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# ON POINCARÉ AND LOGARITHMIC SOBOLEV INEQUALITIES FOR A CLASS OF SINGULAR GIBBS MEASURES

DJALIL CHAFAÏ AND JOSEPH LEHEC

**ABSTRACT.** This note, mostly expository, is devoted to Poincaré and log-Sobolev inequalities for a class of Boltzmann–Gibbs measures with singular interaction. Such measures allow to model one-dimensional particles with confinement and singular pair interaction. The functional inequalities come from convexity. We prove and characterize optimality in the case of quadratic confinement via a factorization of the measure. This optimality phenomenon holds for all beta Hermite ensembles including the Gaussian unitary ensemble, a famous exactly solvable model of random matrix theory. We further explore exact solvability by reviewing the relation to Dyson–Ornstein–Uhlenbeck diffusion dynamics admitting the Hermite–Lassalle orthogonal polynomials as a complete set of eigenfunctions. We also discuss the consequence of the log-Sobolev inequality in terms of concentration of measure for Lipschitz functions such as maxima and linear statistics.

## CONTENTS

1. Introduction	1
1.1. Functional inequalities and concentration of measure	3
1.2. Dynamics	4
1.3. Hermite–Lassalle orthogonal polynomials	6
1.4. Comments and open questions	7
2. Useful or beautiful facts	8
2.1. Random matrices, GUE, and beta Hermite ensemble	8
2.2. Isotropy of beta Hermite ensembles	8
2.3. Log-concavity and curvature	10
2.4. Factorization by projection	10
3. Proofs	11
3.1. Proof of Theorems 1.1 and 1.2	11
3.2. Proof of Theorem 1.3 and comments on optimality	13
3.3. Proof of Corollary 1.4 and comments on concentration	16
3.4. Proof of Theorem 1.5	17
References	18

## 1. INTRODUCTION

The aim of this note is first to provide synthetic exposition gathering material from several distant sources, and second to provide extensions and novelty about optimality.

Let  $n \in \{1, 2, \dots\}$ . For a given  $\rho \in \mathbb{R}$ , we say that a function  $\phi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is  $\rho$ -convex when  $x \mapsto \phi(x) - \rho|x|^2/2$  is convex, where  $|x| := \sqrt{x_1^2 + \dots + x_n^2}$  is the Euclidean norm. In particular a 0-convex function is just a convex function. An equivalent condition

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is that for all  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ ,

$$\phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\phi(x) + \lambda\phi(y) - \frac{\rho \lambda(1 - \lambda)}{2}|y - x|^2.$$

If  $\phi$  is  $\mathcal{C}^2$ -smooth on its domain then this is yet equivalent to  $\text{Hess}(\phi) \geq \rho I_n$  as quadratic forms, pointwise, where  $I_n$  is the identity:  $\langle \text{Hess}(\phi)(x)y, y \rangle \geq \rho|x|^2$  for all  $x, y \in \mathbb{R}^n$ .

Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $W : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  be two functions, called “confinement potential” and “interaction potential” respectively. We assume that

- $V$  is  $\rho$ -convex for some  $\rho > 0$ ;
- $W$  is convex with domain  $(0; +\infty)$ . In particular  $W \equiv +\infty$  on  $(-\infty; 0]$ .

The energy of a configuration  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is

$$U(x) = V(x_1, \dots, x_n) + \sum_{i < j} W(x_i - x_j) = V(x) + U_W(x) \in \mathbb{R} \cup \{+\infty\}.$$

The nature of  $W$  gives that  $U(x)$  is finite if and only if  $x$  belongs to the “Weyl chamber”

$$D = \{x \in \mathbb{R}^n : x_1 > \dots > x_n\}.$$

Assuming that

$$Z_\mu = \int_{\mathbb{R}^n} e^{-U(x_1, \dots, x_n)} dx_1 \dots dx_n < \infty$$

we define a probability measure  $\mu$  on  $\mathbb{R}^n$  by

$$\mu(dx) = \frac{e^{-U(x_1, \dots, x_n)}}{Z_\mu} dx. \quad (1.1)$$

The support of  $\mu$  is  $\overline{D} = \{x \in \mathbb{R}^n : x_1 \geq \dots \geq x_n\}$ . Note that if

$$W(u) = \begin{cases} -\beta \log u, & \text{if } u > 0 \\ +\infty & \text{otherwise} \end{cases} \quad (1.2)$$

where  $\beta$  is a positive parameter, and if  $X$  is a random vector of  $\mathbb{R}^n$  distributed according to  $\mu$ , then for every  $\sigma > 0$ , the scaled random vector  $\sigma X$  follows the law  $\mu$  with same  $W$  but with  $V$  replaced by  $V(\cdot/\sigma)$ .

Following Edelman [34], the beta Hermite ensemble corresponds to the case

$$V(x) = \frac{n}{2}|x|^2 = \frac{n}{2}(x_1^2 + \dots + x_n^2),$$

and  $W$  given by (1.2). In this case  $\mu$  rewrites using a Vandermonde determinant as

$$d\mu(x) = \frac{e^{-\frac{n}{2}|x|^2}}{Z_\mu} \prod_{i < j} (x_i - x_j)^\beta \mathbb{1}_{\{x_1 \geq \dots \geq x_n\}} dx. \quad (1.3)$$

The normalizing constant  $Z_\mu$  can be explicitly computed in terms of Gamma functions by reduction to a classical Selberg integral, but this is useless for our purposes in this work. The Gaussian unitary ensemble (GUE) of Dyson [37] corresponds to  $\beta = 2$ , namely

$$d\mu(x) = \frac{e^{-\frac{n}{2}|x|^2}}{Z_\mu} \prod_{i < j} (x_i - x_j)^2 \mathbb{1}_{\{x_1 \geq \dots \geq x_n\}} dx. \quad (1.4)$$

Note that on  $\mathbb{R}^n$  the density of the beta Hermite ensemble (1.3) with respect to the Gaussian law  $\mathcal{N}(0, \frac{1}{n}I_n)$  is equal up to a multiplicative constant to  $\prod_{i < j} (x_i - x_j)^\beta$  times the indicator function of the Weyl chamber. The cases  $\beta = 1$  and  $\beta = 4$  are known as the Gaussian orthogonal ensemble (GOE) and the Gaussian symplectic ensemble (GSE).

Let  $L^2(\mu)$  be the Lebesgue space of measurable functions from  $\mathbb{R}^n$  to  $\mathbb{R}$  which are square integrable with respect to  $\mu$ . Let  $H^1(\mu)$  be the Sobolev space of functions in  $L^2(\mu)$  with weak derivative in  $L^2(\mu)$  in the sense of Schwartz distributions.

We provide in Section 2 some useful or beautiful facts about (1.1), (1.3), and (1.4).

**1.1. Functional inequalities and concentration of measure.** Given  $f \in L^2(\mu)$  we define the variance of  $f$  with respect to  $\mu$  by

$$\text{var}_\mu(f) = \int_{\mathbb{R}^n} f^2 d\mu - \left( \int_{\mathbb{R}^n} f d\mu \right)^2.$$

If additionally  $f \geq 0$ , then we define similarly the entropy of  $f$  with respect to  $\mu$  by

$$\text{ent}_\mu(f) = \int_{\mathbb{R}^n} f \log f d\mu - \left( \int_{\mathbb{R}^n} f d\mu \right) \log \left( \int_{\mathbb{R}^n} f d\mu \right).$$

**Theorem 1.1** (Poincaré inequality). *Let  $\mu$  be as in (1.1). For all  $f \in H^1(\mu)$ ,*

$$\text{var}_\mu(f) \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

*This holds in particular with  $\rho = n$  for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

**Theorem 1.2** (Log-Sobolev inequality). *Let  $\mu$  be as in (1.1). For all  $f \in H^1(\mu)$ ,*

$$\text{ent}_\mu(f^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

*This holds in particular with  $\rho = n$  for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

**Theorem 1.3** (Optimality for Poincaré and log-Sobolev inequalities). *Let  $\mu$  be as in (1.1). Assume that  $V$  is quadratic:  $V(x) = \rho|x|^2/2$  for some  $\rho > 0$ . This is in particular the case for the beta Hermite ensemble (1.3) for all  $\beta > 0$ . Then equality is achieved in the Poincaré inequality of Theorem 1.1 for*

$$f : x \in \mathbb{R}^n \mapsto \lambda(x_1 + \cdots + x_n) + c, \quad \lambda, c \in \mathbb{R}.$$

*Moreover equality is achieved in the logarithmic Sobolev inequality of Theorem 1.2 for*

$$f : x \in \mathbb{R}^n \mapsto e^{\lambda(x_1 + \cdots + x_n) + c}, \quad \lambda, c \in \mathbb{R}.$$

*Lastly, in both cases these are the only extremal functions.*

Theorems 1.1 and 1.2 are proved in Section 3.1 and Theorem 1.3 in Section 3.2.

Poincaré and logarithmic Sobolev inequalities for beta ensembles are already known in the literature about random matrix theory, see for instance [1, 38] and references therein. However the optimality that we point out here seems to be new.

The following corollary of Theorem 1.2 provides concentration of measure around the mean for Lipschitz functions, including linear statistics and maximum.

**Corollary 1.4** (Gaussian concentration inequality for Lipschitz functions). *Let  $\mu$  be as in (1.1). For every Lipschitz function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  and for all real parameter  $r > 0$ ,*

$$\mu \left( \left| F - \int F d\mu \right| \geq r \right) \leq 2 \exp \left( -\frac{\rho}{\|F\|_{\text{Lip}}^2} \frac{r^2}{2} \right). \quad (1.5)$$

*In particular for any measurable  $f : \mathbb{R} \rightarrow \mathbb{R}$  and all  $r > 0$ , with  $L_n(f)(x) = \frac{1}{n} \sum_{i=1}^n f(x_i)$ ,*

$$\mu \left( \left| L_n(f) - \int L_n(f) d\mu \right| \geq r \right) \leq 2 \exp \left( -n \frac{\rho}{\|f\|_{\text{Lip}}^2} \frac{r^2}{2} \right). \quad (1.6)$$

*Additionally, for all  $r > 0$ ,*

$$\mu \left( \left| x_1 - \int x_1 \mu(dx) \right| \geq r \right) \leq 2 \exp \left( -\rho \frac{r^2}{2} \right). \quad (1.7)$$

*This holds in particular with  $\rho = n$  for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

The proof of Corollary 1.4 and some additional comments are given in Section 3.3.

The scale in (1.7) is not optimal for the beta Hermite ensemble, the largest particle is actually more concentrated than what is predicted by Corollary 1.4. Indeed, it is proved for instance in [63] that  $n^{2/3}(\lambda_1 - 2)$  converges in law as  $n$  tends to infinity to a Tracy–Widom distribution of parameter  $\beta$ . In particular fluctuations of  $\lambda_1$  are of order  $n^{-2/3}$ , whereas (1.7) only predicts an upper bound of order  $n^{-1/2}$ . See also [4, 56, 57] for a concentration property that matches the correct order of fluctuations.

Note also that (1.6) allows to get concentration for the Cauchy–Stieltjes transform of the empirical distribution by taking  $f$  equal to the real or imaginary part of  $x \mapsto 1/(x - z)$  where  $z = a + ib$  is a fixed complex parameter with  $b > 0$ .

The function  $(x_1, \dots, x_n) \mapsto L_n(f)(x) = \frac{1}{n} \sum_{i=1}^n f(x_i)$  is called a linear statistics. The inequality (1.6) appears for the spectrum of random matrix models in the work of Guionnet and Zeitouni [44] via the logarithmic Sobolev inequality, see also [45] and [1, Section 4.4.2], and [1, Exercise 4.4.33] for beta ensembles. The exponential speed  $n^2$  in (1.6) is optimal according to the large deviation principle satisfied by  $L_n$  under  $\mu$  established by Ben Arous and Guionnet [10] for the GUE, see [25] and references therein for the general case (1.1). Concentration inequalities and logarithmic Sobolev inequalities for spectra of some random matrix models at the correct scale can also be obtained using coupling methods or exact decompositions, see for instance [60, 61] and references therein.

Many proofs involve the following simple transportation facts:

$$\mathcal{N}(0, n^{-1}I_n) \xrightarrow{\text{Caffarelli}} \mu \xrightarrow{x_1 + \dots + x_n} \mathcal{N}(0, 1)$$

and

$$\mathcal{N}(0, n^{-1}I_n) \xrightarrow{x_1 + \dots + x_n} \mathcal{N}(0, 1)$$

and

$$\text{Law}(H) \xrightarrow{\text{Spectrum}} \mu \xrightarrow{x_1 + \dots + x_n} \mathcal{N}(0, 1)$$

and

$$\text{Law}(H) \xrightarrow{\text{Trace}} \mathcal{N}(0, 1)$$

where  $H$  is a random Hermitian matrix as in Theorem 2.1 or Theorem 2.2.

**1.2. Dynamics.** Let us assume in this section that the functions  $V$  and  $W$  are smooth on  $\mathbb{R}^n$  and  $(0, +\infty)$  respectively. Then the energy  $U$  is smooth on its domain  $D$ . Fix  $X_0 \in D$  and consider the overdamped Langevin diffusion associated to the potential  $U$  starting from  $X_0$ , solving the stochastic differential equation

$$X_t = X_0 + \sqrt{2}B_t - \int_0^t \nabla U(X_s) ds + \Phi_t, \quad t \geq 0, \quad (1.8)$$

where  $(B_t)_{t \geq 0}$  is a standard Brownian Motion of  $\mathbb{R}^n$ , and where  $\Phi_t$  is a reflection at the boundary of  $D$  which constrains the process  $X$  to stay in  $D$ . More precisely

$$\Phi_t = - \int_0^t \mathbf{n}_s L(ds)$$

where  $L$  is a random measure depending on  $X$  and supported on  $\{t \geq 0: X_t \in \partial D\}$  and where  $\mathbf{n}_t$  is an outer unit normal to the boundary of  $D$  at  $X_t$  for every  $t$  in the support of  $L$ . The process  $L$  is called the “local time” at the boundary of  $D$ . The stochastic differential equation (1.8) writes equivalently

$$dX_t = \sqrt{2}dB_t - \nabla U(X_t)dt - \mathbf{n}_t L(dt).$$

It is not obvious that equation (1.8) admits a solution. Such diffusions with reflecting boundary conditions were first considered by Tanaka. He proved in [66] that if  $\nabla U$  is globally Lipschitz on  $D$  and grows at most linearly at infinity then (1.8) does admit a unique strong solution.

If it exists, the solution is a Markov process. Its generator is the operator  $G$  where

$$G = \Delta - \langle \nabla U, \nabla \rangle = \sum_{i=1}^n \partial_{x_i}^2 - \sum_{i=1}^n (\partial_{x_i} V)(x) \partial_{x_i} - \sum_{i \neq j} W'(x_i - x_j) \partial_{x_i} \quad (1.9)$$

with Neumann boundary conditions at the boundary of  $D$ . Stokes formula then shows that  $G$  is symmetric in  $L^2(\mu)$ . As a result the measure  $\mu$  is reversible for the process  $(X_t)$ . By integration by parts the density  $f_t$  of  $X_t$  with respect to the Lebesgue measure satisfies the Fokker–Planck equation  $\partial_t f_t = \Delta f_t + \operatorname{div}(f_t \nabla U)$ .

It is common to denote  $X_t = (X_t^1, \dots, X_t^n)$  and to interpret  $X_t^1, \dots, X_t^n$  as interacting particles on the real line experiencing confinement and pairwise interactions. Let us discuss now the particular case of the beta Hermite ensemble (1.3), for which (1.8) rewrites

$$dX_t^i = \sqrt{2} dB_t^i - n X_t^i dt + \beta \sum_{j: j \neq i} \frac{1}{X_t^i - X_t^j} dt, \quad 1 \leq i \leq n \quad (1.10)$$

as long as the particles have not collided. We call this diffusion the Dyson–Ornstein–Uhlenbeck process. Without the confinement term  $-X_t^i dt$  this diffusion is known in the literature as the Dyson Brownian motion. Indeed Dyson proved in [36] the following remarkable fact: if  $(M_t)$  is an Ornstein–Uhlenbeck process taking values in the space of complex Hermitian matrices then the eigenvalues of  $(M_t)$  follow the diffusion (1.10) with parameter  $\beta = 2$ , while if  $(M_t)$  is an Ornstein–Uhlenbeck process taking values in the space of real symmetric matrices then the same holds true with  $\beta = 1$ . Dyson also proved an analogue result for the eigenvalues of a Brownian motion on the unitary group. It is natural to ask whether the repulsion term  $1/(X_t^i - X_t^j)$  is strong enough to actually prevent the collision of particles. This was investigated by Rogers and Shi in [64], see also [1]. They proved that if  $\beta \geq 1$  then there are no collisions: (1.10) admits a unique strong solution and with probability 1, the process  $(X_t)$  stays in the Weyl chamber  $D$  for all time. This means that in that case, Tanaka’s equation (1.8) does admit a unique strong solution, but the reflection at the boundary  $\Phi_t$  is actually identically 0. This critical phenomenon was also observed twenty five years ago by Calogero in [17]. Besides, although it is not explicitly written in Rogers and Shi’s article, when  $\beta < 1$  collisions do occur in finite time, so that the reflection  $\Phi_t$  enters the picture. In that case though, the existence of a process  $(X_t)$  satisfying (1.8) does not follow from Tanaka’s theorem [66], as the potential  $U$  is singular at the boundary of  $D$ . Still (1.8) does admit a unique strong solution. Indeed, this was established by Cépa and Lépingle in [20] using an existence result for multivalued stochastic differential equations due to Cépa [19]. See also the work of Demni [31, 30].

*Long time behavior of the dynamics.* Let us assume that the process (1.8) is well defined. We denote by  $(P_t)$  the associated semigroup: For every test function  $f$

$$P_t f(x) = \mathbb{E}(f(X_t) \mid X_0 = x).$$

Given a probability measure  $\nu$  on  $\mathbb{R}^n$  we denote  $\nu P_t$  the law of the process at time  $t$  when initiated from  $\nu$ . Recall that the measure  $\mu$  is stationary:  $\mu P_t = \mu$  for all time. For all real number  $p \geq 1$ , the  $L^p$  Kantorovich or Wasserstein distance between  $\mu$  and  $\nu$  is

$$W_p(\nu, \mu) = \inf_{\substack{(X,Y) \\ X \sim \nu \\ Y \sim \mu}} \mathbb{E}(|X - Y|^p)^{1/p}. \quad (1.11)$$

Note that  $W_p(\nu, \mu) < \infty$  if  $|\cdot|^p \in L^1(\nu) \cap L^1(\mu)$ . It can be shown that the convergence for  $W_p$  is equivalent to weak convergence together with convergence of  $p$ -th moment. If  $\nu$  has density  $f$  with respect to  $\mu$ , the relative entropy of  $\nu$  with respect to  $\mu$  is

$$H(\nu \mid \mu) = \int_{\mathbb{R}^n} \log f \, d\nu. \quad (1.12)$$

If  $\nu$  is not absolutely continuous we set  $H(\nu \mid \mu) = +\infty$  by convention.

**Theorem 1.5** (Convergence to equilibrium). *For any two probability measures  $\nu_0, \nu_1$  on  $\mathbb{R}^n$  we have, for all  $p \geq 1$  and  $t \geq 0$ , in  $[0, +\infty]$ ,*

$$W_p(\nu_0 P_t, \nu_1 P_t) \leq e^{-\rho t} W_p(\nu_0, \nu_1).$$

*In particular, choosing  $\nu_1 = \mu$  yields*

$$W_p(\nu_0 P_t, \mu) \leq e^{-\rho t} W_p(\nu_0, \mu).$$

*Moreover we also have, for all  $t \geq 0$ ,*

$$H(\nu_0 P_t \mid \mu) \leq e^{-\rho t} H(\nu_0 \mid \mu). \quad (1.13)$$

A proof of Theorem 1.5 is given in Section 3.4.

**1.3. Hermite–Lassalle orthogonal polynomials.** Recall that for all  $n \geq 1$ , the classical Hermite polynomials  $(H_{k_1, \dots, k_n})_{k_1 \geq 0, \dots, k_n \geq 0}$  are the orthogonal polynomials for the standard Gaussian distribution  $\gamma_n$  on  $\mathbb{R}^n$ . The tensor product structure  $\gamma_n = \gamma_1^{\otimes n}$  gives  $H_{k_1, \dots, k_n}(x_1, \dots, x_n) = H_{k_1}(x_1) \cdots H_{k_n}(x_n)$  where  $(H_k)_{k \geq 0}$  are the orthogonal polynomials for the one-dimensional Gaussian distribution  $\gamma_1$ . Among several remarkable characteristic properties, these polynomials satisfy a differential equation which writes

$$LH_{k_1, \dots, k_n} = -(k_1 + \cdots + k_n)H_{k_1, \dots, k_n} \quad \text{where} \quad L = \Delta - \langle x, \nabla \rangle \quad (1.14)$$

is the infinitesimal generator of the Ornstein–Uhlenbeck process, which admits  $\gamma_n$  as a reversible invariant measure. In other words these orthogonal polynomials form a complete set of eigenfunctions of this operator. Such a structure is relatively rare, see [59] for a complete classification when  $n = 1$ .

Lassalle discovered in the 1990s that a very similar phenomenon takes place for beta Hermite ensembles and the Dyson–Ornstein–Uhlenbeck process, provided that we restrict to symmetric polynomials. Observe first that this cannot hold for all polynomials, simply because the infinitesimal generator

$$G = \sum_{i=1}^n \partial_{x_i}^2 - n \sum_{i=1}^n x_i \partial_{x_i} + \beta \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_{x_i}, \quad (1.15)$$

of the Dyson–Ornstein–Uhlenbeck process, which is a special case of (1.9), does not preserve polynomials, for instance we have  $Gx_1 = -nx_1 + \beta \sum_{j \neq 1} \frac{1}{x_1 - x_j}$ . However, rewriting this operator by symmetrization as

$$G = \sum_{i=1}^n \partial_{x_i}^2 - n \sum_{i=1}^n x_i \partial_{x_i} + \frac{\beta}{2} \sum_{i \neq j} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j}), \quad (1.16)$$

it is easily seen that the set of symmetric polynomials in  $n$  variables is left invariant by  $G$ .

Let  $\mu$  be the beta Hermite ensemble defined in (1.3). Lassalle studied in [51] multivariate symmetric polynomials  $(P_{k_1, \dots, k_n})_{k_1 \geq \dots \geq k_n \geq 0}$  which are orthogonal with respect to  $\mu$ . He called them “generalized Hermite” but we decide to call them “Hermite–Lassalle”. For all  $k_1 \geq \dots \geq k_1 \geq 0$  and  $k'_1 \geq \dots \geq k'_n \geq 0$ ,

$$\int P_{k_1, \dots, k_n}(x_1, \dots, x_n) P_{k'_1, \dots, k'_n}(x_1, \dots, x_n) \mu(dx) = \mathbb{1}_{(k_1, \dots, k_n) = (k'_1, \dots, k'_n)}. \quad (1.17)$$

They can be obtained from the standard basis of symmetric polynomials by using the Gram–Schmidt algorithm in the Hilbert space  $L_{\text{sym}}^2(\mu)$  of square integrable symmetric functions. The total degree of  $P_{k_1, \dots, k_n}$  is  $k_1 + \cdots + k_n$ , in particular  $P_{0, \dots, 0}$  is a constant polynomial. The numbering in terms of  $k_1, \dots, k_n$  used in [51] is related to Jack polynomials. Beware that [51] comes without proofs. We refer to [5] for proofs, and to [35] for symbolic computation via Jack polynomials.

The Hermite–Lassalle symmetric polynomials form an orthogonal basis in  $L_{\text{sym}}^2(\mu)$  of eigenfunctions of the Dyson–Ornstein–Uhlenbeck operator  $G$ . Restricted to symmetric

functions, this operator is thus exactly solvable, just like the classical Ornstein–Uhlenbeck operator. Here is the result of Lassalle in [51], see [5] for a proof.

**Theorem 1.6** (Eigenfunctions and eigenvalues). *For all  $n \geq 2$  and  $k_1 \geq \dots \geq k_n \geq 0$ ,*

$$GP_{k_1, \dots, k_n} = -n(k_1 + \dots + k_n)P_{k_1, \dots, k_n}. \quad (1.18)$$

where  $G$  is the operator (1.15).

When  $\beta = 0$  then  $G$  becomes the Ornstein–Uhlenbeck operator. For all  $\beta > 0$ , the spectrum of  $G$  is identical to the one of the Ornstein–Uhlenbeck operator. This can be guessed from the fact that the eigenfunctions are polynomials together with the fact that the interaction term (the non O.-U. part) lowers the degree of polynomials.

The spectral gap of  $G$  in  $L^2_{\text{sym}}(\mu)$  is  $n$ : if  $f \in L^2_{\text{sym}}(\mu)$  is orthogonal to constants then

$$n \int f^2 d\mu \leq - \int f Gf d\mu = \int |\nabla f|^2 d\mu.$$

Theorem 1.1 shows that this inequality holds actually for all  $f$ , not only symmetric ones.

Hermite–Lassalle polynomials can be decomposed in terms of Jack polynomials, and this decomposition generalizes the hypergeometric expansion of classical Hermite polynomials.

**Remark 1.7** (Examples and formulas). It is not difficult to check that up to normalization

$$x_1 + \dots + x_n \quad \text{and} \quad x_1^2 + \dots + x_n^2 - 1 - \beta \frac{n-1}{2}.$$

are Hermite–Lassalle polynomials. In the GUE case,  $\beta = 2$ , Lassalle gave in [51], using Jack polynomials and Schur functions, a formula for  $P_{k_1, \dots, k_n}$  in terms of a ratio of a determinant involving classical Hermite polynomials and a Vandermonde determinant.

**1.4. Comments and open questions.** Regarding functional inequalities, one can probably extend the results to the class of Gaussian  $\varphi$ -Sobolev inequalities such as the Beckner inequality [9], see also [22]. Lassalle has studied not only the beta Hermite ensemble in [51], but also the beta Laguerre ensemble in [53] with density proportional to

$$x \in D \mapsto \prod_{k=1}^n x_k^a e^{-bnx_k} \prod_{i < j} (x_i - x_j)^\beta \mathbb{1}_{x_1 \geq \dots \geq x_n \geq 0},$$

and the beta Jacobi ensemble in [52] with density proportional to

$$x \in D \mapsto \prod_{k=1}^n x_k^{a-1} (1-x_k)^{b-1} \prod_{i < j} (x_i - x_j)^\beta \mathbb{1}_{1 \geq x_1 \geq \dots \geq x_n \geq 0}.$$

It is tempting to study functional inequalities and concentration of measure for these ensembles. The proofs of Lassalle, based on Jack polynomials, are not in [53, 52, 51] but can be found in the work [5] by Baker and Forrester. We refer to [32] for the link with Macdonald polynomials. It is natural (maybe naive) to ask about direct proofs of these results without using Jack polynomials. The study of beta ensembles can be connected to  $H$ -transforms and to the work [42] on Brownian motion in a Weyl chamber, see also [33]. The analogue of the Dyson Brownian motion for the Laguerre ensemble is studied in [14], see also [50, 33, 43, 67]. Tridiagonal matrix models for Dyson Brownian motion are studied in [48].

The natural isometry between  $L^2(\gamma_n)$  and  $L^2(dx)$  leads to associate to the Ornstein–Uhlenbeck operator a real Schrödinger operator which turns out to be the quantum harmonic oscillator. Similarly, the natural isometry between  $L^2_{\text{sym}}(\mu)$  and  $L^2_{\text{sym}}(dx)$  leads to associate to the Dyson–Ornstein–Uhlenbeck operator a real Schrödinger operator known as the Calogero–Moser–Sutherland operator, which is related to radial Dunkl operators, see for instance [65, 32]. The fact that the eigenfunctions of such operators are explicit



and involve polynomials goes back at least to Calogero [17], more than twenty five years before Lassalle!

The factorization phenomenon captured by Lemma 2.6, which is behind the optimality provided by Theorem 1.3, reminds some kind of concentration-compactness related to continuous spins systems as in [21] and [58] for instance. The factorization Lemma 2.6 remains valid for other ensembles such as the Beta-Ginibre ensemble with density proportional to

$$z \in \mathbb{C}^n \mapsto e^{-n \sum_{k=1}^n |z_k|^2} \prod_{j < k} |z_j - z_k|^\beta, \quad (1.19)$$

see [24, Remark 5.4] for the case  $n = \beta = 2$ . However, in contrast with the Beta-Hermite ensemble, the interaction term is not convex in the complex case, and it is not clear at all what are the Poincaré and log-Sobolev constants of the Ginibre ensemble. See [24] for an upper bound and further discussions on the associated dynamics.

## 2. USEFUL OR BEAUTIFUL FACTS

**2.1. Random matrices, GUE, and beta Hermite ensemble.** The following result from random matrix theory goes back to Dyson, see [37, 62, 1, 40].

**Theorem 2.1** (Gaussian random matrices and GUE). *The Gaussian unitary ensemble  $\mu$  defined by (1.4) is the law of the ordered eigenvalues of a random  $n \times n$  Hermitian matrix  $H$  with density proportional to  $h \mapsto e^{-\frac{n}{2} \text{Trace}(h^2)} = e^{-\frac{n}{2} \sum_{i=1}^n h_{ii}^2 - n \sum_{i < j} |h_{ij}|^2}$  in other words the  $n^2$  real random variables  $\{H_{ii}, \Re H_{ij}, \Im H_{ij}\}_{1 \leq i < j \leq n}$  are independent, with  $\Re H_{ij}$  and  $\Im H_{ij} \sim \mathcal{N}(0, 1/(2n))$  for any  $i < j$  and  $H_{ii} \sim \mathcal{N}(0, 1/n)$  for any  $1 \leq i \leq n$ .*

There is an analogue theorem for the GOE case  $\beta = 1$  with random Gaussian real symmetric matrices, and for the GSE case  $\beta = 4$  with random Gaussian quaternion selfdual matrices. The following result holds for all beta Hermite ensemble (1.3), see [34].

**Theorem 2.2** (Tridiagonal random matrix model for beta Hermite ensemble). *The beta Hermite ensemble  $\mu$  defined by (1.3) is the distribution of the ordered eigenvalues of the random tridiagonal symmetric  $n \times n$  matrix*

$$H = \frac{1}{\sqrt{2n}} \begin{pmatrix} \mathcal{N}(0, 2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0, 2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \mathcal{N}(0, 2) & \chi_\beta \\ & & & \chi_\beta & \mathcal{N}(0, 2) \end{pmatrix}$$

where, up to the scaling prefactor  $1/\sqrt{2n}$ , the entries in the upper triangle including the diagonal are independent, follow a Gaussian law  $\mathcal{N}(0, 2)$  on the diagonal, and  $\chi$ -laws just above the diagonal with a decreasing parameter with step  $\beta$  from  $(n-1)\beta$  to  $\beta$ .

In particular the trace follows the Gaussian law  $\mathcal{N}(0, 1)$ . Such random matrix models with independent entries allow notably to compute moments of (1.3) via traces of powers.

**2.2. Isotropy of beta Hermite ensembles.** This helps to understand the structure. Let  $\mu$  be the beta Hermite ensemble (1.3), and let  $\tilde{\mu}$  be the probability measure obtained from  $\mu$  by symmetrizing coordinates: For every test function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we have

$$\int f \, d\tilde{\mu} = \int f_* \, d\mu$$

where  $f_*$  is the symmetrization of  $f$ , defined by

$$f_*(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where  $\Sigma_n$  is the symmetric group of permutations of  $\{1, \dots, n\}$ . Of course the probability measures  $\mu$  and  $\tilde{\mu}$  coincide on symmetric test functions. The probability measure  $\tilde{\mu}$  is by definition invariant by permutation of the coordinates, and its density with respect to the Lebesgue measure is

$$\frac{d\tilde{\mu}}{dx} = \frac{e^{-\frac{n}{2}|x|^2}}{n!Z_\mu} \prod_{i < j} |x_i - x_j|^\beta$$

Note that the support of  $\tilde{\mu}$  is the whole space and that  $\tilde{\mu}$  is not log-concave, even though  $\mu$  is.

**Corollary 2.3** (Isotropy of beta Hermite ensemble). *For every  $1 \leq i \neq j \leq n$ ,*

$$\int x_i d\tilde{\mu} = 0, \quad \int x_i^2 d\tilde{\mu} = \frac{\beta}{2} + \frac{2-\beta}{2n}, \quad \int x_i x_j d\tilde{\mu} = -\frac{\beta}{2n}.$$

*In particular, the law  $\tilde{\mu}$  is asymptotically isotropic.*

Recall that isotropy means zero mean and covariance matrix multiple of the identity.

In the extremal case  $\beta = 0$ , the measure  $\tilde{\mu}$  is the Gaussian law  $\mathcal{N}(0, \frac{1}{n}I_n)$ .

*Proof of Corollary 2.3.* Observe first that if  $X \sim \mu$  then  $\sum X_i$  is a standard Gaussian. This can be seen using Theorem 2.2, and observing that  $\sum X_i$  coincides with the trace of the matrix  $H$ . Actually this is true regardless of the interaction potential  $W$ , see Lemma 2.6 below. In particular

$$\int (x_1 + \dots + x_n) \mu(dx) = 0,$$

hence, by definition  $\tilde{\mu}$ ,

$$\int x_i \tilde{\mu}(dx) = \frac{1}{n} \int (x_1 + \dots + x_n) \mu(dx) = 0,$$

for every  $i \leq n$ . Since  $\sum X_i$  is a standard Gaussian we also have

$$\int (x_1 + \dots + x_n)^2 \mu(dx) = 1. \tag{2.1}$$

Next we compute  $\int |x|^2 d\mu$ . This can be done using Theorem 2.2, namely

$$\int |x|^2 \mu(dx) = \mathbb{E}(\text{Trace}(H^2)) = 1 + \frac{\beta}{n} \sum_{k=1}^{n-1} k = 1 + \frac{(n-1)\beta}{2}. \tag{2.2}$$

Note that the matrix model gives more: indeed, using the algebra of the Gamma laws,

$$\text{Trace}(H^2) \sim \text{Gamma}\left(\frac{n}{2} + \frac{\beta n(n-1)}{4}, \frac{n}{2}\right).$$

Alternatively one can use the fact that the square of the norm  $|\cdot|^2$  is, up to an additive constant, an eigenvector of  $G$ , see Remark 1.7. Namely, recall the definition (1.15) of the operator  $G$  and note that

$$G(|\cdot|^2)(x) = 2n - 2n|x|^2 + 2\beta \sum_{i \neq j} \frac{x_i}{x_i - x_j} = 2n - 2n|x|^2 + n(n-1)\beta.$$

In particular  $G(|\cdot|^2) \in L^2(\mu)$ . Since  $\mu$  is stationary, we then have  $\int G|x|^2 d\mu = 0$ , and we thus recover (2.2).

Combining (2.1) and (2.2) we get

$$\int x_i^2 \tilde{\mu}(dx) = \frac{1}{n} \int |x|^2 \mu(dx) = \frac{\beta}{2} + \frac{2-\beta}{2n},$$

and

$$\int x_i x_j \tilde{\mu}(dx) = \frac{1}{n(n-1)} \int (x_1 + \dots + x_n)^2 - (x_1^2 + \dots + x_n^2) \tilde{\mu}(dx) = -\frac{\beta}{2n}.$$

□

**Remark 2.4** (Mean and covariance of beta Hermite ensembles). Let  $\mu$  and  $\tilde{\mu}$  be as in Corollary 2.3. In contrast with the probability measure  $\tilde{\mu}$ , the probability measure  $\mu$  is log-concave but is not centered, even asymptotically as  $n \rightarrow \infty$ , and this is easily seen from  $0 \notin D$ . Moreover, if  $X_n = (X_{n,1}, \dots, X_{n,n}) \sim \mu$  then the famous Wigner theorem for the beta Hermite ensemble, see for instance [44], states that almost surely and in  $L^1$ , regardless of the way we choose the common probability space,

$$\frac{1}{n} \sum_{i=1}^n \delta_{X_{n,i}} \xrightarrow[n \rightarrow \infty]{\text{weak}} \nu_\beta \quad (2.3)$$

where

$$\begin{aligned} \nu_\beta &= \arg \inf_{\mu} \left( \int \frac{x^2}{2} d\mu(x) - \beta \iint \log(x-y) d\mu(x) d\mu(y) \right) \\ &= \frac{\sqrt{2\beta - x^2}}{\beta\pi} \mathbb{1}_{[-\sqrt{2\beta}, \sqrt{2\beta}]}(x) dx. \end{aligned} \quad (2.4)$$

This follows for instance from a large deviation principle. Moreover it can be shown that  $X_{n,1} \xrightarrow[n \rightarrow \infty]{} -\sqrt{2\beta}$  and  $X_{n,n} \xrightarrow[n \rightarrow \infty]{} \sqrt{2\beta}$ . This suggests in a sense that asymptotically, as  $n \rightarrow \infty$ , the mean is supported by the whole interval  $[-\sqrt{2\beta}, \sqrt{2\beta}]$ . It is quite natural to ask about the asymptotic shape of the covariance matrix of  $\mu$ . Elements of answer can be found in the work of Gustavsson [46].

**2.3. Log-concavity and curvature.** The following Lemma is essentially the key of the proof of Theorem 1.1 and Theorem 1.2.

**Lemma 2.5** (Log-concavity and curvature). *Let  $\mu$  be as in (1.1). Then  $U$  is  $\rho$ -convex. In particular, for the beta Hermite ensemble (1.3), the potential  $U$  is  $n$ -convex, for all  $\beta > 0$ .*

*Proof.* Recall from (1) that  $U(x) = V(x) + U_W(x)$ . Observe that  $U_W$  is convex as a sum of linear maps composed with the convex function  $W$ . Thus, if  $V$  is  $\rho$ -convex then so is  $U$ . □

**2.4. Factorization by projection.** The following factorization lemma is the key of the proof of Theorem 1.3. Let  $u$  be the unit vector of  $\mathbb{R}^n$  given by the diagonal direction:

$$u = \frac{1}{\sqrt{n}}(1, \dots, 1)$$

and let  $\pi$  and  $\pi^\perp$  be the orthogonal projection onto  $\mathbb{R}u$  and  $(\mathbb{R}u)^\perp = \{v \in \mathbb{R}^n : \langle v, u \rangle = 0\}$ .

**Lemma 2.6** (Factorization by projection). *Let  $\mu$  be as in (1.1) and let  $X$  be a random vector distributed according to  $\mu$ . Assume that the confinement potential  $V$  is quadratic:  $V = \rho|\cdot|^2/2$  for some  $\rho > 0$ . Then  $\mu$  has a Gaussian factor in the direction  $u$  in the sense that  $\pi(X) = \langle X, u \rangle u$  and  $\pi^\perp(X)$  are independent and*

$$\langle X, u \rangle \sim \mathcal{N}\left(0, \frac{1}{\rho}\right).$$

Moreover  $\pi^\perp(X)$  has density proportional to  $e^{-U}$  with respect to the Lebesgue measure on  $(\mathbb{R}u)^\perp$ .

In the special case of the beta Hermite ensemble (1.3), the law of  $\langle X, u \rangle = \text{Trace}(H)/\sqrt{n}$  is easily seen on the random matrix model  $H$  provided by theorems 2.1 and 2.2.

An extension of Lemma 2.6 to higher dimensional gases is considered in [26].

*Proof of Lemma 2.6.* Since  $x = \pi(x) + \pi^\perp(x)$  and  $\pi(x) = \langle x, u \rangle u$ , we have

$$|x|^2 = \langle x, u \rangle^2 + |\pi^\perp(x)|^2.$$

Besides it is easily seen that  $U_W(x) = U_W(\pi^\perp(x))$  for all  $x$ , a property which comes from the shift invariance of the interaction energy  $U_W$  along  $\mathbb{R}u$ . Therefore

$$e^{-U(x)} = e^{-\rho \langle x, u \rangle^2 / 2} \times e^{-\rho |\pi^\perp(x)|^2 / 2 - U_W(\pi^\perp(x))} = e^{-\rho \langle x, u \rangle^2 / 2} \times e^{-U(\pi^\perp(x))}.$$

So the density of  $X$  is the product of a function of  $\langle x, u \rangle$  by a function of  $\pi^\perp(x)$ .  $\square$

The result extends naturally by the same proof to the more general quadratic case  $V = \langle Ax, x \rangle$  where  $A$  is a symmetric positive definite  $n \times n$  matrix, provided that the diagonal direction  $u$  is an eigenvector of  $A$ .

**Remark 2.7** (Gaussian factor and orthogonal polynomials). Let  $\mu$  be as in Lemma 2.6. Let  $H_i$  and  $H_j$  be two distinct univariate (Hermite) orthogonal polynomials with respect to the standard Gaussian law  $\mathcal{N}(0, I_n)$ . Then it follows from Lemma 2.6 that the symmetric multivariate polynomials  $H_i(\sqrt{\rho/n}(x_1 + \dots + x_n))$  and  $H_j(\sqrt{\rho/n}(x_1 + \dots + x_n))$  are orthogonal with respect to  $\mu$ . In particular, when  $\rho = n$  and with  $H_i(x) = x$  and  $H_j(x) = x^2 - 1$ , we get that  $x_1 + \dots + x_n$  and  $(x_1 + \dots + x_n)^2 - 1$  are orthogonal for  $\mu$ .

### 3. PROOFS

#### 3.1. Proof of Theorems 1.1 and 1.2.

*Proof of Theorems 1.1 and 1.2.* Let us first mention that Theorem 1.2 actually implies Theorem 1.1. Indeed it is well-known that applying log-Sobolev to a function  $f$  of the form  $f = 1 + \epsilon h$  and letting  $\epsilon$  tend to 0 yields the Poincaré inequality for  $h$ , with half the constant if the log-Sobolev inequality. See for instance [2] or [7] for details.

In the discussion below, we call *potential* of a probability measure  $\mu$  the function  $-\log \rho$ , where  $\rho$  is the density of  $\mu$  with respect to the Lebesgue measure. In view of Lemma 2.5 it is enough to prove that a probability measure  $\mu$  on  $\mathbb{R}^n$  whose potential  $U$  is  $\rho$ -convex for some positive  $\rho$  satisfies the logarithmic Sobolev inequality with constant  $2/\rho$ . This is actually a well-known fact. It can be seen in various ways which we briefly spell out now. Some of these arguments require extra assumptions on  $U$ , namely that the domain of  $U$  equals  $\mathbb{R}^n$  (equivalently  $\mu$  has full support) and that  $U$  is  $\mathcal{C}^2$ -smooth on  $\mathbb{R}^n$ . For this reason we first explain a regularization procedure showing that these hypothesis can be added without loss of generality.

*Regularization procedure.* Let  $\gamma$  be the Gaussian measure whose density is proportional to  $e^{-\rho|x|^2/2}$  and let  $f$  be the density of  $\mu$  with respect to  $\gamma$ . Clearly  $U$  is  $\rho$ -convex if and only if  $\log f$  is concave. Next let  $(Q_t)$  be the Ornstein–Uhlenbeck semigroup having  $\gamma$  as a stationary measure, namely for every test function  $g$

$$Q_t g(x) = \mathbb{E} \left[ g \left( e^{-t} x + \sqrt{1 - e^{-2t}} G \right) \right]$$

where  $G \sim \gamma$ . Since  $\gamma$  is reversible for  $(Q_t)$  the measure  $\mu Q_t$  has density  $Q_t f$  with respect to  $\gamma$ . Moreover the semigroup  $(Q_t)$  satisfies the following property

$$f \text{ log-concave} \quad \Rightarrow \quad Q_t f \text{ log-concave.}$$

This is indeed an easy consequence of the Prékopa–Leindler inequality, see (3.3) below. As a result the potential  $U_t$  of  $\mu Q_t$  is also  $\rho$ -convex. Besides  $U_t$  is clearly  $\mathcal{C}^\infty$  smooth on the whole  $\mathbb{R}^n$ . Lastly since  $\lim_{t \rightarrow 0} Q_t f(x) = f(x)$  for almost every  $x$ , we have  $\mu P_t \rightarrow \mu$  weakly as  $t$  tends to 0. As a result, if  $\mu P_t$  satisfies log-Sobolev with constant  $2/\rho$  for every  $t$ , then so does  $\mu$ .

*First proof: The Brascamp–Lieb inequality.* A theorem due to Brascamp and Lieb [13] states that if the potential of  $\mu$  is smooth and satisfies  $\text{Hess}(U)(x) > 0$  for all  $x \in \mathbb{R}^n$ , then for any  $\mathcal{C}^\infty$  compactly supported test function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have the inequality

$$\text{var}_\mu(f) \leq \int_{\mathbb{R}^n} \langle \text{Hess}(U)^{-1} \nabla f, \nabla f \rangle d\mu.$$

If  $U$  is  $\rho$ -convex then  $\text{Hess}(U)^{-1} \leq (1/\rho)I_n$  and we obtain

$$\text{var}_\mu(f) \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

The extension of this inequality to all  $f \in H^1(\mu)$  follows by truncation and regularization. Note that this method only works for Poincaré. The Brascamp–Lieb inequality does not seem to admit a logarithmic Sobolev inequality counterpart, see [12] for a discussion.

*Second proof: Caffarelli’s contraction theorem.* Again let  $\gamma$  be the Gaussian measure on  $\mathbb{R}^n$  whose density is proportional to  $e^{-\rho|x|^2/2}$ . The theorem of Caffarelli [16, 15] states that if the potential of  $\mu$  is  $\rho$ -convex then the Brenier map from  $\gamma$  to  $\mu$  is 1-Lipschitz. This easily implies that the Poincaré constant of  $\mu$  is at least as good as that of  $\gamma$ , namely  $1/\rho$ . Let us sketch the argument briefly. Let  $T$  be the Brenier map from  $\gamma$  to  $\mu$  and let  $f$  be a smooth function on  $\mathbb{R}^n$ . Using the fact that  $T$  pushes forward  $\gamma$  to  $\mu$ , the Poincaré inequality for  $\gamma$  and the Lipschitz property of  $T$  we get

$$\begin{aligned} \text{var}_\mu(f) &= \text{var}_\gamma(f \circ T) \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla(f \circ T)|^2 d\gamma \\ &\leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 \circ T d\gamma \\ &= \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu. \end{aligned} \tag{3.1}$$

This contraction principle works just the same for log-Sobolev.

*Third proof: The Bakry–Émery criterion.* Assume that  $U$  is finite and smooth on the whole  $\mathbb{R}^n$  and consider the Langevin diffusion

$$dX_t = \sqrt{2} dB_t - \nabla U(X_t) dt.$$

The generator of the diffusion is the operator  $G = \Delta - \langle \nabla U, \nabla \rangle$ . The carré du champ  $\Gamma$  and its iterated version  $\Gamma_2$  are easily computed:

$$\begin{aligned} \Gamma(f, g) &= \frac{1}{2}(G(fg) - fG(g) - gG(f)) = \langle \nabla f, \nabla g \rangle \\ \Gamma_2(f, g) &= \frac{1}{2}(G\Gamma(f, g) - \Gamma(f, Gg) - \Gamma(Gf, g)) \\ &= \text{Tr}(\text{Hess}(f)\text{Hess}(g)) + \langle \text{Hess}(U)\nabla f, \nabla g \rangle. \end{aligned} \tag{3.2}$$

We also set  $\Gamma(f) = \Gamma(f, f)$  and similarly for  $\Gamma_2$ . The hypothesis that  $U$  is  $\rho$ -convex thus implies that

$$\Gamma_2(f) \geq \rho\Gamma(f),$$

for every suitable  $f$ . Actually this inequality is equivalent to the condition that  $U$  is  $\rho$ -convex, as can be seen by plugging in linear functions. In the language of Bakry–Émery, see [6, 2, 7], the diffusion satisfies the curvature dimension criterion  $\text{CD}(\rho, \infty)$ . This criterion implies that the stationary measure  $\mu$  satisfies the following logarithmic Sobolev inequality

$$\text{ent}_\mu(f^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^n} \Gamma(f) d\mu,$$

see [7, Proposition 5.7.1]. Formally this proof also works if  $\mu$  does not have full support by adding a reflection at the boundary, just as in section 1.2. However this poses some

technical issues which are not always easy to overcome. As a matter of fact, diffusions with reflecting boundary conditions are not treated in the book [7].

*Fourth proof: An argument of Bobkov and Ledoux.* This fourth proof is the one that requires the least background. Another nice feature is that the regularization procedure is not needed for this proof. It is based on the Prékopa-Leindler inequality. The latter, which is a functional form of the Brunn–Minkowski inequality, states that if  $f, g, h$  are functions on  $\mathbb{R}^n$  satisfying

$$(1-t)f(x) + tg(y) \leq h((1-t)x + ty)$$

for every  $x, y \in \mathbb{R}^n$  and for some  $t \in [0, 1]$ , then

$$\left( \int e^f dx \right)^{1-t} \left( \int e^g dx \right)^t \leq \int e^h dx. \quad (3.3)$$

We refer to [8] for a nice presentation of this inequality. Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function with compact support, and for  $s > 0$  let  $R_s F$  be the infimum convolution

$$R_s F(x) = \inf_{y \in \mathbb{R}^n} \left\{ F(x+y) + \frac{1}{2s} |y|^2 \right\}.$$

Fix  $t \in (0, 1)$ . Using the  $\rho$ -convexity of  $U$ :

$$(1-t)U(x) + tU(y) \leq U((1-t)x + ty) - \frac{\rho t(1-t)}{2} |x-y|^2,$$

it is easily seen that the functions  $f = R_{t/\rho} F - U$ ,  $g = -U$  and  $h = (1-t)F - U$  satisfy the hypothesis of the Prékopa-Leindler inequality. The conclusion (3.3) rewrites in this case

$$\left( \int_{\mathbb{R}^n} e^{R_{t/\rho} F} d\mu \right)^{1-t} \leq \int_{\mathbb{R}^n} e^{(1-t)F} d\mu. \quad (3.4)$$

It is well-known that  $(R_s)$  solves the Hamilton–Jacobi equation

$$\partial R_s F + \frac{1}{2} |\nabla R_s F|^2 = 0,$$

see for instance [39]. Using this and differentiating the inequality (3.4) at  $t = 0$  yields

$$\text{ent}_\mu(e^F) \leq \frac{1}{2\rho} \int_{\mathbb{R}^n} |\nabla F|^2 e^F d\mu$$

which is equivalent to the desired log-Sobolev inequality. We refer to the article [12] for more details.  $\square$

**3.2. Proof of Theorem 1.3 and comments on optimality.** In the case of the beta ensemble, Theorem 1.6 shows that  $x \in \mathbb{R}^n \mapsto x_1 + \dots + x_n$  is the only symmetric function optimal in the Poincaré inequality, up to additive and multiplicative constants. Our goal now is to study the optimality far beyond this special case.

We have just seen that if a measure  $\mu$  has density  $e^{-\phi}$  where  $\phi$  is  $\rho$ -convex for some  $\rho > 0$  then it satisfies Poincaré with constant  $1/\rho$ . This constant is sharp in the case where  $\mu$  is the Gaussian measure whose density is proportional to  $e^{-\rho|x|^2/2}$ . Indeed the Poincaré constant of that Gaussian measure is equal to  $1/\rho$  and extremal functions are affine functions, see for instance [2, 7]. Similarly, its log-Sobolev constant is  $2/\rho$  and extremal functions are log-affine functions, see [18]. The next lemma asserts that conversely, if the Poincaré constant of  $\mu$  or its log-Sobolev constant matches the bound predicted by the strict convexity of its potential, then  $\mu$  must be Gaussian in some direction.

Recall the notion of having a Gaussian factor in a given direction, used in Lemma 2.6.

**Lemma 3.1.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  with density  $e^{-\phi}$  where  $\phi$  is  $\rho$ -convex for some  $\rho > 0$ , and assume that there exists a non constant function  $f$  such that*

$$\mathrm{var}_\mu(f) = \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu. \quad (3.5)$$

*Then the following properties hold true:*

- (i) *The function  $f$  is affine: there exists a vector  $u$  and a constant  $b$  such that*

$$f(x) = \langle u, x \rangle + b.$$

- (ii) *The measure  $\mu$  has a Gaussian factor of variance  $1/\rho$  in the direction  $u$ .*

*Besides, there is a similar statement for the log-Sobolev inequality: if there exists a non constant function  $f$  such that*

$$\mathrm{ent}_\mu(f^2) = \frac{2}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

*then  $\log f$  is affine and  $\mu$  has a Gaussian factor in the corresponding direction.*

The Poincaré case is contained in the main result of Cheng and Zhou's article [27]. The general result is a consequence of the works of de Philippis and Figalli [29], and also Courtade and Fathi [28]. These authors actually establish a stability estimate for this lemma: if there exists a function  $f$  which is near optimal in Poincaré then  $\mu$  nearly has a Gaussian factor. We sketch a proof of Lemma 3.1 based on their ideas.

*Proof of Lemma 3.1.* We analyze the equality case in the third proof of the main theorem, the one based on Caffarelli's contraction theorem. Recall that the Brenier map  $T$  from  $\gamma$  to  $\mu$  is the gradient of a convex function and that it pushes forward  $\gamma$  to  $\mu$ . Recall also that Caffarelli's theorem asserts that under the hypothesis of the lemma  $T$  is 1-Lipshitz. Therefore  $T$  is differentiable almost everywhere, and its differential is a symmetric matrix satisfying

$$0 \leq (dT)_x \leq I_n \quad (3.6)$$

as quadratic forms. Now, observe that if  $f$  satisfies (3.5) then every inequality in (3.1) must actually be an equality. In particular  $f \circ T$  must be optimal in the Poincaré inequality for  $\gamma$ . This implies that  $f \circ T$  is affine. Also there is equality in the inequality

$$|\nabla(f \circ T)(x)|^2 \leq |\nabla f(Tx)|^2$$

for almost every  $x$ . Because of (3.6) this actually implies that

$$(dT)_x(\nabla f(Tx)) = \nabla f(Tx),$$

for almost every  $x$ . Since  $f \circ T$  is affine the left hand side is constant, and we obtain that  $f$  itself must be affine. Thus, there exists a vector  $u$  and a constant  $b$  such that  $f(x) = \langle u, x \rangle + b$ , and moreover  $(dT)_x(u) = u$  for almost every  $x$ . By a change of variable, we can assume that  $u$  is a multiple of the first coordinate vector. The differential of  $T$  at  $x$  thus has the form

$$(dT)_x = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$$

for almost every  $x$ . Therefore

$$T(x_1, \dots, x_n) = (x_1 + a, S(x_2, \dots, x_n))$$

for some constant  $a$  and some map  $S$  from  $\mathbb{R}^{n-1}$  to itself. This implies that the image  $\mu$  of  $\gamma$  by  $T$  is a product measure, and that the first factor is the Gaussian measure with mean  $a$  and variance  $1/\rho$ . This finishes the proof of the first part of the lemma. The log-Sobolev version can be obtained very similarly and we omit the details.  $\square$

**Remark 3.2** (Alternate proof based on the Bakry–Émery calculus). We spell out briefly an alternative proof of Lemma 3.1 based on the Bakry–Émery calculus starting from a work by Ledoux [54]. Let  $G$ ,  $\Gamma$ , and  $\Gamma_2$  be as in (3.2), and let  $(P_t)_{t \geq 0}$  be the Markov semigroup generated by  $G$ . The usual Bakry–Émery method gives, up to regularity considerations,

$$\mathrm{var}_\mu(f) = \frac{1}{\rho} \int \Gamma f d\mu - \frac{2}{\rho} \int_0^\infty \left( \int (\Gamma_2 - \rho\Gamma)(P_t f) d\mu \right) dt.$$

This is a Taylor–Lagrange formula expressing the “deficit” in the Poincaré inequality. It shows that if  $\Gamma_2 \geq \rho\Gamma$  and  $\mathrm{var}_\mu(f) = \frac{1}{\rho} \int \Gamma f d\mu$  then  $(\Gamma_2 - \rho\Gamma)(P_t f)(x) = 0$  almost everywhere in  $t$  and  $x$ . Up to regularity issues, we get in particular

$$(\Gamma_2 - \rho\Gamma)(f) = 0.$$

In other words, denoting  $\|\cdot\|_{\mathrm{HS}}$  the Hilbert–Schmidt norm,

$$\|\mathrm{Hess}(f)\|_{\mathrm{HS}}^2 + \langle (\mathrm{Hess}(U) - \rho I_n) \nabla f, \nabla f \rangle = 0.$$

Since both term are non negative this actually implies that

$$\mathrm{Hess}(f) = 0 \quad \text{and} \quad \langle (\mathrm{Hess}(U) - \rho I_n) \nabla f, \nabla f \rangle = 0.$$

Thus  $f$  is affine: there exists a unit vector  $u$  and two constants  $\lambda$  and  $c$  such that

$$f(x) = \lambda \langle u, x \rangle + c,$$

and moreover  $\langle \mathrm{Hess}(U)u, u \rangle = \rho$ . Since  $\mathrm{Hess}(U) \geq \rho I_n$  this actually implies that

$$\mathrm{Hess}(U)u = \rho u$$

pointwise. Proceeding as in the proof of Lemma 3.1, we then see that  $\mu$  has a Gaussian factor of variance  $1/\rho$  in the direction  $u$ . There is a similar argument for the log-Sobolev inequality using

$$\mathrm{ent}_\mu(f) = \frac{1}{2\rho} \int \Gamma(\log f) f d\mu - \frac{1}{\rho} \int_0^\infty \left( \int (\Gamma_2 - \rho\Gamma)(\log P_t f) P_t f d\mu \right) dt.$$

This leads to the fact that if  $f$  is optimal in the logarithmic Sobolev inequality then  $f$  is of the form  $f(x) = e^{\lambda \langle u, x \rangle + c}$  and  $\mu$  has a Gaussian factor in the direction  $u$ . As usual, this seductive approach requires to justify rigorously the computations via delicate handling of regularity and smoothing, see for instance [7], [3], and [1, Section 4.4.2].

*Proof of Theorem 1.3.* According to Lemma 2.6, if  $V = \rho |\cdot|^2/2$  for some  $\rho > 0$  then  $\mu$  has a Gaussian factor in the diagonal direction  $u = (1, \dots, 1)/\sqrt{n}$ . As we have seen, this Gaussian satisfies Poincaré with constant  $1/\rho$  and log-Sobolev with constant  $2/\rho$ . Moreover, affine functions are optimal in Poincaré and log-affine functions are optimal in log Sobolev. This shows that we have equality in the Poincaré inequality of Theorem 1.1 for functions  $f$  of the form  $f(x) = \lambda(x_1 + \dots + x_n) + c$  for some constants  $\lambda$  and  $c$ , and equality in the logarithmic Sobolev inequality of Theorem 1.2 for functions whose logarithm is of the preceding type.

Let us now prove that these are the only optimal functions. Assume that  $f$  is non constant and extremal in the Poincaré inequality. Then by Lemma 3.1, there exists a vector  $v$  and a constant  $b$  such that  $f(x) = \langle v, x \rangle + b$ , and moreover  $\mu$  has a Gaussian factor in the direction  $v$ . Since the support of  $\mu$  is the set  $\{x_1 \geq \dots \geq x_n\}$  this can only happen if  $v$  is proportional to the diagonal direction  $u$ , which is the result. The proof for log-Sobolev is similar.  $\square$



### 3.3. Proof of Corollary 1.4 and comments on concentration.

*Proof of Corollary 1.4.* The Gaussian concentration can be deduced from the log-Sobolev inequality via an argument due to Herbst, see for instance [55], which consists in using log-Sobolev with  $f = e^F$  to get the Gaussian upper bound on the Laplace transform

$$\int e^F d\mu \leq \exp \left( \int F d\mu + \frac{\|F\|_{\text{Lip}}^2}{2\rho} \right), \quad (3.7)$$

which leads in turn to the concentration inequality (1.5) via the Markov inequality. Alternatively we can use the intermediate inequality (3.4) obtained in the course of the fourth proof of Theorem 1.2. Indeed applying Jensen's inequality to the right-hand side of (3.4) and letting  $t \rightarrow 1$ , we obtain

$$\int e^{R_{1/\rho} F} d\mu \leq \exp \left( \int F d\mu \right). \quad (3.8)$$

Moreover, if  $F$  is Lipschitz it is easily seen that

$$R_{1/\rho} F \geq F - \frac{1}{2\rho} \|F\|_{\text{Lip}}^2.$$

Plugging this into the previous inequality yields (3.7). Note that a result due to Bobkov and Götze states that (3.8) is equivalent to a Talagrand  $W_2$  transportation inequality for  $\mu$ , see for instance [55] and references therein.

In the case  $F(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n f(x_i) = L_n(f)(x)$  we have

$$\|F\|_{\text{Lip}} \leq \frac{\|f\|_{\text{Lip}}}{\sqrt{n}}$$

so that (1.6) follows from (1.5).

Finally taking  $F(x_1, \dots, x_n) = \max(x_1, \dots, x_n)$  ( $= x_1$  on  $D$ ) in (1.5) gives (1.7).  $\square$

**Remark 3.3** (Concentration of measure in transportation distance). Following Gozlan [41], it is possible to obtain concentration of measure inequalities in Kantorovich–Wasserstein distance  $W_2$  from the Hoffman–Wielandt inequality. Namely, given a Hermitian matrix  $A$ , we let  $x_1(A) \geq \dots \geq x_n(A)$  be the eigenvalues of  $A$ , arranged in decreasing order, and

$$L_A = \frac{1}{n} \sum_{i=1}^n \delta_{x_i(A)}$$

be the corresponding empirical measure. If  $B$  is another Hermitian matrix, we get from the Hoffman–Wielandt inequality

$$nW_2(L_A, L_B)^2 = \sum_{i=1}^n (x_i(A) - x_i(B))^2 \leq \text{Trace}((A - B)^2) = \|A - B\|_{\text{HS}}^2. \quad (3.9)$$

Thanks to the triangle inequality for  $W_2$ , this implies that for every probability measure  $\nu$  on  $\mathbb{R}$  with finite second moment, the Lipschitz constant of the map  $A \mapsto W_2(L_A, \nu)$  with respect to the Hilbert–Schmidt norm is at most  $1/\sqrt{n}$ . If  $G$  is a Gaussian matrix with density proportional to  $e^{-n\text{Trace}(X^2)}$ , the Gaussian concentration inequality then yields

$$\mathbb{P}(|W_2(L_G, \nu) - \mathbb{E}W_2(L_G, \nu)| > r) \leq 2e^{-\frac{n^2}{2}r^2}.$$

Note that  $\nu$  is arbitrary. This inequality can be reformulated as follows: If  $\mu$  is the Gaussian unitary ensemble in  $\mathbb{R}^n$  and  $L$  is the map  $x \in \mathbb{R}^n \mapsto \frac{1}{n} \sum_{i \leq n} \delta_{x_i}$  then for any probability measure  $\nu$  on  $\mathbb{R}$  we have

$$\mu \left( \left| W_2(L, \nu) - \int W_2(L, \nu) d\mu \right| > r \right) \leq 2e^{-\frac{n^2}{2}r^2}.$$

More generally this inequality remains valid when  $\mu$  is the law of the eigenvalues of a random matrix satisfying Gaussian concentration with rate  $n$ . This is the case for instance

if the matrix has independent entries satisfying a logarithmic Sobolev inequality with constant  $1/n$ .

**Remark 3.4** (Proof for GUE/GOE via Hoffmann–Wielandt inequality). For the GUE and the GOE one can give a fifth proof, based on the contraction principle, like the proof using Caffarelli’s theorem above. The Hoffman–Wielandt inequality [47, 49, 11], states that for all  $n \times n$  Hermitian matrices  $A$  and  $B$  with ordered eigenvalues  $x_1(A) \geq \dots \geq x_n(A)$  and  $x_1(B) \geq \dots \geq x_n(B)$  respectively, we have

$$\sum_{i=1}^n (x_i(A) - x_i(B))^2 \leq \sum_{i,j=1}^n |A_{ij} - B_{ij}|^2.$$

In other words the map which associates to an  $n \times n$  Hermitian matrix  $A$  its vector of eigenvalues  $(x_1(A), \dots, x_n(A)) \in \mathbb{R}^n$  is 1-Lipschitz for the Euclidean structure on  $n \times n$  Hermitian matrices, given by  $\langle A, B \rangle = \text{Trace}(AB)$ . On the other hand, as we saw in section 2.1, the Gaussian unitary ensemble is the image by this map of the Gaussian measure on  $\mathbb{H}_n$  whose density is proportional to  $e^{-(n/2)\text{Trace}(H^2)}$ . The Poincaré constant of this Gaussian measure is  $1/n$  so by the contraction principle the Poincaré constant of the GUE is  $1/n$  at most. The argument works similarly for log-Sobolev and for the GOE.

### 3.4. Proof of Theorem 1.5.

*Proof of Theorem 1.5.* The exponential decay of relative entropy (1.13) is a well-known consequence of the logarithmic Sobolev inequality, see for instance [7, Theorem 5.2.1]. The decay in Wasserstein distance follows from the Bakry–Émery machinery, see [7, Theorem 9.7.2]. Alternatively it can be seen using parallel coupling. We explain this argument briefly.

Let  $X$  and  $Y$  be two solutions of the stochastic differential equation (1.8) driven by the same Brownian motion:

$$\begin{aligned} dX_t &= \sqrt{2} dB_t - \nabla U(X_t) dt + d\Phi_t \\ dY_t &= \sqrt{2} dB_t - \nabla U(Y_t) dt + d\Psi_t, \end{aligned}$$

where  $\Phi$  and  $\Psi$  are the reflections at the boundary of the Weyl chamber of  $X$  and  $Y$  respectively, see section 1.2 for a precise definition. Assume additionally that  $X_0 \sim \nu_0$ ,  $Y_0 \sim \nu_1$  and that

$$\mathbb{E}(|X_0 - Y_0|^p) = W_p(\nu_0, \nu_1)^p.$$

Observe that

$$d|X_t - Y_t|^2 = -2\langle X_t - Y_t, \nabla U(X_t) - \nabla U(Y_t) \rangle dt + 2\langle X_t - Y_t, d\Phi_t \rangle + 2\langle Y_t - X_t, d\Psi_t \rangle.$$

Since  $U$  is  $\rho$ -convex  $\langle X_t - Y_t, \nabla U(X_t) - \nabla U(Y_t) \rangle \geq \rho|X_t - Y_t|^2$ . Besides  $d\Phi_t = -n_t dL_t$  where  $L$  is the local time of  $X$  at the boundary of the Weyl chamber  $D$  and  $n_t$  is an outer unit normal at  $X_t$ . Since  $Y_t \in D$  and since  $D$  is convex we get in particular  $\langle X_t - Y_t, d\Phi_t \rangle \leq 0$  for all  $t$ , and similarly  $\langle Y_t - X_t, d\Psi_t \rangle \leq 0$ . Thus  $d|X_t - Y_t|^2 \leq -2\rho|X_t - Y_t|^2 dt$ , hence

$$|X_t - Y_t| \leq e^{-\rho t} |X_0 - Y_0|.$$

Taking the  $p$ -th power and expectation we get, in  $[0, +\infty]$ ,

$$\mathbb{E}[|X_t - Y_t|^p]^{1/p} \leq e^{-\rho t} \mathbb{E}(|X_0 - Y_0|^p)^{1/p} = e^{-\rho t} W_p(\nu_0, \nu_1).$$

Moreover since  $X_t \sim \nu_0 P_t$  and  $Y_t \sim \nu_1 P_t$  we have by definition of  $W_p$

$$W_p(\nu_0 P_t, \nu_1 P_t) \leq \mathbb{E}[|X_t - Y_t|^p]^{1/p}.$$

Hence the result.  $\square$

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