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Abstract

The purpose of this paper is to establish a topological relation between several known production models, precisely a link between $\mathbb{B}$-convex and Cobb-Douglas production models. The framework is based on the algebraic structures of the technology sets, issued from data envelopment, respecting either the assumption of constant elasticity of substitution and transformation (CES-CET) or α-returns to scale. It is shown that the Painlevé-Kuratowski limit of the CES-CET technology provides either $\mathbb{B}$-convex or inverse $\mathbb{B}$-convex technologies. Also, α-returns to scale models have topological limits relevant with constant return to scale $\mathbb{B}$-convex (or inverse $\mathbb{B}$-convex) technologies.

Keywords: Non-parametric production models, Painlevé-Kuratowski limit, lattice, CES-CET model, generalized convexity, α-returns to scale.

1 Introduction

Traditionally there exist two basic approaches to estimate a production technology over a sector of the economy. The first is based on the econometric estimation of the production frontier, which involves a parametric specification of some functional form to describe the frontier of the technology.

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The second approach is based upon operation research methods and non-parametric models that do not specify a functional form of the production technology. In their papers, Charnes, Cooper and Rhodes [15] and Banker, Charnes, Cooper [6] show how to determine the efficient observed production units in a sample of firms operating on the same sector of the economy. In their approach the production set is derived from the convex hull of all production vectors representing each firm. Using a linear programming the measure of technical efficiency can be computed to compare the decision making units and to determine the efficient ones. Implicitly this yields an estimation of the production frontier.

From Charnes *et al.* [15] and Banker *et al.* [6], several extensions of the non-parametric production model have been proposed with respect to the data envelopment analysis (DEA). A piecewise Cobb-Douglas envelopment was introduced by Charnes *et al.* [16] and Banker and Maindiratta [7]. In Färe, Grosskopf and Njinkeu [19] a CES-CET (Constant-Elasticity-Substitution-Transformation) was investigated. The point-wise limit of the CES-CET model were analyzed in a production context by Post [23] from the transformations proposed in Aczél [1], Avriel [4] and Ben-Tal [9]. A relaxation of the CES-CET model was proposed in Boussemart *et al.* [10]. This model involves a structure of α-returns to scale where the returns to scale of the technology can be either increasing or decreasing according to the choice of some parameters. More recently, some classes of path-connected semi-lattice production models were introduced by Briec and Horvath [13] and extended by Briec and Liang [14]. These models are called $\mathbb{B}$-convex and are issued from the upper (or lower) limit of the convex hull of a finite number of points.

It is worth mentioning that the aforementioned data envelopment techniques rely on the transformation of input/output vectors. As advocated by Post [23], data transformation is crucial because it limits the number of observations to be non attainable, that is, to avoid input/output combinations to be far-off the envelopment of the data – this is particularly challenging when the samples are of limited size. Also, data transformation allows a more important quantity of data to be exploitable, and as a consequence, accurate indicators of technical efficiency may be derived.

This paper shows that, given an observed set of decision making units, the Painlevé-Kuratowski limit of a sequence of CES-CET models yields either a $\mathbb{B}$-convex or a Cobb-Douglas non-parametric estimation of the technology. The same holds true for a sequence of non-parametric technology satisfying an alpha-returns to scale assumption. The suitable sequence of generalized means (power means) is derived from the transformation scheme suggested by Ben-Tal [9]. Precisely, for any given bijection $\varphi : K \to \mathbb{R}$, one can define over K the operations defined, for all $\lambda, \mu \in K$, by

$$\lambda \overset{\varphi}{+} \mu = \varphi^{-1}\left(\varphi(\lambda) + \varphi(\mu)\right) \text{ and } \lambda \overset{\varphi}{\cdot} \mu = \varphi^{-1}\left(\varphi(\lambda) \cdot \varphi(\mu)\right).$$

Let $\mathbf{u}, \mathbf{v} \in K^d$ and $\lambda \in K$ such that $\Phi(\mathbf{u}) = (\varphi(u_1), \ldots, \varphi(u_d))$, then

$$\lambda \overset{\Phi}{+} \mathbf{u} = \Phi^{-1}\left(\Phi(\mathbf{u}) + \Phi(\mathbf{v})\right) \text{ and } \lambda \overset{\Phi}{\cdot} \mu = \Phi^{-1}\left(\varphi(\lambda) \cdot \Phi(\mathbf{u})\right).$$

From Ben-Tal [9] the set $\varphi^{-1}(\mathbb{R})$ endowed with the algebraic operators $\overset{\varphi}{+}$ and $\overset{\varphi}{\cdot}$ is a scalar field. To establish the final results, the convergence of the sequence of the generalized convex hull of a finite number of points plays a crucial role. Indeed, defining the Φ-sum of a non-empty set $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\} \subset K^d$ as

$$\overset{\Phi}{\sum_{k=1,\ldots,\ell}} \mathbf{u}_k = \Phi^{-1}\left(\sum_{k=1,\ldots,\ell} \Phi(\mathbf{u}_k)\right)$$

allows the φ-convex hull of E to be defined as follows,

$$Co^{\varphi}(E) = \left\{\Phi^{-1}\Big(\sum_{k=1,\ldots,\ell} s_k \Phi\big(\mathbf{u}_k\big)\Big) : \sum_{k=1,\ldots,\ell} s_k = 1,\ s_k \geq 0\right\}.$$

The φ-convex hull of E provides, for any given well-defined function φ, such as the power function, some distortion (contraction) of E. In this respect, Briec and Horvath [13] define $\mathbb{B}$-convex sets as upper limit of $Co^{\varphi}(E)$. Based on this generalized convex hull of a set of points, it is shown that either CES-CET or Cobb-Douglas non-parametric estimation of technologies are limit cases of φ-convex hulls.

The paper is organized as follows. Section 2 presents the standard DEA model. The CES-CET and Cobb-Douglas DEA models are also presented. Section 3 focuses on the notion of generalized convexity and power means. Section 4 establishes some key results concerning the convergence of a generalized convex hull. A notion of limit set is also derived with a typology of those limits. Section 5 deals with $\mathbb{B}$-convex production technologies and Section 6 exhibits the results for the limit of α-returns to scale models. Section 7 closes the paper.

2 The Non-Parametric Production Model

The mathematical tools presented in the Introduction can now be applied to production models. Subsections 1, 2 and 3 are devoted to the exposition of the basic concepts: the production technology, the methods used to estimate the production frontier, and by the way, the technology set.

2.1 The Background of the Production Model

We first define the notations used in this section. Let $\mathbb{R}^d_+$ be the non negative d-dimensional Euclidean space. For $\mathbf{z}, \mathbf{w} \in \mathbb{R}^d_+$, we denote $\mathbf{z} \leq \mathbf{w}$ if, and only if, $z_i \leq w_i$ for all $i \in [d]$ where $[d] = \{1, \ldots, d\}$. For all $m, n \in \mathbb{N}$, such that $d = m + n$, a production technology transforms inputs $\mathbf{x} = (x_1, \ldots, x_m)$ into outputs $\mathbf{y} = (y_1, \ldots, y_n)$. The set $T \subset \mathbb{R}^{m+n}_+$ of all input-output vectors that are feasible is called the production set. It is defined as follows:

$$T = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}_+ : \mathbf{x} \text{ can produce } \mathbf{y}\right\}.$$

T can also be characterized by an input correspondence $L : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$ and an output correspondence $P : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$ respectively defined by

$$L(\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^m : (\mathbf{x}, \mathbf{y}) \in T\} \text{ and } P(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : (\mathbf{x}, \mathbf{y}) \in T\}.$$

The multivalued map P to each element $\mathbf{x}$ of $\mathbb{R}^m_+$ a subset $P(\mathbf{x})$ of $\mathbb{R}^n_+$. The production set T can be identified with its graph, that is:

$$T = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ : \mathbf{x} \in L(\mathbf{y})\right\} = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ : \mathbf{y} \in P(\mathbf{x})\right\}.$$

The inverse of P is the input correspondence L defined by $\mathbf{x} \in L(\mathbf{y})$ if and only if $\mathbf{y} \in P(\mathbf{x})$. The sets $P(\mathbf{x})$ are the values of P while the sets $L(\mathbf{y})$ are the fibers of P. The image of a subset A of $\mathbb{R}^m_+$ by P is the set $P(A) = \bigcup_{\mathbf{x} \in A} P(\mathbf{x})$. Finally, let us denote

$$K = \mathbb{R}^m_+ \times (-\mathbb{R}^n_+).$$

In the remainder, this set will be called the free disposal cone. It plays an important role to characterize the free disposal assumption defined below. There are some assumptions that can be made on the production technology, see Shephard [24].
T1: T is a closed set.
T2: T is a bounded set, *i.e* for any $\mathbf{z} \in T$, $(\mathbf{z} - K) \cap T$ is bounded.
T3: T is strongly disposable, *i.e.* $T = (T + K) \cap \mathbb{R}^{m+n}_+$.

Assumptions T1-T3 define a convex technology with freely disposable inputs and outputs. The following subsection presents a classical way to estimate the production technology.

2.2 Non-Parametric Convex and Non-Convex Technology

Following the works initiated by Farrell [20], Charnes *et al.* [15] and Banker *et al.* [6], the production set is traditionally defined by the convex hull that contains all the observations under a free disposal assumption. Suppose that $A = \{(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_\ell, \mathbf{y}_\ell)\} \subset \mathbb{R}^{m+n}_+$ is a finite set of ℓ production vectors. Let

$Co(A)$ denotes the convex hull of A. From Banker *et al.* [6], the production set under an assumption of variable returns to scale is defined by,

$$T_{DEA}(A) = (Co(A) + K) \cap \mathbb{R}_+^{m+n},$$

or equivalently, for any given vector $\mathbf{t} = (t_1, \ldots, t_\ell)$, by

$$T_{DEA}(A) = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{m+n} : \mathbf{x} \geq \sum_{k=1}^{\ell} t_k\, \mathbf{x}_k, \mathbf{y} \leq \sum_{k=1}^{\ell} t_k\, \mathbf{y}_k, \mathbf{t} \geq 0, \sum_{k=1}^{\ell} t_k = 1\Big\}.$$

This approach is the so-called DEA method (Data Envelopment Analysis) that leads to an operational definition of the production set. This subset represents some kind of convex hull of the observed production vectors. In line with Charnes *et al.* [15], under an assumption of constant returns to scale, the production set can also be represented by the smallest convex cone containing all the observed firms. In such a case the constraint $\sum_{k=1}^{\ell} t_k = 1$ is dropped from the above model. Technical efficiency can be measured by introducing the usual concept of input distance function and finding the closest point to any observed firms on the boundary of the production set. Along this line, the problem of efficiency measurement can be readily solved by linear programming. Among the most usual measures of technical efficiency, the Farrell efficiency measure (see Farrell [20] and Debreu [17]) is essentially the inverse of Shephard's distance function (Shephard [24], pp.6). The input Farrell efficiency measure is the map $E_{in} : \mathbb{R}_+^{m+n} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ defined as follows:

$$E_{in}(\mathbf{x}, \mathbf{y}) = \inf \Big\{\lambda \geq 0 : \big(\lambda\mathbf{x}, \mathbf{y}\big) \in T\Big\}.$$

It measures the greatest contraction of an input vector until to reach the isoquant of the input correspondence, and can be computed by linear programming. In the output case, the output Farrell efficiency measure is the map $E_{out} : \mathbb{R}_+^{m+n} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ defined as:

$$E_{out}(\mathbf{x}, \mathbf{y}) = \sup \Big\{\theta \geq 0 : \big(\mathbf{x}, \theta\mathbf{y}\big) \in T\Big\}.$$

It is also possible to exogenously set input and outputs to measure efficiency [8]. It is possible to provide a non-parametric estimation that does not postulate the convexity of the technology. It is the FDH approach developed by Deprins, Simar and Tulkens [18] – FDH stands for Free Disposal Hull. The FDH hull of a data set yields the following non-parametric production set,

$$T_{FDH}(A) = (A + K) \cap \mathbb{R}_+^{n+m}.$$

The main difference to the convex non-parametric technology is that $\mathbf{t} \in \{0, 1\}^\ell$. The FDH technology is non-convex, in general, but it only postulates the free disposal assumption. Shephard's distance function can also be computed over the FDH production set by enumeration, see Tulkens and Vanden Eeckaut [25]. One can also consider mixed approaches combining both DEA and FDH approaches (see Podinovski [22]). The next section presents the parametric viewpoint to estimate the production set.

2.3 The CES-CET and Cobb-Douglas Models

This subsection focuses on a modification of the Constant Elasticity of Substitution (CES)-Constant Elasticity of Transformation (CET) model introduced by Färe *et al.* [19] and extended by Boussemart *et al.* [10]. It consists of two parts: the output part is characterized by a Constant Elasticity of Transformation formula and the input part is characterized by a Constant Elasticity of Substitution formula.

This CES-CET model can be seen as a generalization of the traditional linear models proposed by Charnes *et al.* [15] and Banker *et al.* [6]. Moreover, it admits as a limiting case the multiplicative model proposed by Charnes *et al.* [16], which is also discussed in the next subsection.

To do that, let us set $d = m + n$. Suppose that $r > 0$ and let us consider the map $\Phi_r : \mathbb{R}^d_+ \longrightarrow \mathbb{R}^d_+$ defined as:

$$\Phi_r(\mathbf{u}) = (u_1^r, \ldots, u_d^r) \,.$$

This function is a bijective function from $\mathbb{R}^d_+$ to itself and its reciprocal is defined on $\mathbb{R}^d_+$ as:

$$\Phi_r^{-1}(\mathbf{u}) = \left(u_1^{1/r}, \ldots, u_d^{1/r}\right).$$

If $r < 0$ the map Φ_r is a bijective function from $\mathbb{R}^d_{++}$ to itself. For the sake of simplicity suppose that $A = \{(\mathbf{x}_k, \mathbf{y}_k) : k \in [\ell]\} \subset \mathbb{R}^{m+n}_{++}$. Moreover, let us denote $\Delta_\ell^{(r)}$ the Φ_r simplex defined by:

$$\Delta_\ell^{(r)} = \big\{(t_1, \ldots, t_\ell) \in \mathbb{R}^\ell_+ : \sum_{k\in[\ell]} {t_k}^r = 1\big\}.$$

Now, let us consider the following set:

$$T^{(r)}_{CES}(A) = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}_{++} : \mathbf{x} \geq \Phi_r^{-1}\big(\sum_{k\in[\ell]} t_k \Phi_r(\mathbf{x}_k)\big), \\ \mathbf{y} \leq \Phi_r^{-1}\big(\sum_{k\in[\ell]} t_k \Phi_r(\mathbf{y}_k)\big), \mathbf{t} \in \Delta_\ell^{(r)}\Big\}.$$

Accordingly, the input Farrell efficiency measure may be computed as follows:

$$
\begin{aligned}
E_{in}(\mathbf{x},\mathbf{y}) = \inf \lambda \\
s.t.\ \lambda x_i \geq \Big(\sum_{k\in[\ell]} t_k {x_{ki}}^r\Big)^{\frac{1}{r}} \quad i=1,\ldots,m \\
y_j \leq \Big(\sum_{k\in[\ell]} t_k {y_{kj}}^r\Big)^{\frac{1}{r}} \quad j=1,\ldots,n \\
\sum_{k\in[\ell]} {t_k}^r = 1,\ \mathbf{t} \geq 0.
\end{aligned}
$$

It is then straightforward to convert the above program into a linear program.

Based on Charnes *et al.* [16], we now consider the piecewise Cobb-Douglass (CD) model. Let us define the map $\Phi_0 : \mathbb{R}^d_{++} \longrightarrow \mathbb{R}^d_{++}$ defined as:

$$\Phi_0(\mathbf{u}) = (\ln(u_1), \ldots, \ln(u_d)).$$

This function is a bijective function from $\mathbb{R}^d_{++}$ to itself and its reciprocal is defined on $\mathbb{R}^d_{++}$ by:

$$\Phi_0^{-1}(\mathbf{u}) = (\exp(u_1), \ldots, \exp(u_d)).$$

The map Φ_0^{-1} is a bijective function from $\mathbb{R}^d_{++}$ to itself. Again, in order to simplify the notations, let us denote $\Delta_\ell^{(0)}$ the Φ_0 simplex defined by:

$$\Delta_\ell^{(0)} = \Big\{(\lambda_1, \ldots, \lambda_\ell) \in \mathbb{R}^\ell_{++} : \prod_{k\in[\ell]} \lambda_k = 1\Big\}.$$

Therefore, the Cobb-Douglas technology is defined by:

$$T_{CD}(A) = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{m+n}_{++} : \mathbf{x} \geq \prod_{k\in[\ell]} \mathbf{x}_k^{\lambda_k},\ \mathbf{y} \leq \prod_{k\in[\ell]} \mathbf{y}_k^{\lambda_\ell}, \lambda \in \Delta_\ell^{(0)}\Big\}.$$

The program solving for the technical efficiency in the Cobb-Douglas case is:

$$
\begin{aligned}
E_{in}(\mathbf{x},\mathbf{y}) = \inf\ \lambda \\
s.t.\ \lambda\mathbf{x} \geq \prod_{k\in[\ell]} \mathbf{x}_k^{\lambda_k} \\
\mathbf{y} \leq \prod_{k\in[\ell]} \mathbf{y}_k^{\lambda_k} \\
\sum_{k\in[\ell]} \lambda_k = 1, \lambda_k \geq 0.
\end{aligned}
$$

Applying a log-linear transformation to this program yields a linear program.

3 Isomorphism of Vector Space Structures

This section introduces a notion of generalized convexity based on some particular algebraic operators. These preliminary properties were established and analyzed in details by Ben-Tal [9].

3.1 Isomorphism of a Vector Space Structure

Given an algebraic structure on a set X and a bijection $\Phi : Z \longrightarrow X$ one can transport on Z the structure of X *via* Φ and with that structure on Z, Φ becomes an isomorphism if the initial structure on X is algebraic, an homeomorphism if the initial structure is topological and so on. Let d be a positive integer and let $\Phi : E \longrightarrow \mathbb{R}^d$ be a bijective map, where E is an arbitrary set. From Ben-Tal [9] we consider on E the algebraic operators $\overset{\Phi}{+}$ and $\overset{\Phi}{\cdot}$ defined for all $\mathbf{u}, \mathbf{v} \in E$ and for all $\alpha \in \mathbb{R}$ by:

$$\mathbf{u} \overset{\Phi}{+} \mathbf{v} = \Phi^{-1}\big(\Phi(\mathbf{u}) + \Phi(\mathbf{v})\big)$$
$$\alpha \overset{\Phi}{\cdot} \mathbf{u} = \Phi^{-1}\big(\alpha \cdot \Phi(\mathbf{u})\big).$$

The subset E endowed with these algebraic operators has some properties very similar to those of a vector space. Indeed, let K be an arbitrary nonempty set and let $\varphi : K \longrightarrow \mathbb{R}$ be a bijective function. One can define over K the operations defined for all $\lambda, \mu \in K$ by

$$\lambda \overset{\varphi}{+} \mu = \varphi^{-1}\left(\varphi(\lambda) + \varphi(\mu)\right)$$
$$\lambda \overset{\varphi}{\cdot} \mu = \varphi^{-1}\left(\varphi(\lambda) \cdot \varphi(\mu)\right).$$

From Ben-Tal [9] the set $\varphi^{-1}(\mathbb{R})$ endowed with the algebraic operators $\overset{\varphi}{+}$ and $\overset{\varphi}{\cdot}$ is a scalar field.

A vector space can then be constructed as the cartesian product of an isomorphic transformation of the scalar field $\mathbb{R}$, that is K^d, in the case where the bijective map Φ is defined for all $\mathbf{u} \in \mathbb{R}^d$ and all $\mathbf{u} \in E = K^d$ by:

$$\Phi(\mathbf{u}) = (\varphi(u_1), \ldots, \varphi(u_d)) \text{ and } \Phi^{-1}(\mathbf{u}) = \left(\varphi^{-1}(u_1), \ldots, \varphi^{-1}(u_d)\right).$$

It follows that $K = \varphi^{-1}(\mathbb{R})$ is endowed with a total order defined by:

$$\lambda \overset{\varphi}{\leq} \mu \iff \varphi(\lambda) \leq \varphi(\mu).$$

Obviously $\left(K^d, \overset{\varphi}{+}, \overset{\varphi}{\cdot}\right)$ is a vector space where the algebraic operators $\overset{\varphi}{+}$ and $\overset{\varphi}{\cdot}$ are those defined above. It is then clear that if $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is a basis of $\mathbb{R}^d$ then $B^\varphi = \Phi^{-1}(B) = \{\Phi^{-1}(\mathbf{v}_1), \ldots, \Phi^{-1}(\mathbf{v}_d)\}$ is a basis of the vector space $\left(K^d, \overset{\varphi}{+}, \overset{\varphi}{\cdot}\right)$.

One can then define some convexity notion, from the algebraic operators $\overset{\varphi}{+}$ and $\overset{\varphi}{\cdot}$ defined over K. Notice that the scalar field the algebraic structure is based upon may not be $\mathbb{R}$. However, it is shown below that such a formulation also yields a number of geometrical properties, in particular when φ is a bijection from $\mathbb{R}$ to itself.

Definition 3.1.1 *Let φ be a bijective map defined from a nonempty set K to $\mathbb{R}$. A subset E of K^d is φ-convex if for all $\mathbf{u}, \mathbf{v} \in E$ and all $t_1, t_2 \in \varphi^{-1}([0,1])$, with $t_1 \overset{\varphi}{+} t_2 = \varphi^{-1}(1)$ we have $t_1 \overset{\varphi}{\cdot} \mathbf{u} \overset{\varphi}{+} t_2 \overset{\varphi}{\cdot} \mathbf{v} \in E$.*

Now, we can define the convex hull of a finite number of points in K^d.

Definition 3.1.2 *Let φ be a bijective map defined from a nonempty set K to $\mathbb{R}$. Let $E = \{\mathbf{u}_1, \dots, \mathbf{u}_\ell\} \subset K^d$. The subset*

$$Co^\varphi(E) = \left\{ \overset{\Phi}{\sum_{k\in[\ell]}} t_k \overset{\Phi}{\cdot} \mathbf{u}_k : \overset{\varphi}{\sum_{k\in[\ell]}} t_k = \varphi^{-1}(1), t_k \overset{\varphi}{\geq} \varphi^{-1}(0), k \in [\ell] \right\}$$

is the φ-convex hull of E.

This definition means $Co^\varphi(E)$ is the smallest φ-convex set which contains E. A proof can be found in Briec and Horvath [12]. The convex hull defined in Definition 3.1.2 can be written in a mixed form. Such a particularity will be of importance in the remainder of the paper. The φ-convex hull of E, $Co^\varphi(E)$, is said to be expressed in mixed form if, for $\mathbf{s} = \Phi(\mathbf{t})$

$$Co^\varphi(E) = \Big\{ \Phi^{-1}\Big(\sum_{k\in[\ell]} s_k \Phi\big(\mathbf{u}_k\big) \Big) : \sum_{k\in[\ell]} s_k = 1, \mathbf{s} \geq 0 \Big\}.$$

3.2 The Example of Power functions

In this subsection, the concepts developed above are applied to a special transformation of a real scalar field, which is the usual power function.

For all $r \in]0, +\infty[$, let $\varphi_r : \mathbb{R} \longrightarrow \mathbb{R}$ be the map defined by:

$$\varphi_r(\lambda) = \begin{cases} \lambda^r & \text{if } \lambda \geq 0 \\ -|\lambda|^r & \text{if } \lambda \leq 0. \end{cases}$$

For all $r \neq 0$, the inverse map is $\varphi_r^{-1} = \varphi_{\frac{1}{r}}$. Let $\mathbb{Z}$ be the set of integers. If $r \in 2\mathbb{Z}+1$, then $\varphi_r(u) = u^r$, for all $u \in \mathbb{R}$. In the following subsection, we successively distinguish several cases. It is first quite straightforward to state that: (i) φ_r is defined over $\mathbb{R}$; (ii) φ_r is continuous over $\mathbb{R}$; (iii) φ_r is bijective.

Throughout the section, for any vector $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d$ we use the following notations:

$$\Phi_r(\mathbf{u}) = \big(\varphi_r(u_1), \dots, \varphi_r(u_d)\big).$$

If $\mathbf{u} \in \mathbb{R}^d_+$, then

$$\Phi_r(\mathbf{u}) = \big({u_1}^r, \dots, {u_d}^r\big) = \mathbf{u}^r.$$

It is then natural to introduce the following algebraic operations over $\mathbb{R}^n$:

$$\mathbf{u} \overset{r}{+} \mathbf{v} = \Phi_r^{-1}\big(\Phi_r(\mathbf{u}) + \Phi_r(\mathbf{v})\big)$$
$$\lambda \overset{r}{\cdot} \mathbf{u} = \Phi_r^{-1}\left(\varphi_r(\lambda)\Phi_r(\mathbf{u})\right).$$

Let us consider $E = \{\mathbf{u}_1, \dots, \mathbf{u}_\ell\} \subset \mathbb{R}^d$. The φ_r-convex hull of the set E is:

$$Co^{\varphi_r}(E) = \Big\{ \sum_{k\in[\ell]}^{\varphi_r} t_k \overset{r}{\cdot} \mathbf{u}_k : \sum_{k\in[\ell]}^{\varphi_r} t_k = 1,\ \mathbf{t} \geq 0 \Big\}.$$

If $E \subset \mathbb{R}^d_+$, then:

$$Co^{\varphi_r}(E) = \Big\{ \big(\sum_{k\in[\ell]} {t_k}^r {\mathbf{u}_k}^r\big)^{\frac{1}{r}} : \big(\sum_{k\in[\ell]} {t_k}^r\big)^{\frac{1}{r}} = 1,\ \mathbf{t} \geq 0 \Big\}.$$

Let us focus on the case $r \in]-\infty, 0[$. The map $u \longrightarrow u^r$ is not defined at point $u = 0$. Thus, it is not possible to construct a bijective map $\mathbb{R}$ to itself. Set $K = \{\infty\} \cup \mathbb{R} \setminus \{0\}$. For all $r \in]-\infty, 0[$ we consider the function $\bar{\varphi}_r$ defined by:

$$\bar{\varphi}_r(\lambda) = \begin{cases} \lambda^r & \text{if } \lambda > 0 \\ -(|\lambda|)^r & \text{if } \lambda < 0 \\ 0 & \text{if } \lambda = +\infty. \end{cases}$$

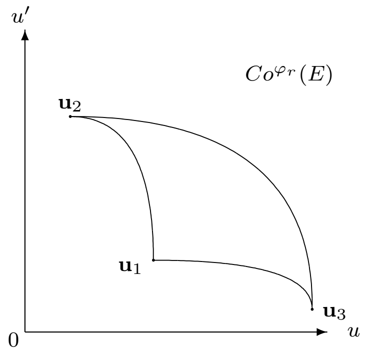
Clearly, the function $\bar{\varphi}_r$ is a bijective function from K to $\mathbb{R}$ and therefore it is an isomorphism by construction. Moreover, let us construct the bijective function $\bar{\Phi}_r : K^d \longrightarrow \mathbb{R}^d$, defined by $\bar{\Phi}_r(\mathbf{u}) = (\bar{\varphi}_r(u_1), \dots, \bar{\varphi}_r(u_d))$.
For all $r < 0$, let us consider the algebraic operators $\overset{r}{+}$ and $\overset{r}{\cdot}$ defined by:

$$\mathbf{u} \overset{r}{+} \mathbf{v} = \bar{\Phi}_r^{-1}\left(\bar{\Phi}_r(\mathbf{u}) + \bar{\Phi}_r(\mathbf{v})\right)$$
$$\lambda \overset{r}{\cdot} \mathbf{u} = \bar{\Phi}_r^{-1}\left(\bar{\varphi}_r(\lambda) \cdot \bar{\Phi}_r(\mathbf{u})\right).$$

Then $\left(K^d, \overset{r}{+}, \overset{r}{\cdot}\right)$ is a $\bar{\Phi}_r$-vector space. One can remark that if $r < 0$ then $\Phi_r = \Phi_{-1}(\Phi_{|r|})$. Hence, if $E \subset \mathbb{R}^d_{++}$, then we also have:

$$Co^{\varphi_r}(E) = \Phi^{-1}\big(Co^{\varphi_r}(\Phi^{-1}(E))\big).$$

Now, to depict the geometrical form of the convex hull induced by the power function, it is useful to distinguish the cases $r > 1$ and $r < 1$. If $r = 1$, one retrieves the standard convex hull. The curvature of the "facets" changes with respect to r. This is is depicted in Figure 3.2.1 and 3.2.2, when $r > 1$ and $r < 1$ respectively. The shaded lines represents the usual convex hull, *i.e.* when $r = 1$.



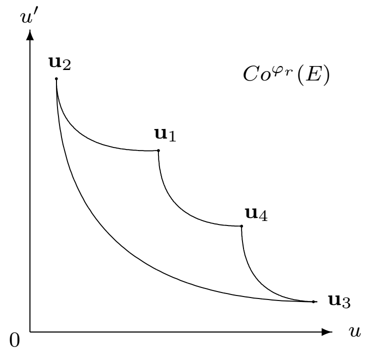


Figure 3.2.1 Convex hull for $r > 1$. **Figure 3.2.2** Convex hull for $r < 1$.

4 Power Functions and Limit Sets

This section introduces the notion of a limit set when $r \longrightarrow r_0 \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. In particular the geometric deformation of the φ_r-convex hull with respect to r is studied. To simplify the notations, let us denote $Co^r(E) = Co^{\varphi_r}(E)$ for all finite subsets E of $\mathbb{R}^d$.

The Painlevé-Kuratowski upper limit of the sequence of sets $(\,Co^r(E)\,)_{r\in\mathbb{N}}$, where E is finite, will be denoted by $Co^\infty(E)$. By definition, a $\mathbb{B}$-polytope is a set of the form $Co^\infty(E)$ for some finite subset of $\mathbb{R}^d$.

We will see that in $\mathbb{R}^d_+$ the upper-limit is in fact a limit and that the elements of $Co^\infty(E)$ have a simple analytic description. The Painlevé-Kuratowski lower [upper] limit of the sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ is denoted $Li_{n\to\infty}E_n$ [$Ls_{n\to\infty}E_n$]. For a set of points p for which there exists a sequence $\{p_n\}$ of points such that $p_n \in E_n$ for all n and $p = \lim_{n\to\infty} p_n$, a sequence $\{E_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^m$ is said to converge, in the Painlevé-Kuratowski sense, to a set E if $Ls_{n\to\infty}E_n = E = Li_{n\to\infty}E_n$, in which case we write $E = Lim_{n\to\infty}E_n$. Our first statement, Lemma 4.1.1, gives a simple algebraic description of $Co^\infty(E)$; it has been extended to arbitrary sets by Briec and Horvath [12].

4.1 Typology of Limit Sets

We denote by $\bigvee_{k=1}^{\ell} \mathbf{u}_k$ the least upper bound of $\mathbf{u}_1, \ldots, \mathbf{u}_\ell \in \mathbb{R}^d$, that is:

$$\bigvee_{k=1}^{\ell} \mathbf{u}_k = \left(\max\{u_{11}, \ldots, u_{\ell 1}\}, \ldots, \max\{u_{1d}, \ldots, u_{\ell d}\}\right).$$

The following result is an immediate adaptation of the result established by Briec [11].

Lemma 4.1.1 *Let $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ be a finite subset of $\mathbb{R}^d_+$. For all positive real number r let $E^{(r)} = \{\mathbf{u}_1^{(r)}, \ldots, \mathbf{u}_\ell^{(r)}\}$ be a finite collection of ℓ vectors in $\mathbb{R}^d_+$.*

(a) If there exists an increasing sequence $\{r_s\}_{s\in\mathbb{N}}$ *of positive real numbers such that* $\lim_{s\longrightarrow\infty} r_s = \infty$ *and* $\lim_{s\longrightarrow\infty} \mathbf{u}_k^{(r_s)} = \mathbf{u}_k$ *with* $k = 1,\dots,\ell$*, then:*

$$\lim_{s\longrightarrow\infty} \sum_{k\in[\ell]}^{\varphi_{r_s}} \mathbf{u}_k^{(r_s)} = \bigvee_{k\in[\ell]} \mathbf{u}_k.$$

(b) If for $k = 1,\dots,\ell$ $\lim_{s\longrightarrow\infty} \mathbf{u}_k^{(r_s)} = \mathbf{u}_k$*, then*

$$Co^{\infty}(E) = Lim_{s\to\infty} Co^{r_s}(E^{(r_s)}) = \Big\{ \bigvee_{k\in[\ell]} t_k \mathbf{u}_k : \max_k t_k = 1, t_k \geq 0 \Big\}.$$

Our first result, Lemma 4.1.1, gives a simple algebraic description of $Co^{\infty}(E)$. All these properties are linked to the notion of $\mathbb{B}$-convexity, defined in Briec and Horvath [12]. A subset C of $\mathbb{R}^d_+$ is $\mathbb{B}$-convex, if for all subsets E of C, $Co^{\infty}(E) \subset C$. A similar result can be obtained in the case where $\lim_{s\longrightarrow\infty} r_s = -\infty$. In such a case one should, however, assume that E is a subset of $\mathbb{R}^d_{++}$.

We denote by $\bigwedge_{k=1}^{\ell} \mathbf{u}_k$ the least upper bound of $\mathbf{u}_1,\dots,\mathbf{u}_\ell \in \mathbb{R}^d$, that is:

$$\bigwedge_{k=1}^{\ell} \mathbf{u}_k = \left(\min\{u_{11},\dots,u_{\ell 1}\},\dots,\min\{u_{1d},\dots,u_{\ell d}\}\right).$$

The result is a straightforward consequence of that obtained by Adilov and Yesilce [3], where a suitable notion of $\mathbb{B}^{-1}$-convexity was introduced.

Lemma 4.1.2 *Let* $E = \{\mathbf{u}_1,\dots,\mathbf{u}_\ell\}$ *be a finite subset of* $\mathbb{R}^d_{++}$*. For all real number* r *let* $E^{(r)} = \{\mathbf{u}_1^{(r)},\dots,\mathbf{u}_\ell^{(r)}\}$ *be a finite collection of* ℓ *vectors in* $\mathbb{R}^d_{++}$*.*
(a) If there exists a decreasing sequence $\{r_s\}_{s\in\mathbb{N}}$ *of real numbers such that* $\lim_{s\longrightarrow\infty} r_s = -\infty$ *and* $\lim_{s\longrightarrow\infty} \mathbf{u}_k^{(r_s)} = \mathbf{u}_k$ *for* $k = 1,\dots,\ell$*, then:*

$$\lim_{s\longrightarrow\infty} \sum_{k\in[\ell]}^{\varphi_{r_s}} \mathbf{u}_k^{(r_s)} = \bigwedge_{k\in[\ell]} \mathbf{u}_k.$$

(b) If for $k = 1,\dots,\ell$ $\lim_{s\longrightarrow\infty} \mathbf{u}_k^{(r_s)} = \mathbf{u}_k$*, then*

$$Lim_{s\to\infty} Co^{r_s}(E^{(r_s)}) = \Big\{ \bigwedge_{k\in[\ell]} t_k \mathbf{u}_k : \min_k t_k = 1, t_k \geq 1 \Big\} = Co^{-\infty}(E).$$

In the following lines, we consider the case where $r_s \longrightarrow 0$. For this purpose, let us denote,

$$Co^{0}(E) = \Big\{ \prod_{k\in[\ell]} {\mathbf{u}_k}^{\lambda_k} : \sum_k \lambda_k = 1, \lambda_k \geq 0 \Big\},$$

for all subset E of $\mathbb{R}^d_{++}$.

Lemma 4.1.3 *Let $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ be a finite subset of $\mathbb{R}^d_{++}$. For all real number r let $E^{(r)} = \{\mathbf{u}_1^{(r)}, \ldots, \mathbf{u}_\ell^{(r)}\}$ be a finite collection of ℓ vectors in $\mathbb{R}^d_{++}$ and let $\lambda^{(r)} \in \mathbb{R}^\ell_+$ be an element of $\Delta_\ell^{(1)}$.*
(a) If there exists some $\lambda \in \mathbb{R}^\ell_+$ and a decreasing sequence $\{r_s\}_{s\in\mathbb{N}}$ of positive real numbers such that $\lim_{s\longrightarrow\infty} r_s = 0^+$, $\lim_{s\longrightarrow\infty} \lambda^{(r_s)} = \lambda$ and $\lim_{s\longrightarrow\infty} \mathbf{u}_k^{(r_s)} = \mathbf{u}_k$ then:

$$\lim_{s\longrightarrow\infty} {\Phi_{r_s}}^{-1}\Big(\sum_{k\in[\ell]} \lambda_k^{(r_s)} \Phi_{r_s}\big(\mathbf{u}_k^{(r_s)}\big)\Big) = \prod_{k\in[\ell]} \mathbf{u}_k^{\lambda_k}.$$

(b) If for $k = 1, \ldots, \ell$ $\lim_{s\longrightarrow\infty} \mathbf{u}_k^{(r_s)} = \mathbf{u}_k$, then

$$Lim_{s\to\infty} Co^{r_s}(E^{(r_s)}) = \Big\{ \prod_{k\in[\ell]} {\mathbf{u}_k}^{\lambda_k} : \sum_k \lambda_k = 1, \lambda_k \geq 0 \Big\} = Co^0(E).$$

Proof: (a) For all $j \in [d]$, we have:

$$\Big[{\Phi_{r_s}}^{-1}\Big(\sum_{k\in[\ell]} \lambda_k \Phi_{r_s}\big(\mathbf{u}_k^{(r_s)}\big)\Big)\Big]_j = \Big(\sum_{k\in[\ell]} \lambda_k \big(\mathbf{u}_k^{(r_s)}\big)^{r_s}\Big)^{\frac{1}{r_s}}.$$

Since $\Delta_\ell^{(1)}$ is a closed set, $\lambda \in \Delta_\ell^{(1)}$. Hence, $\sum_k \lambda_k = 1$. Taking the logarithm and applying the Lhôspital rule for $r_s \longrightarrow 0^+$ yields the desired result.

(b) We first remark that setting $\varphi_r(t_k) = \lambda_k$ for all $k \in [\ell]$ yields:

$$Co^{r_s}(E) = \Big\{{\Phi_{r_s}}^{-1}\Big(\sum_{k\in[\ell]} \lambda_k \Phi_r\big(\mathbf{u}_k^{(r_s)}\big)\Big) : \sum_k \lambda_k = 1, \lambda \geq 0\Big\}.$$

We first establish that $Co^0(E) = \Big\{ \prod_{k\in[\ell]} {\mathbf{u}_k}^{\lambda_k} : \sum_k \lambda_k = 1, \lambda_k \geq 0\Big\} \subset Li_{s\longrightarrow\infty} Co^{r_s}(E)$. Let $\mathbf{v} = \prod_{k\in[\ell]} {\mathbf{u}_k}^{\lambda_k}$ with $\lambda_1, \ldots, \lambda_\ell \in [0, 1]$ and $\sum_{k\in[\ell]} \lambda_k = 1$. Define $\mathbf{v}^{(r_s)} \in Co^{r_s}(E)$ by:

$$\mathbf{v}^{(r_s)} = {\Phi_{r_s}}^{-1}\left(\lambda_1 \Phi_{r_s}(\mathbf{u}_1^{(r_s)}) + \cdots + \lambda_\ell \Phi_{r_s}(\mathbf{u}_\ell^{(r_s)})\right).$$

Since $\mathbf{u}_1^{(r_s)}, \ldots, \mathbf{u}_\ell^{(r_s)} \in \mathbb{R}^d_{++}$ we deduce from (a) that:

$$\lim_{s\longrightarrow\infty} \mathbf{v}^{(r_s)} = \mathbf{v}.$$

This completes the first part of the proof. Next, we establish that $Ls_{s\longrightarrow\infty} Co^{r_s}(E) \subset Co^0(E)$. Take $\mathbf{v} \in Ls_{s\longrightarrow\infty} Co^{r_s}(E)$. There is an increasing sequence $\{s_l\}_{l\in\mathbb{N}}$ and a sequence of points $\{\mathbf{v}_l\}_{l\in\mathbb{N}}$ such that $\mathbf{v}_l \in Co^{r_{s_l}}(E^{(r_{s_l})})$ and $\lim_{l\longrightarrow\infty} \mathbf{v}_l = \mathbf{v}$. Each $\mathbf{v}_l$ being in $Co^{r_{s_l}}(E^{(r_{s_l})})$, we can write:

$$\mathbf{v}_l = {\Phi_{r_{s_l}}}^{-1}\left(\lambda_{l1} \Phi_{r_{s_l}}(\mathbf{u}_1^{(r_{s_l})}) + \cdots + \lambda_{l\ell} \Phi_{r_{s_l}}(\mathbf{u}_\ell^{(r_{s_l})})\right).$$

Since $\lambda_{l1}, \ldots, \lambda_{l\ell} \in [0,1]$ one can extract a subsequence $(\lambda_{l_q})_{q\in\mathbb{N}}$ that converges to a point $\lambda^* = (\lambda_1^*, \ldots, \lambda_\ell^*) \in [0,1]^\ell$. From (a) we deduce that: $\mathbf{u} = \prod_{k=1}^{\ell} \mathbf{u}_k^{\lambda_k^*}$ with $\sum_{k\in[\ell]} \lambda_k^* = 1$. The first and the second part of the proof show that:

$$Ls_{s\longrightarrow\infty} Co^{r_s}(E^{(r_s)}) \subset Co^0(E) \subset Li_{s\longrightarrow\infty} Co^{r_s}(E^{(r_s)}).$$

This completes the proof since we always have the inclusion $Li_{s\longrightarrow\infty} Co^{r_s}(E^{(r_s)}) \subset Ls_{s\longrightarrow\infty} Co^{r_s}(E^{(r_s)})$. $\square$

The following table provides a synthesis of the limit sets with respect to the form of the vector space structure.

Table 4. 1: Typology of the limit sets

	Convex Hull Limit
$r_s \longrightarrow 0$	$Co^0(E) = \left\{ \prod_{k=1}^{\ell} \mathbf{u}_k{}^{\lambda_k} : \sum_{k=1}^{\ell} \lambda_k = 1, \lambda_k \geq 0, k \in [\ell] \right\}$
$r_s \longrightarrow +\infty$	$Co^{\infty}(E) = \left\{ \bigvee_{k=1}^{\ell} t_k \mathbf{u}_k : \bigvee_{k=1}^{\ell} t_k = 1, t_k \geq 0, k \in [\ell] \right\}$
$r_s \longrightarrow -\infty$	$Co^{-\infty}(E) = \left\{ \bigwedge_{k=1}^{\ell} t_k \mathbf{u}_k : \bigwedge_{k=1}^{\ell} t_k = 1, t_k \geq 0, k \in [\ell] \right\}$

For all finite and nonempty set E contained in $\mathbb{R}^d$, $Co^r(E)$ belongs to $\mathcal{K}(\mathbb{R}^d)$, the space of nonempty compact subsets of $\mathbb{R}^d$, which is metrizable by the Hausdorff metric:

$$D_H(C_1, C_2) = \inf \Big\{ \varepsilon > 0 : C_1 \subset \bigcup_{\mathbf{u}\in C_2} B(\mathbf{u}, \varepsilon), \text{ and } C_2 \subset \bigcup_{\mathbf{u}\in C_1} B(\mathbf{u}, \varepsilon) \Big\},$$

where $B(\mathbf{u}, \varepsilon)$ is the ball of center $\mathbf{u}$ and radius ε. In the remainder, we assume that $E \subset \mathbb{R}^d_{++}$. Notice, however, that the case where $s \longrightarrow \infty$ holds true when $E \subset \mathbb{R}^d_+$ (see Briec and Horvath [12]).

Lemma 4.1.4 *Let $\bar{r} \in \{-\infty, 0, \infty\}$. Let $\{r_s\}_{s\in\mathbb{N}}$ be a sequence of real numbers which converges to $\bar{r}$. For all finite nonempty subsets E of $\mathbb{R}^d_{++}$, the sequence $\{Co^{r_s}(E)\}_{s\in\mathbb{N}}$ converges to $Co^{\bar{r}}(E)$ in $\mathcal{K}(\mathbb{R}^d)$, with respect to the Hausdorff metric.*

Proof: First, remark that since $E \subset \mathbb{R}^d_{++}$, $Co^r(E)$ is well defined for all $r \in [-\infty, +\infty]$. Choose $\delta > 0$ such that $E \subset [0, \delta]^d$; we have $\Phi_{r_s}(E) \subset [0, \delta^{2r_s+1}]^d$, and therefore also $Co(\Phi_{r_s}(E)) \subset [0, \delta^{2r_s+1}]^d$. Taking the inverse image by Φ_{r_s} yields $Co^{r_s}(E) \subset [0, \delta]^d$; all the terms of the sequence

$\{Co^{r_s}(E)\}_{s\in\mathbb{N}}$ are contained in the compact set $[0,\delta]^d$. To conclude, recall that on compact metric spaces, Painlevé-Kuratowski convergence of a sequence of compact sets implies convergence in the Hausdorff metric. $\square$

It is noteworthy that in the three cases the limit set reduces to a singleton. The following figures depict the geometric form of the string joining two points with respect to the parameter r of the power function. Figure 4.1 depicts the case where $\mathbf{u}_1$ and $\mathbf{u}_2$ are not in relation (not ordered).

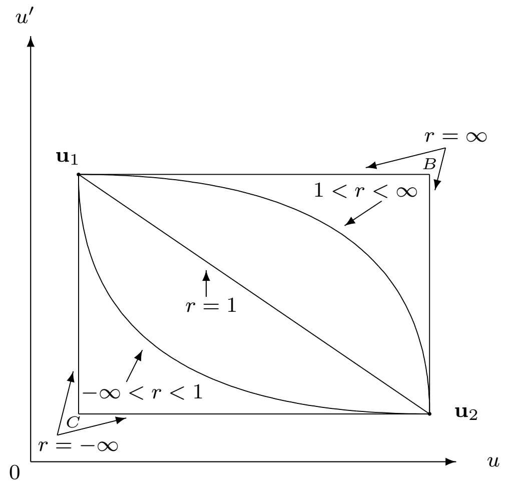


Figure 4.1 String joining $\mathbf{u}_1$ and $\mathbf{u}_2$ when $\mathbf{u}_1$ and $\mathbf{u}_2$ are not in relation.

The maximum-semi-lattice hull $Co^{\infty}(\mathbf{u}_1,\mathbf{u}_2)$ is the broken line joining the points $\mathbf{u}_1$, B and $\mathbf{u}_2$. The minimum-semi-lattice hull $Co^{-\infty}(\mathbf{u}_1,\mathbf{u}_2)$ is the broken line joining the points $\mathbf{u}_1$, C and $\mathbf{u}_2$. The intermediary strings corresponding to $-\infty < r < 1$, $r = 1$, $1 < r < \infty$ are included in the rectangle $\mathbf{u}_1 C \mathbf{u}_2 B$. In particular, the limit set in mixed form $Co^0(\mathbf{u}_1,\mathbf{u}_2)$ is between the string $r = 1$ and $r = -\infty$. Figures 4.2 and 4.3 depict two cases where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in relation.

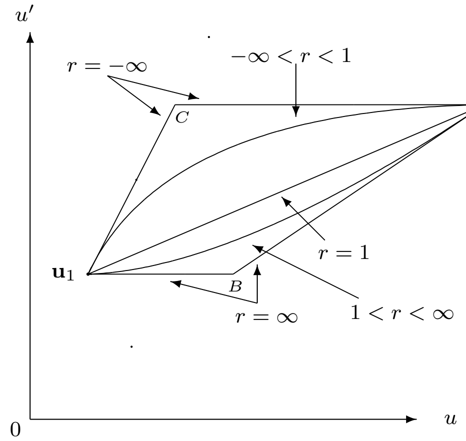


Figure 4.2 $\mathbf{u}_2$ under the ray spanned from $\mathbf{u}_1$.

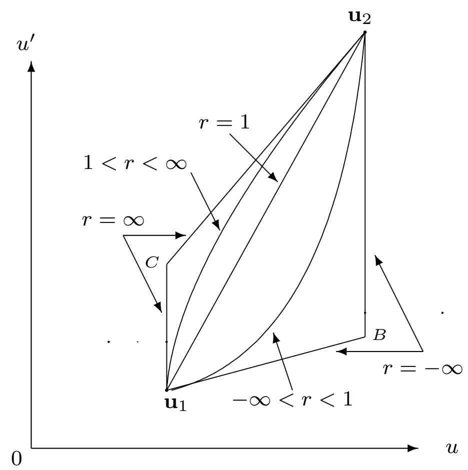


Figure 4.3 $\mathbf{u}_2$ upper the ray generated by $\mathbf{u}_1$.

In Figure 4.2 we consider the situation where $\mathbf{u}_1 \leq \mathbf{u}_2$ and $\mathbf{u}_2$ is under the ray spanned from $\mathbf{u}_1$ that is $\{\lambda \mathbf{u}_1 : \lambda \geq 1\}$. In the case $r = \infty$, the limit set $Co^{\infty}(\mathbf{u}_1, \mathbf{u}_2)$ is the broken line joining the points $\mathbf{u}_1, B$ and $\mathbf{u}_2$. If $r = -\infty$ then the limit set $Co^{-\infty}(\mathbf{u}_1, \mathbf{u}_2)$ is the broken line joining the points $\mathbf{u}_1, C$ and $\mathbf{u}_2$. If $r = 1$, then one retrieves the usual convex hull between the points $\mathbf{u}_1$ and $\mathbf{u}_2$. The intermediary cases $-\infty < r < 1$ and $1 < r < \infty$ are respectively represented between the string $r = 1$ and $r = -\infty$ on the one hand and between the string $r = 1$ and $r = \infty$ on the other hand. The string $Co^0(\mathbf{u}_1, \mathbf{u}_2)$ in the mixed case $r \longrightarrow 0$ can also be represented between the string $r = 1$ and $r = -\infty$.
Figure 4.3 depicts the case where $\mathbf{u}_1 \leq \mathbf{u}_2$ and $\mathbf{u}_2$ is above the ray spanned from $\mathbf{u}_1$. It follows that the geometric form of $Co^{-\infty}(\mathbf{u}_1, \mathbf{u}_2)$ and $Co^{\infty}(\mathbf{u}_1, \mathbf{u}_2)$ are significantly modified. The string $Co^{-\infty}(\mathbf{u}_1, \mathbf{u}_2)$ is then the broken line joining $\mathbf{u}_1, B$ and $\mathbf{u}_2$. Moreover $Co^{\infty}(\mathbf{u}_1, \mathbf{u}_2)$ is the broken line joining $\mathbf{u}_1, C$ and $\mathbf{u}_2$. Geometrically, comparing to Figure 4.3, the respective positions of the maximum and minimum envelopments are reversed. The same holds considering the intermediary cases $-\infty < r < 1$ and $1 < r < \infty$. Of course the limit set $Co^0(\mathbf{u}_1, \mathbf{u}_2)$ could be represented between the string $r = 1$ and $r = -\infty$. Figure 4.2 and 4.3 show that the relative position of $\mathbf{u}_1$ and $\mathbf{u}_2$ has a strong implication on the curvature of the string joining them.

4.2 Some General Properties

Proposition 4.2.1 *Let $\{C_n\}_{n \in \mathbb{N}}$, be a sequence of compact sets of points of $\mathbb{R}^d$ which converges, in the Painlevé-Kuratowski sense, to a set C of $\mathbb{R}^d$, that is $Lim_{n \longrightarrow \infty} C_n = C$. Then for all closed subset K of $\mathbb{R}^d$, we have:*

$$Lim_{n \longrightarrow \infty}(C_n + K) = C + K.$$

Proof: Suppose that $\mathbf{w} \in Ls_{n \longrightarrow \infty}(C_n + K)$. We first prove that $\mathbf{w} \in C + K$. By hypothesis there is a sequence $\{\mathbf{w}_{n_k}\}_{k \in \mathbb{N}}$ such that $\mathbf{w}_{n_k} \in C_{n_k} + K$ for all natural numbers k and $\lim_{k \longrightarrow \infty} \mathbf{w}_{n_k} = \mathbf{w}$. By definition, for all k there exists $(\mathbf{u}_{n_k}, \mathbf{v}_{n_k}) \in C_{n_k} \times K$ with $\mathbf{w}_{n_k} = \mathbf{u}_{n_k} + \mathbf{v}_{n_k}$. Since $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of compact subsets of $\mathbb{R}^d$ its Painlevé-Kuratowski limit C is closed, bounded and therefore compact. However, on compact metric spaces, Painlevé-Kuratowski convergence of a sequence of compact sets implies convergence in the Hausdorff metric. Consequently, the sequence $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ is bounded. Moreover, since $\{\mathbf{w}_{n_k}\}_{k \in \mathbb{N}}$ is a convergent sequence it is also bounded and it follows that the sequence $\{\mathbf{v}_{n_k}\}_{k \in \mathbb{N}}$ is bounded. Therefore one can extract from the sequence $\{\mathbf{u}_{n_k}, \mathbf{v}_{n_k}\}_{n \in \mathbb{N}}$ a subsequence $\{\mathbf{u}_{n_{k_l}}, \mathbf{v}_{nk_l}\}_{l \in \mathbb{N}}$ which converges to some $(\mathbf{u}_\star, \mathbf{v}_\star) \in \mathbb{R}^d \times \mathbb{R}^d$. By hypothesis $\mathbf{u}_\star \in C$ and since K is closed $\mathbf{v}_\star \in K$. Moreover, $\lim_{l \longrightarrow \infty} \mathbf{u}_{n_{k_l}} + \mathbf{v}_{n_{k_l}} = \mathbf{u}_\star + \mathbf{v}_\star = \mathbf{w}$. Hence, $\mathbf{w} \in C + K$ which proves the first inclusion. Conversely, if $\mathbf{w} \in C + K$, there exists $(\mathbf{u}, \mathbf{v}) \in C \times K$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Moreover, there exists a sequence $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ with $u_n \in C_n$ for all n and such that $\lim_{n \longrightarrow \infty} \mathbf{u}_n = \mathbf{u}$. Hence $\mathbf{w} = \lim_{n \longrightarrow \infty}(\mathbf{u}_n + \mathbf{v})$ and it follows that $\mathbf{w} \in Li_{n \longrightarrow \infty}(C_n + K)$. Hence, we deduce that

$$Ls_{n \longrightarrow \infty}(C_n + K) \subset C + K \subset Li_{n \longrightarrow \infty}(C_n + K).$$

Consequently, since $Li_{n \longrightarrow \infty}(C_n + K) \subset Ls_{n \longrightarrow \infty}(C_n + K)$, we deduce that:

$$Ls_{n \longrightarrow \infty}(C_n + K) = Li_{n \longrightarrow \infty}(C_n + K) = Lim_{n \longrightarrow \infty}(C_n + K) = C + K. \quad \square$$

Given a subset C of $\mathbb{R}^d$, $Hh(C)$ denotes the set homogeneously spanned from C, equivalently $Hh(C) = \{\lambda \mathbf{v} : \mathbf{v} \in C, \lambda \in \mathbb{R}\}$.

Proposition 4.2.2 *Let $\{C_n\}_{n \in \mathbb{N}}$, be a sequence of compact sets of points of $\mathbb{R}^d$ which converges, in the Painlevé-Kuratowski sense, to a set C of $\mathbb{R}^d$, that is $Lim_{n \longrightarrow \infty} C_n = C$. Then, we have:*

$$Lim_{n \longrightarrow \infty} Hh(C_n) = Hh(C).$$

Proof: Suppose that $\mathbf{w} \in Ls_{n \longrightarrow \infty} Hh(C_n)$. We first prove that $\mathbf{w} \in Hh(C)$. By hypothesis there is a sequence $\{\mathbf{w}_{n_k}\}_{k \in \mathbb{N}}$ such that $\mathbf{w}_{n_k} \in Hh(C_{n_k})$ for all natural numbers k and $\lim_{k \longrightarrow \infty} \mathbf{w}_{n_k} = \mathbf{w}$. By definition for all k there exists $(\mathbf{u}_{n_k}, \lambda_{n_k}) \in C_{n_k} \times \mathbb{R}$ with $\mathbf{w}_{n_k} = \lambda_{n_k} \mathbf{u}_{n_k}$. Since $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of compact subsets of $\mathbb{R}^d$ its Painlevé-Kuratowski limit C is closed, bounded and thereby compact. However, on compact metric spaces, Painlevé-Kuratowski convergence of a sequence of compact sets implies convergence in the Hausdorff metric. Consequently, the sequence $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ is bounded. Moreover, since $\{\mathbf{w}_{n_k}\}_{k \in \mathbb{N}}$ is a convergent sequence it is also bounded and it follows that the real sequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ is bounded. Therefore one can extract from the sequence $\{\mathbf{u}_{n_k}, \lambda_{n_k}\}_{n \in \mathbb{N}}$ a subsequence $\{\mathbf{u}_{n_{k_l}}, \lambda_{nk_l}\}_{l \in \mathbb{N}}$ which converges to some $(\mathbf{u}_\star, \lambda_\star) \in \mathbb{R}^d \times \mathbb{R}$. By hypothesis $\mathbf{u}_\star \in C$ and $\lambda_\star \in \mathbb{R}$. Moreover, $\lim_{l \longrightarrow \infty} \lambda_{n_{k_l}} \mathbf{u}_{n_{k_l}} = \lambda_\star \mathbf{u}_\star = \mathbf{w}$. Hence, $\mathbf{w} \in Hh(C)$ which proves the first inclusion. Conversely, if $\mathbf{w} \in Hh(C)$, there exists $(\mathbf{u}, \lambda) \in C \times \mathbb{R}$ such that $\mathbf{w} = \lambda \mathbf{u}$. Moreover, there exists a sequence $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ with $u_n \in C_n$ for all n and such that $\lim_{n \longrightarrow \infty} \mathbf{u}_n = \mathbf{u}$. Hence $\mathbf{w} = \lim_{n \longrightarrow \infty} \lambda \mathbf{u}_n$ and it follows that $\mathbf{w} \in Li_{n \longrightarrow \infty} Hh(C_n)$. Hence, we deduce that

$$Ls_{n \longrightarrow \infty} Hh(C_n) \subset Hh(C) \subset Li_{n \longrightarrow \infty} Hh(C_n).$$

Consequently, since $Li_{n \longrightarrow \infty} Hh(C_n) \subset Ls_{n \longrightarrow \infty} Hh(C_n)$, we have

$$Ls_{n \longrightarrow \infty} Hh(C_n) = Li_{n \longrightarrow \infty} Hh(C_n) = Lim_{n \longrightarrow \infty} Hh(C_n) = Hh(C). \quad \square$$

Proposition 4.2.3 *Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of compact sets of points of $\mathbb{R}^d_+$ which converges, in the Painlevé-Kuratowski sense, to a set C of $\mathbb{R}^d_+$, that is $Lim_{n\longrightarrow\infty}C_n = C$. Let $K = \mathbb{R}^{d_1}_+ \times (-\mathbb{R}^{d_2}_+)$ with $d_1 + d_2 = d$. Then we have:*

$$Lim_{n\longrightarrow\infty}\left[(C_n + K)\cap\mathbb{R}^d_+\right] = (C+K)\cap\mathbb{R}^d_+.$$

Proof: For the sake of simplicity, let us denote $D_n = C_n + K$ for all n and $D = C + K$. From Proposition 4.2.1, $Lim_{n\longrightarrow\infty}D_n = D$. Suppose that $\mathbf{w} \in Ls_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+)$. We first prove that $\mathbf{w} \in D\cap\mathbb{R}^d_+$. By hypothesis there is a sequence $\{\mathbf{w}_{n_k}\}_{k\in\mathbb{N}}$ such that $\mathbf{w}_{n_k} \in D_{n_k}\cap\mathbb{R}^d_+$ for all natural numbers k and $\lim_{k\longrightarrow\infty}\mathbf{w}_{n_k} = \mathbf{w}$. Since $\mathbf{w}_{n_k} \in D_{n_k}\cap\mathbb{R}^d_+$, $\mathbf{w}_{n_k} \in D_{n_k}$. It follows that $\mathbf{w} = \lim_{k\longrightarrow\infty}\mathbf{w}_{n_k} \in Ls_{n\longrightarrow\infty}D_n$. Moreover, since $\mathbb{R}^d_+$ is closed then $\mathbf{w} \in \mathbb{R}^d_+$. Hence, $\mathbf{w} \in D\cap\mathbb{R}^d_+$, which proves the first inclusion. Suppose now that $\mathbf{w} \in D\cap\mathbb{R}^d_+$. By definition, for all n there exists $\mathbf{w}_n \in D_n$ such that $\lim_{n\longrightarrow\infty}\mathbf{w}_n = \mathbf{w}$. For all n, $D_n\cap\mathbb{R}^d_+ \neq \emptyset$. Since $\mathbb{R}^d_+$ is closed and C_n is compact, it follows that $D_n\cap\mathbb{R}^d_+$ is a nonempty closed subset of $\mathbb{R}^d$. Consequently, for all natural numbers n, there exists some $\bar{\mathbf{w}}_n \in D_n\cap\mathbb{R}^d_+$ such that:

$$\|\mathbf{w}_n - \bar{\mathbf{w}}_n\| = \min_{\mathbf{v}\in D_n\cap\mathbb{R}^d_+}\|\mathbf{w}_n - \mathbf{v}\|,$$

where $\|\cdot\|$ is the Euclidean norm. It is easy to show that, since $\mathbf{w}_n \in D_n = C_n + K$, we have for all n and all $i \in [d]$:

$$\bar{w}_{ni} = \begin{cases} w_{ni} & \text{if} \quad 1 \le i \le d_1 \\ \max\{0, w_{ni}\} & \text{if} \quad d_1 + 1 \le i \le d_1 + d_2 \end{cases}$$

Since $\mathbf{w} \in \mathbb{R}^d_+$, we have $\|\mathbf{w} - \bar{\mathbf{w}}_n\| \le \|\mathbf{w} - \mathbf{w}_n\|$. It follows that $\lim_{n\longrightarrow\infty}\bar{\mathbf{w}}_n = \mathbf{w}$. Hence, $\mathbf{w} \in Li_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+)$. Therefore, we deduce that:

$$Ls_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+) \subset D\cap\mathbb{R}^d_+ \subset Li_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+).$$

Consequently, since $Li_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+) \subset Ls_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+)$, we have

$$Ls_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+) = Li_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+) = Lim_{n\longrightarrow\infty}(D_n\cap\mathbb{R}^d_+) = D\cap\mathbb{R}^d_+. \quad \square$$

Proposition 4.2.4 *Let $\{r_s\}_{s\in\mathbb{N}}$ be an increasing sequence on real numbers. Let $\{T^{(r_s)}\}_{s\in\mathbb{N}}$ be a sequence of production sets of $\mathbb{R}^{m+n}_+$, closed and free disposable for all natural numbers k. If $T = Lim_{s\longrightarrow\infty}T^{(r_s)}$, then T is closed and satisfies a free disposal assumption.*

Proof: The Painlevé-Kuratowski limit of a sequence of sets is closed. Therefore T is also closed. Let $K = \mathbb{R}^m_+ \times (-\mathbb{R}^n_+)$ be the free disposal cone. K

is closed, moreover $\mathbb{R}_+^{m+n}$ is closed and convex. From Propositions 4.2.1 and 4.2.3 we have:

$$Lim_{s\longrightarrow\infty}\Big[(T^{(r_s)}+K)\cap\mathbb{R}_+^{m+n}\Big]=(T+K)\cap\mathbb{R}_+^{m+n}.$$

Since by hypothesis, $T^{(r_s)}$ is free disposable for all s, $(T^{(r_s)}+K)\cap\mathbb{R}_+^{m+n}=T^{(r_s)}$. It follows that

$$T=Lim_{s\longrightarrow\infty}T^{(r_s)}=Lim_{s\longrightarrow\infty}\Big[(T^{(r_s)}+K)\cap\mathbb{R}_+^{m+n}\Big]=(T+K)\cap\mathbb{R}_+^{m+n}.$$

Thus T is free disposable which ends the proof. □

Notice that if the condition T2, holds for a sequence of production sets, it may not be true for the limit set. Indeed, the Painlevé-Kuratowski limit of a sequence of bounded sets may not be bounded.

5 $\mathbb{B}$-convex Production Technologies and Painlevé-Kuratowski Limit of CES-CET Models

5.1 $\mathbb{B}$-convex Models

We come now to the introduction of $\mathbb{B}$-convexity which was defined by Briec and Horvath [12]. A subset $E\subset\mathbb{R}_+^d$ is said to be a $\mathbb{B}$-convex set, if for all $\mathbf{u},\mathbf{w}\in E$, and all $t\in[0,1]$ $\mathbf{u}\vee t\mathbf{w}\in E$. The basic properties of $\mathbb{B}$-convex sets are analyzed in Briec and Horvath [12]. From this definition a set C such that for all $\mathbf{u},\mathbf{w}\in C$ for all $s,t\geq 0$, $s\mathbf{u}\vee t\mathbf{w}\in C$ is called a $\mathbb{B}$-convex cone.

Along this line, a notion of $\mathbb{B}$-convex hull can be provided. Let $A=\{\mathbf{z}_1,\ldots,\mathbf{z}_\ell\}\subset\mathbb{R}_+^d$ then the set,

$$\mathbb{B}(A)=\Big\{\bigvee_{k\in[\ell]}t_k\mathbf{z}_k,\mathbf{t}\geq 0,\max_{k\in[\ell]}t_k=1\Big\},$$

is called the $\mathbb{B}$-convex hull of A. Paralleling this definition of $\mathbb{B}$-convexity, inverse $\mathbb{B}$-convexity (denoted by $\mathbb{B}^{-1}$-convexity) is obtained from usual convexity making the formal substitution $+\mapsto\min$. It is shown in the remainder of this section that $\mathbb{B}^{-1}$-convex sets can be derived from $\mathbb{B}$-convex sets *via* a suitable bijective function. This mean that these notions are identical making a lexical change based on the formal substitution max $\to$ min. Hence, all the results satisfy by $\mathbb{B}$-convex sets can be transposed to $\mathbb{B}^{-1}$-convex sets *via* a suitable bijective function, see *e.g.* Adilov and Yesilce [3] and Adilov and Rubinov [2] for $\mathbb{B}^{-1}$ maps and $\mathbb{B}$ maps, respectively. Let $E\subset(\mathbb{R}_{++}\cup\{+\infty\})^d$. E is $\mathbb{B}^{-1}$-convex, if for all $u,z\in E$ and for all $t\in[1,+\infty]$ we have $u\wedge tz\in E$.

Inverse $\mathbb{B}$-convex sets are isomorphically linked to $\mathbb{B}$-convex sets. To see that let $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_{++}\cup\{+\infty\}$ be the inverse map defined by $\varphi(\alpha)=\frac{1}{\alpha}$. A subset $E \subset \mathbb{R}^d_{++}$ is a $\mathbb{B}^{-1}$-convex set if, and only if, $L = \phi^{-1}(E)$ is a $\mathbb{B}$-convex set, where:

$$\phi(z_1, \dots, z_d) = (\varphi(z_1), \dots, \varphi(z_d)) .$$

In other words, a subset $E \subset (\mathbb{R}_{++} \cup \{+\infty\})^d$ is $\mathbb{B}^{-1}$-convex if and only if its inverse is $\mathbb{B}$-convex. Though the respective geometric representation of $\mathbb{B}$-convex sets and $\mathbb{B}^{-1}$-convex sets are different, they are both linked through a bijective function over $(\mathbb{R}_{++} \cup \{+\infty\})^d$. We then provide the following definition. For all $A = \{\mathbf{z}_1, \dots, \mathbf{z}_\ell\} \subset (\mathbb{R}_{++} \cup \{+\infty\})^d$, the set,

$$\mathbb{B}^{-1}(A) = \left\{ \bigwedge_{k \in [\ell]} s_k \mathbf{z}_k, \min_{k \in [\ell]} s_k = 1,\ \mathbf{s} \geq 0 \right\},$$

is called the inverse $\mathbb{B}$-convex hull of A.

Accordingly, one can expose the $\mathbb{B}$-convex non-parametric model introduced by Briec and Horvath [13]. We consider a collection $A = \big\{(\mathbf{x}_k, \mathbf{y}_k) : k \in [\ell]\big\}$ of ℓ observed firms. The subset of $\mathbb{R}^{m+n}_+$ defined by,

$$T_{\max}(A) = \big(\mathbb{B}(A) + K\big) \cap \mathbb{R}^{m+n}_+,$$

is called a $\mathbb{B}$-convex non-parametric estimation of the production technology. One can equivalently write:

$$T_{\max}(A) = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}_+ : \mathbf{x} \geq \bigvee_{k \in [\ell]} t_k \mathbf{x}_k, \mathbf{y} \leq \bigvee_{k \in [\ell]} t_k \mathbf{y}_k, \max_{k \in [\ell]} t_k = 1, \mathbf{t} \geq 0\Big\}.$$

Similarly, one can define a $\mathbb{B}^{-1}$-convex production model defined by Briec and Liang [14]. It is derived by analogy to the DEA model and the $\mathbb{B}$-convex structure proposed in the previous section. Let $A = \big\{(\mathbf{x}_k, \mathbf{y}_k) : k \in [\ell]\big\} \subset \mathbb{R}^{m+n}_+$ a collection of ℓ observed production vectors. The subset

$$T_{\min}(A) = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}_+ : \mathbf{x} \geq \bigwedge_{k \in [\ell]} s_k \, \mathbf{x}_k, \mathbf{y} \leq \bigwedge_{k \in [\ell]} s_k \, \mathbf{y}_k, \min_{k \in [\ell]} s_k = 1,\ \mathbf{s} \geq 0\Big\}$$

is called the $\mathbb{B}^{-1}$-convex non-parametric estimation of the production technology. Note that if $A \subset \mathbb{R}^{m+n}_{++}$ then its $\mathbb{B}^{-1}$-convex hull $\mathbb{B}^{-1}(A)$ is well defined and one can equivalently write:

$$T_{\min}(A) = \big(\mathbb{B}^{-1}(A) + K\big) \cap \mathbb{R}^{m+n}_+.$$

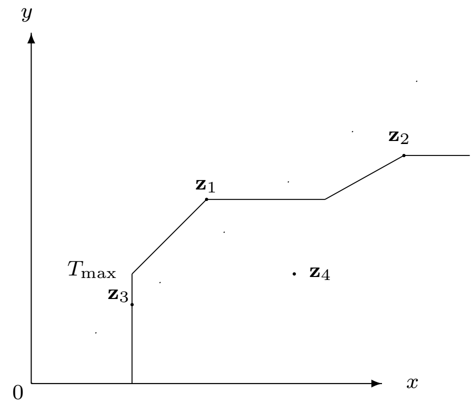


Figure 5.1 $\mathbb{B}$-convex estimation

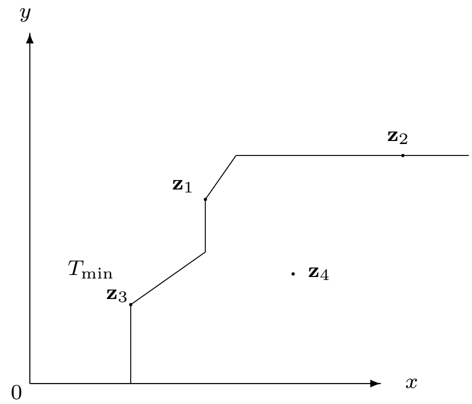


Figure 5.2 Inverse $\mathbb{B}$-convex estimation

5.2 Painlevé-Kuratowski Limit

From Briec and Horvath [12] and Proposition 4.1.1, if $Lim_{s\longrightarrow\infty}r_s = +\infty$, then:

$$\mathbb{B}(A) = Lim_{s\longrightarrow\infty}Co^{r_s}(A). \tag{5.1}$$

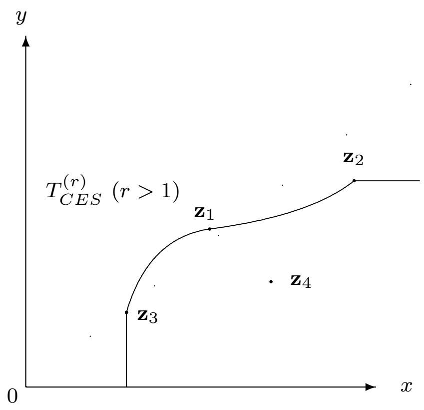
Moreover, from Proposition 4.1.2 and from Adilov and Yesilce [3], if $A \subset \mathbb{R}^d_{++}$, and if $\lim_{s\longrightarrow\infty} r_s = -\infty$, then:

$$\mathbb{B}^{-1}(A) = Lim_{s\longrightarrow\infty}Co^{r_s}(A). \tag{5.2}$$

Notice also that for all $r \in \mathbb{R}\backslash\{0\}$:

$$T^{(r)}_{CES}(A) = (Co^r(A) + K) \cap \mathbb{R}^{m+n}_+. \tag{5.3}$$

From Figures 4.1 and 4.2, it is possible to depict the geometric form of the production set with respect to r. An eyeball shows that when the parameter r respectively tends toward $+\infty$ and $-\infty$, then the geometric deformations yields figures 5.1 and 5.2.



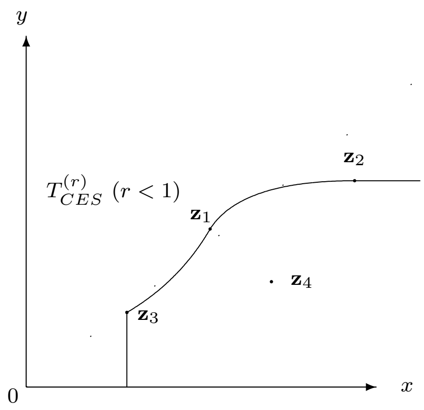


Figure 5.3 CES enveloppement in the case $r > 1$ **Figure 5.4** CES enveloppement in the case $r < 1$

The following result establishes that the Painlevé-Kuratowski limit of the CES-CET production technology is the $\mathbb{B}$-convex technology when the parameter r tends toward $+\infty$.

Proposition 5.2.1 *Let* $A = \{(\mathbf{x}_1, \mathbf{y}_1), \dots, (\mathbf{x}_\ell, \mathbf{y}_\ell)\}$ *be a finite number of production vectors of* $\mathbb{R}^{m+n}_+$*. Let* $T^{(r)}_{CES}(A)$ *be the CES piecewise estimation of the production technology with respect to* A*. Suppose that* $\{r_s\}_{s\in\mathbb{N}}$ *is an increasing sequence of real numbers such that* $\lim_{s\longrightarrow\infty} r_s = +\infty$*. Then:*

$$T_{\max}(A) = Lim_{s\longrightarrow\infty}T^{(r_s)}_{CES}(A).$$

Proof: From equation (5.3):

$$Lim_{s \longrightarrow \infty} T_{CES}^{(r_s)}(A) = Lim_{s \longrightarrow \infty}(Co^{r_s}(A) + K) \cap \mathbb{R}_+^{m+n}.$$

From equation (5.1) and Proposition 4.2.1:

$$Lim_{r \longrightarrow \infty}(Co^{r_s}(A) + K) = \mathbb{B}(A) + K.$$

Moreover for all r:

$$T_{CES}^{(r_s)}(A) = (Co^{r_s}(A) + K) \cap \mathbb{R}_+^{m+n}.$$

Hence, from Proposition 4.2.3:

$$Lim_{s \longrightarrow \infty} T_{CES}^{(r_s)}(A) = (\mathbb{B}(A) + K) \cap \mathbb{R}_+^{m+n} = T_{\max}(A). \ \Box$$

In the next statement, it is established that the Painlevé-Kuratowski limit of the CES-CET production technology is the $\mathbb{B}^{-1}$-convex technology when the parameter r tends toward $-\infty$. In addition, when the parameter r tends toward 0, it is the piecewise Cobb-Douglas model. Notice that one should assume that $A \subset \mathbb{R}_{++}^{m+n}$.

Proposition 5.2.2 *Let $A = \{(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_\ell, \mathbf{y}_\ell)\}$ be a finite number of production vectors of $\mathbb{R}_{++}^{m+n}$. Let $T_{CES}^{(r)}(A)$ be the CES piecewise estimation of the production technology with respect to A.*
(a) Suppose that $\{r_s\}_{s \in \mathbb{N}}$ is a decreasing sequence of real numbers such that $\lim_{s \longrightarrow \infty} r_s = -\infty$. *Then:*

$$T_{\min}(A) = Lim_{s \longrightarrow \infty} T_{CES}^{(r_s)}(A).$$

(b) Suppose that $\{r_s\}_{s \in \mathbb{N}}$ is a sequence of real numbers such that $\lim_{s \longrightarrow \infty} r_s = 0$. *Then:*

$$T_{CD}(A) = Lim_{s \longrightarrow \infty} T_{CES}^{(r_s)}(A).$$

Proof: (a) From equation (5.2) and Proposition 4.2.1:

$$Lim_{s \longrightarrow \infty}(Co^{r_s}(A) + K) = \mathbb{B}^{-1}(A) + K.$$

Moreover for all s:

$$T_{CES}^{(r_s)}(A) = (Co^{r_s}(A) + K) \cap \mathbb{R}_+^{m+n}.$$

Hence, from Proposition 4.2.3

$$Lim_{s \longrightarrow \infty} T_{CES}^{(r_s)}(A) = (\mathbb{B}^{-1}(A) + K) \cap \mathbb{R}_+^{m+n} = T_{\min}(A).$$

(b) From Lemma 4.1.3 and Proposition 4.2.1:

$$Ls_{s \longrightarrow \infty}(Co^{r_s}(A) + K) = Co^0(A) + K.$$

Hence, from Proposition 4.2.3

$$Lim_{s \longrightarrow \infty}(Co^{r_s}(A) + K) \cap \mathbb{R}_+^{m+n} = (Co^0(A) + K) \cap \mathbb{R}_+^{m+n} = T_{CD}(A). \ \Box$$

6 Limit of α-returns to scale Models

We investigate the modification of the Constant Elasticity of Substitution (CES)-Constant Elasticity of Transformation (CET) model of Färe *et al.* [19], extended by Boussemart *et al.* [10], by introducing the so-called α-returns to scale model. It consists in two parts: the output part is characterized by a Constant Elasticity of Transformation formula and the input part is characterized by a Constant Elasticity of Substitution formula. This model can be seen as a generalization of the traditional constant returns to scale linear models proposed by Charnes *et al.* [15]. As in the earlier section it will be shown that, it admits as a limiting case a variant of the multiplicative model proposed by Banker and Maindiratta [7], which is also discussed in the next subsection.

For the sake of simplicity, assume that $A = \{(\mathbf{x}_k, \mathbf{y}_k) : k \in [\ell]\} \subset \mathbb{R}^{m+n}_{++}$. Now, let us consider the following set:

$$T^{(q,r)}_{\text{alpha}}(A) = \Big\{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \geq \Phi_q^{-1}\big(\sum_{k\in[\ell]} t_k \Phi_q(\mathbf{x}_k)\big), \\ \mathbf{y} \leq \Phi_r^{-1}\big(\sum_{k\in[\ell]} t_k \Phi_r(\mathbf{y}_k)\big), \mathbf{t} \geq 0\Big\}, \tag{6.1}$$

where $qr > 0$. This production model slightly extends the one proposed by Boussemart *et al.* [10] because it allows negative power means. However, it is assumed that r and q have the same sign. Notice that, compared with the CES-CET model, the variable returns to scale constraint $\sum_{k\in[\ell]} t_k = 1$ is dropped.

In the following, we consider the notion of α-returns to scale proposed by Boussemart *et al.* [10]. We say that a technology T satisfies α-returns to scale if for all $\lambda > 0$:

$$(\mathbf{x}, \mathbf{y}) \in T \quad \Longrightarrow \quad (\lambda \mathbf{x}, \lambda^\alpha \mathbf{y}) \in T.$$

Assuming that r and q may be jointly negative yields the following result.

Proposition 6.0.3 *Let $A = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{k\in[\ell]} \subset \mathbb{R}^{m+n}_{++}$ be a set of ℓ observed production vectors. Suppose that $qr > 0$, the production technology $T^{(q,r)}_{\text{alpha}}(A)$ defined in (6.1) satisfies α-returns to scale with $\alpha = q/r$.*

The proof is identical to the one given in Boussemart *et al.* [10]. Note that the CES-CET model as defined by Färe *et al.* [19] does not satisfy q/r-returns to scale because of the constraint $\sum_{k=1}^{\ell} t_k = 1$.

6.1 The Constant Returns to Scale Case

For all subsets E of $\mathbb{R}^{m+n}_+$, let us denote $Hh_+(E) = \{t\mathbf{u} : \mathbf{u} \in E, t \geq 0\}$, that is the conical hull of E. If $r = q$, then by construction, we have:

$$T^{(r,r)}_{\text{alpha}}(A) = \big(Hh_+(Co^r(A)) + K\big) \cap \mathbb{R}^{m+n}_+.$$

Briec and Horvath [13] propose a CRS $\mathbb{B}$-convex model defined as follows:

$$T^c_{\max} = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{m+n}_+ : \mathbf{x} \geq \bigvee_{k\in[\ell]} t_k\mathbf{x}_k, \mathbf{y} \leq \bigvee_{k\in[\ell]} t_k\mathbf{y}_k, \mathbf{t} \geq 0\Big\}.$$

Similarly, a $\mathbb{B}^{-1}$-convex production model may be defined following Briec and Liang [14]. It is constructed by analogy to the DEA model and the $\mathbb{B}$-convex structure proposed in the previous section. The subset,

$$T^c_{\min}(A) = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{m+n}_+ : \mathbf{x} \geq \bigwedge_{k\in[\ell]} s_k\,\mathbf{x}_k, \mathbf{y} \leq \bigwedge_{k\in[\ell]} s_k\,\mathbf{y}_k, \mathbf{s} \geq 0\Big\},$$

is called the CRS $\mathbb{B}^{-1}$-convex non-parametric estimation of the production technology. It is obtained by dropping the constraint $\min_k s_k = 1$ from the initial model.

We have by construction:

$$T^c_{\max}(A) = \big(Hh_+(Co^{\infty}(A)) + K\big) \cap \mathbb{R}^{m+n}_+,$$

and

$$T^c_{\min}(A) = \big(Hh_+(Co^{-\infty}(A)) + K\big) \cap \mathbb{R}^{m+n}_+.$$

Proposition 6.1.1 *Let* $A = \{(\mathbf{x}_1,\mathbf{y}_1),\ldots,(\mathbf{x}_\ell,\mathbf{y}_\ell)\}$ *be a finite number of production vectors of* $\mathbb{R}^{m+n}_+$*. Let* $T^{(r)}_{CES}(A)$ *be the CES piecewise estimation of the production technology with respect to* A*.*
(a) *Suppose that* $\{r_s\}_{s\in\mathbb{N}}$ *is an increasing sequence of real numbers such that* $\lim_{s\longrightarrow\infty} r_s = \infty$*. Then:*

$$T^c_{\max}(A) = Lim_{s\longrightarrow\infty} T^{(r_s,r_s)}_{\text{alpha}}(A).$$

(b) *Suppose that* $A \subset \mathbb{R}^{m+n}_{++}$ *and* $\{r_s\}_{s\in\mathbb{N}}$ *is a decreasing sequence of real numbers such that* $\lim_{s\longrightarrow\infty} r_s = -\infty$*. Then:*

$$T^c_{\min}(A) = Lim_{s\longrightarrow\infty} T^{(r_s,r_s)}_{\text{alpha}}(A).$$

Proof: (a) From Proposition 4.2.4:

$$Lim_{s\longrightarrow\infty} Hh_+(Co^{r_s}(A)) = Hh_+(\mathbb{B}(A)).$$

Hence, from Proposition 4.2.1:

$$Lim_{s\longrightarrow\infty} Hh_+\big((Co^{r_s}(A)) + K\big) = Hh_+(\mathbb{B}(A)) + K.$$

Proposition 4.2.3 yields:

$$Lim_{s\longrightarrow\infty} Hh_+\big((Co^{r_s}(A)) + K\big) \cap \mathbb{R}^{m+n}_+ = (Hh_+(\mathbb{B}(A)) + K) \cap \mathbb{R}^{m+n}_+ = T_{\max}(A).$$

(b) The proof is similar. $\square$

Remark that since we consider a situation where $q_s = r_s$ for all s, then $\alpha = 1$ for the corresponding CES-CET production model. This is the reason why we retrieve the constant returns to scale assumption of the $\mathbb{B}$-convex models.

6.2 α-returns to Scale Case

We first notice that if $q = \alpha r$, then $\Phi_q = \Phi_\alpha \Phi_r$. Hence, the constraint

$$\mathbf{x} \geq \Phi_q^{-1}\big(\sum_{k \in [\ell]} t_k \Phi_q(\mathbf{x}_k)\big)$$

can be rewritten:

$$\mathbf{x}^\alpha \geq \Phi_r^{-1}\big(\sum_{k \in [\ell]} t_k \Phi_r(\mathbf{x}_k^\alpha)\big).$$

Let $\Psi_\alpha : \mathbb{R}_+^m \times \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^m \times \mathbb{R}_+^n$ be the map defined by $\Psi_\alpha(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^\alpha, \mathbf{y})$. This map is an homeomorphism (a continuous bijective map whose reciprocal is also continuous) and its reciprocal is ${\Psi_\alpha}^{-1}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^{\frac{1}{\alpha}}, \mathbf{y})$. It follows that:

$$T_{\text{alpha}}^{(q,r)}(A) = \{(\mathbf{x}, \mathbf{y}) : \Psi_\alpha(\mathbf{x}, \mathbf{y}) \in T_{\text{alpha}}^{(\alpha r, r)}(A)\}.$$

Hence:

$$T_{\text{alpha}}^{(\alpha r, r)}(A) = {\Psi_\alpha}^{-1}\Big(T_{\text{alpha}}^{(q,r)}(A)\Big).$$

Equivalently:

$$T_{\text{alpha}}^{(\alpha r, r)}(A) = {\Psi_\alpha}^{-1}\Big(\big(Co^r(\Psi_\alpha(A)) + K\big) \cap \mathbb{R}_+^{m+n}\Big).$$

Let us consider the two following models:

$$T_{\max}^{\alpha}(A) = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{m+n} : \mathbf{x} \geq \bigvee_{k \in [\ell]} {t_k}^{\frac{1}{\alpha}} \mathbf{x}_k, \mathbf{y} \leq \bigvee_{k \in [\ell]} t_k \, \mathbf{y}_k, \mathbf{t} \geq 0\Big\}$$

and

$$T_{\min}^{\alpha}(A) = \Big\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{m+n} : \mathbf{x} \geq \bigwedge_{k \in [\ell]} s_k^{\frac{1}{\alpha}} \, \mathbf{x}_k, \mathbf{y} \leq \bigwedge_{k \in [\ell]} s_k \, \mathbf{y}_k, \mathbf{s} \geq 0\Big\}.$$

An exercise of calculus shows that:

$$T_{\max}^{\alpha}(A) = \Big({\Psi_\alpha}^{-1}\big(\mathbb{B}(\Psi_\alpha(A))\big) + K\Big) \cap \mathbb{R}_+^{m+n}$$

and

$$T_{\min}^{\alpha}(A) = \Big({\Psi_\alpha}^{-1}\big(\mathbb{B}^{-1}(\Psi_\alpha(A))\big) + K\Big) \cap \mathbb{R}_+^{m+n}.$$

Suppose now that $\{r_s\}_{s \in \mathbb{N}}$ is an increasing sequence of real numbers such that $\lim_{s \longrightarrow \infty} r_s = \infty$, then

$$Lim_{s \longrightarrow \infty} Co^{r_s}(\Psi_\alpha(A)) = \mathbb{B}(\Psi_\alpha(A)).$$

If $\{r_s\}_{s \in \mathbb{N}}$ is a decreasing sequence of real numbers such that $\lim_{s \longrightarrow -\infty} r_s = \infty$, then

$$Lim_{s \longrightarrow \infty} Co^{r_s}(\Psi_\alpha(A)) = \mathbb{B}^{-1}(\Psi_\alpha(A)).$$

We can then deduce the following result.

Proposition 6.2.1 *Let* $A = \{(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_\ell, \mathbf{y}_\ell)\}$ *be a finite number of production vectors of* $\mathbb{R}_+^{m+n}$. *For all* $r > 0$, *let* $T_{\text{alpha}}^{(\alpha r, r)}(A)$ *be a piecewise estimation of the production technology with respect to* A *satisfying an assumption of* α*-returns to scale.*
(a) *Suppose that* $A \subset \mathbb{R}_{++}^{m+n}$ *and that* $\{r_s\}_{s\in\mathbb{N}}$ *is an increasing sequence of real numbers such that* $\lim_{s\longrightarrow\infty} r_s = \infty$. *Then:*

$$T_{\max}^{\alpha}(A) = Lim_{s\longrightarrow\infty} T_{\text{alpha}}^{(\alpha r_s, r_s)}(A).$$

(b) *Suppose that* $A \subset \mathbb{R}_{++}^{m+n}$ *and that* $\{r_s\}_{s\in\mathbb{N}}$ *is a decreasing sequence of real numbers such that* $\lim_{s\longrightarrow\infty} r_s = -\infty$. *Then:*

$$T_{\min}^{\alpha}(A) = Lim_{s\longrightarrow\infty} T_{\text{alpha}}^{(\alpha r_s, r_s)}(A).$$

Proof: Let $\{C_{r_s}\}_{s\in\mathbb{N}}$ be a sequence of compact convex sets of $\mathbb{R}_+^{m+n}$ which converges in the Painlevé-Kuratowski sense to $C \subset \mathbb{R}_+^{m+n}$, then, since Ψ_α is an homeomorphism, it follows that $\{\Psi_\alpha(C_{r_s})\}_{s\in\mathbb{N}}$ converges in the Painlevé-Kuratowski sense to $\Psi_\alpha(C)$. Using Propositions 4.2.1 and 4.2.3, the proof of (a) and (b) follow. $\Box$

7 Conclusion

In this paper, we have provided a generalization of the traditional DEA models thanks to the seminal works of Avriel [4] and Ben-Tal [9].

The first generalization is based on the power mean (*i.e.* generalized mean) initiated by Hardy, Littlewood and Polya [21]. Non-parametric technologies as well as CES-CET Cobb-Douglas technologies are obtained from the generalized mean. Accordingly, linear programs related to those non-parametric models are derived in order to compute technical efficiency.

The second generalization is built on some limiting cases of convex hulls of isomorphisms due to Ben-Tal [9], the so-called $\mathbb{B}$-convex sets introduced by Briec and Horvath [13]. It is shown that α-returns to scale models (increasing or constant returns to scale) are particular cases of technologies inherent to semi-lattice structures being either $\mathbb{B}$-convex sets or inverse $\mathbb{B}$-convex sets.

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