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Research Article

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Abstract: We study the Cauchy problem for a system of equations corresponding to a singular limit of radiative hydrodynamics, namely, the 3D radiative compressible Euler system coupled to an electromagnetic field. Assuming smallness hypotheses for the data, we prove that the problem admits a unique global smooth solution and study its asymptotics.

Keywords: Compressible, Euler, radiation hydrodynamics

MSC 2010: 35Q30, 76N10

1 Introduction

In [3], after the studies of Lowrie, Morel and Hittinger [15], and Buet and Després [5], we considered a singular limit for a compressible inviscid radiative flow, where the motion of the fluid is given by the Euler system for the evolution of the density $\varrho = \varrho(t, x)$, the velocity field $\vec{u} = \vec{u}(t, x)$ and the absolute temperature $\vartheta = \vartheta(t, x)$, and where radiation is described in the limit by an extra temperature $T_r = T_r(t, x)$. All of these quantities are functions of the time t and the Eulerian spatial coordinate $x \in \mathbb{R}^3$.

In [3] we proved that the associated Cauchy problem admits a unique global smooth solution, provided that the data are small enough perturbations of a constant state.

In [4] we coupled the previous model to the electromagnetic field through the so-called magnetohydrodynamic (MHD) approximation, in presence of thermal and radiative dissipation. Hereafter, we consider the perfect non-isentropic Euler–Maxwell system and we also consider a radiative coupling through a pure convective transport equation for the radiation (without dissipation). Then we deal with a pure hyperbolic system with partial relaxation (damping on velocity).

More specifically the system of equations to be studied for the unknowns $(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})$ reads

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x(p + p_r) = -\varrho(\vec{E} + \vec{u}\times\vec{B}) - \nu\varrho\vec{u}, \tag{1.2}$$

$$\partial_t(\varrho E) + \mathrm{div}_x((\varrho E + p)\vec{u}) + \vec{u}\cdot\nabla_x p_r = -\sigma_a(a\vartheta^4 - E_r) - \varrho\vec{E}\cdot\vec{u}, \tag{1.3}$$

$$\partial_t E_r + \mathrm{div}_x(E_r\vec{u}) + p_r\,\mathrm{div}_x\,\vec{u} = -\sigma_a(E_r - a\vartheta^4), \tag{1.4}$$

$$\partial_t\vec{B} + \mathrm{curl}_x\,\vec{E} = 0, \tag{1.5}$$

$$\partial_t\vec{E} - \mathrm{curl}_x\,\vec{B} = \varrho\vec{u}, \tag{1.6}$$

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$$\operatorname{div}_x \vec{B} = 0, \tag{1.7}$$

$$\operatorname{div}_x \vec{E} = \overline{\varrho} - \varrho, \tag{1.8}$$

where ϱ is the density, $\vec{u}$ the velocity, ϑ the temperature of matter, $E = \frac{1}{2}|\vec{u}|^2 + e(\varrho, \vartheta)$ is the total mechanical energy, E_r is the radiative energy related to the temperature of radiation T_r by $E_r = aT_r^4$, and p_r is the radiative pressure given by $p_r = \frac{1}{3}aT_r^4 = \frac{1}{3}E_r$, with $a > 0$. Finally, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic induction.

We assume that the pressure $p(\varrho, \vartheta)$ and the internal energy $e(\varrho, \vartheta)$ are positive smooth functions of their arguments, with

$$C_v := \frac{\partial e}{\partial \vartheta} > 0, \quad \frac{\partial p}{\partial \varrho} > 0,$$

and we also suppose for simplicity that $\nu = \frac{1}{\tau}$ (where $\tau > 0$ is a momentum-relaxation time) and μ, σ_a and a are positive constants.

A simplification appears if one observes that, provided that equations (1.7) and (1.8) are satisfied at $t = 0$, they are satisfied for any time $t > 0$, and consequently they can be discarded from the analysis below.

Notice that the reduced system (1.1)–(1.4) is the non-equilibrium regime of radiation hydrodynamics, introduced by Lowrie, Morel and Hittinger [15] and, more recently, by Buet and Després [5], and studied mathematically by Blanc, Ducomet and Nečasová [3]. Extending this last analysis, our goal in this work is to prove global existence of solutions for system (1.1)–(1.8) when the data are sufficiently close to an equilibrium state, and study their large time behavior.

For the sake of completeness, we mention that related non-isentropic Euler–Maxwell systems have been the subject of a number of studies in the recent past. Let us quote the recent works [9, 10, 12, 14, 18, 21].

In the following, we show that the ideas used by Ueda, Wang and Kawashima in [19, 20] in the isentropic case can be extended to the (radiative) non-isentropic system (1.1)–(1.6). To this end, we follow the following plan. In Section 2 we present the main results and then, in Section 3, we prove the well-posedness of system (1.1)–(1.6). Finally, in Section 4, we prove the large time asymptotics of the solution.

2 Main results

We are going to prove that system (1.1)–(1.8) has a global smooth solution close to any equilibrium state. Namely, we have the following theorem.

Theorem 2.1. *Let $(\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\vec{B}}, 0)$ be a constant state, with $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{E_r} > 0$, and compatibility condition $\overline{E_r} = a\overline{\vartheta}^4$, and suppose that $d \geq 3$. There exists $\varepsilon > 0$ such that for any initial state $(\varrho_0, \vec{u}_0, \vartheta_0, E_r^0, \vec{B}_0, \vec{E}_0)$ satisfying*

$$\operatorname{div}_x \vec{B}_0 = \varrho_0 - \overline{\varrho}, \quad \operatorname{div}_x \vec{B}_0 = 0, \quad (\varrho_0 - \overline{\varrho}, \vec{u}_0, \vartheta_0 - \overline{\vartheta}, E_{r0} - \overline{E_r}, \vec{B}_0 - \overline{\vec{B}}, \vec{E}_0) \in H^d$$

and

$$\|(\varrho_0, \vec{u}_0, \vartheta_0, E_r^0, \vec{B}_0, \vec{E}_0) - (\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\vec{B}}, 0)\|_{H^d} \leq \varepsilon,$$

there exists a unique global solution $(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})$ to (1.1)–(1.8) such that

$$(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r}, \vec{B} - \overline{\vec{B}}, \vec{E}) \in C([0, +\infty); H^d) \cap C^1 g([0, +\infty); H^{d-1}).$$

In addition, this solution satisfies the following energy inequality:

$$\begin{aligned}\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r}, \vec{B} - \overline{\vec{B}}, \vec{E})(t)\|_{H^d} &+ \int_0^t \big(\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r})(\tau)\|_{H^d}^2 + \|\nabla_x \vec{B}(\tau)\|_{H^{d-2}}^2 + \|\vec{E}(\tau)\|_{H^{d-1}}^2\big)\, d\tau \\ &\leq C\|(\varrho_0 - \overline{\varrho}, 0, \vartheta_0 - \overline{\vartheta}, E_r^0 - \overline{E_r}, \vec{B}_0 - \overline{\vec{B}}, \vec{E}_0)\|_{H^d}^2\end{aligned} \tag{2.1}$$

for some constant $C > 0$ which does not depend on t.

The large time behavior of the solution is described as follows.

Theorem 2.2. *Let $d \geq 3$. The unique global solution $(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})$ to (1.1)–(1.8), defined in Theorem 2.1, converges to the constant state $(\overline{\varrho}, \vec{0}, \overline{\vartheta}, \overline{E_r}, \overline{\vec{B}}, \vec{0})$ uniformly in $x \in \mathbb{R}^3$ as $t \to \infty$. More precisely,*

$$\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r}, \vec{E})(t)\|_{W^{d-2,\infty}} \to 0 \quad \text{as } t \to \infty.$$

Moreover, if $d \geq 4$, then

$$\|(\vec{B} - \overline{\vec{B}})(t)\|_{W^{d-4,\infty}} \to 0 \quad \text{as } t \to \infty,$$

Remark 2.3. Note that, due to lack of dissipation by viscous, thermal and radiative fluxes, the Kawashima–Shizuta stability criterion (see [17] and [1]) is not satisfied for the system under study, and the techniques of [13] relying on the existence of a compensating matrix do not apply. However, we will check that radiative sources play the role of relaxation terms for the temperature and radiative energy and this will lead to global existence for the system.

3 Global existence

3.1 A priori estimates

Multiplying (1.2) by $\vec{u}$, (1.5) by $\vec{B}$, (1.6) by $\vec{E}$ and adding the result to equations (1.3) and (1.4), we get the total energy conservation law

$$\partial_t\Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e + E_r + \frac{1}{2}(|\vec{B}|^2 + |\vec{E}|^2)\Big) + \operatorname{div}_x\big((\varrho E + E_r)\vec{u} + (p + p_r)\vec{u} + \vec{E} \times \vec{B}\big) = 0. \tag{3.1}$$

Introducing the entropy s of the fluid by the Gibbs law $\vartheta ds = de + pd(\frac{1}{\varrho})$ and denoting by $S_r := \frac{4}{3}aT_r^3$ the radiative entropy, equation (1.4) is rewritten as

$$\partial_t S_r + \operatorname{div}_x(S_r\vec{u}) = -\sigma_a \frac{E_r - a\vartheta^4}{T_r}. \tag{3.2}$$

The internal energy equation is

$$\partial_t(\varrho e) + \operatorname{div}_x(\varrho e\vec{u}) + p \operatorname{div}_x \vec{u} - \nu\varrho|\vec{u}|^2 = -\sigma_a(a\vartheta^4 - E_r),$$

and by dividing it by ϑ, we get the entropy equation for matter

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s\vec{u}) - \frac{\nu}{\vartheta}|\vec{u}|^2 = -\sigma_a \frac{a\vartheta^4 - E_r}{\vartheta}. \tag{3.3}$$

So adding (3.3) and (3.2), we obtain

$$\partial_t(\varrho s + S_r) + \operatorname{div}_x((\varrho s + S_r)\vec{u}) = \frac{a\sigma_a}{\vartheta T_r}(\vartheta - T_r)^2(\vartheta + T_r)(\vartheta^2 + T_r^2) + \frac{\nu}{\vartheta}|\vec{u}|^2. \tag{3.4}$$

By subtracting (3.4) from (3.1) and using the conservation of mass, we get

$$\begin{aligned}\partial_t\Big(\frac{1}{2}\varrho|\vec{u}|^2 &+ H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) + H_{r,\overline{\vartheta}}(T_r) + \frac{1}{2}(|\vec{B} - \overline{\vec{B}}|^2 + |\vec{E}|^2)\Big) \\ &= \operatorname{div}_x\big((\varrho E + E_r)\vec{u} + (p + p_r)\vec{u} + \overline{\vartheta}(\varrho s + S_r)\vec{u}\big) - \overline{\vartheta}\frac{a\sigma_a}{\vartheta T_r}(\vartheta - T_r)^2(\vartheta + T_r)(\vartheta^2 + T_r^2) - \frac{\nu}{\vartheta}|\vec{u}|^2.\end{aligned}$$

By introducing the Helmholtz functions

$$H_{\overline{\vartheta}}(\varrho, \vartheta) := \varrho(e - \overline{\vartheta}s) \quad \text{and} \quad H_{r,\overline{\vartheta}}(T_r) := E_r - \overline{\vartheta}S_r,$$

we check that the quantities $H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$ and $H_{r,\overline{\vartheta}}(T_r) - H_{r,\overline{\vartheta}}(\overline{T}_r)$ are non-negative and strictly coercive functions reaching zero minima at the equilibrium state $(\overline{\varrho}, \overline{\vartheta}, \overline{E}_r)$.

Lemma 3.1. *Let $\overline{\varrho}$ and $\overline{\vartheta} = \overline{T}_r$ be given positive constants. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the sets defined by*

$$\mathcal{O}_1 := \Big\{(\varrho, \vartheta) \in \mathbb{R}^2 : \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta}\Big\}, \quad \mathcal{O}_2 := \Big\{T_r \in \mathbb{R} : \frac{\overline{T}_r}{2} < T_r < 2\overline{T}_r\Big\}.$$

Then there exist positive constants $C_{1,2}(\overline{\varrho}, \overline{\vartheta})$ and $C_{3,4}(\overline{T}_r)$ such that

$$C_1(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2) \leq H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \leq C_2(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2) \tag{3.5}$$

for all $(\varrho, \vartheta) \in \mathcal{O}_1$, and

$$C_3|T_r - \overline{T}_r|^2 \leq H_{r,\overline{\vartheta}}(T_r) - H_{r,\overline{\vartheta}}(\overline{T}_r) \leq C_4|T_r - \overline{T}_r|^2$$

for all $T_r \in \mathcal{O}_2$.

Proof. The first assertion is proved in [8], and we only sketch the proof for convenience. According to the decomposition

$$\varrho \to H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) = \mathcal{F}(\varrho) + \mathcal{G}(\varrho),$$

where

$$\mathcal{F}(\varrho) = H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \quad \text{and} \quad \mathcal{G}(\varrho) = H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}),$$

one checks that $\mathcal{F}$ is strictly convex and reaches a zero minimum at $\overline{\varrho}$, while $\mathcal{G}$ is strictly decreasing for $\vartheta < \overline{\vartheta}$ and strictly increasing for $\vartheta > \overline{\vartheta}$, according to the standard thermodynamic stability properties, see [8]. Computing the derivatives of $H_{\overline{\vartheta}}$ leads directly to estimate (3.5).

The second assertion follows from the properties of

$$x \mapsto H_{r,\overline{\vartheta}}(x) - H_{r,\overline{\vartheta}}(T_r) = ax^3\Big(x - \frac{4}{3}\overline{\vartheta}\Big) + \frac{a}{3}\overline{\vartheta}^4. \qquad \square$$

Using the previous entropy properties, we have the following energy estimate.

Proposition 3.2. *Let the assumptions of Theorem 2.1 be satisfied with*

$$V = (\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E}) \quad \text{and} \quad \overline{V} = (\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\vec{B}}, 0).$$

Consider a solution $(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})$ of system (1.1)–(1.3) on $[0, t]$, for some $t > 0$. Then, for a constant $C_0 > 0$, one gets

$$\|V(t) - \overline{V}\|_{L^2}^2 + \int_0^t \|\vec{u}(\tau)\|_{L^2}^2 \, d\tau \leq C_0\|V_0 - \overline{V}\|_{L^2}^2. \tag{3.6}$$

Proof. We define

$$\eta(t, x) = H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) + H_{r,\overline{\vartheta}}(T_r),$$

multiply (3.4) by $\overline{\vartheta}$, and subtract the result to (3.1). By integrating over $[0, t] \times \mathbb{R}^3$, we find

$$\int_{\mathbb{R}^3} \frac{1}{2}\varrho|\vec{u}|^2 + \eta(t, x) + \frac{1}{2}|\vec{B} - \overline{\vec{B}}|^2 + \frac{1}{2}|\vec{E}|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \frac{\overline{\vartheta}}{\vartheta}\nu|\vec{u}|^2 \leq \int_{\mathbb{R}^3} \frac{1}{2}\varrho_0|\vec{u}_0|^2(t) + \eta(0, x) + \frac{1}{2}|\vec{B}_0 - \overline{\vec{B}}|^2 + \frac{1}{2}|\vec{E}_0|^2 \, dx.$$

Applying Lemma 3.1 yields (3.6). $\square$

By defining, for any $d \geq 3$, the auxiliary quantities

$$E(t) := \sup_{0 \leq \tau \leq t} \|(\varrho - \overline{\varrho}, \vec{u}, \vec{B} - \overline{\vec{B}}, \vec{E})(\tau)\|_{W^{1,\infty}},$$

$$F(t) := \sup_{0 \leq \tau \leq t} \|(V - \overline{V})(\tau)\|_{H^d},$$

$$I^2(t) := \int_0^t \|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^\infty}^2 \, d\tau$$

and

$$D^2(t) := \int_0^t \Big(\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(\tau)\|_{H^d}^2 + \|\vec{E}(\tau)\|_{H^{d-1}}^2 + \|\partial_x \vec{B}(\tau)\|_{H^{d-2}}^2 \Big)\, d\tau,$$

we can bound the spatial derivatives as follows.

Proposition 3.3. *Assume that the hypotheses of Theorem 2.1 are satisfied. Then, for $C_0 > 0$, we have*

$$\|\partial_x V(t)\|_{H^{d-1}}^2 + \int_0^t \|\partial_x \vec{u}(\tau)\|_{H^{d-1}}^2\, d\tau \le C_0 \|\partial_x V_0\|_{H^{d-1}}^2 + C_0\big(E(t)D(t)^2 + F(t)I(t)D(t)\big).$$

Proof. By rewriting system (1.1)–(1.6) in the form

$$\begin{cases} \partial_t \varrho + \vec{u} \cdot \nabla_x \varrho + \varrho \operatorname{div}_x \vec{u} = 0, \\ \partial_t \vec{u} + (\vec{u} \cdot \nabla_x)\vec{u} + \dfrac{p_\varrho}{\varrho} \nabla_x \varrho + \dfrac{p_\vartheta}{\varrho} \nabla_x \vartheta + \dfrac{1}{3a\varrho} \nabla_x E_r + \vec{E} + \vec{u} \times \overline{\vec{B}} + \nu \vec{u} = -\vec{u} \times (\vec{B} - \overline{\vec{B}}), \\ \partial_t \vartheta + (\vec{u} \cdot \nabla_x)\vartheta + \dfrac{\vartheta p_\vartheta}{\varrho C_v} \operatorname{div}_x \vec{u} = -\dfrac{\sigma_a}{\varrho C_v}(a\vartheta^4 - E_r), \\ \partial_t E_r + (\vec{u} \cdot \nabla_x) E_r + \dfrac{4}{3} E_r \operatorname{div}_x \vec{u} = -\sigma_a (E_r - a\vartheta^4), \\ \partial_t \vec{B} + \operatorname{curl}_x \vec{E} = 0, \\ \partial_t \vec{E} - \operatorname{curl}_x \vec{B} - \overline{\varrho} \vec{u} = (\varrho - \overline{\varrho}) \vec{u}, \end{cases} \tag{3.7}$$

and applying ∂_x^ℓ to this system, we get

$$\begin{aligned} &\partial_t(\partial_x^\ell \varrho) + (\vec{u} \cdot \nabla_x)\partial_x^\ell \varrho + \varrho \operatorname{div}_x \partial_x^\ell \vec{u} = F_1^\ell, \\ &\partial_t(\partial_x^\ell \vec{u}) + (\vec{u} \cdot \nabla_x)\partial_x^\ell \vec{u} + \frac{p_\varrho}{\varrho} \nabla_x \partial_x^\ell \varrho + \frac{p_\vartheta}{\varrho} \nabla_x \partial_x^\ell \vartheta + \frac{1}{3a\varrho} \nabla_x \partial_x^\ell E_r + \partial_x^\ell \vec{E} + \partial_x^\ell \vec{u} \times \overline{\vec{B}} + \nu \partial_x^\ell \vec{u} = -\partial_x^\ell[\vec{u} \times (\vec{B} - \overline{\vec{B}})] + F_2^\ell, \\ &\partial_t(\partial_x^\ell \vartheta) + (\vec{u} \cdot \nabla_x)\partial_x^\ell \vartheta + \frac{\vartheta p_\vartheta}{\varrho C_v} \operatorname{div}_x \partial_x^\ell \vec{u} = -\partial_x^\ell\Big[\frac{\sigma_a}{\varrho C_v}(a\vartheta^4 - E_r)\Big] + F_3^\ell, \\ &\partial_t(\partial_x^\ell E_r) + (\vec{u} \cdot \nabla_x)\partial_x^\ell E_r + \frac{4}{3} E_r \operatorname{div}_x \partial_x^\ell \vec{u} = -\partial_x^\ell[\sigma_a(E_r - a\vartheta^4)] + F_4^\ell, \\ &\partial_t(\partial_x^\ell \vec{B}) + \operatorname{curl}_x \partial_x^\ell \vec{E} = 0, \\ &\partial_t(\partial_x^\ell \vec{E}) - \operatorname{curl}_x \partial_x^\ell \vec{B} - \overline{\varrho} \partial_x^\ell \vec{u} = \partial_x^\ell[(\varrho - \overline{\varrho})\vec{u}], \end{aligned}$$

where

$$\begin{aligned} F_1^\ell &:= -[\partial_x^\ell, \vec{u} \cdot \nabla_x]\vec{u} - [\partial_x^\ell, \varrho \operatorname{div}_x]\vec{u}, \\ F_2^\ell &:= -[\partial_x^\ell, \vec{u} \cdot \nabla_x]\vec{u} - \Big[\partial_x^\ell, \frac{p_\varrho}{\varrho} \nabla_x\Big]\varrho - \Big[\partial_x^\ell, \frac{p_\vartheta}{\varrho} \nabla_x\Big]\vartheta - \Big[\partial_x^\ell, \frac{1}{3a\varrho} \nabla_x\Big] E_r, \\ F_3^\ell &:= -[\partial_x^\ell, \vec{u} \cdot \nabla_x]\vartheta - \Big[\partial_x^\ell, \frac{\vartheta p_\vartheta}{\varrho C_v} \operatorname{div}_x\Big]\vec{u}, \\ F_4^\ell &:= -[\partial_x^\ell, \vec{u} \cdot \nabla_x] E_r - \Big[\partial_x^\ell, \frac{4}{3} E_r \operatorname{div}_x\Big]\vec{u}. \end{aligned}$$

Then, by taking the scalar product of each of the previous equations, respectively, by

$$\frac{p_\varrho}{\varrho^2} \partial_x^\ell \varrho, \quad \partial_x^\ell \vec{u}, \quad \frac{C_v}{\vartheta} \partial_x^\ell \vartheta, \quad \frac{1}{4a\varrho E_r} \partial_x^\ell E_r, \quad \partial_x^\ell \vec{B} \quad \text{and} \quad \partial_x^\ell \vec{E},$$

and adding the resulting equations, we get

$$\partial_t \mathcal{E}^\ell + \operatorname{div}_x \vec{\mathcal{F}}^\ell + \nu(\partial_x^\ell \vec{u})^2 = \mathcal{R}^\ell + \mathcal{S}^\ell, \tag{3.8}$$

where

$$\begin{aligned}
\mathcal{E}^\ell &:= \frac{1}{2}(\partial_x^\ell \vec{u})^2 + \frac{1}{2}\frac{p_\varrho}{\varrho}(\partial_x^\ell \varrho)^2 + \frac{1}{2}\frac{C_v}{\vartheta}(\partial_x^\ell \vartheta)^2 + \frac{1}{2}\frac{1}{4a\varrho E_r}(\partial_x^\ell E_r)^2 + \frac{1}{2}(\partial_x^\ell \vec{E})^2 + \frac{1}{2}(\partial_x^\ell \vec{B})^2, \\
\vec{\mathcal{F}}^\ell &:= \Big(\frac{p_\varrho}{\varrho}\partial_x^\ell \varrho + \frac{p_\vartheta}{\varrho}\partial_x^\ell \vartheta + \frac{1}{3a\varrho}\partial_x^\ell E_r\Big)\partial_x^\ell \vec{u} + \frac{1}{2}\Big((\partial_x^\ell \vec{u})^2 + \frac{p_\varrho}{\varrho}(\partial_x^\ell \varrho)^2 + \frac{C_v}{\vartheta}(\partial_x^\ell \vartheta)^2 + \frac{1}{4a\varrho E_r}(\partial_x^\ell E_r)^2\Big)\vec{u}, \\
\mathcal{R}^\ell &:= \frac{1}{2}\Big[\frac{p_\varrho}{\varrho^2}\Big]_t(\partial_x^\ell \varrho)^2 + \frac{1}{2}\Big[\frac{C_v}{\vartheta}\Big]_t(\partial_x^\ell \vartheta)^2 + \frac{1}{2}\Big[\frac{1}{4a\varrho E_r}\Big]_t(\partial_x^\ell E_r)^2 + \frac{1}{2}\operatorname{div}_x\Big(\frac{p_\varrho}{\varrho^2}\vec{u}\Big)(\partial_x^\ell \varrho)^2 \\
&\quad + \frac{1}{2}\operatorname{div}_x \vec{u}(\partial_x^\ell \vec{u})^2 + \frac{1}{2}\operatorname{div}_x\Big(\frac{C_v}{\vartheta}\vec{u}\Big)(\partial_x^\ell \vartheta)^2 + \frac{1}{2}\operatorname{div}_x\Big(\frac{1}{4a\varrho E_r}\vec{u}\Big)(\partial_x^\ell E_r)^2 \\
&\quad + \nabla_x\Big(\frac{p_\varrho}{\varrho}\Big)\partial_x^\ell \varrho \partial_x^\ell \vec{u} + \nabla_x\Big(\frac{p_\vartheta}{\varrho}\Big)\partial_x^\ell \vartheta \partial_x^\ell \vec{u} + \nabla_x\Big(\frac{1}{3a\varrho}\Big)\partial_x^\ell E_r \partial_x^\ell \vec{u} \\
&\quad + \frac{p_\varrho}{\varrho^2}\partial_x^\ell \varrho\, F_1^\ell + \partial_x^\ell \vec{u}\, F_2^\ell + \frac{C_v}{\vartheta}\partial_x^\ell \vartheta\, F_3^\ell + \partial_x^\ell E_r\, F_4^\ell + \overline{\varrho}\partial_x^\ell \vec{E}\cdot\partial_x^\ell \vec{u}, \\
\mathcal{S}^\ell &:= -\partial_x^\ell \vec{u}\cdot\partial_x^\ell[\vec{u}\times(\vec{B}-\overline{\vec{B}})] - \frac{C_v}{\vartheta}\partial_x^\ell \vartheta\,\partial_x^\ell\Big[\frac{\sigma_a}{\varrho C_v}(a\vartheta^4 - E_r)\Big] - \frac{1}{4a\varrho E_r}\partial_x^\ell E_r\,\partial_x^\ell[\sigma_a(E_r - a\vartheta^4)] + \partial_x^\ell \vec{E}\,\partial_x^\ell[(\varrho-\overline{\varrho})\vec{u}].
\end{aligned}$$

By integrating (3.8) on space, one gets

$$\partial_t \int_{\mathbb{R}^3} \mathcal{E}^\ell\, dx + \|\partial_x^\ell \vec{u}\|_{L^2}^2 \le \int_{\mathbb{R}^3} (|\mathcal{R}^\ell| + |\mathcal{S}^\ell|)\, dx.$$

Integrating now with respect to t and summing on ℓ, with $|\ell| \le d$, yields

$$\|\partial_x V(t)\|_{H^{d-1}}^2 + \int_0^t \|\partial_x \vec{u}(\tau)\|_{H^{d-1}}^2\, d\tau \le C_0\|\partial_x V_0\|_{H^{d-1}}^2 + C_0\sum_{|\ell|=1}^{d}\int_{\mathbb{R}^3}(|\mathcal{R}^\ell| + |\mathcal{S}^\ell|)\, dx.$$

By observing that

$$|\partial_t \varrho| \le C|\partial_x \varrho|, \quad |\partial_t \vartheta| \le C(|\partial_x \varrho| + |\partial_x \vartheta| + |\partial_x E_r||\Delta\vartheta|) \quad \text{and} \quad |\partial_t E_r| \le C(|\partial_x \varrho| + |\partial_x \vartheta| + |\partial_x E_r|),$$

and using the commutator estimates (see the Moser-type calculus inequalities in [16])

$$\|(F_1^\ell, F_2^\ell, F_3^\ell, F_4^\ell)\|_{L^2} \le \|\partial_x(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^\ell(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^2}^2,$$

we see that

$$|\mathcal{R}^\ell| \le C(\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty})\|\partial_x^\ell(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^2}^2.$$

Then integrating with respect to time gives

$$\begin{aligned}
\int_0^t |\mathcal{R}^\ell(\tau)|\, d\tau &\le C\sup_{0\le\tau\le t}\{\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty}\}\int_0^t \|\partial_x^\ell(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^2}^2\, d\tau \\
&\le CE(t)D^2(t)
\end{aligned}$$

for any $|\ell| \le d$. Similarly, we estimate

$$\begin{aligned}
|\mathcal{S}^\ell| &\le C\|\partial_x^\ell \vec{u}\|_{L^2}^2\|\partial_x^\ell[\vec{u}\times(\vec{B}-\overline{\vec{B}})]\|_{L^2}^2 + C\|\partial_x^\ell \vartheta\|_{L^2}^2\Big\|\partial_x^\ell\Big[\frac{\sigma_a}{\varrho C_v}(a\vartheta^4 - E_r)\Big]\Big\|_{L^2}^2 \\
&\quad + C\|\partial_x^\ell E_r\|_{L^2}^2\|\partial_x^\ell[\sigma_a(E_r - a\vartheta^4)]\|_{L^2}^2 + C\|\partial_x^\ell \vec{E}\|_{L^2}^2\|\partial_x^\ell[(\varrho-\overline{\varrho})\vec{u}]\|_{L^2}^2.
\end{aligned}$$

Then we get

$$\begin{aligned}
|\mathcal{S}^\ell| &\le C\|\vec{B}-\overline{\vec{B}}\|_{L^\infty}\|\partial_x^\ell \vec{u}\|_{L^2}^2 + C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^\ell(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^2}\|\partial_x^\ell(\vec{B}, \vec{E})\|_{L^\infty} \\
&\quad + C(\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty})\|\partial_x^\ell(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^2}^2.
\end{aligned}$$

Then integrating with respect to time yields

$$\begin{aligned}\int_0^t |\mathbb{S}^\ell(\tau)|\,d\tau &\le C \sup_{0\le\tau\le t}\|(\vec B-\overline{\vec B})(\tau)\|_{L^\infty}\int_0^t \|\partial_x^\ell \vec u(\tau)\|_{L^2}^2\,d\tau + C\sup_{0\le\tau\le t}\|\partial_x^\ell(\vec B,\vec E)(\tau)\|_{L^2}\\
&\quad\times\int_0^t \big\|(\varrho-\overline{\varrho},\vec u,\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(\tau)\big\|_{L^\infty}\|\partial_x^\ell(\varrho-\overline{\varrho},\vec u,\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(\tau)\|_{L^2}\,d\tau\\
&\quad+C\sup_{0\le\tau\le t}\{\|\partial_x\varrho\|_{L^\infty}+\|\partial_x\vec u\|_{L^\infty}+\|\partial_x\vartheta\|_{L^\infty}+\|\partial_x E_r\|_{L^\infty}(\tau)\}\int_0^t\big\|\partial_x^\ell(\varrho-\overline{\varrho},\vec u,\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\big\|_{L^2}^2\,d\tau\\
&\le C\big(E(t)D^2(t)+F(t)I(t)D(t)\big)\end{aligned}$$

for any $|\ell|\le d$. □

The above results, together with (3.6), allow us to derive the following energy bound.

Corollary 3.4. *Assume that the assumptions of Proposition 3.2 are satisfied. Then*

$$\|(V-\overline{V})(t)\|_{H^d}^2+\int_0^t\|\vec u(\tau)\|_{H^d}^2\,d\tau\le C\|(V-\overline{V})(0)\|_{H^d}^2+C\big(E(t)D(t)^2+F(t)I(t)D(t)\big). \tag{3.9}$$

Our goal is now to derive bounds for the integrals in the right- and left-hand sides of equation (3.9). For this purpose we adapt the results of Ueda, Wang and Kawashima [20].

Lemma 3.5. *Under the assumptions of Theorem 2.1, and supposing that $d\ge 3$, we have the following estimate for any $\varepsilon>0$:*

$$\begin{aligned}&\int_0^t\big(\big\|(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(\tau)\big\|_{H^d}^2+\|\vec E(\tau)\|_{H^{d-1}}^2\big)\,d\tau\\
&\qquad\le\varepsilon\int_0^t\|\partial_x\vec B(\tau)\|_{H^{d-2}}^2\,d\tau+C_\varepsilon\{\|V_0-\overline{V}\|_{H^{d-1}}^2+E(t)D(t)^2+F(t)I(t)D(t)\}.\end{aligned}\tag{3.10}$$

Proof. We linearize the principal part of system (1.1)–(1.3) as follows:

$$\partial_t\varrho+\overline{\varrho}\,\mathrm{div}_x\,\vec u=g_1,\tag{3.11}$$

$$\partial_t\vec u+\overline{a}_1\nabla_x\varrho+\overline{a}_2\,\nabla_x\vartheta+\overline{a}_3\nabla_x E_r+\vec E+\vec u\times\overline{\vec B}+\nu\vec u=g_2,\tag{3.12}$$

$$\partial_t\vartheta+\overline{b}_1\,\mathrm{div}_x\,\vec u+\overline{b}_2(\vartheta-\overline{\vartheta})=g_3,\tag{3.13}$$

$$\partial_t E_r+\overline{c}_1\,\mathrm{div}_x\,\vec u+\overline{c}_3(E_r-\overline{E}_r)=g_4,\tag{3.14}$$

$$\partial_t\vec B+\mathrm{curl}_x\,\vec E=0,\tag{3.15}$$

$$\partial_t\vec E-\mathrm{curl}_x\,\vec B-\overline{\varrho}\vec u=g_5,\tag{3.16}$$

with coefficients

$$\begin{aligned}&a_1(\varrho,\vartheta)=\frac{p_\varrho}{\varrho},\quad a_2(\varrho,\vartheta)=\frac{p_\vartheta}{\varrho},\quad a_3(\varrho,\vartheta)=\frac{1}{3\varrho},\quad \overline{a}_j=a_j(\overline{\varrho},\overline{\vartheta}),\\
&b_1(\varrho,\vartheta)=\frac{\vartheta p_\vartheta}{\varrho C_v},\quad b_2(\varrho,\vartheta,E_r)=\frac{a\sigma_a}{\varrho C_v}(\vartheta^2+\overline{\vartheta}^2)(\vartheta+\overline{\vartheta}),\quad b_3(\varrho,\vartheta,E_r)=\frac{a\sigma_a}{\varrho C_v},\quad \overline{b}_j=b_j(\overline{\varrho},\overline{\vartheta}),\\
&c_1(\varrho,\vartheta,E_r)=\frac{4}{3}E_r,\quad c_2(\varrho,\vartheta,E_r)=a\sigma_a(\vartheta^2+\overline{\vartheta}^2)(\vartheta+\overline{\vartheta}),\quad c_3(\varrho,\vartheta,E_r)=\sigma_a,\quad \overline{c}_j=c_j(\overline{\varrho},\overline{\vartheta}),\end{aligned}$$

and sources

$$\begin{aligned}g_1&:=-\{\vec u\cdot\nabla_x\varrho+(\varrho-\overline{\varrho})\,\mathrm{div}_x\,\vec u\},\\
g_2&:=-\Big\{(\vec u\cdot\nabla_x)\vec u+(a_1-\overline{a}_1)\nabla_x\varrho+(a_2-\overline{a}_2)\nabla_x\vartheta+(a_3-\overline{a}_3)\nabla_x E_r+\vec u\times(\vec B-\overline{\vec B})\Big\},\end{aligned}$$

$$
\begin{aligned}
g_3 &:= -\Big\{(\vec{u}\cdot\nabla_x)\vartheta + (b_1 - \overline{b}_1)\operatorname{div}_x \vec{u} + (b_2 - \overline{b}_2)(\vartheta - \overline{\vartheta}) + b_3(E_r - \overline{E}_r)\Big\},\\
g_4 &:= -\Big\{(\vec{u}\cdot\nabla_x)E_r + (\overline{c}_1 - c_1)\operatorname{div}_x \vec{u} + c_2(\vartheta - \overline{\vartheta}) + (c_3 - \overline{c}_3)(E_r - \overline{E}_r)\Big\}
\end{aligned}
$$

and

$$g_5 = (\varrho - \overline{\varrho})\vec{u}.$$

By multiplying (3.11) by $-\overline{a}_1 \operatorname{div}_x \vec{u}$, (3.12) by $\overline{a}_1 \nabla_x\varrho + \overline{a}_2 \nabla_x\vartheta + \overline{a}_3 \nabla_x E_r + \vec{E}$, (3.13) by $-\overline{a}_2 \operatorname{div}_x \vec{u} + \vartheta - \overline{\vartheta}$, (3.14) by $-\overline{a}_3 \operatorname{div}_x \vec{u} + E_r - \overline{E}_r$, (3.15) by 1, (3.16) by $\vec{u}$ and summing up, we get

$$
\begin{aligned}
&\overline{a}_1(\nabla_x\varrho\vec{u}_t - \varrho_t \operatorname{div}_x \vec{u}) + \overline{a}_2(\nabla_x\vartheta\vec{u}_t - \vartheta_t \operatorname{div}_x \vec{u}) + \overline{a}_3(\nabla_x E_r\vec{u}_t - (E_r)_t \operatorname{div}_x \vec{u})\\
&\quad + \vec{E}\vec{u}_t + \vec{E}_t\vec{u} + \Big\{\frac{1}{2}[(\vartheta - \overline{\vartheta})^2 + (E_r - \overline{E}_r)^2]\Big\}_t + (\overline{a}_1\nabla_x\varrho + \overline{a}_2\nabla_x\vartheta + \overline{a}_3\nabla_x E_r + \vec{E})^2\\
&\quad + (\overline{a}_1 \nabla_x\varrho + \overline{a}_2 \nabla_x\vartheta + \overline{a}_3 \nabla_x E_r + \vec{E})(\vec{u}\times\overline{\vec{B}} + \nu\vec{u}) + \overline{b}_2(\vartheta - \overline{\vartheta})^2 + \overline{c}_3(E_r - \overline{E}_r)^2\\
&\quad + \overline{b}_1(\vartheta - \overline{\vartheta})\operatorname{div}_x \vec{u} + \overline{c}_1(E_r - \overline{E}_r)\operatorname{div}_x \vec{u} + (\overline{a}_3\overline{c}_2 - \overline{a}_2\overline{b}_2)(\vartheta - \overline{\vartheta})\operatorname{div}_x \vec{u}\\
&\quad + (\overline{a}_2\overline{b}_3 - \overline{a}_3\overline{c}_3)(E_r - \overline{E}_r)\operatorname{div}_x \vec{u} - \vec{u}\ \operatorname{curl}_x \vec{B} - \overline{\varrho}\vec{u}^2 - (\operatorname{div}_x \vec{u})^2[\overline{a}_1 + \overline{a}_2 + \overline{a}_3] = G_1^0,
\end{aligned}
\tag{3.17}
$$

where

$$G_1^0 := -\overline{a}_1 g_1 \operatorname{div}_x \vec{u} + [\overline{a}_1\nabla_x\varrho + \overline{a}_2\nabla_x\vartheta + \overline{a}_3\nabla_x E_r + \vec{E}]g_2 - [\overline{a}_2 + \vartheta - \overline{\vartheta}]\operatorname{div}_x \vec{u}g_3 - [\overline{a}_3 + E_r - \overline{E}_r]\operatorname{div}_x \vec{u}g_4 + g_5\vec{u}.$$

By rearranging the left-hand side of (3.17), we get

$$\{H_1^0\}_t + \operatorname{div}_x \vec{F}_1^0 + D_1^0 = M_1^0 + G_1^0, \tag{3.18}$$

where

$$
\begin{aligned}
H_1^0 &= -[\overline{a}_1(\varrho - \overline{\varrho}) + \overline{a}_2(\vartheta - \overline{\vartheta}) + \overline{a}_3(E_r - \overline{E}_r)]\operatorname{div}_x \vec{u} + \vec{E}\cdot\vec{u} + \frac{1}{2}[(\vartheta - \overline{\vartheta})^2 + (E_r - \overline{E}_r)^2],\\
\vec{F}_1^0 &= [\overline{a}_1(\varrho - \overline{\varrho}) + \overline{a}_2(\vartheta - \overline{\vartheta}) + \overline{a}_3(E_r - \overline{E}_r)]\vec{u}_t - 2[\overline{a}_1(\varrho - \overline{\varrho}) + \overline{a}_2(\vartheta - \overline{\vartheta}) + \overline{a}_3(E_r - \overline{E}_r)]\vec{E}\\
&\quad + (\overline{a}_3\overline{c}_2 - \overline{a}_2\overline{b}_2 + \overline{b}_1)(\vartheta - \overline{\vartheta})\vec{u} + (\overline{a}_2\overline{b}_3 - \overline{a}_3\overline{c}_3 + \overline{c}_1)(E_r - \overline{E}_r)\vec{u},\\
D_1^0 &= \overline{a}_1^2|\nabla_x\varrho|^2 + \overline{a}_2^2|\nabla_x\vartheta|^2 + \overline{a}_3^2|\nabla_x E_r|^2 + |\vec{E}|^2 + 2\overline{a}_1(\varrho - \overline{\varrho})^2 + \overline{b}_2(\vartheta - \overline{\vartheta})^2 + \overline{c}_3(E_r - \overline{E}_r)^2,\\
M_1^0 &= -\Big\{2\overline{a}_1\overline{a}_2\nabla_x\varrho\cdot\nabla_x\vartheta + 2\overline{a}_1\overline{a}_3\nabla_x\varrho\cdot\nabla_x E_r + 2\overline{a}_2\overline{a}_3\nabla_x\vartheta\cdot\nabla_x E_r + 2\overline{a}_2(\varrho - \overline{\varrho})(\vartheta - \overline{\vartheta})\\
&\quad + 2\overline{a}_2(\varrho - \overline{\varrho})(E_r - \overline{E}_r) + (\overline{a}_1 \nabla_x\varrho + \overline{a}_2 \nabla_x\vartheta + \overline{a}_3 \nabla_x E_r + \vec{E})(\vec{u}\times\overline{\vec{B}} + \nu\vec{u}) - \vec{u}\ \operatorname{curl}_x \vec{B} - \overline{\varrho}\vec{u}^2\\
&\quad - (\operatorname{div}_x \vec{u})^2[\overline{a}_1 + \overline{a}_2 + \overline{a}_3] - (\overline{a}_3\overline{c}_2 - \overline{a}_2\overline{b}_2 + \overline{b}_1)\nabla_x\vartheta\cdot\vec{u} - (\overline{a}_2\overline{b}_3 - \overline{a}_3\overline{c}_3 + \overline{c}_1)\nabla_x E_r\cdot\vec{u}\Big\}.
\end{aligned}
$$

Integrating (3.18) over space and using Young's inequality yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} H_1^0\,dx + C(\|\varrho\|_{L^2}^2 + \|\nabla_x\vartheta\|_{L^2}^2 + \|\nabla_x E_r\|_{L^2}^2 + \|\vec{E}\|_{L^2}^2 + \|\varrho - \overline{\varrho}\|_{L^2}^2)\\
&\qquad \le \varepsilon\|\partial_x\vec{B}\|_{L^2}^2 + C_\varepsilon(\|\vec{u}\|_{H^1}^2 + \|\vartheta - \overline{\vartheta}\|_{H^1}^2 + \|E_r - \overline{E}_r\|_{H^1}^2) + \int_{\mathbb{R}^3}|G_1^0|\,dx.
\end{aligned}
$$

In fact, in the same way one obtains estimates for the derivatives of V. Namely, applying ∂_x^ℓ to system (3.11)–(3.16), we get

$$\{H_1^\ell\}_t + \operatorname{div}_x \vec{F}_1^\ell + D_1^\ell = M_1^\ell + G_1^\ell, \tag{3.19}$$

where

$$
\begin{aligned}
H_1^\ell &= -[\overline{a}_1\partial_x^\ell(\varrho - \overline{\varrho}) + \overline{a}_2\partial_x^\ell(\vartheta - \overline{\vartheta}) + \overline{a}_3\partial_x^\ell(E_r - \overline{E}_r)]\operatorname{div}_x \partial_x^\ell\vec{u} + \partial_x^\ell\vec{E}\cdot\partial_x^\ell\vec{u} + \frac{1}{2}[(\partial_x^\ell\vartheta)^2 + (\partial_x^\ell E_r)^2],\\
\vec{F}_1^\ell &= [\overline{a}_1\partial_x^\ell(\varrho - \overline{\varrho}) + \overline{a}_2\partial_x^\ell(\vartheta - \overline{\vartheta}) + \overline{a}_3\partial_x^\ell(E_r - \overline{E}_r)]\vec{u}_t\\
&\quad + (\overline{a}_3\overline{c}_2 - \overline{a}_2\overline{b}_2 + \overline{b}_1)\partial_x^\ell\vartheta\,\partial_x^\ell\vec{u} + (\overline{a}_2\overline{b}_3 - \overline{a}_3\overline{c}_3 + \overline{c}_1)\partial_x^\ell E_r\,\partial_x^\ell\vec{u}\\
&\quad - 2[\overline{a}_1\partial_x^\ell(\varrho - \overline{\varrho}) + \overline{a}_2\partial_x^\ell(\vartheta - \overline{\vartheta}) + \overline{a}_3\partial_x^\ell(E_r - \overline{E}_r)]\partial_x^\ell\vec{E} + \partial_x^\ell\vec{u}\times\partial_x^\ell(\vec{B} - \overline{\vec{B}}),
\end{aligned}
$$

$$
\begin{aligned}
D_1^\ell &= \overline{a}_1^2|\nabla_x\partial_x^\ell\varrho|^2 + \overline{a}_2^2|\partial_x^\ell\nabla_x\vartheta|^2 + \overline{a}_3^2|\partial_x^\ell\nabla_x E_r|^2 + |\partial_x^\ell\vec{E}|^2 + 2\overline{a}_1(\partial_x^\ell(\varrho-\overline{\varrho}))^2 + \overline{b}_2(\partial_x^\ell\vartheta)^2 + \overline{c}_3(\partial_x^\ell E_r)^2,\\
M_1^\ell &= -\Big\{2\overline{a}_1\overline{a}_2\nabla_x\partial_x^\ell\varrho\cdot\nabla_x\partial_x^\ell\vartheta + 2\overline{a}_1\overline{a}_3\nabla_x\partial_x^\ell\varrho\cdot\nabla_x\partial_x^\ell E_r + 2\overline{a}_2\overline{a}_3\nabla_x\partial_x^\ell\vartheta\cdot\nabla_x\partial_x^\ell E_r + 2\overline{a}_2\partial_x^\ell(\varrho-\overline{\varrho})\partial_x^\ell(\vartheta-\overline{\vartheta})\\
&\qquad + 2\overline{a}_2\partial_x^\ell(\varrho-\overline{\varrho})\partial_x^\ell(E_r-\overline{E}_r) + (\overline{a}_1\nabla_x\partial_x^\ell\varrho + \overline{a}_2\nabla_x\partial_x^\ell\vartheta + \overline{a}_3\nabla_x\partial_x^\ell E_r + \partial_x^\ell\vec{E})(\partial_x^\ell\vec{u}\times\overline{\vec{B}} + \nu\partial_x^\ell\vec{u})\\
&\qquad - (\overline{a}_3\overline{c}_2 - \overline{a}_2\overline{b}_2 + \overline{b}_1)\nabla_x\partial_x^\ell\vartheta\cdot\partial_x^\ell\vec{u} - (\overline{a}_2\overline{b}_3 - \overline{a}_3\overline{c}_3 + \overline{c}_1)\nabla_x\partial_x^\ell E_r\cdot\partial_x^\ell\vec{u}\\
&\qquad - \operatorname{curl}_x\partial_x^\ell\vec{u}\,\partial_x^\ell(\vec{B}-\overline{\vec{B}}) - \overline{\varrho}(\partial_x^\ell\vec{u})^2 - (\operatorname{div}_x\partial_x^\ell\vec{u})^2[\overline{a}_1+\overline{a}_2+\overline{a}_3]\Big\},\\
G_1^\ell &= -\overline{a}_1\partial_x^\ell g_1\operatorname{div}_x\partial_x^\ell\vec{u} + [\overline{a}_1\nabla_x\partial_x^\ell\varrho + \overline{a}_2\nabla_x\partial_x^\ell\vartheta + \overline{a}_3\nabla_x\partial_x^\ell E_r + \partial_x^\ell\vec{E}]\partial_x^\ell g_2\\
&\qquad - \overline{a}_2\partial_x^\ell g_3\operatorname{div}_x\partial_x^\ell\vec{u} - \overline{a}_3\partial_x^\ell g_4\operatorname{div}_x\partial_x^\ell\vec{u} + \partial_x^\ell g_5\partial_x^\ell\vec{u} + \partial_x^\ell g_3\partial_x^\ell\vartheta + \partial_x^\ell g_4\partial_x^\ell E_r.
\end{aligned}
$$

Integrating (3.19) over space and time yields

$$
\begin{aligned}
&\int_{\mathbb{R}^3} H_1^\ell(t)\,dx - \int_{\mathbb{R}^3} H_1^\ell(0)\,dx + C\int_0^t (\|\nabla_x\partial_x^\ell\varrho\|_{L^2}^2 + \|\nabla_x\partial_x^\ell\vartheta\|_{L^2}^2 + \|\nabla_x\partial_x^\ell E_r\|_{L^2}^2 + \|\partial_x^\ell\vec{E}\|_{L^2}^2)\,d\tau\\
&\qquad + C\int_0^t (\|\partial_x^\ell(\varrho-\overline{\varrho})\|_{L^2}^2 + \|\partial_x^\ell(\vartheta-\overline{\vartheta})\|_{L^2}^2 + \|\partial_x^\ell(E_r-\overline{E}_r)\|_{L^2}^2)\,d\tau\\
&\le \varepsilon\int_0^t \|\partial_x^\ell(\vec{B}-\overline{\vec{B}})\|_{L^2}^2\,d\tau + C_\varepsilon\int_0^t (\|\partial_x^\ell\vec{u}\|_{H^1}^2 + \|\partial_x^\ell(\vartheta-\overline{\vartheta})\|_{H^1}^2 + \|\partial_x^\ell(E_r-\overline{E}_r)\|_{H^1}^2)\,d\tau + \int_0^t\int_{\mathbb{R}^3} |G_1^\ell|\,dx\,d\tau. \qquad (3.20)
\end{aligned}
$$

By observing that

$$
\Big|\int_{\mathbb{R}^3} H_1^\ell(t)\,dx\Big| \le C(\|\partial_x^\ell(\varrho-\overline{\varrho})\|_{L^2}^2 + \|\partial_x^\ell(\vartheta-\overline{\vartheta})\|_{L^2}^2 + \|\partial_x^\ell(E_r-\overline{E}_r)\|_{L^2}^2 + \|\partial_x^\ell\vec{u}\|_{H^1}^2),
$$

and summing (3.20) on ℓ for $1 \le \ell \le d-1$, we get

$$
\begin{aligned}
&\int_0^t (\|(\varrho-\overline{\varrho}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)(\tau)\|_{H^d}^2 + \|\vec{E}(\tau)\|_{H^{d-1}}^2)\,d\tau\\
&\qquad \le C_\varepsilon\|(V-\overline{V})(0)\|_{H^d}^2 + \varepsilon\int_0^t \|\partial_x\vec{B}(\tau)\|_{H^{d-2}}^2\,d\tau + C_\varepsilon(E(t)D^2(t) + F(t)I(t)D(t)) + \sum_{|\ell|=1}^{d-1}\int_0^t\int_{\mathbb{R}^3} |G_1^\ell(\tau)|\,dx\,d\tau,
\end{aligned}
$$

where we used Corollary 3.4.

Let us estimate the last integral in (3.20). We have

$$
\begin{cases}
\|\partial_x^\ell g_1\|_{L^2} \le C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2},\\
\|\partial_x^\ell g_2\|_{L^2} \le C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2}\\
\qquad\qquad + C\|\vec{B}-\overline{\vec{B}}\|_{L^\infty}\|\partial_x^\ell\vec{u}\|_{L^2} + C\|\partial_x^\ell(\vec{B}-\overline{\vec{B}})\|_{L^2}\|\vec{u}\|_{L^\infty},\\
\|\partial_x^\ell g_3\|_{L^2} \le C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2}\\
\qquad\qquad + C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+2}(\vartheta, E_r)\|_{L^2},\\
\|\partial_x^\ell g_4\|_{L^2} \le C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2}\\
\qquad\qquad + C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+2}(\vartheta, E_r)\|_{L^2},\\
\|\partial_x^\ell g_5\|_{L^2} \le C\|(\varrho-\overline{\varrho}, \vec{u}, \vartheta-\overline{\vartheta}, E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^\ell(\varrho,\vec{u})\|_{L^2}
\end{cases}
\qquad (3.21)
$$

for $1 \le |\ell| \le d-1$. Then

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^3} |G_1^\ell(\tau)|\,dx\,d\tau &\le C\|\partial_x^{\ell+1}\vec{u}\|_{L^2}\|\partial_x^\ell g_1\|_{L^2} + C(\|\partial_x^{\ell+1}\varrho\|_{L^2} + \|\partial_x^{\ell+1}\vartheta\|_{L^2} + \|\partial_x^{\ell+1}E_r\|_{L^2} + \|\partial_x^\ell\vec{E}\|_{L^2})\|\partial_x^\ell g_2\|_{L^2}\\
&\qquad + C\|\partial_x^{\ell+1}\vec{u}\|_{L^2}\|\partial_x^\ell g_3\|_{L^2} + C\|\partial_x^{\ell+1}\vec{u}\|_{L^2}\|\partial_x^\ell g_4\|_{L^2} + C\|\partial_x^\ell\vec{u}\|_{L^2}\|\partial_x^\ell g_5\|_{L^2}.
\end{aligned}
$$

Plugging bounds (3.21) into the last inequality gives

$$\sum_{|\ell|=1}^{d-1}\int_0^t\int_{\mathbb{R}^3}|G_1^\ell(\tau)|\,dx\,d\tau \le CE(t)D^2(t),$$

which completes the proof of Lemma 3.5. □

Finally, we check from [20, Lemma 4.4] that the following result for the Maxwell's system holds true for our system with a similar proof.

Lemma 3.6. *Under the assumptions of Theorem 2.1, and supposing that $d \ge 3$, for any $\varepsilon > 0$, the following estimate (here, we set $V = (\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})^T$) holds:*

$$\int_0^t \|\partial_x \vec{B}(\tau)\|^2_{H^{s-2}}\,d\tau \le C\|V_0 - \overline{V}\|^2_{H^{s-1}} + C\int_0^t \|\partial_x \vec{E}(\tau)\|^2_{H^{s-2}}\,d\tau + C\big(E(t)D(t)^2 + F(t)I(t)D(t)\big). \tag{3.22}$$

Proof. By applying ∂_x^ℓ to (1.5) and (1.6), multiplying, respectively, by $-\operatorname{curl}_x \partial_x^\ell \vec{B}$ and $\operatorname{curl}_x \partial_x^\ell \vec{E}$, and adding the resulting equations, we get

$$-(\partial_x^\ell \vec{E}\cdot \operatorname{curl}_x \partial_x^\ell \vec{B})_t + |\operatorname{curl}_x \partial_x^\ell \vec{B}|^2 - \operatorname{div}_x(\partial_x^\ell \vec{E}\times \partial_x^\ell \vec{B}_t) = M_2^\ell + G_2^\ell,$$

where

$$M_2^\ell = -\overline{\varrho}\partial_x^\ell \vec{u}\cdot \operatorname{curl}_x \partial_x^\ell \vec{B} + |\operatorname{curl}_x \partial_x^\ell \vec{E}|^2$$

and

$$G_2^\ell = -\partial_x^\ell((\varrho - \overline{\varrho})\vec{u})\cdot \operatorname{curl}_x \partial_x^\ell \vec{B}.$$

Integrating in space gives

$$-\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\ell \vec{E}\cdot \operatorname{curl}_x \partial_x^\ell \vec{B}\,dx + C\|\operatorname{curl}_x \partial_x^\ell \vec{B}\|^2_{L^2} \le \|\operatorname{curl}_x \partial_x^\ell \vec{E}\|^2_{L^2} + \|\partial_x^\ell \vec{u}\|^2_{L^2} + \int_{\mathbb{R}^3}|G_2^\ell|\,dx.$$

By integrating on time and summing for $1 \le |\ell| \le d-2$, we have

$$\begin{aligned}\int_0^t \|\partial_x \vec{B}\|^2_{H^{d-2}}\,dt &\le C\|(V-\overline{V})(t)\|_{H^{d-1}} + C\|(V-\overline{V})(0)\|_{H^{d-1}} + C\int_0^t \|\partial_x \vec{E}\|^2_{H^{d-2}}\,dt\\ &\quad + C\int_0^t \|\vec{u}\|^2_{H^{d-2}}\,dt + C\sum_{|\ell|=0}^{d-2}\int_{\mathbb{R}^3}|G_2^\ell(\tau)|\,dx\,d\tau\\ &\le C\|(V-\overline{V})(0)\|_{H^{d-1}} + C\int_0^t \|\partial_x \vec{E}|^2_{H^{d-2}}\,dt + C\big(E(t)D(t)^2 + F(t)I(t)D(t)\big),\end{aligned}$$

where we used the bound

$$\sum_{|\ell|=1}^{d-1}\int_0^t\int_{\mathbb{R}^3}|G_2^\ell(\tau)|\,dx\,d\tau \le CE(t)D^2(t),$$

obtained in the same way as in the proof of Lemma 3.5. The proof of Lemma 3.6 is completed. □

We are now in position to conclude with the proofs of Theorems 2.1 and 2.2.

3.2 Proof of Theorem 2.1

We first point out that local existence for the hyperbolic system (1.1)–(1.6) may be proved using standard fixed-point methods. We refer to [16] for the proof.

Now, by plugging (3.22) into (3.10) with ε small enough, we get

$$\int_0^t \left(\|(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{H^d}^2+\|\vec{E}(\tau)\|_{H^{d-1}}^2\right)d\tau \le C\{\|V_0-\overline{V}\|_{H^{d-1}}^2+E(t)D(t)^2+F(t)I(t)D(t)\}. \tag{3.23}$$

Putting this last estimate into (3.22) yields

$$\int_0^t \|\partial_x\vec{B}(\tau)\|_{H^{s-2}}^2\,d\tau \le C\|V_0-\overline{V}\|_{H^s}^2+C(E(t)D(t)^2+F(t)I(t)D(t)). \tag{3.24}$$

Then, from (3.10), (3.23) and (3.24), we get

$$\begin{aligned}\|(V-\overline{V})(t)\|_{H^d}^2&+\int_0^t\left(\|(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(\tau)\|_{H^d}^2+\|\vec{E}(\tau)\|_{H^{d-1}}^2+\|\partial_x\vec{B}(\tau)\|_{H^{d-2}}^2\right)d\tau\\&\le C\|V_0-\overline{V}\|_{H^d}^2+C(E(t)D(t)^2+F(t)I(t)D(t))\end{aligned}$$

or, equivalently,

$$F(t)^2+D(t)^2\le C\|V_0-\overline{V}\|_{H^d}^2+C(E(t)D(t)^2+F(t)I(t)D(t)).$$

Now, by observing that, provided $d\ge 3$, one has $\|(V-\overline{V})(t)\|_{H^d}\le E(t)\le CF(t)$, and, provided $d\ge 2$, one has $I(t)\le CD(t)$ for some positive constant C, we see that

$$F(t)^2+D(t)^2\le C\|V_0-\overline{V}\|_{H^d}^2+CF(t)D(t)^2.$$

In order to prove global existence, we argue by contradiction, and assume that $T_c>0$ is the maximum time existence. Then we necessarily have

$$\lim_{t\to T_c}N(t)=+\infty,$$

where $N(t)$ is defined by

$$N(t):=(F(t)^2+D(t)^2)^{1/2}.$$

Thus, we are left to prove that N is bounded. For this purpose, we use the argument used in [3]. After the previous calculation, we have

$$N(t)^2\le C(\|V_0-\overline{V}\|_{H^d}^2+N(t)^3)\quad\text{for all } T\in[0,T_c]. \tag{3.25}$$

Hence, setting $\|V_0-\overline{V}\|_{H^d}=\varepsilon$, we have

$$\frac{N(t)^2}{\varepsilon^2+N(t)^3}\le C.$$

By studying the variation of $\phi(N)=N^2/(\varepsilon^2+N^3)$, we see that $\phi'(0)=0$, and that ϕ is increasing on the interval $[0,(2\varepsilon^2)^{1/3}]$ and decreasing on the interval $[(2\varepsilon^2)^{1/3},+\infty)$. Hence,

$$\max\phi=\phi((2\varepsilon^2)^{1/3})=\frac{1}{3}\left(\frac{2}{\varepsilon}\right)^{2/3}.$$

Hence, we can choose ε small enough to have $\phi(N)\le C$ for all $N\in[0,N^*]$, where $N^*>0$, and we see that $N\le N^*$, which contradicts (3.25).

4 Large time behavior

We have the following analogue of Proposition 3.2 for time derivatives.

Corollary 4.1. *Let the assumptions of Theorem 2.1 be satisfied, and consider the solution* $V:=(\varrho,\vec{u},\vartheta,E_r,\vec{B},\vec{E})$ *of system* (1.1)–(1.3) *on* $[0,t]$, *for some* $t>0$. *Then, for a constant* $C_0>0$, *one gets*

$$\|\partial_tV(t)\|_{H^{d-1}}^2+\int_0^t\left(\|\partial_t(\varrho,\vec{u},\vartheta,E_r)(\tau)\|_{H^{d-1}}^2+\|\partial_t(\vec{B},\vec{E})(\tau)\|_{H^{d-2}}^2\right)d\tau\le C_0\|V_0-\overline{V}\|_{H^d}^2. \tag{4.1}$$

Proof. By using system (3.7), we see that

$$\|\partial_t V\|_{H^{d-1}} \le C\|V - \overline{V}\|_{H^d},$$
$$\|\partial_t(\varrho, \vec{u}, \vartheta, E_r)\|_{H^{d-1}} \le \big\|\partial_x(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})\big\|_{H^{d-1}} + C\big\|(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})\big\|_{H^{d-1}}$$

and

$$\|\partial_t(\vec{B}, \vec{E})\|_{H^{d-2}} \le \|\partial_x(\vec{B}, \vec{E})\|_{H^{d-2}} + C\|\vec{u}\|_{H^{d-1}}.$$

Then, for $d \ge 3$, using the uniform estimate $\|V - \overline{V}\|^2_{H^d} \le C$ of Theorem 2.1, we get estimate (4.1). □

4.1 Proof of Theorem 2.2

By using Corollary 4.1, we get

$$\begin{aligned}\int_0^\infty \Big|\frac{d}{dt}\big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|_{H^{d-1}}\Big|\,dt &\le 2\int_0^\infty \big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|_{H^{d-1}}\|\partial_t(\varrho, \vec{u}, \vartheta, E_r)(t)\|_{H^{d-1}}\,dt \\ &\le \int_0^\infty \|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\|^2_{H^{d-1}} + \|\partial_t(\varrho, \vec{u}, \vartheta, E_r)(t)\|^2_{H^{d-1}}\,dt \\ &\le C_0\|V_0 - \overline{V}\|^2_{H^d}.\end{aligned}$$

This implies that

$$t \mapsto \big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|^2_{H^{d-1}} \in L^1(0, \infty)$$

and

$$t \mapsto \frac{d}{dt}\big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|_{H^{d-1}} \in L^1(0, \infty),$$

and then

$$\big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|_{H^{d-1}} \to 0 \quad \text{when } t \to \infty.$$

Now, by applying the Gagliardo–Nirenberg inequality and (2.1), we get

$$\big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|_{W^{d-2,\infty}} \le \big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|^{1/4}_{H^{d-2}}\|\partial^2_x(\varrho, \vec{u}, \vartheta, E_r)(t)\|^{3/4}_{H^{d-2}}.$$

So

$$\big\|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\big\|_{W^{d-2,\infty}} \to 0 \quad \text{when } t \to \infty.$$

Similarly,

$$t \mapsto \|\vec{E}(t)\|^2_{H^{d-1}} \in L^1(0, \infty) \quad \text{and} \quad t \mapsto \frac{d}{dt}\|\vec{E}(t)\|_{H^{d-1}} \in L^1(0, \infty),$$

and then

$$\|\vec{E}(t)\|_{W^{d-1,\infty}} \to 0 \quad \text{when } t \to \infty.$$

Finally,

$$t \mapsto \|\partial_x\vec{B}(t)\|^2_{H^{d-3}} \in L^1(0, \infty) \quad \text{and} \quad t \mapsto \frac{d}{dt}\|\partial_x\vec{B}(t)\|_{H^{d-3}} \in L^1(0, \infty).$$

Then, arguing as before,

$$\|(\vec{B} - \overline{\vec{B}})(t)\|_{W^{d-4,\infty}} \le \|(\vec{B} - \overline{\vec{B}})(t)\|^{1/4}_{H^{d-4}}\|\partial^2_x\vec{B}(t)\|^{3/4}_{H^{d-3}}.$$

So

$$\|(\vec{B} - \overline{\vec{B}})\|_{W^{d-4,\infty}} \to 0 \quad \text{when } t \to \infty,$$

which completes the proof.

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