullet$ of free $\widetilde{R}$-module is *sublinear* if for all i the differential $\mathbb{F}_i \to \mathbb{F}_{i-1}$ is linear or constant in $y_0, \ldots, y_n$.

With this definition, we have:

Corollary 2.10. *Assuming that R is a graded commutative noetherian ring, let I be an R-ideal generated by $n+1$ elements all homogeneous of the same degree δ and such that* $\mathrm{depth}(I,R) = n$. *Then the ideal* $\mathrm{Sym}(I)$ *admits a sublinear free resolution over* $\widetilde{R}$.

Proof. This follows from the description of the morphisms of the resolution ($\mathbb{F}_\bullet$) of $\mathrm{Sym}(I)$ in Theorem 2.4. Indeed, for any $i \geq 1$, the morphisms $\delta_i : B_i \to B_{i-1}$ and $\alpha_i : B_i \to \widetilde{F_{i+1}}$ are linear or constant in the variables $y_0, \ldots, y_n$, see the respective definitions of δ_i and α_i in (2.3) and (2.4). $\square$

This latter result is sharp in the following sense. If $\mathrm{depth}(I,R) < n+1$, then the resolution of $\mathbb{X}$ might not be sublinear as shown in the following example. This example was explained to us by Aldo Conca.

Example 2.11. In $\mathbb{P}^3$, consider the zero locus Z of the ideal $I_Z = (-x_2^3x_3 + x_3^4, -x_2^4 - x_3^4, -x_1x_3^3 - x_3^4, x_2^2x_3^2 + x_3^4)$. We have $\mathrm{depth}(I_Z, R) = 2$ over the ring $R = \mathrm{k}[x_0, \ldots, x_3]$, $\dim(Z) = 1$, and a minimal graded free resolution of $I_{\mathbb{P}(I_Z)}$ over the ring $\widetilde{R} = R[y_0, \ldots, y_3]$ reads:

$$0 \to \widetilde{R}(-5,-3) \to \begin{array}{c}\widetilde{R}(-5,-2)\\ \oplus \\ \widetilde{R}(-4,-3)^3\end{array} \to \begin{array}{c}\widetilde{R}(-4,-1)\\ \oplus \\ \widetilde{R}(-3,-2)^3\\ \oplus \\ \widetilde{R}(-4,-2)\\ \oplus \\ \widetilde{R}(-3,-3)\end{array} \xrightarrow{l_2} \begin{array}{c}\widetilde{R}(-1,-1)\\ \oplus \\ \widetilde{R}(-2,-1)^2\\ \oplus \\ \widetilde{R}(-3,-1)\end{array} \to I_{\mathbb{P}(I_Z)} \to 0$$

where, as above, we wrote the shift in the y variables in the right position. As one can check from a computation (via a computer algebra system), the morphism l_2 has several quadratic entries in y, hence the resolution of $I_{\mathbb{P}(I_Z)}$ is not sublinear.

3. Locally free resolution of $\mathbb{P}(\mathcal{I})$

To finish, let us come back to our initial geometric motivation. Fix a field k and let X be a regular quasi-projective variety over k. Let $\mathcal{L}$ be a line bundle over X and let V be an $(n+1)$-dimensional subspace of $\mathrm{H}^0(X,\mathcal{L})$. The image of the *evaluation map* $\mathrm{V}\otimes\mathcal{L}^\vee \to \mathcal{O}_X$ is an ideal sheaf $\mathcal{I}_Z$ of a closed subscheme Z in X. Given a basis $(\phi_0,\dots,\phi_n)$ of V, this provides a rational map $\Phi : X \dashrightarrow \mathbb{P}(\mathrm{V})$ sending $x\in X$ to $\big(\phi_0(x):\ldots:\phi_n(x)\big)$ and defined away from Z.

Denoting $\mathbb{X}$ the projectivization $\mathrm{Proj}\,\big(\mathrm{Sym}(\mathcal{I}_Z)\big)$ of the ideal sheaf $\mathcal{I}_Z$, the surjection $\mathrm{V}\otimes\mathcal{L}^\vee \to \mathcal{I}_Z$ induces a closed embedding $\mathbb{X}\hookrightarrow\mathbb{P}^n_X$. This is just the geometric counterpart of the fact that $\mathrm{Sym}(\mathcal{I}_Z)$ is an ideal sheaf of $\mathrm{Sym}(\mathrm{V}\otimes\mathcal{L}^\vee)$ with the notation of Section 2.

We describe now a locally free resolution of the ideal sheaf $\mathcal{I}_{\mathbb{X}}$ in $\mathbb{P}^n_X$ assuming that Z has depth n in X. To this end, let

$$(\mathcal{F}_\bullet) \qquad 0 \longrightarrow \mathcal{F}_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_0$$

be a locally free resolution of $\mathcal{O}_Z\otimes\mathcal{L} = (\mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Z)\otimes\mathcal{L}$. Remark that $(\mathcal{F}_\bullet)$ has lenght n in view of the Auslander-Buchsbaum formula and because X is regular. Let also $p:\mathbb{P}^n_X\to X$ be the projective bundle map, ξ be the first Chern class $c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)$ of $\mathcal{O}_{\mathbb{P}^n_X}(1)$ and, depending on the context, η be either $c_1(\mathcal{L})$ or the pull back $p^*c_1(\mathcal{L})$ of $c_1(\mathcal{L})$ by p. Given the variables $y_0,\dots,y_n$ of $\mathbb{P}^n_X$, let Ψ be the map

$$\Psi \ : \qquad \widetilde{\mathcal{F}_1} = \mathrm{V}\otimes\mathcal{O}_{\mathbb{P}^n_X} \xrightarrow{\begin{pmatrix}\phi_0 & \ldots & \phi_n\\ y_0 & \ldots & y_n\end{pmatrix}} \widetilde{\mathcal{G}} = \begin{array}{c}\mathcal{O}_{\mathbb{P}^n_X}(\eta)\\ \oplus\\ \mathcal{O}_{\mathbb{P}^n_X}(\xi)\end{array}$$

Lemma 3.1. *Assuming Z has depth n, a locally free resolution of* $\ker(\Psi)$ *reads:*

$$(\mathcal{B}_\bullet) \qquad 0 \longrightarrow \mathcal{B}_{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_2} \mathcal{B}_1 \longrightarrow \ker(\Psi) \longrightarrow 0.$$

where $\mathcal{B}_i = \wedge^{i+2}\widetilde{\mathcal{F}_1}\otimes_{\widetilde{R}} D_{i-1}\widetilde{\mathcal{G}}^*\big(\det(\widetilde{\mathcal{G}}^*)\big)$ *for* $1\le i\le n-1$.

Proof. The arguments are the same as in the proof of Proposition 2.2. ☐

Now, as in Subsection 2.2, we define a mapping cone from $(\mathcal{B}_\bullet)$ and

$$(\widetilde{\mathcal{F}_\bullet}) \qquad \ldots \xrightarrow{\widetilde{d_4}} \widetilde{\mathcal{F}_3} \xrightarrow{\widetilde{d_3}} \widetilde{\mathcal{F}_2}.$$

Lemma 3.2. *For each $1 \leq i \leq n-1$, there exists a morphism $\alpha_i : \mathcal{B}_i \to \widetilde{\mathcal{F}_{i+1}}$ such that the collection of the morphisms $\alpha_1, \dots \alpha_{n-1}$ defines a map of complexes between $(\mathcal{B}_\bullet)$ and $(\widetilde{\mathcal{F}_\bullet})$.*

Proof. Let first define α_1. To this end, consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \ker(\Psi) \arrow[r] & \widetilde{\mathcal{F}_1} \arrow[r, "\Psi"] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^n_X}(\eta) \oplus \mathcal{O}_{\mathbb{P}^n_X}(\xi) \arrow[d, "pr_1"] \\ 0 \arrow[r] & \ker(\widetilde{\Phi}) \arrow[r] & \widetilde{\mathcal{F}_1} \arrow[r, "\widetilde{\Phi}"] & \mathcal{O}_{\mathbb{P}^n_X}(\eta) \end{tikzcd}$$

where pr_1 is the first projection. Since $(\mathcal{B}_\bullet)$ resolves $\ker(\Psi)$ and $(\widetilde{\mathcal{F}_\bullet})$ resolves $\ker(\widetilde{\Phi})$, we have a morphism $\ker(\Psi) \to \ker(\widetilde{\Phi})$ and thus by composition a morphism $\mathcal{B}_1 \to \ker(\widetilde{\Phi})$. Hence we are looking for a lift α_1 of the latter morphism as in the following diagram:

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d, dotted, "\alpha_1"'] \arrow[dr] & \\ \widetilde{\mathcal{F}_2} \arrow[r, two heads] & \ker(\widetilde{\Phi}). \end{tikzcd}$$

The existence of α_1 follows from the vanishing of the sheaf $\mathrm{Ext}^1\left(\mathcal{B}_1, \ker(\widetilde{d_2})\right)$ since, by using the resolution

$$\dots \xrightarrow{\widetilde{d_4}} \widetilde{\mathcal{F}_3} \longrightarrow \ker(\widetilde{d_2})$$

of $\ker(\widetilde{d_2})$ and by diagram chasing, $\mathrm{Ext}^1(\mathcal{B}_1, \ker(\widetilde{d_2})) = 0$. In turn, to prove this vanishing, by diagram chasing it suffices to show that

$$\mathrm{Ext}^1(\mathcal{B}_1, \widetilde{\mathcal{F}_3}) = \mathrm{Ext}^2(\mathcal{B}_1, \widetilde{\mathcal{F}_4}) = \dots = \mathrm{Ext}^{n-2}(\mathcal{B}_1, \widetilde{\mathcal{F}_n}) = 0.$$

These latter vanishings hold since for any $1 \leq j \leq n-2$,

$$\mathrm{Ext}^j(\mathcal{B}_1, \widetilde{\mathcal{F}_{j+2}}) \simeq \mathrm{H}^j(\mathbb{P}^n_X, \mathcal{B}_1^\vee \otimes \widetilde{\mathcal{F}_{j+2}}) = 0$$

in view of the projection formula and cohomological vanishings of the projective space, see [Har77, III.5, III.9.3].

The existence of the morphisms $\alpha_2, \dots, \alpha_{n-2}$ follows then from a finite induction (the base case being the existence of α_1 above). Indeed, for each $2 \leq i \leq n-2$, assuming the existence of a morphism $\alpha_{i-1} : \mathcal{B}_{i-1} \to \widetilde{\mathcal{F}_i}$ that commutes with the associated morphisms δ and $\widetilde{d}$ in $(\mathcal{B}_\bullet)$ and $(\widetilde{\mathcal{F}_\bullet})$ (induction hypothesis), we have a morphism $\mathcal{B}_i \to \ker(\widetilde{d_i})$ and we are looking for a lift α_i of the latter morphism as in the following diagram:

$$\begin{tikzcd} \mathcal{B}_i \arrow[d, dotted, "\alpha_i"'] \arrow[dr] & \\ \widetilde{\mathcal{F}_{i+1}} \arrow[r, two heads] & \ker(\widetilde{d_i}). \end{tikzcd}$$

As in the base case, the existence of this lift α_i follows from a diagram chasing that insures the vanishing of $\mathrm{Ext}^1\left(\mathcal{B}_i, \ker(d_{i+1})\right)$ since

$$\mathrm{Ext}^j(\mathcal{B}_i, \widetilde{\mathcal{F}_{j+i+1}}) \simeq \mathrm{H}^j(\mathbb{P}^n_X, \mathcal{B}_i^\vee \otimes \widetilde{\mathcal{F}_{j+i+1}}) = 0 \text{ for } 1 \leq j \leq n-i-1.$$

The last morphism $\alpha_{n-1} : \mathcal{B}_{n-1} \to \widetilde{\mathcal{F}_n}$ is then built from the morphism $\mathcal{B}_{i-1} \to \ker(\widetilde{d_{i-1}})$ and the identification $0 \to \widetilde{\mathcal{F}_n} \to \ker(\widetilde{d_{i-1}}) \to 0$. $\square$

Theorem 3.3. *Under the assumption that* $\operatorname{depth}(Z) = n$*, a locally free resolution of the ideal sheaf* $\mathcal{I}_{\mathbb{X}}$ *of* $\mathbb{X} = \mathbb{P}(\mathcal{I}_Z)$ *reads:*

$$0 \longrightarrow \mathcal{B}_{n-1} \xrightarrow{\begin{pmatrix} \delta_{n-1} \\ \alpha_{n-1} \end{pmatrix}} \begin{matrix} \mathcal{B}_{n-2} \\ \oplus \\ \widetilde{\mathcal{F}_n} \end{matrix} \xrightarrow{\begin{pmatrix} \delta_{n-2} & 0 \\ \alpha_{n-2} & \widetilde{d_n} \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} \delta_2 & 0 \\ \alpha_2 & \widetilde{d_4} \end{pmatrix}} \begin{matrix} \mathcal{B}_1 \\ \oplus \\ \widetilde{\mathcal{F}_3} \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_1 & \widetilde{d_3} \end{pmatrix}} \widetilde{\mathcal{F}_2} \longrightarrow \mathcal{O}_{\mathbb{P}^n_X}. \tag{3.1}$$

Proof. To show that (3.1) is a locally free resolution, we have to show that it is a resolution over every affine open set of $\mathbb{P}^n_X$. To this end, consider the restriction of (3.1) to such an affine open set U. Since the formation of the symmetric algebra commutes with base change, we have that this restriction coincides with the resolution obtained in the affine case in Theorem 2.4. Thus, this restriction provides a free resolution over U which concludes the proof. $\square$

References

[BC05] L. Busé and M. Chardin. Implicitizing rational hypersurfaces using approximation complexes. *J. of Symbolic Computation*, 40:1150–1168, 2005. ↑1

[BJ03] L. Busé and J.-P. Jouanolou. On the closed image of a rational map and the implicitization problem. *J. of Algebra*, 265:312–357, 2003. ↑1

[Bou70] N. Bourbaki. *Algèbre, Livre 1 3*. Élements de Mathématique. Springer-Verlag Berlin Heidelberg, 2007 (1970). ↑3

[BV88] W. Bruns and U. Vetter. *Determinantal Rings*. Lecture Notes in Mathematics 1327. Springer-Verlag Berlin Heidelberg, 1988. ↑5

[CU02] M. Chardin and B. Ulrich. Liaison and Castelnuovo-Mumford regularity. *Amer. J. Math.*, 124:1103–1124, 2002. ↑1

[Dol11] I.V. Dolgachev. *Classical Algebraic Geometry: a modern view*. Cambridge Univ. Press, 2011. ↑3

[Eis95] D. Eisenbud. *Commutative algebra, with a view toward algebraic geometry*. Graduate Texts in Mathematics. Springer, 1995. ↑2, ↑4, ↑5

[Eis05] D. Eisenbud. *The Geometry of Syzygies*. Graduate Texts in Mathematics. Springer-Verlag New York, 2005. ↑4

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977. ↑11

[Mic64] A. Micali. Sur les algèbre universelles. *Ann. Inst. Fourier*, 14:33–87, 1964. ↑1

[PRV01] I. Pan, F. Ronga, and T. Vust. Transformations birationnelles quadratiques de l'espace projectif complexe à trois dimensions. *Ann. Inst. Fourier*, 51:1153–1187, 2001. ↑2

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