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Abstract

Cooperation between firms can never improve the technical efficiency of any coalition of firms. This standard result of the productivity measurement literature is based on the directional distance function computed on firm groups. Directional distance functions are usually defined on the standard sum of input/output vectors. In this paper, the aggregation of input/output vectors is generalized thanks to an isomorphism in order to capture three results: the cooperation improves technical efficiency ; the cooperation reduces technical efficiency ; and finally the cooperation between firms yields no variation of technical efficiency, *i.e.*, the distance function is quasi linear. The improvement of technical efficiency is shown to be compatible with semilattice technologies. In this case, the firms merge according to their inputs only because constraints are imposed on outputs, and conversely, they may merge according to the outputs they can produce because some limitations are imposed on the use of inputs.

Keywords: Aggregation, Cooperative games, Distance functions, Productivity, Technical efficiency.

JEL Codes: D21, D24.

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1 Introduction

The cooperation between firms can never improve the technical efficiency of any given coalition (industry). This impossibility has become a standard result in the productivity measurement literature, see Briec, Dervaux and Leleu (2003) or Färe, Grosskopf and Zelenyuk (2008). The *firm game* is the transferable utility game (TU-game) that exhibits this impossibility (see Briec and Mussard, 2014), in other words, the core interior of the firm game is empty. The result is based on the directional distance functions applied to the technology of the industry, which relies on the standard sum of sets (technologies of the firms), see *e.g.* Färe, Grosskopf and Li (1992) and Li and Ng (1995). Li and Ng (1995) point out that the standard sum of sets can yield different results, in particular, it depends on the convexity of the technology set. Li and Ng's (1995) result follows the one due to Førsund and Hjalmarsson (1979) in which inputs/outputs are simply averaged thanks to an arithmetic mean. Li and Ng (1995) show that this particular aggregation provides a bad representation of the technical efficiency of the industry. In other words, the choice of the aggregation process, for aggregating input/outputs vectors or equivalently aggregating technologies of firms, has a crucial impact on the productivity measures. Since then, this question has been pending for a long time, and from the best of our knowledge, no attempt has been made to overcome this problem of aggregation, so that the standard sum is always used in the efficiency literature to analyze the cooperation between firms.

In cooperative games, Lozano (2012) show that firm groups may take benefit from cooperation when they share data about inputs and outputs. The firms have also the possibility to merge in a so-called *production games* (Lozano, 2013). In *firm games*, Briec and Mussard (2014) show that any given firm coalition may always improve its allocative efficiency if the input/output vectors are simply aggregated with the standard sum. However, the impossibility outlines before is always met, *i.e.*, the inefficiency of the industry is always greater than the sum of the inefficiencies of each firm. In other words, the *technical bias* that represents the difference between the two aforementioned inefficiencies, is always positive. Coalitions of firms are said to be sub-efficient because their cooperation increases the technical inefficiency of the group. As a consequence, the core of the firm game is empty: no firm can improve its technical efficiency by joining any given coalition.

In Data Envelopment Analysis (DEA), Post (2001) suggests a concave transformation of the input/output vectors in order to limit the number of observations to be non attainable, that is, input/output combinations being outside the envelopment of the data that characterizes the production technology. This point is crucial because decisions are taken on the basis of samples. If the sample size is low and if an important quantity of data is not exploitable by the current mathematical techniques, then it is difficult to propose accurate indicators for taking decisions. We aim at extending Post's (2001) suggestion about data transformation in a cooperative game framework, without taking recourse to DEA. The interesting feature relying on the change of variables of input/output vectors is the derivation of a flexible aggregation process, an *aggregator* from now on, that captures either improvement or decline of technical efficiency of firm coalitions.

The aggregation of technologies is generalized thanks to an aggregator, precisely a Φ-aggregator, inspired from Ben-Tal (1977) who studied the algebraic properties of the aggregation process Φ underlying the generalized mean introduced by Hardy, Littlewood and Pólya (1934) and characterized by Aczél (1966) and Eichorn (1979). The aggregation bias, *i.e.* the difference between the inefficiency of the firm coalition and the sum of each firm's inefficiency, takes different forms with respect to the nature of the isomorphism Φ. (i) The aggregation bias is null: there is no variation of technical efficiency inherent to the cooperation. We retrieve the well-known result due to Briec *et al.* (2003) and Färe *et al.* (2008) as a special case: the directional distance function is (quasi) linear. (ii) The bias is positive, so that the cooperation between firms is impossible. (iii) The bias is negative: the aggregate inefficiency of any given coalition of firms decreases with cooperation. The core of the firm game may be non void in this case. For that purpose, we begin with a general (non specified) distance function and a general aggregation process in order to understand the implications of a null bias when inputs and outputs are transformed by a general isomorphism Φ. The conclusion is clear, the distance function is quasi linear, *i.e.* there is no gain/loss for a firm to join a coalition. Consequently, the employ of a Φ-aggregator is robust for the measurement of group (technical) efficiency for any given distance functions. Also, postulating the homogeneity property of the distance function – such as the well-known directional distance function, introduced by Chambers, Chung and Färe (1996, 1998) – the Φ-aggregator is found to be the generalized mean.

On the basis of the generalized mean aggregator, two limit cases – in the neighborhood of infinity – are introduced. The first one enables Kholi's (1983) technology to be characterized for a group of firms. It is a particular $\mathbb{B}$-convex technology, introduced by Briec and Horvath (2009). This coalitional technology is shown to be consistent with the traditional assumptions of the literature. It is a compact upper semilattice respecting a free disposal assumption. The second one is a particular $\mathbb{B}^{-1}$-convex technology introduced by Briec and Liang (2011). It is a compact lower semilattice also relevant with a free disposal assumption. It is shown that those two aggregated technology sets enable the paradox of the positive technical bias to be solved. Indeed, the negative bias is obtained by specifying two firm games, the input fixed firm game and the output fixed firm game. The input fixed firm game postulates that the cooperation between firms is related to the use of inputs only, because some constrains of production are imposed on the industrial sector in order to limit the number of outputs (pollution limitations). On the contrary, the output fixed firm games is defined on the possibility to improve the amount of outputs when the firms are limited by a given amount of inputs (resource limitations). Those results are derived thanks to the directional distance function applied on the aggregated data. The input [output] firm game defined on upper semilattice technologies yields a negative [positive] bias. On the contrary, the output [input] firm game defined on lower semilattice technologies yields a negative [positive] bias. Finally, if the aggregation bias is supposed to be submodular, then the core of the game is always non empty, *i.e.*, the joint cooperation improves the technical efficiency (negative bias).

The outline of the paper is as follows. Section 2 sets the notations. Section 3 defines the firm game and the directional distance function. Section 4 is devoted to the characterization of the aggregated technology when the data are transformed by an isomorphism *à la* Ben-Tal (1977). Then, the results about the exact aggregation are exposed for the directional distance function. Section 5 explores the negative bias supported by the directional distance function. Section 6 introduces semilattice technologies, where it is shown that coalitions of firms may increase their technical efficiency either by putting in common their inputs or their outputs (input/output firm games). Section 7 closes the article.

2 Setup

The set of firms (players) is $\mathcal{K} := \{1, \ldots, |\mathcal{K}|\}$, where $|\mathcal{K}| \equiv \#\{\mathcal{K}\}$. The subsets of the grand coalition $\mathcal{K}$ are denoted by $\mathcal{S}$. A transferable utility game, *i.e.* a TU-game, is a pair $(\mathcal{K}, v)$, where v is defined as $v : 2^{|\mathcal{K}|} \to \mathbb{R}_+$ such that $v(\emptyset) := 0$, with $\mathbb{R}_+$ the non-negative part of the real line and $\mathbb{R}_{++}$ its positive part (with $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ its n-dimensional representation). The set of all maps v is denoted Γ, such that $v(\mathcal{S})$ provides the worth of coalition $\mathcal{S}$. A valued solution $\varphi(v)$ is the *pay-off* vector of the TU-game $(\mathcal{K}, v)$ that is a $|\mathcal{K}|$-dimensional real vector that represents what the firms could take benefit from cooperation. The valued solution of the TU-game is assumed to satisfy the standard axioms.

Linearity: $\varphi(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \varphi(v_1) + \alpha_2 \varphi(v_2)$, for all maps $v_1, v_2 \in \Gamma$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Symmetry: for any given pay-off vector $\varphi = (\varphi_1, \ldots, \varphi_k, \ldots, \varphi_{|\mathcal{K}|})$, then $\varphi_k(v) = \varphi_{\pi(k)}(v)$ for all permutations, where a permutation is given by $v(\pi(\mathcal{S})) = v(\mathcal{S})$ for all $\mathcal{S} \subseteq \mathcal{K}$ and $v \in \Gamma$.

Efficiency: $\sum_{k \in \mathcal{K}} \varphi_k(v) = v(\mathcal{K})$, for all $v \in \Gamma$.

Let $x \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^m_+$ be the input and output vectors, respectively. The technology T of the firms satisfies the following basic assumptions:

(T1): $(0_n, 0_m) \in T$, $(0_n, y) \in T \Longrightarrow y = 0_m$ *i.e.*, no free lunch;

(T2): the set $A(x) = \{(u, y) \in T : u \leqslant x\}$ of dominating observations is bounded $\forall x \in \mathbb{R}^n_+$, *i.e.*, infinite outputs cannot be obtained from a finite input vector;

(T3): T is closed;

(T4): $\forall z = (x, y) \in T$, $(x, -y) \leqslant (u, -v) \Longrightarrow (u, v) \in T$, *i.e.*, fewer outputs can always be produced with more inputs, and inversely;

(T5): $\forall \beta \geq 0$, if $(x, y) \in T$ then $(\beta x, \beta y) \in T$, *i.e.* the technology satisfies constant returns to scale.

Given a production set one can define an input correspondence $L : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$ and an output correspondence $P : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$ such that:

$$T = \big\{(x, y) \in \mathbb{R}^{n+m}_+ : x \in L(y)\big\} = \big\{(x, y) \in \mathbb{R}^{n+m}_+ : y \in P(x)\big\}. \qquad (2.1)$$

The literature on technology aggregation (see *e.g.* Li and Ng, 1995) defines the technology of the grand coalition as a standard sum of input and output vectors. Let $(x^k, y^k) \in \mathbb{R}^{n+m}_+$ be the input-output vectors of firm k whose technology is T^k. Following this specification, the technology of any given coalition $\mathcal{S}$ is the standard sum of the technologies T^k of each firm $k \in \mathcal{S}$:

$$T^{\mathcal{S}} := \sum_{k \in \mathcal{S}} T^k = \Big\{\big(\sum_{k \in \mathcal{S}} x^k, \sum_{k \in \mathcal{S}} y^k\big) : (x^k, y^k) \in T^k, k \in \mathcal{S}\Big\}. \qquad (2.2)$$

In the remainder of the paper, 0_n $[0_m]$ is the n-dimensional [m-dimensional] vector of zeros, $\mathbb{1}_n$ the n-dimensional vector of ones, $\mathbb{N}$ the set of (strictly) positive integers, and finally $\geq$ $(\leq)$ denotes inequalities over scalars and $\geqslant$ $(\leqslant)$ over vectors.

3 Directional Distance Functions and Firm Game

The directional distance function introduced by Chambers, Chung and Färe (1996, 1998)[1] $D_T : \mathbb{R}^{n+m}_+ \times \mathbb{R}^{n+m}_+ \longrightarrow \mathbb{R}_+$ involving a simultaneous input and

[1] See also Chambers and Färe (1998) and Chambers (2002) for more details on directional distance functions.

output variation in the direction of a pre-assigned vector $g = (g_i, g_o) \in \mathbb{R}_+^{n+m}$ is defined as:

$$D_T(x, y; g) = \sup_{\delta} \{\delta \in \mathbb{R} : (x - \delta g_i, y + \delta g_o) \in T\}. \quad (3.1)$$

In the sequel, the directional distance is such that $D_T(x, y; g) \geq 0$, *i.e.*, the cases of infeasibilities for which $(x, y) \notin T$ are not reported. For a group of $|\mathcal{K}|$ firms with technology T^k, the technical aggregation bias is defined as follows (see Briec, Dervaux and Leleu, 2003):

$$AB(\mathcal{K}; g) := D_T\left(\sum_{k\in\mathcal{K}}(x^k, y^k); g\right) - \sum_{k\in\mathcal{K}} D_{T^k}(x^k, y^k; g). \quad (3.2)$$

It provides the loss of technical efficiency due to the cooperation between the firms of group $\mathcal{K}$. The aggregation bias may be nil, in this case the exact aggregation condition is:

$$D_T\left(\sum_{k\in\mathcal{K}}(x^k, y^k); g\right) = \sum_{k\in\mathcal{K}} D_{T^k}(x^k, y^k; g). \quad (3.3)$$

Under the assumptions (T1)-(T4), the exact aggregation is possible whenever:

(i) the technologies T^k are identical and the input set is one-dimensional;
(ii) the firms use the same technique and (T5)-(T6) hold.

Actually, those results (see Briec *et al.*, 2003) are merely dependent on the structure of the aggregation process, *i.e.* the standard sum used to describe the aggregated technology of the group of firms. Indeed, the standard sum is used in a cooperative game, the so-called *firm game*, in order to aggregate inputs and outputs.

Definition 3.1 *A firm game is a collection* $\{\mathcal{K}, v(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}\}$ *such that:*

$(\mathbf{x}, \mathbf{y}) : 2^{\mathcal{K}} \to \mathbb{R}_+^{n+m}\ \forall \mathcal{S} \in 2^{\mathcal{K}},\ \mathcal{S} \neq \emptyset,$ *where* $(\mathbf{x}, \mathbf{y})(\mathcal{S}) := \sum_{k\in\mathcal{S}} (x^k, y^k) \in T^{\mathcal{S}},$
with $v : 2^{\mathcal{K}} \to \mathbb{R}_+^{n+m} \to \mathbb{R}_+\ ,\ v(\mathcal{S}) := D_{T^{\mathcal{S}}} \circ (\mathbf{x}, \mathbf{y})(\mathcal{S}),$
and $(\mathbf{x}, \mathbf{y})(\emptyset) := 0,\ \ v(\emptyset) := 0$ *by convention.*

The game $v(\mathcal{S})$ provides the value of the directional distance function $D_{T^{\mathcal{S}}}(\cdot)$ related to any given firm coalition $\mathcal{S}$ with technology $T^{\mathcal{S}}$:

$$v(\mathcal{S}) = D_{T^{\mathcal{S}}}\left(\sum_{k\in\mathcal{S}} (x^k, y^k)\,; g\right), \text{ for all } \mathcal{S} \subseteq \mathcal{K}. \quad (3.4)$$

On this basis, the technical aggregation bias is non negative. As suggested by Post (2001), it seems that the way the inputs/outputs are aggregated has serious implications on the efficiency measures. Precisely, the measurement of technical efficiency and its bias depend on particular aggregators.

4 Technology Aggregators: characterization

The literature on firm groups outlines an impossibility according to the directional distance function: the collaboration between firms cannot improve the technical efficiency of the group. This impossibility was proven independently by Briec, Dervaux and Leleu (2003) and Färe, Grosskopf and Zelenyuk (2008). Their result may be rewritten by choosing a general distance function denoted $f_S : \mathbb{R}_+^{n+m} \times \mathbb{R}_+^{n+m} \longrightarrow \mathbb{R}_+$. The game $v^f(\mathcal{S})$ is the distance function for any given coalition $\mathcal{S}$, such that $v^f(\mathcal{S}) \equiv f_S$ for all $\mathcal{S} \subseteq \mathcal{K}$. Following this notation, setting f the directional distance function, their result is:

$$v^f(\mathcal{K}) - \sum_{k\in\mathcal{K}} v^f(\{k\}) = AB(\mathcal{K}; g) \geq 0, \tag{4.1}$$

where the distance f gauges technical efficiency (the distance between one point and the frontier of the technology). The distance of the group is, for any given type of technology, always greater than the sum of the individual distances. The same conclusion holds for all possible coalitions $\mathcal{S} \subseteq \mathcal{K}$ if f is also chosen to be the directional distance function (see Briec and Mussard, 2014):

$$v^f(\mathcal{S}) - \sum_{k\in\mathcal{S}} v^f(\{k\}) = AB(\mathcal{S}; g) \geq 0. \tag{4.2}$$

This result reports a sub-efficiency related to the cooperation between firms. It is inherent to the standard additive form of the aggregation of vectors. However, following Ben-Tal (1977), Eichorn (1979) or Blackorby and Russell (1999), there exist many other forms of aggregation. For instance, Blackorby *et al.* (1981) characterize an isomorphism widely employed in welfare economics, specially for welfare, inequality and poverty indices. In the following lines, the proposed isomorphism is directly inspired from Ben-Tal (1977) and Blackorby *et al.* (1981) who studied the algebraic properties of the aggregator underlying the generalized sum (mean).

4.1 Generalized Sum

Let d be a positive integer and let $\Phi : X \longrightarrow \mathbb{R}^d$ be a bijective map, where X is an arbitrary set. From Ben-Tal (1977) we consider on X the algebraic operators $\overset{\Phi}{+}$ and $\overset{\Phi}{\cdot}$ defined for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$ by:

$$x \overset{\Phi}{+} y = \Phi^{-1}\big(\Phi(x) + \Phi(y)\big) \quad \text{and} \quad \alpha \overset{\Phi}{\cdot} x = \Phi^{-1}\big(\alpha \cdot \Phi(x)\big). \tag{4.3}$$

The Φ-sum, denoted $\overset{\Phi}{\sum}$, of $(x_1, \cdots, x_d) \in \mathbb{R}^n$ is defined by[2]

$$\overset{\Phi}{\sum_{i\in[d]}} x_i = \Phi^{-1}\Big(\sum_{i\in[d]} \Phi(x_i)\Big). \tag{4.4}$$

[2]For ease of exposition, for all $d \in \mathbb{N}$, $[d] := \{1, \ldots, d\}$.

The subset X endowed with these algebraic operators has some properties very similar to those of a vector space. Indeed, let E be an arbitrary nonempty set and let $\phi : E \longrightarrow \mathbb{R}$ be an isomorphism. One can define over E the operations defined $\forall \lambda, \mu \in E$ by:

$$\lambda \overset{\phi}{+} \mu = \phi^{-1}(\phi(\lambda) + \phi(\mu)) \quad \text{and} \quad \lambda \overset{\phi}{\cdot} \mu = \phi^{-1}(\phi(\lambda).\phi(\mu)). \tag{4.5}$$

From Ben-Tal (1977) the set $\phi(\mathbb{R})$ endowed with the algebraic operators $\overset{\phi}{+}$ and $\overset{\phi}{\cdot}$ is a scalar field. A vector space can then be constructed as the Cartesian product of an isomorphic transformation of the scalar field $\mathbb{R}$, that is E^d, in the case where the bijective map Φ is defined for all $u \in \mathbb{R}^d$ and all $x \in X = E^d$ by:

$$\Phi(x) = (\phi(x_1), \ldots, \phi(x_d)) \quad \text{and} \quad \Phi^{-1}(u) = \left(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d)\right). \tag{4.6}$$

It follows that $E = \phi^{-1}(\mathbb{R})$ is endowed with a total order defined by:

$$\lambda \overset{\phi}{\leq} \mu \iff \phi(\lambda) \leq \phi(\mu). \tag{4.7}$$

Obviously $(E^d, \overset{\phi}{+}, \overset{\phi}{\cdot})$ is a vector space where the algebraic operators $\overset{\phi}{+}$ and $\overset{\phi}{\cdot}$ are those defined above. It is then clear that if $B = \{v_1, \ldots, v_d\}$ is a basis of $\mathbb{R}^d$ then $B^\phi := \Phi^{-1}(B) = \{\Phi^{-1}(v_1), \ldots, \Phi^{-1}(v_d)\}$ is a basis of the vector space $(E^d, \overset{\phi}{+}, \overset{\phi}{\cdot})$.

4.2 Characterization of the aggregator

We first begin our investigations with a general (non specified) multidimensional aggregator. It is a map that transforms the data, *i.e.*, a function whose images are monotonic transformations of inputs and outputs. Let Φ be a general aggregator (isomorphism) such that $\Phi^{-1} : \mathbb{R}^d_+ \longrightarrow E^d_+$ with $d \in \mathbb{N}$:

$$\sum_{k \in S}^{\Phi} z^k := \Phi^{-1}\left(\sum_{k \in S} \Phi(z^k)\right). \tag{4.8}$$

In the following, we say that Φ is a canonical ϕ-isomorphism if there exists a real valued bijective map $\phi : E_+ \longrightarrow \mathbb{R}_+$ such that for all $z \in \mathbb{R}^d_+$:

$$\Phi(z) := (\phi(z_1), \ldots, \phi(z_d)). \tag{4.9}$$

By definition, the Φ-sum of the production technologies is:

$$\sum_{k \in S}^{\Phi} T^k = \Big\{ \big(\sum_{k \in S}^{\Phi} x^k, \sum_{k \in S}^{\Phi} y^k \big) : (x^k, y^k) \in T^k, k \in S \Big\}. \tag{4.10}$$

In this respect, the technology of the coalition S is defined as follows.

Definition 4.1 – **Coalitional Technology** (CT) *The aggregated technologies $T_\Phi^{\mathcal{S}}$, for all $\mathcal{S} \subseteq \mathcal{K}$ such that $|\mathcal{S}| \geq 1$, are defined as follows:*

$$T_\Phi^{\mathcal{S}} := \sum_{k\in\mathcal{S}}^{\Phi} T^k.$$

Note that whenever $|\mathcal{S}| = 1$, $T_\Phi^{\mathcal{S}} = T^k$. The distance function f_S of coalition $\mathcal{S} \subseteq \mathcal{K}$ yields the distance between one point and the frontier of the technology $T_\Phi^{\mathcal{S}}$:

$$v^f(\mathcal{S}) \equiv f_S\left(\sum_{k\in\mathcal{S}}^{\Phi} x^k, \sum_{k\in\mathcal{S}}^{\Phi} y^k\right). \tag{4.11}$$

The distance function exists because some firms merge in order to improve their technical efficiency, the so-called *firm game*.

Definition 4.2 *A firm game defined on the algebraic operators $\overset{\Phi}{+}$ and $\overset{\Phi}{\cdot}$ is a collection $\{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi\}$ such that the coalitional technology is defined as follows:*
$(\mathbf{x}, \mathbf{y}) : 2^{|\mathcal{K}|} \rightarrow E_+^{n+m}$ *for all* $\mathcal{S} \in 2^{|\mathcal{K}|}$, *such that* $|\mathcal{S}| \geq 1$, *where*
$(\mathbf{x}, \mathbf{y})(\mathcal{S}) := \left(\sum_{k\in\mathcal{S}}^{\Phi} x^k, \sum_{k\in\mathcal{S}}^{\Phi} y^k\right) \in T_\Phi^{\mathcal{S}}$, *with*
$f^{\mathcal{S}} : 2^{|\mathcal{K}|} \rightarrow E_+^{n+m} \rightarrow \mathbb{R}_+$, $v^f(\mathcal{S}) := f^{\mathcal{S}} \circ (\mathbf{x}, \mathbf{y})$, *and*
$(\mathbf{x}, \mathbf{y})(\emptyset) := 0_{n+m}$, $v^f(\emptyset) = f^{\mathcal{S}} \circ (\mathbf{x}, \mathbf{y})(\emptyset) = 0$ *by convention.*

The technical aggregation bias of any given coalition is then modeled thanks to the firm game. A negative [positive] bias is a *sub*[*super*]*additive game* defined on the basis of the general aggregation process Φ. A negative [positive] bias represents an improvement [decline] of technical efficiency due to the cooperation between firms. The technical aggregation bias (Φ-bias for short) of a canonical isomorphism Φ defined with respect to a real valued isomorphism ϕ is defined as, for all $\mathcal{S} \subseteq \mathcal{K}$ such that $|\mathcal{S}| \geq 2$:

$$AB_\Phi(\mathcal{S}) := f_S\left(\sum_{k\in\mathcal{S}}^{\Phi} x^k, \sum_{k\in\mathcal{S}}^{\Phi} y^k\right) - \sum_{k\in\mathcal{S}}^{\phi} f_k(x^k, y^k). \tag{4.12}$$

Definition 4.3 – **Φ-sub[super]additivity** (SUB$_\Phi$[SUP$_\Phi$]) – *Let $\mathcal{R}, \mathcal{S} \subseteq \mathcal{K}$ such that $\mathcal{R} \cap \mathcal{S} = \emptyset$. Let $\Phi : E_+^{n+m} \longrightarrow \mathbb{R}_+^{n+m}$ be a canonical ϕ-isomorphism. A firm game $\{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi\}$ is defined to be sub*[*super*]*additive if:*

$$AB_\Phi(\mathcal{S} \cup \mathcal{R}) \leq \; [\geq] \; 0 \iff v^f(\mathcal{S} \cup \mathcal{R}) \leq \; [\geq] \; v^f(\mathcal{S}) \overset{\phi}{+} v^f(\mathcal{R}).$$

The technical aggregation bias is equivalently rewritten as, for all $\mathcal{S} \subseteq \mathcal{K}$ such that $|\mathcal{S}| \geq 2$:

$$AB_\Phi(\mathcal{S}) = v^f(\mathcal{S}) - \sum_{k\in\mathcal{S}}^{\phi} v^f(\{k\}). \tag{4.13}$$

We first examine the existence of an aggregator when the Φ-bias is null. The first result shows that the transformation of the data, generated by any given aggregator Φ, yields a linear distance function defined up to the map ϕ^{-1}. This is a generalization of the untransformed case in which the distance function is linear, see Briec *et al.* (2003) and Färe *et al.* (2008).

Proposition 4.1 *Let $\Phi : E_+^{n+m} \longrightarrow \mathbb{R}_+^{n+m}$ be a canonical ϕ-isomorphism. Under* (T1)-(T4) *and* (CT), *for all firm games $\{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi\}$, the following implication holds:*

$$\left[AB_\Phi(\mathcal{S}) = 0,\ \forall \mathcal{S} \subseteq \mathcal{K}\right] \implies \left[v^f(\mathcal{S}) = \phi^{-1}\left(\sum\nolimits_{k\in\mathcal{S}} \mathbf{c}\cdot\Phi(z^k) + \sum\nolimits_{k\in\mathcal{S}} c_k\right)\right].$$

Proof:
Let $AB_\Phi(\mathcal{S}) = 0$, such that $\Phi : E_+^{n+m} \longrightarrow \mathbb{R}_+^{n+m}$ with its one-dimensional representation $\phi : \mathbb{R}_+ \longrightarrow E_+$. For all $\mathcal{S} \subseteq \mathcal{K}$ such that $|\mathcal{S}| \geq 2$:

$$AB_\Phi(\mathcal{S}) = 0 \iff f_{\mathcal{S}}\left(\sum_{k\in\mathcal{S}}^{\Phi}(x^k, y^k)\right) = \sum_{k\in\mathcal{S}}^{\phi}\left[f_k(x^k, y^k)\right].$$

Thus,

$$f_{\mathcal{S}}\left[\Phi^{-1}\left(\sum_{k\in\mathcal{S}}\Phi(x^k, y^k)\right)\right] = \phi^{-1}\left(\sum_{k\in\mathcal{S}}\phi\left[f_k(x^k, y^k)\right]\right).$$

Let us denote the vector $z^k := (x^k, y^k) \in \mathbb{R}_+^{n+m}$ such that $u^k := \Phi(z^k)$, thus:

$$\phi\left(f_{\mathcal{S}}\left[\Phi^{-1}\left(\sum_{k\in\mathcal{S}} u^k\right)\right]\right) = \sum_{k\in\mathcal{S}}\phi\left[f_k(z^k)\right].$$

Set $\phi \circ f_{\mathcal{S}} =: \phi_{\mathcal{S}}$ and $\phi \circ f_k =: \phi_k$, for all $k \in \{1, \ldots, |\mathcal{S}|\}$. As $z^k = \Phi^{-1}(u^k)$, we get:

$$\phi_{\mathcal{S}}\left[\Phi^{-1}\left(\sum_{k\in\mathcal{S}} u^k\right)\right] = \sum_{k\in\mathcal{S}}\phi_k\left[\Phi^{-1}(u^k)\right].$$

We recognize the well-known Pexider's equation of solution (see Aczél, 1966, p.141):

$$\phi_{\mathcal{S}} \circ \Phi^{-1}\left(\sum_{k\in\mathcal{S}} u^k\right) = \mathbf{c}\cdot\left(\sum_{k\in\mathcal{S}} u^k\right) + \sum_{k\in\mathcal{S}} c_k\ ;$$
$$\phi_k \circ \Phi^{-1}(u^k) = \mathbf{c}\cdot u^k + c_k\ ,$$

where the vector $\mathbf{c} \in E_+^{n+m}$ and the constants $c_k \in E_+$ are set to be non-negative in order to get well-defined distance functions (being non-negative). The solution can be rewritten in a general setting as:

$$\phi \circ f_k(z^k) = \mathbf{c}\cdot\Phi(z^k) + c_k\ ,\ \forall k \in \{1, \ldots, |\mathcal{S}|\},$$

and,

$$\phi \circ f_S \left(\sum_{k \in S}^{\Phi} z^k \right) = \mathbf{c} \cdot \sum_{k \in S} \Phi(z^k) + \sum_{k \in S} c_k \ , \ \forall S \subseteq \mathcal{K}.$$

Thus,

$$v^f(\mathcal{S}) = f_S \left(\sum_{k \in S}^{\Phi} z^k \right) = \phi^{-1} \left(\mathbf{c} \cdot \sum_{k \in S} \Phi\left(z^k\right) + \sum_{k \in S} c_k \right) \ , \ \forall S \subseteq \mathcal{K}.$$

■

The result proves that the transformation of the data thanks to the Φ-aggregator enables the standard interpretation to be retrieved as a particular case, that is, the distance function is linear (up to the map ϕ^{-1}), as shown by Briec *et al.* (2003) and Färe *et al.* (2008) with the standard sum of sets.

Now, we impose more structure to the distance function in order to characterize the Φ-aggregator. The homogeneity of degree one is known to be well suited for the measure of technical efficiency, see Chambers, Chung and Färe (1996, 1998)[3]: $f_k(\lambda x^k, \lambda y^k) = \lambda f_k(x^k, y^k)$ for all $\lambda \geq 0$. We show that if the technical Φ-bias is null, then Φ is found to be quasi-linear.[4]

Proposition 4.2 *Under the assumptions* (T1)-(T4) *and* (CT), *for all firm games* $\{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi\}$ *such that* f^S *is homogeneous of degree one, if* $\Phi : E_+^{n+m} \longrightarrow \mathbb{R}^{n+m}$ *is a* ϕ*-canonical isomorphism, then the following are equivalent:*
(i) $AB_\Phi(\mathcal{S}) = 0$.
(ii) $v^f(\mathcal{S}) = \left(\sum_{k \in S} \mathbf{c} \cdot (z^k)^\tau\right)^{\frac{1}{\tau}}$, *for some* $\tau \neq 0$.

Proof:
[(i) $\Longrightarrow$ (ii)]. From Proposition 4.1, when the Φ-bias is null, setting $\mathbf{c} := (b_1, \ldots, b_{n+m})$, we get:

$$v^f(\mathcal{S}) = \phi^{-1} \left(\sum_{k \in S} \mathbf{c} \cdot \Phi(z^k) + \sum_{k \in S} c_k \right) = \phi^{-1} \left(\sum_{k \in S} \sum_{\ell=1}^{n+m} b_\ell \phi(z_\ell^k) + \sum_{k \in S} c_k \right).$$

If the distance function is homogeneous of degree one, then the function $\phi^{-1}\left(\sum_{k \in S} \sum_{\ell=1}^{n+m} b_\ell \phi(\lambda z_\ell^k) + \sum_{k \in S} c_k\right)$ inherits the homogeneity property of v^f (of degree one). Hence, for $\lambda \geq 0$, the following relation holds,

$$\lambda \phi^{-1} \left(\sum_{k \in S} \sum_{\ell=1}^{n+m} b_\ell \phi(z_\ell^k) + \sum_{k \in S} c_k \right) = \phi^{-1} \left(\sum_{k \in S} \sum_{\ell=1}^{n+m} b_\ell \phi(\lambda z_\ell^k) + \sum_{k \in S} c_k \right),$$

[3] See also Chambers and Färe (1998) and Chambers (2002).

[4] The homogeneity of degree one of the distance function is sometimes associated with the constant returns to scale hypothesis (T5). This is the case for instance for the directional distance function.

if and only if $\sum_{k\in S} c_k = 0$ and ϕ^{-1} is homogeneous of degree $\tau \neq 0$. Then,

$$\phi^{-1}(\lambda t) = t^{\tau}\phi^{-1}(\lambda) =: t^{\tau}\kappa_1,$$

and,

$$\phi^{-1}(\lambda t) = \lambda^{\tau}\phi^{-1}(t) =: \kappa_2\phi^{-1}(t).$$

Thus,

$$\phi^{-1}(t) = t^{\tau}\frac{\kappa_1}{\kappa_2} =: \kappa t^{\tau},\ \kappa_2 \neq 0,\ \tau \neq 0,$$

with $\kappa \geq 0$ in order to get a non-negative distance function. If the distance function $v^f(S)$ is supposed to be homogeneous of degree one, then the distance function is a mean of order $\tau \neq 0$ and the aggregator ϕ is a power function.

[(ii) $\Longrightarrow$ (i)]. Since $\Phi(x) = (x_1^{\tau}, \ldots, x_{n+m}^{\tau})$ for $\tau \neq 0$, we deduce from the functional form of $v^f(S)$ that for all $k \in S$ and $\kappa \geq 0$:

$$v^f(\{k\}) = \kappa \mathrm{c} \cdot z^k,$$

and the implication follows. ∎

In sum, the transformation of the data, thanks to the isomorphisms Φ, enables the usual aggregation bias to be linked with the quasi-linearity of some homogeneous distance functions f_S. A similar result based on the directional distance function was formerly found by Briec *et al.* (2003) in the case where the data are not transformed, *i.e.*, Φ would be reduced to the identity map in our framework. In this case, the core of the firm game would be represented by one point. Indeed, the core of the firm game is:

$$C_{\Phi} := \left\{\varphi \in E_+^{|\mathcal{K}|} : \sum_{k\in S}\varphi_k \leq v^f(S), \forall S \subset \mathcal{K}\right\} \bigcap \left\{\sum_{k\in\mathcal{K}}\varphi_k = v^f(\mathcal{K})\right\}. \quad (4.14)$$

Proposition 4.3 *Under the assumptions* (T1)-(T4), (CT) *and* (SUB$_\Phi$), *for all firm games* $\{\mathcal{K}, v^f(S) : S \subseteq \mathcal{K}, \Phi\}$ *such that* $\Phi = Id_{E_+^d}$, *the following are equivalent:*

(i) $AB_{\Phi}(S) = 0$.

(ii) $C_{\Phi} = \{v^f(\{1\}), \ldots, v^f(\{|\mathcal{K}|\})\}$, $\overset{\circ}{C}_{\Phi} = \emptyset$.

Proof:

[(i) $\Longrightarrow$ (ii)]: Let $\varphi_k := v^f(\{k\})$. Note that $AB_{\Phi}(S) = 0$ implies that $\sum_{k\in S}\varphi_k = v(S)$ for all $S \subseteq \mathcal{K}$. The vector $\varphi := (\varphi_1, \ldots, \varphi_{|\mathcal{K}|})$ is an imputation since it respects $\varphi_k \leq v^f(\{k\})$ and $\sum_{k\in\mathcal{K}}\varphi_k = v(\mathcal{K})$. As a consequence, since $\varphi_k = v^f(\{k\})$, then φ is the core. The core C_{Φ} is then reduced to one point and so $\overset{\circ}{C}_{\Phi} = \emptyset$.

[(ii) $\Longrightarrow$ (i)]: If the pay-off point $\{v^f(\{1\}), \ldots, v^f(\{|\mathcal{K}|\})\}$ is the core, then by definition, Eq.(4.14), we get that $\sum_{k\in S}\varphi_k \leq v^f(S)$, for all $S \subset \mathcal{K}$. Since

$v^f(\{k\}) = \varphi_k$, then $\sum_{k\in\mathcal{S}} v^f(\{k\}) \leq v^f(\mathcal{S})$, for all $\mathcal{S} \subset \mathcal{K}$. By subadditivity, we get $\sum_{k\in\mathcal{S}} v^f(\{k\}) \geq v^f(\mathcal{S})$, and so $AB(\mathcal{S}) = 0$ for all $\mathcal{S} \subseteq \mathcal{K}$. ■

Finally, we have seen that the use of an aggregator Φ being a power function, in order to deal with heterogeneous firms, allows well-defined homogeneous distance functions f_S to be derived when the aggregation bias AB_Φ is null. As a consequence, it is of interest to test whether the aggregator is compatible with the celebrated directional distance function to gauge technical efficiency.

5 Biases of the Directional Distance Function

We show in this section, with numerical examples, that the aggregator is enough flexible to capture different technical biases (positive, negative and null) with the directional distance function. Before, we specify the power function characterized in the previous section, and we subsequently define the resulting aggregator (the generalized mean) and the aggregated directional distance function.

5.1 Power Functions

For all $\alpha \in]0, +\infty[$, let $\phi_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be the map defined by:

$$\phi_\alpha(\lambda) = \begin{cases} \lambda^\alpha & \text{if } \lambda \geq 0 \\ -|\lambda|^\alpha & \text{if } \lambda \leq 0. \end{cases} \tag{5.1}$$

For all $\alpha \neq 0$, the reciprocal map is $\phi_\alpha^{-1} := \phi_{\frac{1}{\alpha}}$. It is first quite straightforward to state that: (i) ϕ_α is defined over $\mathbb{R}_+$; (ii) ϕ_α is continuous over $\mathbb{R}_+$; (iii) ϕ_α is bijective over $\mathbb{R}_+$. Throughout the section, for any vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+$ we use the following notations:

$$\Phi_\alpha(x) = \big(\phi_\alpha(x_1), \ldots, \phi_\alpha(x_d)\big) = \big({x_1}^\alpha, \ldots, {x_d}^\alpha\big) = x^\alpha. \tag{5.2}$$

It is then natural to introduce the following algebraic operation over $\mathbb{R}^n_+$:

$$x \overset{\alpha}{+} y = \Phi_\alpha^{-1}\big(\Phi_\alpha(x) + \Phi_\alpha(y)\big) \quad \text{and} \quad \lambda \overset{\alpha}{\cdot} x = \Phi_\alpha^{-1}\left(\phi_\alpha(\lambda)\Phi_\alpha(x)\right). \tag{5.3}$$

In this case $(\phi_\alpha(\mathbb{R}), +, \cdot)$ is a scalar field since $\phi_\alpha(\mathbb{R}) = \mathbb{R}$.

Let us focus on the case $\alpha \in]-\infty, 0[$. The map $x \mapsto x^\alpha$ is not defined at point $x = 0$. Thus, it is not possible to construct a bijective endomorphism on $\mathbb{R}$. However, it is possible to construct an operation preserving at least associativity. For all $\alpha \in]-\infty, 0[$ we consider the function ϕ_α defined by:

$$\phi_\alpha(\lambda) = \begin{cases} \lambda^\alpha & \text{if } \lambda > 0 \\ -(|\lambda|)^\alpha & \text{if } \lambda < 0 \\ +\infty & \text{if } \lambda = 0. \end{cases} \tag{5.4}$$

In such a case $M := \phi_\alpha(\mathbb{R}) = \mathbb{R}\backslash\{0\} \cup \{+\infty\}$. Moreover, let us construct the application $\Phi_\alpha : \mathbb{R}^d \longrightarrow M^d$, defined by $\Phi_\alpha(x_1, \ldots, x_d) = (\phi_\alpha(x_1), \ldots, \phi_\alpha(x_d))$. For all $\alpha < 0$, let us consider the algebraic operators $\overset{\alpha}{+}$ and $\overset{\alpha}{\cdot}$ defined by:

$$x \overset{\alpha}{+} y = \Phi_\alpha{}^{-1}\left(\Phi_\alpha(x) + \Phi_\alpha(y)\right) \text{ and } \lambda \overset{\alpha}{\cdot} x = \Phi_\alpha{}^{-1}\left(\phi_\alpha(\lambda).\Phi_\alpha(x)\right). \quad (5.5)$$

In such a case $(\mathbb{R}, \overset{\alpha}{+}, \overset{\alpha}{\cdot})$ is not a scalar field because there is not a neutral element. Notice that $(\mathbb{R}, \overset{\alpha}{+}, \overset{\alpha}{\cdot})$ admits 0 as an absorbing element. It is easy to check that for all $\lambda \in \mathbb{R}$, $0 \overset{\alpha}{+} \lambda = 0$. This comes from the fact that for all $\mu \in M$, $\mu + \infty = \infty \in M$. Thus $\left(\mathbb{R}^d, \overset{\alpha}{+}, \overset{\alpha}{\cdot}\right)$ is not a Φ_α-vector space. However, the addition $\overset{\alpha}{+}$ is well defined over $\mathbb{R}^d$ and it is trivial to check that associativity holds. For the purpose of the paper, the fact that M does not contain a neutral element is not a problem since we consider operations defined on $\mathbb{R}^d_+$. If $\alpha = 0$, we denote $\phi_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ the map defined by:

$$\phi_0(\lambda) = \begin{cases} \ln(\lambda) & \text{if } \lambda > 0 \\ -\infty & \text{if } \lambda = 0. \end{cases} \quad (5.6)$$

The reciprocal map is:

$$\phi_0{}^{-1}(\lambda) = \begin{cases} \exp(\lambda) & \text{if } \lambda \in \mathbb{R} \\ 0 & \text{if } \lambda = -\infty. \end{cases} \quad (5.7)$$

It is then possible to construct an algebraic operator summing the elements of $\mathbb{R}^n_+$. 1 is a neutral element of $(\mathbb{R}, \overset{0}{+}, \overset{0}{\cdot})$ and ∞ is an absorbing element.

5.2 The generalized mean

When the distance function is homogeneous of degree one, the aggregator Φ is the generalized mean investigated by Ben-Tal (1977), Eichorn (1979) and Blackorby *et al.* (1981). Then, the aggregation process found in the previous section is defined as follows. Let $\phi_\alpha : \mathbb{R} \longrightarrow M$ be the injective map defined for all $t > 0$ by $\phi_\alpha(t) = t^\alpha$. For all $(t_1, \ldots, t_\ell) \in \mathbb{R}^\ell_+$, an one-dimensional aggregator is given by:

$$\sum_{k \in [\ell]}^{\phi_\alpha} t_k = \begin{cases} \left(\sum_{k\in[\ell]} (t_k)^\alpha\right)^{\frac{1}{\alpha}} & \forall \alpha > 0 \\ \prod_{k\in[\ell]} t_k & \alpha = 0. \end{cases} \quad (5.8)$$

If $\alpha < 0$, then:

$$\sum_{k \in [\ell]}^{\phi_\alpha} t^k = \begin{cases} \left(\sum_{k\in[\ell]} (t_k)^\alpha\right)^{\frac{1}{\alpha}} & \text{if } \min_k t_k > 0 \\ 0 & \text{if } \min_k t_k = 0. \end{cases} \quad (5.9)$$

Note that the well-known arithmetic mean is obtained when $\alpha = 1$. On the other hand, using L'hospital rule yields the geometric mean:

$$\lim_{\alpha \longrightarrow 0} \sum_{k \in [\ell]}^{\phi_\alpha} t_k = \prod_{k \in [\ell]} t_k. \tag{5.10}$$

In order to aggregate technologies (input/output vectors), we take recourse to a multidimensional map, the so-called Φ_α-aggregator.

Definition 5.1 – **Φ_α-Aggregation** – *Let $\Phi_\alpha : \mathbb{R}^d \longrightarrow M^d$ be an injective map defined by:*

$$\Phi_\alpha(z_1, \ldots, z_d) = (\phi_\alpha(z_1), \ldots, \phi_\alpha(z_d)).$$

For all collections $Z = \{z^k : k \in \mathcal{S}\} \in \mathbb{R}_+^d$, a Φ_α-aggregator is given by:

$$\sum_{k \in \mathcal{S}}^{\Phi_\alpha} z^k = \Big(\sum_{k \in \mathcal{S}}^{\phi_\alpha} z_1^k, \ldots, \sum_{k \in \mathcal{S}}^{\phi_\alpha} z_d^k \Big).$$

The definition of the coalitional technology (CT) yields, under the Φ_α-aggregator, the following aggregated technology for all $\mathcal{S} \subseteq \mathcal{K}$ and $|\mathcal{S}| \geq 1$:

$$T_\alpha^{\mathcal{S}} := \sum_{k \in \mathcal{S}}^{\Phi_\alpha} T^k. \tag{5.11}$$

It enables different cases to be captured. When $\alpha = 0$, a multi-output Cobb-Douglas technology is designed. When $\alpha = 1$, we retrieve the well-known aggregation over sets, studied in Li and Ng (1995), Briec *et al.* (2003), Färe *et al.* (2008), Briec and Mussard (2014).

It has been shown in the previous sections that the distance function is a good candidate to measure technical efficiency for a group of firms with respect to the general aggregator Φ. The directional distance function introduced by Chambers, Chung and Färe (1996, 1998) is homogeneous of degree one. With respect to the Φ_α-aggregator, it is given by $D_{T_\alpha^{\mathcal{S}}} : \Phi_\alpha^{-1}(M^{n+m}) \times \Phi_\alpha^{-1}(M^{n+m}) \longrightarrow \mathbb{R}_+$ involving a simultaneous input and output variation in the direction of a pre-assigned vector $g = (g_i, g_o) \in \mathbb{R}_+^n \times \mathbb{R}_+^m$. Following the *firm game*, Definition 4.2 and the Φ_α-aggregator, the directional distance function is expressed as,

$$v_\alpha(\mathcal{S}) := D_{T_\alpha^{\mathcal{S}}} \left(\sum_{k \in \mathcal{S}}^{\Phi_\alpha} x^k, \sum_{k \in \mathcal{S}}^{\Phi_\alpha} y^k; g \right), \tag{5.12}$$

and the aggregation bias by,

$$AB_\alpha(\mathcal{S}) := v_\alpha(\mathcal{S}) - \sum_{k \in \mathcal{S}}^{\phi_\alpha} v_\alpha(\{k\}), \ \forall \mathcal{S} \subseteq \mathcal{K}, \ |\mathcal{S}| \geq 2. \tag{5.13}$$

By generalizing the technologies thanks to the Φ_α aggregator, the resulting aggregated technology is enough flexible to cover positive aggregation bias as well as negative ones. A negative bias indicates that some firm coalitions improve their technical efficiency, whereas this result has been found to be impossible under the standard sum of the firms' technologies.

5.3 Negative technical bias: examples

We show that the transformation of the data thanks to Φ_α enables the technical bias inherent to the directional distance function to be designed as negative (positive), *i.e.*, the improvement (decline) of technical efficiency of the firm group is due to the cooperation between firms.

Example 5.1 *Suppose that* $\mathcal{K} = \{1,2\}$ *and that* $T^k = \{(x,y) \in \mathbb{R}^3_+ : y^\alpha - (x_1)^\alpha - (x_2)^\alpha \leq 0\}$ *for* $k = 1,2$. *Assume moreover that for* $k = 1,2$ *we have* $(x_1, y_1) = (2,1,1)$ *and* $(x_2, y_2) = (1,2,1)$. *In the following, it is shown that* $T^1 = T^2$, *it is possible to find some* α *such that* $AB_\alpha(S;g) \lesseqgtr 0$.
For all $\alpha > 0$, *let us consider the isomorphism* $\phi_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by,*

$$\phi_\alpha(\lambda) = \begin{cases} \lambda^\alpha & \text{if } \lambda \geq 0 \\ -(-\lambda)^\alpha & \text{if } \lambda < 0 \end{cases}$$

and its inverse is defined by,

$$\phi_\alpha^{-1}(\lambda) = \begin{cases} \lambda^{\frac{1}{\alpha}} & \text{if } \lambda \geq 0 \\ -(-\lambda)^{\frac{1}{\alpha}} & \text{if } \lambda < 0. \end{cases}$$

We have by definition:

$$\sum_{k=1,2}^{\Phi_\alpha} T^k_\alpha = \Big((T^1)^\alpha + (T^2)^\alpha\Big)^{\frac{1}{\alpha}}.$$

Set $T^0 = T^1 = T^2$. *By construction* T^0 *is quasi linear and satisfies a constant returns to scale assumption. We obtain from Briec, Dervaux et Leleu (2003):*

$$\sum_{k=1,2}^{\Phi_\alpha} T^k_\alpha = T^0.$$

Setting $g = (1,1,0)$, *we have* $D_{T^0}(2,1,1;1,1,0) = 1$ *and* $D_{T^0}(1,2,1;1,1,0) = 1$. *It follows that:*

$$\sum_{k=1,2}^{\alpha} D_{T^k_\alpha}(x^k, y^k; 1,1,0) = \Big((D_{T^0}(2,1,1;1,1,0))^\alpha + (D_{T^0}(1,2,1;1,1,0))^\alpha\Big)^{\frac{1}{\alpha}} = 2^{\frac{1}{\alpha}}.$$

Moreover,

$$\sum_{k=1,2}^{\Phi_\alpha} (x^k, y^k) = \Big((2^\alpha + 1^\alpha)^{\frac{1}{\alpha}}, (1^\alpha + 2^\alpha)^{\frac{1}{\alpha}}, (1^\alpha + 1^\alpha)^{\frac{1}{\alpha}}\Big) = \Big((1+2^\alpha)^{\frac{1}{\alpha}}, (1+2^\alpha)^{\frac{1}{\alpha}}, 2^{\frac{1}{\alpha}}\Big).$$

The input set is by definition $L^0(2^{\frac{1}{\alpha}}) = \{(x_1, x_2) \in \mathbb{R}^2_+ : (x_1)^\alpha + (x_2)^\alpha \geq 2^{\frac{1}{\alpha}}\}$. *It follows that:*

$$D_{T^S_\alpha}\Big(\sum_{k=1,2}^{\Phi_\alpha} (x^k, y^k); 1, 1, 0\Big) = D_{T^S_\alpha}\Big((1+2^\alpha)^{\frac{1}{\alpha}}, (1+2^\alpha)^{\frac{1}{\alpha}}, 2^{\frac{1}{\alpha}}; 1, 1, 0\Big) = (1+2^\alpha)^{\frac{1}{\alpha}} - 1.$$

Thus,

$$AB_\alpha = (1 + 2^\alpha)^{\frac{1}{\alpha}} - 1 - 2^{\frac{1}{\alpha}}.$$

- *If* $\alpha = 1$, *we have*
$$AB_\alpha = 3 - 1 - 2 = 0.$$
- *If* $\alpha = 1/2$, *we have*
$$AB_\alpha = (1 + \sqrt{2})^2 - 1 - 2^2 = (1 + \sqrt{2})^2 - 5 > 0.$$
- *If* $\alpha = 2$, *we have*
$$AB_\alpha = \sqrt{5} - 1 - \sqrt{2} < 0.$$

It is also possible to show that, when $\alpha = 0$, that the technical bias is either positive or negative.

Example 5.2 *Let* $z_1 = \mathbb{1}_d$ *and* S *a coalition of two firms,* $k = 1, 2$, *such that* $T^S_\alpha = T^1_\alpha$. *Set* $\alpha = 0$, *we show in the following that* $AB_0(S; g) \lesseqgtr 0$. *We get, for* $\alpha = 0$:

$$AB_0(S; g) = D_{T^S_0}\left(\prod_{k=1,2} z_k; g\right) - D_{T^1_0}(z_1; g) \cdot D_{T^2_0}(z_2; g).$$

Since $T^S_0 = T^1_0$, *it comes that*

$$D_{T^S_0}\left(\prod_{k=1,2} z_k; g\right) = D_{T^1_0}(z_1; g).$$

By definition, $D_{T^S_0}(\cdot) \geq 0$. *If* $D_{T^2_0}(z_2; g) \gtreqless 1$ *then* $AB_0(S; g) \lesseqgtr 0$.

Those examples show, for some values of α, that the cooperation between firms provides either an improvement of technical efficiency (negative bias), a constant technical efficiency (null bias) or finally a decrease of technical efficiency (positive bias). In what follows, the values of α are defined at the neighborhood of infinity in order to capture negative biases associated with semilattice technology sets.

6 Input/output fixed firm games: limit cases

We introduce input/output firm games in order to exhibit the conditions allowing for negative biases to be conceived. Those games rely on semilattice technologies. Solutions inside the core are characterized, so that the core is partitioned with respect to either negative biases or positive ones.

6.1 Aggregated technologies

The negative bias represents the improvement of technical efficiency for any given coalition of firms. The demonstration of its existence is made with the directional distance function either output oriented or input oriented. We investigate aggregate technologies that are either defined on the maximum available input-output combination or on the minimum one, respectively. In the maximum case, any coalition of firms takes benefit from cooperation since its technical efficiency is improved, in other terms, the game is subadditive (SUB_Φ), defined from now onwards (SUB_{Φ_α}) – see Definitions 4.3 and 5.1.

For that purpose, the ϕ_α-aggregator is defined for particular values of α. First note that for all $u \in \mathbb{R}^d_+$:

$$\sum_{\ell\in[d]}^{\phi_\alpha} u_\ell := \begin{cases} \min\limits_{\ell\in[d]} u_\ell & \text{if } \alpha = -\infty \\ \max\limits_{\ell\in[d]} u_\ell & \text{if } \alpha = \infty. \end{cases} \tag{6.1}$$

Notice that by construction $\sum_\ell^{\phi_\alpha} u_\ell = \phi_\alpha^{-1}\Big(\sum_\ell \phi_\alpha(u_\ell)\Big)$ when $\alpha \notin \{-\infty, \infty\}$. In such a case, Blackorby *et al.* (1981) axiomatically characterize this aggregator[5]. This notation is justified by the fact that:

$$\begin{array}{ll} \lim_{\alpha\longrightarrow-\infty} \sum_{\ell\in[d]}^{\phi_\alpha} u_\ell = \min\limits_{\ell\in[d]} u_\ell & \text{if } \alpha = -\infty \\ \lim_{\alpha\longrightarrow+\infty} \sum_{\ell\in[d]}^{\phi_\alpha} u_\ell = \max\limits_{\ell\in[d]} u_\ell & \text{if } \alpha = \infty. \end{array} \tag{6.2}$$

Notice also that in the case where $\alpha = -\infty$, if there is some ℓ with $u_\ell = 0$, then $\phi_\alpha(u_\ell) = +\infty$ and it follows that $\sum_{\ell\in[d]}^{\phi_\alpha} u_\ell = \min\limits_{\ell\in[d]} u_\ell = 0$.

Definition 6.1 – **Semilattice Aggregators** – *For all collections $Z = \{z^k : k \in S\} \in \mathbb{R}^d_+$ an upper semilattice-aggregator is given by:*

$$\sum_{k\in S}^{\Phi_\infty} z^k = \bigvee_{k\in S} z^k = \big(\max\{z_1^1, \ldots, z_1^{|S|}\}, \ldots, \max\{z_d^1, \ldots, z_d^{|S|}\}\big).$$

A lower-semilattice aggregator is given by:

$$\sum_{k\in S}^{\Phi_{-\infty}} z^k = \bigwedge_{k\in S} z^k = \big(\min\{z_1^1, \ldots, z_1^{|S|}\}, \ldots, \min\{z_d^1, \ldots, z_d^{|S|}\}\big).$$

In the sequel, we define the notions of semilattice with respect to the usual partial order defined over $\mathbb{R}^{n+m}_+$.

[5] It is a generalized quasi-linear function that respects continuity, monotonicity, separability and symmetry.

Definition 6.2 – **Semilattices** – *A subset A of $\mathbb{R}^{n+m}$ is an upper semilattice if for all $z, z' \in A$ we have $z \vee z' \in A$. A subset B of $\mathbb{R}^{n+m}$ is a lower semilattice if for all $z, z' \in B$ we have $z \wedge z' \in B$.*

Example 6.1 *Let us recall the notion of $\mathbb{B}$-convex ($\mathbb{B}^{-1}$-convex) sets. A $\mathbb{B}$-convex hull of a set $A = \{z_1, \ldots, z_{|S|}\} \subset \mathbb{R}_+^{n+m}$ is*

$$\mathbb{B}(A) = \Big\{ \bigvee_{k \in S} t_k z_k, \max_{k=1,\ldots,|S|} t_k = 1,\ t \geqslant 0 \Big\}.$$

The $\mathbb{B}^{-1}$-convex hull of a set A is given by:

$$\mathbb{B}^{-1}(A) = \left\{ \bigwedge_{k \in S} s_k z_k, \min_{k=1,\ldots,|S|} s_k = 1,\ s \geqslant 0 \right\}.$$

The $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex technologies are given by:

$$T_{\max}^{S} = \Big\{ (x,y) \in \mathbb{R}_+^{m+n} : x \geqslant \bigvee_{k \in S} t_k x_k, y \leqslant \bigvee_{k \in S} t_k y_k, \max_{k \in S} t_k = 1,\ t \geqslant 0 \Big\},$$

$$T_{\min}^{S} = \Big\{ (x,y) \in \mathbb{R}_+^{m+n} : x \geqslant \bigwedge_{k \in S} s_k x_k, y \leqslant \bigwedge_{k \in S} s_k y_k, \min_{k \in S} s_k = 1,\ s \geqslant 0 \Big\}.$$

$\mathbb{B}$-convex technologies belong to the class of Kohli technologies analyzed by Kohli (1983). These technologies exhibit output complementarity in the production. Inverse $\mathbb{B}$-convex technologies are related to Leontief production functions because they imply input complementarity of production factors. They are, however, defined in a multi-output context. Let us remark that the free disposal assumption can be represented thanks to the free disposal cone $K := \mathbb{R}_+^m \times (-\mathbb{R}_+^n)$. In this respect any technology respecting the free disposal assumption may be rewritten as: $T = (A + K) \cap \mathbb{R}_+^{m+n}$. As a consequence, $\mathbb{B}$-convex and $\mathbb{B}^{-1}$-convex technologies are also given by:

$$T_{\max}^{S} = (\mathbb{B}(A) + K) \cap \mathbb{R}_+^{m+n}\ ;\ T_{\min}^{S} = \big(\mathbb{B}^{-1}(A) + K\big) \cap \mathbb{R}_+^{n+m}.$$

These are represented in Figures 1a and 1b.

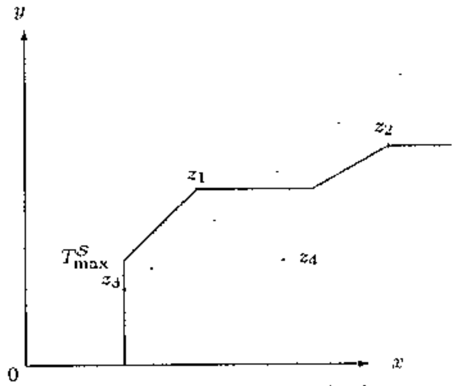

Figure 1.a *$\mathbb{B}$-convex Technology*

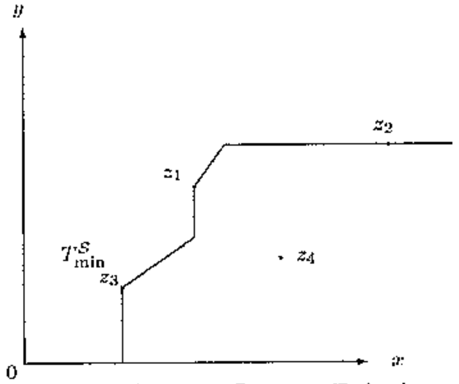

Figure 1.b *Inverse $\mathbb{B}$-convex Technology*

In what follows, we analyze the properties of the aggregated technologies T^S_∞ and $T^S_{-\infty}$, which are semilattice technologies:

$$\begin{array}{ll} T^S_\infty := \overset{\phi_\alpha}{\sum}_{k\in S} T^k = \bigvee_{k\in S} T^k & \text{if } \alpha = +\infty \\ T^S_{-\infty} := \overset{\phi_\alpha}{\sum}_{k\in S} T^k = \bigwedge_{k\in S} T^k & \text{if } \alpha = -\infty. \end{array} \tag{6.3}$$

Proposition 6.1 *We have the two following properties:*
(i) *Suppose that for all $k \in S$, T^k is an upper semilattice. Then $\bigvee_{k\in S} T^k$ is an upper semilattice.*
(ii) *Suppose that for all $k \in S$, T^k is a lower semilattice. Then $\bigwedge_{k\in S} T^k$ is a lower semilattice.*

Proof:
(i) Suppose that $z, w \in \bigvee_{k\in S} T^k$. We need to prove that $z \vee w \in \bigvee_{k\in S} T^k$. By hypothesis, for all $k \in S$ there is some $z^k, w^k \in T^k$ such that $z = \bigvee_{k\in S} z^k$ and $w = \bigvee_{k\in S} w^k$. Therefore,

$$z \vee w = (\bigvee_{k\in S} z^k) \vee (\bigvee_{k\in S} w^k) = \bigvee_{k\in S} (z^k \vee w^k).$$

Since for all k, T^k is an upper semilattice, it follows that $z^k \vee w^k \in T^k$. Thus $z \vee w \in \bigvee_{k\in S} T^k$, and $\bigvee_{k\in S} T^k$ is an upper semilattice. The proof of (ii) is similar. ∎

Now, we prove that the aggregated technology respects the usual assumptions (T1)-(T5) – see Section 2. The first one (T1) displays the *no free lunch* assumption, *i.e.* $(0_n, 0_m) \in T^k$ and $(0_n, y^k) \in T^k \Longrightarrow y^k = 0_m$.

Proposition 6.2 *We have the two following properties:*
(i) *If for all $k \in S$, T^k satisfies the no free lunch assumption* (T1), *then $\bigvee_{k\in S} T^k$ satisfies* (T1).
(ii) *If for all $k \in S$, T^k satisfies the no free lunch assumption* (T1), *then $\bigwedge_{k\in S} T^k$ satisfies* (T1).

Proof:
Straightforward. ∎

Now, we show that infinite outputs cannot be obtained from a finite input vector (T2), *i.e.* the set $A(x^k) = \{(u^k, y^k) \in T^k : u^k \leqslant x^k\}$ of dominating observations is bounded $\forall x^k \in \mathbb{R}^n_+$.

Proposition 6.3 *We have the two following properties:*
(i) *If for all $k \in S$, T^k satisfies* (T2), *then $\bigvee_{k\in S} T^k$ satisfies* (T2).
(ii) *If for all $k \in S$, T^k satisfies* (T2), *then $\bigwedge_{k\in S} T^k$ satisfies* (T2).

Proof:
(i) From (T2) each set $A(x^k)$ is assumed to be bounded. Hence, $A(\bigvee_{k\in\mathcal{S}} x^k)$ is also bounded, and so, $\bigvee_{k\in\mathcal{S}} T^k$ respects (T2). The same holds true for (ii). ■

The aggregated semilattice technology is shown to be closed (T3).

Proposition 6.4 *For all $k \in \mathcal{S}$, let T^k be* (i) *an upper semilattice or* (ii) *a lower semilattice, then two following properties hold true, respectively.*
(i) *For all $k \in \mathcal{S}$, if T^k satisfies the closedness assumption* (T3), *then $\bigvee_{k\in\mathcal{S}} T^k$ satisfies* (T3).
(ii) *For all $k \in \mathcal{S}$, if T^k satisfies the closedness assumption* (T3), *then $\bigwedge_{k\in\mathcal{S}} T^k$ satisfies* (T3).

Proof:
(i) Assume by contradiction that $\bigvee_{k\in\mathcal{S}} T^k$ is open. Hence, there is some $w, z \in \mathbb{R}_+^{n+m}$ such that $(w, z) \notin \bigvee_{k\in\mathcal{S}} T^k$. From Proposition 6.1, $\bigvee_{k\in\mathcal{S}} T^k$ is an upper semilattice. Then, $\bigvee_{k\in\mathcal{S}}(w^k \vee z^k) \notin \bigvee_{k\in\mathcal{S}} T^k$. Hence, the map $w^k \mapsto \bigvee_{k\in\mathcal{S}} w_k$ is not defined and not continuous on some intervals, that is, it exists some $k \in \mathcal{S}$ and at least one $i \in \{1, \ldots, d\}$ such that $\max_{k\in\mathcal{S}} w_i^k$ is not defined. Then, for $w_k = (x^k, y^k)$ it exists $x_i^k \in [\bar{x}_i^k, \bar{x}_i^k + \epsilon]$ or $y_i^k \in [\bar{y}_i^k, \bar{y}_i^k + \epsilon]$ with $\epsilon > 0$ such that $\max_{k\in\mathcal{S}} w_i^k$ is not defined. In such a case, $x^k \vee y^k \notin T^k$, and T^k is open on some intervals $[\bar{x}_i^k, \bar{x}_i^k + \epsilon]$ or $[\bar{y}_i^k, \bar{y}_i^k + \epsilon]$. (ii) *Mutatis mutandis* (i). ■

We can also prove that the aggregated semilattice technology respects the free disposal assumption (T4).

Proposition 6.5 *We have the two following properties:*
(i) *If for all $k \in \mathcal{S}$, T^k satisfies a free disposal assumption* (T4), *then $\bigvee_{k\in\mathcal{S}} T^k$ satisfies* (T4).
(ii) *If for all $k \in \mathcal{S}$, T^k satisfies a free disposal assumption* (T4), *then $\bigwedge_{k\in\mathcal{S}} T^k$ satisfies* (T4).

Proof:
(i) Suppose that $z = (x, y) \in \bigvee_{k\in\mathcal{S}} T^k$ and let $z' = (x', y') \in \mathbb{R}_+^{n+m}$ such that $x' \geqslant x$ and $y' \leqslant y$. We need to prove that $z' \in \bigvee_{k\in\mathcal{S}} T^k$. By hypothesis one can find $(z^1, \ldots, z^{|\mathcal{S}|}) \in \bigvee_{k\in\mathcal{S}} T^k$ such that $z = \bigvee_{k\in\mathcal{S}} z^k$. If $y' \leqslant y$, then there is some $v \in \mathbb{R}_+^m$ such that $y' = y - v$. Moreover $y - v = (\bigvee_{k\in\mathcal{S}} y^k) - v = \bigvee_{k\in\mathcal{S}}(y^k - v)$. However, $y - v \geqslant 0_m$, $y - v = (y - v) \vee 0_m$. Consequently,

$$y - v = \Big(\bigvee_{k\in\mathcal{S}}(y^k - v)\Big) \vee 0_m = \bigvee_{k\in\mathcal{S}} \big[(y^k - v) \vee 0_m\big].$$

For all k, since $y^k \geqslant 0_m$, $(y^k - v) \vee 0_m \leqslant y^k \vee 0_m = y^k$. Similarly, if $x' \geqslant x$, then there is some $u \in \mathbb{R}_+^n$ such that $x' = x + u = (\bigvee_{k\in\mathcal{S}} x^k) + u = \bigvee_{k\in\mathcal{S}}(x^k + u)$.

However, since each T^k satisfies a free disposal assumption the inequalities $(y^k - v) \vee 0_m \leqslant y^k$ and $x^k + u \geqslant x^k$ implies that $(x^k + u, (y^k - v) \vee 0_m) \in T^k$. Hence $z' = (x', y') \in \bigvee_{k \in \mathcal{S}} T^k$, which ends the proof. (ii) The proof is similar except that one should use the distributivity of the operation $\vee$ on $\wedge$. Suppose that $z = (x, y) \in \bigwedge_{k \in \mathcal{S}} T^k$ and let $z' = (x', y') \in \mathbb{R}_+^{n+m}$ such that $x' \geqslant x$ and $y' \leqslant y$. By hypothesis one can find $(z^1, \ldots, z^{|\mathcal{S}|}) \in \bigwedge_{k \in \mathcal{S}} T^k$ such that $z = \bigwedge_{k \in \mathcal{S}} z^k$. Paralleling the proof above, if $y' \leqslant y$, then there is some $v \in \mathbb{R}_+^m$ such that $y' = y - v$. Moreover $y - v = (\bigwedge_{k \in \mathcal{S}} y^k) - v = \bigwedge_{k \in \mathcal{S}} (y^k - v)$. Since, $y - v \geqslant 0_m$, $y - v = (y - v) \vee 0$. Therefore,

$$y - v = \Big(\bigwedge_{k \in \mathcal{S}} (y^k - v) \Big) \vee 0_m = \bigwedge_{k \in \mathcal{S}} \big[(y^k - v) \vee 0_m \big].$$

For all k, since $y^k \geqslant 0_m$ one has $(y^k - v) \vee 0_m \leqslant y^k \vee 0_m = y^k$. Moreover, if $x' \geqslant x$, then there is some $u \in \mathbb{R}_+^n$ such that $x' = x + u = (\bigwedge_{k \in \mathcal{S}} x^k) + u = \bigwedge_{k \in \mathcal{S}} (x^k + u)$. However, since each T^k satisfies a free disposal assumption $(y^k - v) \vee 0_m \leqslant y^k$ and $x^k + u \geqslant x^k$ implies that $z' = (x', y') \in \bigwedge_{k \in \mathcal{S}} T^k$, which ends the proof. ∎

The constant returns to scale assumption (T5) is also respected by the aggregated semilattice technology. Recall that the technology T^k satisfies constant returns to scale if $\forall \beta \geq 0$, $(x^k, y^k) \in T^k$ implies $(\beta x^k, \beta y^k) \in T^k$.

Proposition 6.6 *The two following properties hold true:*
(i) *If for all $k \in \mathcal{S}$, T^k is an upper semilattice and satisfies the constant returns to scale assumption* (T5)*, then $\bigvee_{k \in \mathcal{S}} T^k$ is an upper semilattice respecting* (T5)*.*
(ii) *If for all $k \in \mathcal{S}$, T^k is a lower semilattice and satisfies the constant returns to scale assumption* (T5)*, then $\bigwedge_{k \in \mathcal{S}} T^k$ is a lower semilattice respecting* (T5)*.*

Proof:
(i) Suppose that $z, w \in \bigvee_{k \in \mathcal{S}} T^k$. We have to prove that $\beta z \vee \beta w \in \bigvee_{k \in \mathcal{S}} T^k$ for some $\beta \geq 0$. For all $k \in \mathcal{S}$ choose some $z^k, w^k \in T^k$ such that $z = \bigvee_{k \in \mathcal{S}} z^k$ and $w = \bigvee_{k \in \mathcal{S}} w^k$. Therefore,

$$\beta z \vee \beta w = \Big(\bigvee_{k \in \mathcal{S}} \beta z^k \Big) \vee \Big(\bigvee_{k \in \mathcal{S}} \beta w^k \Big) = \bigvee_{k \in \mathcal{S}} (\beta z^k \vee \beta w^k).$$

By (T5), since T^k is an upper semilattice for all $k \in \mathcal{S}$, we get that $\beta z^k \vee \beta w^k \in T^k$. Thus $\beta z \vee \beta w \in \bigvee_{k \in \mathcal{S}} T^k$, and so $z \vee w \in \bigvee_{k \in \mathcal{S}} T^k$. Consequently, $\bigvee_{k \in \mathcal{S}} T^k$ is an upper semilattice satisfying (T5). (ii) *Mutatis mutandis* (i). ∎

A last property of equal technology will be useful in the games defined below. Inside a coalition, when the technologies of the firms are the same, the aggregated technology of the coalition inherits the same technology.

Proposition 6.7 *We have the two following properties:*
(i) *Suppose that for all $k \in \mathcal{S}$, $T^k = T$ is an upper semilattice. Then $\bigvee_{k\in\mathcal{S}} T^k = T$.*
(ii) *Suppose that for all $k \in \mathcal{S}$, $T^k = T$ is a lower semilattice. Then $\bigwedge_{k\in\mathcal{S}} T^k = T$.*

Proof:
(i) By hypothesis $(0_n, 0_m) \in T^k$ for all k. Therefore, for all k, $T^k = T \subset \bigvee_{k\in\mathcal{S}} T^k$. Let us show the converse. Assume that $z \in T$. It follows that for all k there are some $z^k \in T^k$, such that $z = \bigvee_{k\in\mathcal{S}} z^k$. Since $T = T^k$, $z^k \in T$ for all k. Since T is an upper semilattice it follows that $\bigvee_{k\in\mathcal{S}} z^k \in T$. Therefore $\bigvee_{k\in\mathcal{S}} T^k \subset T$. Consequently, $\bigvee_{k\in\mathcal{S}} T^k = T$.
(ii) Assume that $z \in T$. It follows that for all k there are some $z^k \in T^k$, such that $z = \bigwedge_{k\in\mathcal{S}} z^k$. Since $T = T^k$, $z^k \in T$ for all k. Since T is a lower semilattice it follows that $\bigwedge_{k\in\mathcal{S}} z^k \in T$. Therefore $\bigwedge_{k\in\mathcal{S}} T^k \subset T$. Let us prove the converse inclusion, Suppose that, $z \in T$. Since $T = T^k$ for all k, $z \in T^k$. Obviously $z = \bigwedge_{k\in\mathcal{S}} z^k$. Hence $z \in \bigwedge_{k\in\mathcal{S}} T^k$ and it follows that $T \subset \bigwedge_{k\in\mathcal{S}} T^k$ which proves the converse inclusion. ∎

Finally, those properties indicate that the semilattice structure respects the traditional assumptions (T1)-(T5) used in the literature.

6.2 Games

Since the Φ_∞- and $\Phi_{-\infty}$-aggregators provide semilattice aggregated technologies with desirable assumptions, we can now investigate their implications on firm games. Some restrictions may be imposed on inputs and outputs in order to clearly identify negative and positive technical biases.

Definition 6.3 – **Input/Output fixed firm games** – *Let the firm game be $\{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi_\alpha\}$.*
(i) *An input fixed firm game is given by,*

$$\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{x}) := \{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi_\alpha\},$$

where for all $k \in \mathcal{S}$, $x^k = \bar{x}$.
(ii) *An output fixed firm game is given by,*

$$\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{y}) := \{\mathcal{K}, v^f(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi_\alpha\},$$

where for all $k \in \mathcal{S}$, $y^k = \bar{y}$.

(i) In input fixed firm games $\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{x})$, coalitions of firms are based on possible outputs to be produced whereas the amount of inputs is limited to $\bar{x}$ for all possible coalitions. This may arise when the firms of specific sectors are constrained by the amount of their inputs, for instance a maximum is imposed to respect environmental norms.

(ii) In output fixed firm games $\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{y})$, coalitions are formed on the basis of inputs only, whereas the amount of outputs for each coalition is constrained by $\bar{y}$, which may represent a production quota.

The core of those firm games is:

$$C_\alpha := \left\{ \varphi \in M^{|\mathcal{K}|} : \sum_{k \in \mathcal{S}} \varphi_k \leq v_\alpha(\mathcal{S}), \forall \mathcal{S} \subset \mathcal{K} \right\} \bigcap \left\{ \sum_{k \in \mathcal{K}} \varphi_k = v_\alpha(\mathcal{K}) \right\}. \quad (6.4)$$

The technical biases inherent to the input fixed firm game are the following.

Proposition 6.8 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$, then we have, respectively:*

(i) $[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{x}) \wedge (\alpha = +\infty)] \implies [AB_\alpha(\mathcal{S}; g) \leq 0]$.

(ii) $[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{x}) \wedge (\alpha = -\infty)] \implies [AB_\alpha(\mathcal{S}; g) \geq 0]$.

Proof:

(i) If $\alpha = \infty$, then $\overset{\Phi_\infty}{\sum}_{k \in \mathcal{S}} T^k = \bigvee_{k \in \mathcal{S}} T^k$ is an upper semilattice (see Proposition 6.1). From Proposition 6.5 this set satisfies the free disposal assumption (T4), and moreover $T = \bigvee_{k \in \mathcal{S}} T^k$ (see Proposition 6.7). It follows that the directional distance function is weakly monotonic on T, that is, $(x, y), (u, v) \in T$ such that $u \leqslant y$ and $v \geqslant x$ imply that $D_T(u, v; g) \geq D_T(x, y; g)$. Since this is an input fixed game, we have $x^k = \bar{x}$, for all $k \in \mathcal{S}$. Hence, for all $\mathcal{S} \subseteq \mathcal{K}$, we have $\bigvee_{k \in \mathcal{S}} x^k = \bar{x}$. Moreover, $\bigvee_{k \in \mathcal{S}} y^k \geqslant y^k$ for all k. From weak monotonicity, we have $D_T(\bar{x}, \bigvee_{k \in \mathcal{S}} y^k; g) \leq D_T(x^k, y^k; g) = D_T(\bar{x}, y^k; g)$. However, since $T^k = T$, we have for all $k \in \mathcal{S}$:

$$D_T\left(\bigvee_{k \in \mathcal{S}} x^k, \bigvee_{k \in \mathcal{S}} y^k; g\right) = D_T(\bar{x}, \bigvee_{k \in \mathcal{S}} y^k; g) \leq D_T(x^k, y^k; g) = D_{T^k}(x^k, y^k; g).$$

This implies that $D_T(\bigvee_{k \in \mathcal{S}} x^k, \bigvee_{k \in \mathcal{S}} y^k; g) \leq \max_{k \in \mathcal{S}} D_{T^k}(x^k, y^k; g)$ which proves (i).

(ii) If $\alpha = -\infty$, then $\overset{\Phi_{-\infty}}{\sum}_{k \in \mathcal{S}} T^k = \bigwedge_{k \in \mathcal{S}} T^k = T$ is a lower semilattice by Proposition 6.1. This set satisfies the free disposal assumption (T4) from Proposition 6.5. It follows that the directional distance function is weakly monotonic on $\bigwedge_{k \in \mathcal{S}} T^k = T$ by Proposition 6.7. Then $(x, y), (u, v) \in T$, such that $u \leqslant y$ and $v \geqslant x$ imply that $D_T(u, v; g) \geq D_T(x, y; g)$. Since this is an input fixed game, we have $x^k = \bar{x}$, for all $k \in \mathcal{S}$. Hence, for all $\mathcal{S} \subseteq \mathcal{K}$, we have $\bigwedge_{k \in \mathcal{S}} x^k = \bar{x}$. Moreover, $\bigwedge_{k \in \mathcal{S}} y^k \leqslant y^k$ for all k. From weak monotonicity, we have $D_T(\bar{x}, \bigwedge_{k \in \mathcal{S}} y^k; g) \leq D_T(x^k, y^k; g) = D_T(\bar{x}, y^k; g)$. Moreover, since $T^k = T$, we have for all $k \in \mathcal{S}$:

$$D_T\left(\bigwedge_{k \in \mathcal{S}} x^k, \bigwedge_{k \in \mathcal{S}} y^k; g\right) = D_T(\bar{x}, \bigwedge_{k \in \mathcal{S}} y^k; g) \geq D_T(x^k, y^k; g) = D_{T^k}(x^k, y^k; g).$$

This implies that $D_T(\bigwedge_{k\in\mathcal{S}} x^k, \bigwedge_{k\in\mathcal{S}} y^k; g) \geq \min_{k\in\mathcal{S}} D_{T^k}(x^k, y^k; g)$ which proves (ii). ■

The technical biases inherent to the output fixed firm game are the following.

Proposition 6.9 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$, then we have, respectively:*

(i) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{y}) \wedge (\alpha = +\infty)] \implies [AB_\alpha(\mathcal{S}; g) \geq 0]$

(ii) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{y}) \wedge (\alpha = -\infty)] \implies [AB_\alpha(\mathcal{S}; g) \leq 0]$.

Proof:

(i) If $\alpha = \infty$, then $\sum_{k\in\mathcal{S}}^{\Phi_\infty} T^k = \bigvee_{k\in\mathcal{S}} T^k = T$ is an upper semilattice respecting the free disposal assumption (T4), see Propositions 6.1, 6.5 and 6.7. Thereby, the directional distance function is weakly monotonic on T, that is, $(x, y), (u, v) \in T$, such that $u \leqslant y$ and $v \geqslant x$ imply that $D_T(u, v; g) \geq D_T(x, y; g)$. Since this is an output fixed firm game, we have $y^k = \bar{y}$, for all $k \in \mathcal{S}$, and so $\bigvee_{k\in\mathcal{S}} y^k = \bar{y}$ for all $k \in \mathcal{S}$. Moreover, $\bigvee_{k\in\mathcal{S}} x^k \geqslant x^k$ for all $k \in \mathcal{S}$. From weak monotonicity, we have $D_T(\bigvee_{k\in\mathcal{S}} x^k, \bar{y}; g) \geq D_T(x^k, y^k; g) = D_T(x^k, \bar{y}; g)$. Since $T^k = T$, we have for all $k \in \mathcal{S}$:

$$D_T\left(\bigvee_{k\in\mathcal{S}} x^k, \bar{y}; g\right) \geq D_T(x^k, y^k; g) = D_{T^k}(x^k, \bar{y}; g).$$

Consequently, $D_T(\bigvee_{k\in\mathcal{S}} x^k, \bar{y}; g) \geq \max_{k\in\mathcal{S}} D_{T^k}(x^k, y^k; g)$ which proves (i).

(ii) If $\alpha = -\infty$, then $\sum_{k\in\mathcal{S}}^{\Phi_{-\infty}} T^k = \bigwedge_{k\in\mathcal{S}} T^k = T$ satisfies the free disposal assumption, see Propositions 6.1, 6.5 and 6.7. The directional distance function is then weakly monotonic on T, in other words $(x, y), (u, v) \in T$ such that $u \leqslant y$ and $v \geqslant x$ imply that $D_T(u, v; g) \geq D_T(x, y; g)$. Since this is an output fixed firm game, we have $y^k = \bar{y}$ and so $\bigwedge_{k\in\mathcal{S}} y^k = \bar{y}$ for all $k \in \mathcal{S} \subseteq \mathcal{K}$. Moreover, $\bigwedge_{k\in\mathcal{S}} x^k \leqslant x^k$ for all $k \in \mathcal{S}$. From weak monotonicity, we have $D_T(\bigwedge_{k\in\mathcal{S}} x^k, \bar{y}; g) \leq D_T(x^k, y^k; g)$. Since $T^k = T$, we have for all $k \in \mathcal{S}$:

$$D_T\left(\bigwedge_{k\in\mathcal{S}} x^k, \bar{y}; g\right) \leq D_T(x^k, y^k; g) = D_{T^k}(x^k, \bar{y}; g).$$

This yields $D_T(\bigwedge_{k\in\mathcal{S}} x^k, \bar{y}; g) \leq \min_{k\in\mathcal{S}} D_{T^k}(x^k, y^k; g)$ which proves (ii). ■

The previous results indicate that the data aggregation process Φ_∞ (respectively $\Phi_{-\infty}$) is relevant with a negative bias that embodies an improvement of technical efficiency in the input fixed firm game (respectively in the output fixed firm game).

6.3 Core of firm games and core partitions

The negative bias or equivalently the α-subadditivity (SUB_α) is not sufficient to avoid the vacuity of the core. Briec and Mussard (2014) investigate the submodularity of the technical bias inherent to allocative firm games in order to find super-efficient firm groups. The submodularity of the firm game is defined as follows.

Definition 6.4 – **Submodularity** – *For all firm games* $\{\mathcal{K}, v^J(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{K}, \Phi_\alpha\}$, *such that* $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{K}$ *with* $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$, *the game is submodular (or concave) if:*

$$v_\alpha(\mathcal{S}_1 \cup \mathcal{S}_2) \leq v_\alpha(\mathcal{S}_1) + v_\alpha(\mathcal{S}_2) - v_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2).$$

In the same manner, the submodularity of the aggregation bias is, for $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{K}$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$,

$$AB_\alpha(\mathcal{S}_1 \cup \mathcal{S}_2) \leq AB_\alpha(\mathcal{S}_1) + AB_\alpha(\mathcal{S}_2) - AB_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2). \tag{6.5}$$

The submodularity displays the following interpretation: the loss of technical efficiency due to the cooperation between two coalitions is no higher than the aggregated loss of the coalitions taking into account the loss of the joint cooperation $AB_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2)$. It is shown below that the submodularity of the aggregation bias is closely related to that of the game $v_\alpha(\cdot)$.

Proposition 6.10 *If* T^k *is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that* $T^k = T$ *for all* $k \in \mathcal{S} \subseteq \mathcal{K}$, *then we have, respectively:*

(i) $[(AB_{+\infty}$ is submodular$)] \Longrightarrow [v_{+\infty}$ is submodular$] \Longrightarrow [\varphi \in \mathring{\mathcal{C}}_\infty \neq \emptyset]$.

(ii) *If* $\min_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} v_{-\infty(\{k\})} = \min_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} v_{-\infty(\{k\})}$ *then:*

$[(AB_{-\infty}$ is submodular$)] \Longrightarrow [v_{-\infty}$ is submodular$] \Longrightarrow [\varphi \in \mathring{\mathcal{C}}_{-\infty} \neq \emptyset]$.

Proof:

(i) Let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{K}$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$. Since $T^k = T$ for all $k \in \mathcal{S}$, then:

$$T = T^{\mathcal{S}_1} = T^{\mathcal{S}_2} = T^{\mathcal{S}_1 \cap \mathcal{S}_2} = T^{\mathcal{S}_1 \cup \mathcal{S}_2}.$$

Let $z^k = (x^k, y^k) \in \mathbb{R}_+^{n+m}$, the submodularity of the technical bias is given by:

$$AB_\alpha(\mathcal{S}_1 \cup \mathcal{S}_2) \leq AB_\alpha(\mathcal{S}_1) + AB_\alpha(\mathcal{S}_2) - AB_\alpha(\mathcal{S}_1 \cap \mathcal{S}_2).$$

This entails that:

$$D_T\left(\bigvee_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} z^k; g\right) - \bigvee_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} D_T(z^k; g) + D_T\left(\bigvee_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} z^k; g\right) - \bigvee_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} D_T(z^k; g)$$

$$\leq D_T\left(\bigvee_{k \in \mathcal{S}_1} z^k; g\right) - \bigvee_{k \in \mathcal{S}_1} D_T(z^k; g) + D_T\left(\bigvee_{k \in \mathcal{S}_2} z^k; g\right) - \bigvee_{k \in \mathcal{S}_2} D_T(z^k; g).$$

Note that for $\mathcal{R} = \mathcal{S}_1, \mathcal{S}_2$:

$$\max_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} D_T(z^k; g) \leq \max_{k \in \mathcal{R}} D_T(z^k; g) \leq \max_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} D_T(z^k; g).$$

Hence, the game $v_\alpha(\cdot)$ represented by the characteristic function $D_T(\cdot)$ is concave, that is, the distance function is submodular:

$$D_T\left(\bigvee_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} z^k; g\right) \leq D_T\left(\bigvee_{k \in \mathcal{S}_1} z^k; g\right) + D_T\left(\bigvee_{k \in \mathcal{S}_2} z^k; g\right) - D_T\left(\bigvee_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} z^k; g\right).$$

To find the previous relation, note that three cases have to be considered: either the maximum distance $v_\alpha(\{k\})$ is such that $k \in \{\mathcal{S}_1 \setminus \mathcal{S}_2\}$, or $k \in \{\mathcal{S}_2 \setminus \mathcal{S}_1\}$ or finally $k \in \mathcal{S}_1 \cap \mathcal{S}_2$. By definition, the pay-off vector φ satisfies *linearity*, *symmetry*, and *efficiency* (see Section 2). By Shapley (1972), the core interior $\overset{\circ}{\mathcal{C}}_{+\infty}$ is therefore non empty.

(ii) In the lower semilattice case, we have for $\mathcal{R} = \mathcal{S}_1, \mathcal{S}_2$:

$$\min_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} D_T(z^k; g) \geq \min_{k \in \mathcal{R}} D_T(z^k; g) \geq \min_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} D_T(z^k; g).$$

As a consequence, it is easy to show that the previous condition is not sufficient to ensure the submodularity of $v_{-\infty}$. However, if $\min_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} v_{-\infty}(\{k\}) = \min_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} v_{-\infty}(\{k\})$ then the submodularity of $v_{-\infty}$ follows,

$$D_T\left(\bigwedge_{k \in \mathcal{S}_1 \cup \mathcal{S}_2} z^k; g\right) \leq D_T\left(\bigwedge_{k \in \mathcal{S}_1} z^k; g\right) + D_T\left(\bigwedge_{k \in \mathcal{S}_2} z^k; g\right) - D_T\left(\bigwedge_{k \in \mathcal{S}_1 \cap \mathcal{S}_2} z^k; g\right).$$

Then, the game $v_{-\infty}(\cdot)$ is concave (submodular). Moreover, since by definition the pay-off vector φ satisfies *linearity*, *symmetry*, and *efficiency* (see Section 2), then by Shapley (1972), the core interior $\overset{\circ}{\mathcal{C}}_{-\infty}$ is non void. ∎

The last proposition is interesting since it allows to get the non vacuity of the core. However, it does not tell us the whole story of the sign of the aggregation bias. Indeed, from the previous result, it is clear that the core of the firm game may be non void and in the same time the aggregation bias may be positive or negative as can be seen in Propositions 6.8 and 6.9. In order to get a clear result about the non vacuity of core and about the sign of the technical bias, we introduce a partition of the core. The first core displays the set of imputations φ inherent to a positive bias,

$$\mathcal{C}_{AB_\alpha \geq 0} := \left\{ \varphi \in M^{|\mathcal{K}|} \,\middle|\, \begin{array}{ll} \sum_{k \in \mathcal{S}} \varphi_k & \leq v_\alpha(\mathcal{S}), \forall \mathcal{S} \subset \mathcal{K} \\ AB_\alpha(\mathcal{S}) & \geq 0,\ \forall \mathcal{S} \subset \mathcal{K} \end{array} \right\} \bigcap \left\{ \sum_{k \in \mathcal{K}} \varphi_k = v_\alpha(\mathcal{K}) \right\}.$$

The second core yields the set of pay-off vectors φ inherent to a negative bias,

$$\mathcal{C}_{AB_\alpha \leq 0} := \left\{ \varphi \in M^{|\mathcal{K}|} \,\middle|\, \begin{array}{ll} \sum_{k \in \mathcal{S}} \varphi_k & \leq v_\alpha(\mathcal{S}), \forall \mathcal{S} \subset \mathcal{K} \\ AB_\alpha(\mathcal{S}) & \leq 0,\ \forall \mathcal{S} \subset \mathcal{K} \end{array} \right\} \bigcap \left\{ \sum_{k \in \mathcal{K}} \varphi_k = v_\alpha(\mathcal{K}) \right\}.$$

It is obvious that $\mathcal{C}_\alpha = \mathcal{C}_{AB_\alpha \leq 0} \cup \mathcal{C}_{AB_\alpha \geq 0}$. Consequently, either in the input or the output fixed firm game, whenever the game v_α is submodular, it is always possible to find a solution in the core interior with improvement of technical efficiency.

Corollary 6.1 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subsetneq \mathcal{K}$, then we have, respectively:*

(i) $[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{x}) \wedge (\alpha = +\infty) \wedge (v_\alpha \text{ is submodular})] \Longrightarrow [\varphi \in \mathring{\mathcal{C}}_{AB_{+\infty} \leq 0}]$.

(ii) $[\mathcal{I}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{x}) \wedge (\alpha = -\infty) \wedge (v_\alpha \text{ is submodular})] \Longrightarrow [\varphi \in \mathring{\mathcal{C}}_{AB_{-\infty} \geq 0}]$.

Corollary 6.2 *If T^k is an upper semilattice (lower semilattice respectively) respecting* (T1)-(T4) *such that $T^k = T$ for all $k \in \mathcal{S} \subsetneq \mathcal{K}$, then we have, respectively:*

(i) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{y}) \wedge (\alpha = +\infty) \wedge (v_\alpha \text{ is submodular})] \Longrightarrow [\varphi \in \mathring{\mathcal{C}}_{AB_{+\infty} \geq 0}]$.

(ii) $[\mathcal{O}(\mathcal{K}, v_\alpha(\mathcal{S}), \bar{y}) \wedge (\alpha = -\infty) \wedge (v_\alpha \text{ is submodular})] \Longrightarrow [\varphi \in \mathring{\mathcal{C}}_{AB_{-\infty} \leq 0}]$.

Proof:
Both corollaries are deduced from Propositions 6.8, 6.9 and 6.10. ∎

7 Conclusion

The data transformation suggested by Post (2001), in order to gauge technical efficiency, is of real interest to improve the accuracy of the mathematical tools usually employed in the literature. In the same manner, we first characterize an aggregator Φ relevant with general distance functions, and subsequently relevant with directional distance functions. This aggregator reduces, under the homogeneity property, to the power mean Φ_α, which is a good candidate to deal with aggregated technologies.

The improvement of (the aggregated) technical efficiency of a group of firms has been shown to be no more impossible. The aggregation bias may be negative and the result belongs to the core interior of the firm game. This approach generalizes the firm game, introduced by Briec and Mussard (2014), defined under the standard sum of technology sets. The aggregation bias, issued from the Φ_α-aggregator, takes different values: positive, negative or zero. Then, the cooperation between firms entails all possible cases, especially with aggregated semilattice technologies that respect all desirable assumptions.

Our result can be extended to DEA frameworks in order to check the convergence rate of the Φ_α-aggregator. It is possible to show that the aggregation bias may be negative for technology subsets associated with values of

α that do not necessarily tend to infinity, as in Example 5.1. Indeed, the directional distance functions may be computed by solving the following linear program for all $\mathcal{S} \subseteq \mathcal{K}$ and for all $(x_{\mathcal{S}}, y_{\mathcal{S}}) \in \mathbb{R}_+^{n+m}$:

$$\begin{aligned} \hat{v}_\alpha(\mathcal{S}) &:= \max \delta_{\mathcal{S}} \\ \text{s.t. : } x_{\mathcal{S}} - \delta_{\mathcal{S}} g_i &\geqslant \sum_{k\in\mathcal{S}}^{\Phi_\alpha} \theta_k x^k \\ y_{\mathcal{S}} + \delta_{\mathcal{S}} g_o &\leqslant \sum_{k\in\mathcal{S}}^{\Phi_\alpha} \theta_k x^k \\ \sum_{k\in\mathcal{S}}^{\Phi_\alpha} \theta_k &= 1 \ , \ \theta_k \geq 0. \end{aligned}$$

Choosing a scalar b enough large, the determination of the optimal value α^* allowing for the improvement of the technical efficiency in at least one [resp. for all] coalition[s] is the following.

Algorithm for the convergence of α

- Loop to $\alpha \in [-b, b]$;
  - $\hookrightarrow$ Compute $\hat{v}_\alpha(\mathcal{S})$ for all $\mathcal{S} \subseteq \mathcal{K}$;
  - $\hookrightarrow$ Compute $AB_\alpha(\mathcal{S}) = \hat{v}_\alpha(\mathcal{S}) - \sum_{k\in\mathcal{S}}^{\phi_\alpha} \hat{v}_\alpha(\{k\})$;
    - $\mapsto$ If $AB_\alpha(\mathcal{S}) \leq 0$ for at least one given $\mathcal{S}$ [for all $\mathcal{S}$] then $\alpha^* = \alpha$: End α ;
    - $\mapsto$ Else : change α ;
- End α.

Finally, the aggregated technology inherent to the data transformation, embodied by the Φ_α-aggregator, is closely connected to the best input/output realizations of the firms of the sector (group), *i.e.*, the so-called input/output fixed firm games. This result is in line with the literature on firm concentrations such as monopolization of industries. In our findings, the improvement of technical efficiency due to cooperation is, by duality between cost functions and distance functions, a cost reduction. The fusion of the firms may be viewed as the purchase of the firms of the group realized by the most efficient one. Empirically, the purchase of the firms may be allowed by the Competition Authority insofar the cost reduction implied by the fusion improves the well-being of the consumers. In this case, the well-known cost-benefit trade-offs are of interest, as in the celebrated result of Williamson (1968).

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