Flag structures on real 3-manifolds

Elisha Falbel, Jose Veloso

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E. Falbel and J. M. Veloso

Abstract

We define flag structures on a real three manifold M as the choice of two complex lines on the complexified tangent space at each point of M. We suppose that the plane field defined by the complex lines is a contact plane and construct an adapted connection on an appropriate principal bundle. This includes path geometries and CR structures as special cases. We prove that the null curvature models are given by totally real submanifolds in the flag space $\mathbf{SL}(3, \mathbb{C})/B$, where B is the subgroup of upper triangular matrices. We also define a global invariant which is analogous to the Chern-Simons secondary class invariant for three manifolds with a Riemannian structure and to the Burns-Epstein invariant in the case of CR structures. It turns out to be constant on homotopy classes of totally real immersions in flag space.

1 Introduction

Path geometries and CR structures on real three manifolds were studied by Elie Cartan in a long series of papers (see [Ca, C] and [B] for a beautiful account of this work). Both geometries have models which are obtained through real forms of a complex group. More precisely, the group $\mathbf{SL}(3, \mathbb{C})$ acts by projective transformations on both points in $\mathbb{P}(\mathbb{C}^3)$ and its lines viewed as $\mathbb{P}(\mathbb{C}^{3*})$. The space of flags $F \subset \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\mathbb{C}^{3*})$ of lines containing points is described as the homegeneous space $\mathbf{SL}(3, \mathbb{C})/B$ where B is the subgroup of upper triangular matrices. The path geometry of the flag space is defined by the two projections onto points and lines in projective space. Indeed, the kernels of the differentials of each projection define two complex lines in the tangent space of the flag space at each point. It turns out that the planes generated in this way form a contact plane field.

The two real models appear as closed orbits of the two non-compact real forms $\mathbf{SL}(3, \mathbb{R})$ and $\mathbf{SU}(2, 1)$ in the space of flags. In this work we define a structure over a real manifold which interpolates between these two geometries. Namely, the structure is a choice of two complex lines in the complexified tangent space at each point. We call it a *flag structure*. Path geometry and CR geometry correspond to a particular choice of lines adapted to the real structure of the real forms. Flat structures modeled on $\mathbf{SU}(2, 1)$ are known as spherical CR structures and have been studied since Cartan. But it is not known to what extent a 3-manifold may be equipped with such a structure (see [BS1, S, DF] and their references). A flat path structure on a hyperbolic manifold was recently constructed in [FS] where it is called a flag structure (as in [Ba] where flat path structures on Seifert manifolds arise from representations of surface groups into $\mathbf{PSL}(3, \mathbb{R})$). The use of configurations of flags associated to triangulations of 3-manifolds to obtain information on representation spaces is another theme related to this work (see [BFG] and its references).

Non-flat flag structures are abundant. In fact, any 3-manifold has a real path structure and also a CR structure. We develop the equivalence problem for complex structures on a real three manifold in the first sections. We obtain an adapted connection in an appropriate bundle over the manifold which has structure group B. In particular we characterize null curvature structures as those which can be locally embedded in the flag space F and inherit the flag structure from the path structure of F. Real path structures and spherical CR structures of null curvature have a unique embedding up to translations by elements of $\mathbf{SL}(3,\mathbb{C})$.

We define, using the adapted connection, a secondary invariant in the case the bundle has a global section. The construction follows the same idea as in [BE] for the case of CR structures. Computing the first variation of the invariant, we obtain, in particular, that it is invariant under deformations which are obtained from deformations of totally real immersions into flag space.

Here is a more detailed account of each section. In section 2 we give the definition of flag structure and the classical examples of CR and real path geometry. Those two geometries have been studied for a long time and very readable accounts are in [J] for the CR case and [IL] for real path geometry. The basic examples of embedded totally real submanifolds in flag space are described in 2.2. Two families of homogeneous examples which are neither CR nor real path structures are described in the end of the section.

In section 3 we define a coframe bundle Y over a $\mathbb{C}^*$-bundle E over the real manifold adapted to the structure. The $\mathbb{C}^*$-bundle is the set of contact forms at each point. It is a complex line bundle with its zero section removed. The coframe bundle Y is then constructed over this bundle and the final descrition is given in Proposition 3. As a principal bundle over the real manifold, Y has structure group the group B of upper triangular matrices (up to a finite cover).

Section 4 is the technical core of the paper. We construct the connection forms and curvature forms which will be intepreted as a Cartan connection in the following section. The construction follows Cartan's technique (see [C, CM] and [J]) but it is engineered to include at the same time path geometries and CR structures. The definition of Cartan connection we give (Definition 5.1 in section 5) is slightly more general than usual. We don't impose that the tangent space of the fiber bundle be isomorphic to the Lie algebra. This allows more flexibility and we are able to prove Theorem 5.2 which puts together the construction in section 4 into a Lie algebra valued connection form. The characterization of null curvature structures is done in Theorem 5.3 which identifies them locally as totally real embeddings into flag space. Finally we prove a rigidity theorem (Theorem 5.4) which states that the only CR or real path structures which admit local embeddings into the flag space are the flat ones.

In section 6 we introduce our $\mathbb{R}$-valued global invariant for structures in the case the bundle Y is trivial. This follows Chern-Simons construction and coincides with Burns-Epstein invariant in the case of CR structures. We do not attempt here to extend the construction to more refined versions as in [CL] or [BHR] for the CR case. Also missing in this work is the application to second order differential equations which we plan to develop in a sequel of this paper. The first variation formula (Proposition 6.3) shows that the critical points of the invariant occur at zero curvature structures. In particular, using Theorem 5.3, we thus prove that totally real immersions are critical points with respect to the invariant. On the other hand, it follows from Gromov's h-principle techniques (Theorem 1.4 pg. 245 in [Fo]) that every real 3-manifold admits a totally real immersion in $\mathbb{C}^3$ and therefore into the flag space (for the number of isotopy classes of totally real immersions see [Bo]). One can obtain in this way a set of numbers associated to a compact 3-manifold corresponding to the values of the global invariant on the space of totally real immersions with trivial bundle Y up to homotopy.

Section 7 concerns a natural reduction of the flag structure (which we call pseudo-flag structure). Namely, when one choses a contact form, the structure group can be reduced and one can construct a bundle X with an adapted connection. In the case of CR structures this reduction (called pseudo-Hermitian structures) was throroughly analyzed in [W] and we obtain, analogously, a particular embedding $X \to Y$ which allows one to describe our global invariant in terms of easier local data of the reduced bundle (see Proposition 7.2).

In the last section we explicitly compute examples of homogeneous structures on $\mathbf{SU}(2)$. The curvatures for a family of them are non-zero and therefore, applying Theorem 5.3, they cannot be obtained as totally real manifolds in flag space. It would be interesting to understand if one can embed a flag structure in higher dimensional flag spaces.

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2 Flag structures in dimension 3

Le M be a real three dimensional manifold and $T^{\mathbb{C}} = TM \otimes \mathbb{C}$ be its complexified tangent bundle.

Definition 2.1 *A flag structure on M is a choice of two sub-bundles T^1 and T^2 in $T^{\mathbb{C}}$ such that $T^1 \cap T^2 = \{0\}$ and such that $T^1 \oplus T^2$ is a contact distribution.*

The condition that $T^1 \oplus T^2$ be a contact distribution means that, locally, there exists a one form $\theta \in T^*M \otimes \mathbb{C}$ such that $\ker\theta = T^1 \oplus T^2$ and $d\theta \wedge \theta$ is never zero.

This definition contains two special cases, namely,

- CR structures, which arise when $T^2 = \overline{T^1}$.
- Path geometries, which are defined when T^1 and T^2 are complexifications of real one dimensional sub-bundles of TM.

Path geometries are treated in detail in section 8.6 of [IL] and in [BGH] where the relation to second order differential equations is also explained. CR structures in three dimensional manifolds were studied by Cartan ([C], see also [J]) and his solution of the equivalence problem in dimension 3 was generalized to higher dimensions in [CM]. The goal in this paper is to use the same formalism for both geometries. They appear as real forms of a complex path geometry. Indeed, one can define a complex path geometry on complex manifolds of dimension three whose zero curvature model is the homogeneous flag space $\mathbf{SL}(3,\mathbb{C})/B$ (where B is the Borel subgroup of upper triangular matrices). The group

$\mathbf{SL}(3,\mathbb{C})$ acts on this space transitively and the real geometries associated to it correspond to the two closed orbits of the two non-compact real forms $\mathbf{SL}(3,\mathbb{R})$ and $\mathbf{SU}(2,1)$.

2.1 The flag space

The model cases of a flag structure arise when we consider certain real three dimensional submanifolds of the space F of complete flags (that is, lines and planes containing them) in $\mathbb{C}^3$. The group $\mathbf{SL}(3,\mathbb{C})$ acts on the space of flags with isotropy B, the Borel group of upper triangular matrices. We can describe therefore the space of flags as the homogeneous space $F = \mathbf{SL}(3,\mathbb{C})/B$. The space of flags is equipped with two projections. One projects the line of a flag into $\mathbb{P}(\mathbb{C}^3)$ on one hand and, on the other, the plane into $\mathbb{P}(\mathbb{C}^{3*})$ viewed as a kernel of a linear form.

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ \mathbb{P}(\mathbb{C}^3) & & \mathbb{P}(\mathbb{C}^{3*}) \end{array}$$

The two projections correspond to the projections into $\mathbb{P}(\mathbb{C}^3) = \mathbf{SL}(3,\mathbb{C})/P_1$ and $\mathbb{P}(\mathbb{C}^{3*}) = \mathbf{SL}(3,\mathbb{C})/P_2$ where P_1 and P_2 are two different parabolic subgroups which fix, respectively, the line and the plane of the flag fixed by the Borel subgroup..

The projections define a complex contact distribution on the tangent space of F generated by the tangent spaces to each of the fibers. One can embed F into $\mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\mathbb{C}^{3*})$ as the set of pairs $(z,l) \in \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\mathbb{C}^{3*})$ satisfying the incidence relation $l(z) = 0$.

Another description of the contact distribution can be given using explicitly the Lie algebra structure of $\mathbf{SL}(3,\mathbb{C})$. Indeed, the Lie algebra of $\mathbf{SL}(3,\mathbb{C})$ decomposes in the following direct sum of vector subspaces:

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}.$$

That is the graded decomposition of $\mathfrak{sl}(3,\mathbb{C})$ where $\mathfrak{b} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$ corresponds to upper triangular matrices. The tangent space of $\mathbf{SL}(3,\mathbb{C})/B$ at $[B]$ is identified to

$$\mathfrak{sl}(3,\mathbb{C})/\mathfrak{b} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}.$$

Moreover the choice of the reference flag whose isotropy is B defines a decomposition of $\mathfrak{g}^{-1} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, with $\mathfrak{g}^{-2} = [\mathfrak{t}_1, \mathfrak{t}_2]$, corresponding to the two parabolic subgroups with Lie algebras $\mathfrak{p}_1 = \mathfrak{t}_1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$ and $\mathfrak{p}_2 = \mathfrak{t}_2 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$.

One propagates the decomposition of the tangent space at $[B]$ to the whole flag space by the action of $\mathbf{SL}(3,\mathbb{C})$ to obtain the two field of complex lines $\mathfrak{T}^1$ and $\mathfrak{T}^2$.

2.2 Real submanifolds in flag space

In this section we assume M is a three dimensional real manifold. Let $\varphi : M \to \mathbf{SL}(3,\mathbb{C})/B$ be an embedding and $\varphi_* : TM \to T\mathbf{SL}(3,\mathbb{C})/B$ be its differential. One can extend this map to $\varphi_* : TM^{\mathbb{C}} \to T\mathbf{SL}(3,\mathbb{C})/B$ by $\varphi_*(iv) = J\varphi_*(v)$ where J is the complex structure on the tangent space of the complex manifold $\mathbf{SL}(3,\mathbb{C})/B$ (which is just multiplication by i in matrix coordinates).

Definition 2.2 *An embedding* $\varphi : M \to \mathbf{SL}(3,\mathbb{C})/B$ *is totally real if, for every* $p \in M$,

$$\varphi_* : T_pM^{\mathbb{C}} \to T_pSL(3,\mathbb{C})/B$$

is an isomorphism.

One can define then the spaces $T^1, T^2 \subset TM^{\mathbb{C}}$ which correspond to $\mathfrak{T}^1$ and $\mathfrak{T}^2$ on the flag space $\mathbf{SL}(3,\mathbb{C})/B$. This defines a flag structure on any real 3-manifold M with a totally real embedding into $\mathbf{SL}(3,\mathbb{C})/B$.

There are two fundamental examples. They are both described as the unique closed orbit in the flag space by the action of a non-compact real form of $\mathbf{SL}(3,\mathbb{C})$.

2.2.1 Spherical CR geometry and $\mathbf{SU}(2,1)$

Spherical CR geometry is modeled on the sphere $\mathbb{S}^3$ equipped with a natural $\mathbf{PU}(2,1)$ action. Consider the group $\mathrm{U}(2,1)$ preserving the Hermitian form $\langle z, w\rangle = w^* J z$ defined on $\mathbb{C}^3$ by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the following cones in $\mathbb{C}^3$:

$$V_0 = \left\{ z \in \mathbb{C}^3 - \{0\} \quad : \quad \langle z, z\rangle = 0 \ \right\},$$

$$V_- = \left\{ z \in \mathbb{C}^3 \quad : \quad \langle z, z\rangle < 0 \ \right\}.$$

Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}(\mathbb{C}^3)$ be the canonical projection. Then $\mathbb{H}^2_{\mathbb{C}} = \pi(V_-)$ is the complex hyperbolic space and its boundary is

$$\mathbb{S}^3 = \pi(V_0) = \{[x, y, z] \in \mathbb{CP}^2 \mid x\bar{z} + |y|^2 + z\bar{x} = 0 \ \}.$$

The group of biholomorphic transformations of $\mathbb{H}^2_{\mathbb{C}}$ is then $\mathbf{PU}(2,1)$, the projectivization of $\mathrm{U}(2,1)$. Observe that this group also acts on $\mathbb{S}^3$.

An element $x \in \mathbb{S}^3$ gives rise to a flag in F where the line corresponds to the unique complex line tangent to $\mathbb{S}^3$ at x. More explicitly, we let

$$\begin{array}{rccc} \varphi_{CR} : & \mathbb{S}^3 & \to & F \\ & x & \mapsto & (x, \langle \,.\, , x \,\rangle) \end{array}$$

2.2.2 Real path geometry and $\mathbf{SL}(3,\mathbb{R})$

Flat path geometry is the geometry of real flags in $\mathbb{R}^3$. That is the geometry of the space of all couples $[p, l]$ where $p \in \mathbb{R}P^2$ and l is a real projective line containing p. The space of flags is identified to the quotient

$$\mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}}$$

where $B_{\mathbb{R}}$ is the Borel group of all real upper triangular matrices. The inclusion

$$\varphi_{\mathbb{R}} : \mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}} \to \mathbf{SL}(3,\mathbb{C})/B$$

is clearly a totally real embedding.

2.2.3 Homogeneous immersions of $\mathbf{SU}(2)$

There exists two families:

1. Let $x_0 = [1, 0, 0] \in \mathbb{P}(\mathbb{C}^3)$ and define for a fixed $l \in \mathbb{P}(\mathbb{C}^{3*})$ such that $l(x_0) = 0$ the embedding. Define the family of embeddings:

$$\begin{aligned} \varphi_l \, : \, \mathbf{SU}(2) &\rightarrow F \\ g &\mapsto (gx_0, g^* l) \end{aligned}$$

   where $g \in \mathbf{SU}(2) \subset \mathbf{SU}(2, 1)$. Each choice of $l \in \mathbb{P}(\mathbb{C}^{3*})$ defines a closed orbit in the space of flags by the action of $\mathbf{SU}(2)$ and therefore this family has $\mathbb{P}(\mathbb{C}^2)$ as parameter space. It can be seen as a deformation of the spherical CR structure on the sphere to a family of $\mathbf{SU}(2)$ invariant flag structures. The CR embedding is obtained when $l = \langle \, . \, , x_0 \, \rangle$.

2. The other family arises from a deformation of the real flag space. Let

$$\begin{aligned} \varphi_l \, : \, \mathbb{SO}(3) &\rightarrow F \\ g &\mapsto (gx_0, g^* l) \end{aligned}$$

   where $g \in \mathbb{SO}(3) \subset \mathbf{SL}(3, \mathbb{R})$.

Again, each choice of $l \in \mathbb{P}(\mathbb{C}^{3*})$ defines a closed orbit in the space of flags by the action of $\mathbb{SO}(3)$ and therefore this family has $\mathbb{P}(\mathbb{C}^2)$ as parameter space. But there exists an isotropy. Namely, the orbit of x_0 is $\mathbb{P}(\mathbb{R}^3)$ and has isotropy $\mathbb{O}(2)$. The deformation space in this case is the quotient $\mathbb{O}(2) \setminus \mathbb{P}(\mathbb{C}^2)$ which is a segment. The structures on $\mathbf{SU}(2)$ are obtained considering the twofold cover of $\mathbb{SO}(3)$.

3 The $\mathbb{C}^*$-bundle of contact forms and an adapted coframe bundle

We start the construction of a canonical $\mathbb{C}^*$-bundle over a real three manifold equipped with a flag structure.

We consider the forms θ on $T^{\mathbb{C}}$ such that $\ker \theta = T^1 \oplus T^2$. Define E to be the $\mathbb{C}^*$-bundle of all such forms. This bundle is trivial if and only if there exists a globally defined non-vanishing form θ.

On E we define the tautological form ω. That is $\omega_\theta = \pi^*(\theta)$ where $\pi : E \rightarrow M$ is the natural projection.

Fixing a form θ we next define forms θ^1 and θ^2 on $T^{\mathbb{C}}$ satisfying

$$\theta^1(T^1) \neq 0 \;\text{ and }\; \theta^2(T^2) \neq 0.$$

$$\ker \theta^1 = T^2 \;\text{ and }\; \ker \theta^2 = T^1.$$

Fixing one choice, all others are given by $\theta'^i = a^i\theta^i + v^i\theta$.

We consider the tautological forms defined by the forms above over the line bundle E. That is, for each $\theta \in E$ we let $\omega^i_\theta = \pi^*(\theta^i)$. At each point $\theta \in E$ we have the family of forms defined over TE_θ

$$\omega' = \omega$$

$$\omega'^1 = a^1\omega^1 + v^1\omega$$

$$\omega'^2 = a^2\omega^2 + v^2\omega$$

We may, moreover, suppose that

$$d\theta = \theta^1 \wedge \theta^2 \ \text{ modulo } \theta$$

and therefore

$$d\omega = \omega^1 \wedge \omega^2 \ \text{ modulo } \omega.$$

This imposes that $a^1a^2 = 1$.

Those forms vanish on vertical vectors, that is, vectors in the kernel of the map $TE \to TM$. In order to define non-horizontal 1-forms we let θ be a section of E over M and introduce the coordinate $\lambda \in \mathbb{C}^*$ in E. By abuse of notation, let θ denote the tautological form on the section θ. Therefore the tautological form ω over E is

$$\omega_\lambda = \lambda\theta.$$

Differentiating this formula we obtain

$$d\omega = \omega \wedge \varphi + \omega^1 \wedge \omega^2 \tag{1}$$

where $\varphi = -\frac{d\lambda}{\lambda}$ modulo $\omega, \omega^1, \omega^2$.

Observe that $\frac{d\lambda}{\lambda}$ is a form intrinsically defined on E up to horizontal forms (the minus sign is just a matter of conventions).

For a different choice of forms satisfying the equation we write 1 as

$$\begin{aligned} d\omega &= \omega' \wedge \varphi' + \omega'^1 \wedge \omega'^2 = \omega \wedge \varphi' + (a^1\omega^1 + v^1\omega) \wedge (a^2\omega^2 + v^2\omega) \\ &= \omega \wedge (\varphi' - a^1v^2\omega^1 + a^2v^1\omega^2) + \omega^1 \wedge \omega^2 \end{aligned}$$

it follows $\varphi' = \varphi + a^1v^2\omega^1 - a^2v^1\omega^2 + s\omega$, with $s \in \mathbb{C}$.

We obtain in this way a coframe bundle over E:

$$\omega' = \omega$$

$$\omega'^1 = a^1\omega^1 + v^1\omega$$

$$\omega'^2 = a^2\omega^2 + v^2\omega$$

$$\varphi' = \varphi + a^1v^2\omega^1 - a^2v^1\omega^2 + s\omega$$

$v^1, v^2, s \in \mathbb{C}$ and $a^1, a^2 \in \mathbb{C}^*$ such that $a^1a^2 = 1$.

Definition 3.1 *We denote by Y the coframe bundle $Y \to E$ given by the set of 1-forms $\omega, \omega^1, \omega^2, \varphi$. Two coframes are related by*

$$(\omega', \omega'^1, \omega'^2, \varphi') = (\omega, \omega^1, \omega^2, \varphi) \begin{pmatrix} 1 & v^1 & v^2 & s \\ 0 & a^1 & 0 & a^1 v^2 \\ 0 & 0 & a^2 & -a^2 v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where and $s, v^1, v^2 \in \mathbb{C}$ and $a^1, a^2 \in \mathbb{C}^*$ satisfy $a^1 a^2 = 1$.

The bundle Y can also be fibered over the manifold M. In order to describe the bundle Y as a principal fiber bundle over M observe that choosing a local section θ of E and forms θ^1 and θ^2 on M such that $d\theta = \theta^1 \wedge \theta^2$ one can write a trivialization of the fiber

$$\begin{aligned} \omega &= \lambda\theta \\ \omega^1 &= a^1\theta^1 + v^1\lambda\theta \\ \omega^2 &= a^2\theta^2 + v^2\lambda\theta \\ \varphi &= -\frac{d\lambda}{\lambda} + a^1 v^2 \theta^1 - a^2 v^1 \theta^2 + s\theta, \end{aligned}$$

where $v^1, v^2, s \in \mathbb{C}$ and $a^1, a^2 \in \mathbb{C}^*$ such that $a^1 a^2 = \lambda$. Here the coframe $\omega, \omega^1, \omega^2, \varphi$ is seen as composed of tautological forms.

The group H acting on the right of this bundle is

$$H = \left\{ \begin{pmatrix} \lambda & v^1\lambda & v^2\lambda & s \\ 0 & a^1 & 0 & a^1 v^2 \\ 0 & 0 & a^2 & -a^2 v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } s, v^1, v^2 \in \mathbb{C} \text{ and } a^1, a^2 \in \mathbb{C}^* \text{ satisfy } a^1 a^2 = \lambda \right\}.$$

Consider the homomorphism from the Borel group $B \subset \mathbf{SL}(3, \mathbb{C})$ of upper triangular matrices into H

$$j : B \to H$$

given by

$$\begin{pmatrix} a & c & e \\ 0 & \frac{1}{ab} & d \\ 0 & 0 & b \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{a}{b} & -2a^2 d & \frac{c}{b} & \frac{4e}{b} - 2acd \\ 0 & a^2 b & 0 & abc \\ 0 & 0 & \frac{1}{ab^2} & \frac{2d}{b} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

One verifies that the homomorphism is surjective and its kernel is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ formed by diagonal matrices, so that H is isomorphic to the Borel group of projected upper triangular matrices in $\mathbf{PSL}(3, \mathbb{C})$.

Proposition 3.2 *The bundle $Y \to M$ is a principal bundle with structure group $B/\mathbb{C}^*$ where B is the Borel group of upper triangular matrices.*

4 Construction of connection forms on the bundle Y

The goal of this section is to obtain canonical forms defined on the coframe bundle $Y \to E$. They will give rise to a connection on Y as explained in the next section. The connection will be a form on Y with values in $\mathfrak{sl}(3,\mathbb{C})$ and will not be a Cartan connection as it is usually defined but a slight generalization of it.

A local section of the complexified coframe bundle over M may be given by three forms

$$\theta, \quad \theta^1, \quad \theta^2$$

satisfying $d\theta = \theta^1 \wedge \theta^2$, with $\ker \theta^1 = T^1$ and $\ker \theta^2 = T^2$. They give coordinates on the complexified cotangent bundle over E and, furthermore, we may describe next the tautological forms defined over that bundle in the previous section.

At the point $\lambda\theta \in E$, the coframes of Y are parametrized as follows:

$$\omega = \lambda\theta$$

$$\omega^i = a^i\theta^i + v^i\lambda\theta$$

with $a^1a^2 = \lambda$ and

$$d\omega = \omega^1 \wedge \omega^2 + \omega \wedge \varphi$$

where $\varphi = -\frac{d\lambda}{\lambda} \mod \omega^1, \omega^2, \omega$.

We consider now these forms as tautological forms on the bundle T^*Y. Differentiating ω^1, ω^2 we obtain

$$d\omega^1 = da^1 \wedge \theta^1 + dv^1 \wedge \lambda\theta + a^1 d\theta^1 + v^1 d\,\lambda\theta$$

$$d\omega^2 = da^2 \wedge \theta^2 + dv^2 \wedge \lambda\theta + a^1 d\theta^2 + v^2 d\,\lambda\theta.$$

Observing now that

$$\frac{d\lambda}{\lambda} = \frac{da^1}{a^1} + \frac{da^2}{a^2}$$

we can write (modulo $\omega^1, \omega^2, \omega$)

$$d\omega^1 \equiv \frac{d\lambda}{2\lambda} \wedge \omega^1 + \frac{1}{2}\left(\frac{da^1}{a^1} - \frac{da^2}{a^2}\right) \wedge \omega^1 + dv^1 \wedge \omega$$

$$d\omega^2 \equiv \frac{d\lambda}{2\lambda} \wedge \omega^2 - \frac{1}{2}\left(\frac{da^1}{a^1} - \frac{da^2}{a^2}\right) \wedge \omega^2 + dv^2 \wedge \omega$$

Now, distributing the missing terms in $\omega^1, \omega^2, \omega$ in the last two terms of $d\omega^1$ and $d\omega^2$ and anti-symmetrizing we obtain the following:

Lemma 4.1 *There exists linearly independent forms $\omega_1^1, \varphi^1, \varphi^2$ defined on T^*Y such that*

$$d\omega^1 = \frac{1}{2}\omega^1 \wedge \varphi + \omega^1 \wedge \omega_1^1 + \omega \wedge \varphi^1 \quad \textit{and} \quad d\omega^2 = \frac{1}{2}\omega^2 \wedge \varphi - \omega^2 \wedge \omega_1^1 + \omega \wedge \varphi^2 \qquad (2)$$

Let ${\omega'}_1^1$, φ'^1 and φ'^2 be other forms satisfying equations 2. Taking the difference we obtain for $i = 1, 2$

$$0 = \omega^1 \wedge (\omega_1^1 - \omega_1'^1) + \omega \wedge (\varphi^1 - \varphi'^1)$$

and

$$0 = -\omega^2 \wedge (\omega_1^1 - \omega_1'^1) + \omega \wedge (\varphi^2 - \varphi'^2)$$

Therefore

$$\omega_1^1 - \omega_1'^1 = a\omega$$
$$\varphi^1 - \varphi'^1 = a\omega^1 + b^1\omega$$
$$\varphi^2 - \varphi'^2 = -a\omega^2 + b^2\omega$$

.

Lemma 4.2 *There exists a 1-form ψ such that*

$$d\varphi = \omega^1 \wedge \varphi^2 - \omega^2 \wedge \varphi^1 + \omega \wedge \psi \tag{3}$$

Proof. Differentiating equation

$$d\omega = \omega^1 \wedge \omega^2 + \omega \wedge \varphi$$

and using equations 2 we obtain

$$\omega \wedge \left(d\varphi + \omega^2 \wedge \varphi^1 - \omega^1 \wedge \varphi^2\right) = 0$$

which implies that there exists a form ψ as claimed. □

If other forms ψ', φ'^1 and φ'^2 satisfy 3 then

$$0 = \omega^1 \wedge (\varphi^2 - \varphi'^2) - \omega^2 \wedge (\varphi^1 - \varphi'^1) + \omega \wedge (\psi - \psi')$$

and, therefore, using $\varphi^1 - \varphi'^1 = a\omega^1 + b^1\omega$ and $\varphi^2 - \varphi'^2 = -a\omega^2 + b^2\omega$ we obtain

$$\psi - \psi' = b^2\omega^1 - b^1\omega^2 + c\omega.$$

Our next goal is to fix ω_1^1. For that sake we differentiate equations 2. Equation $dd\omega^1 = 0$ gives

$$\omega^1 \wedge \left(-d\omega_1^1 + \frac{3}{2}\omega^2 \wedge \varphi^1\right) + \omega \wedge \left(-d\varphi^1 + \varphi^1 \wedge \omega_1^1 + \frac{1}{2}\omega^1 \wedge \psi - \frac{1}{2}\varphi^1 \wedge \varphi\right) = 0.$$

Analogously, equation $dd\omega^2 = 0$ gives

$$\omega^2 \wedge \left(d\omega_1^1 - \frac{3}{2}\omega^1 \wedge \varphi^2\right) + \omega \wedge \left(-d\varphi^2 - \varphi^2 \wedge \omega_1^1 + \frac{1}{2}\omega^2 \wedge \psi - \frac{1}{2}\varphi^2 \wedge \varphi\right) = 0.$$

Defining

$$\Omega_1^1 = d\omega_1^1 - \frac{3}{2}\omega^2 \wedge \varphi^1 - \frac{3}{2}\omega^1 \wedge \varphi^2$$

$$\Phi^1 = d\varphi^1 - \varphi^1 \wedge \omega_1^1 - \frac{1}{2}\omega^1 \wedge \psi - \frac{1}{2}\varphi \wedge \varphi^1$$

$$\Phi^2 = d\varphi^2 + \varphi^2 \wedge \omega_1^1 - \frac{1}{2}\omega^2 \wedge \psi - \frac{1}{2}\varphi \wedge \varphi^2$$

So the equations can be written

$$\omega^1 \wedge \Omega_1^1 + \omega \wedge \Phi^1 = 0$$

$$\omega^2 \wedge \Omega_1^1 - \omega \wedge \Phi^2 = 0$$

The first equation implies that $\Omega_1^1 = \omega \wedge \lambda_1^1 + \omega^1 \wedge \mu$, where λ_1^1 and μ are 1-forms so that μ has no terms in ω. From the second equation we obtain that $\mu = S_1\omega^1 + S_2\omega^2$. Substituting in the second we see that $S_1 = 0$. We write therefore

$$\Omega_1^1 = \omega \wedge \lambda_1^1 + S_2\omega^1 \wedge \omega^2.$$

Lemma 4.3 *There exists a unique form ω_1^1 such that $\Omega_1^1 = \omega \wedge \lambda_1^1$ (that is $S_2 = 0$).*

Proof. Computing the difference

$$\Omega_1^1 - \Omega_1'^1 = d(\omega_1^1 - \omega_1'^1) - \frac{3}{2}\omega^2 \wedge (\varphi^1 - \varphi'^1) - \frac{3}{2}\omega^1 \wedge (\varphi^2 - \varphi'^2)$$

and obtain

$$\Omega_1^1 - \Omega_1'^1 = 4a\,\omega^1 \wedge \omega^2 \quad \mod \omega.$$

One can fix therefore a so that Ω_1^1 satisfies the assertion of the lemma. □

4.0.1

As ω_1^1 is fixed we still have the following ambiguities:

$$\varphi^1 - \varphi'^1 = b^1\omega$$

$$\varphi^2 - \varphi'^2 = b^2\omega$$

$$\psi - \psi' = b^2\omega^1 - b^1\omega^2 + c\omega.$$

Lemma 4.4 *There exists unique forms φ^1 and φ^2 such that Ω_1^1 does not contain terms $\omega^i \wedge \omega$, $i = 1, 2$.*

Proof. From the definition of $\Omega_1^1 = d\omega_1^1 - \frac{3}{2}\omega^2 \wedge \varphi^1 - \frac{3}{2}\omega^1 \wedge \varphi^2$ we obtain that

$$\Omega_1^1 - \Omega_1'^1 = -\frac{3}{2}b^1\omega^2 \wedge \omega - \frac{3}{2}b^2\omega^1 \wedge \omega.$$

So we can choose b^1 and b^2 as claimed. □

Observe that we fixed the 1-forms ω_1^1, φ^1 and φ^2 so that

$$d\omega_1^1 - \frac{3}{2}\omega^2 \wedge \varphi^1 - \frac{3}{2}\omega^1 \wedge \varphi^2 = \omega \wedge \lambda_1^1 \tag{4}$$

with $\lambda_1^1 \equiv 0 \mod \varphi, \varphi^1, \varphi^2, \psi$.

4.0.2

It remains to fix ψ.

Lemma 4.5 *There exists a unique 1-form ψ such that Φ^1 does not contain a term $\omega^1 \wedge \omega$.*

Proof. We compute using the definition $\Phi^1 - \Phi'^1 = -\frac{1}{2}\omega^1 \wedge (\psi - \psi') = -\frac{1}{2}c\omega^1 \wedge \omega$ and we can choose a unique $c \in \mathbb{C}$ which proves the lemma. □

4.1 Curvature forms

Curvature forms appear as differentials of connection forms and were used implicitly in the previous paragraphs to fix the connection forms.

In this section we obtain properties of the curvature forms which will be used in the following sections. Substituting $\Omega_1^1 = \omega \wedge \lambda_1^1$ in equation $\omega^1 \wedge \Omega_1^1 + \omega \wedge \Phi^1 = 0$ we obtain that $\omega \wedge (-\omega^1 \wedge \lambda_1^1 + \Phi^1) = 0$ and therefore $\Phi^1 = \omega^1 \wedge \lambda_1^1 + \omega \wedge \nu^1$ for a 1-form ν^1. Observe that $\nu^1 \equiv 0 \mod \omega^2, \varphi, \varphi^1, \varphi^2, \psi$ in view of the last lemma. Analogously, one may write $\Phi^2 = -\omega^2 \wedge \lambda_1^1 + \omega \wedge \nu^2$.

4.1.1

Equation $d(d\varphi) = 0$ obtained differentiating 3 can be simplified to

$$0 = \omega \wedge \left(-d\psi + \varphi \wedge \psi + 2\varphi^1 \wedge \varphi^2 - \omega^1 \wedge \nu^2 + \omega^2 \wedge \nu^1\right).$$

It follows that there exists a 1-form ν such that

$$d\psi - 2\varphi^1 \wedge \varphi^2 - \varphi \wedge \psi + \omega^1 \wedge \nu^2 - \omega^2 \wedge \nu^1 = \nu \wedge \omega. \tag{5}$$

4.1.2

Equation $d(d\omega_1^1) = 0$ obtained differentiating 4 can be simplified to

$$0 = \omega^1 \wedge \omega^2 \wedge 4\lambda_1^1 + \omega \wedge \left(-d\lambda_1^1 + \frac{3}{2}\omega^1 \wedge \nu^2 + \frac{3}{2}\omega^2 \wedge \nu^1 + \varphi \wedge \lambda_1^1\right)$$

which implies, as λ_1^1 does not contain terms in ω that $\lambda_1^1 = 0$ and $\omega \wedge (\omega^1 \wedge \nu^2 + \omega^2 \wedge \nu^1) = 0$. As ν^1 does not have a term in ω^1 it follows from the last equation that ν^2 does not have a term in ω^2 and we conclude that $\nu^1 = Q^1\omega^2$ and $\nu^2 = Q^2\omega^1$ where we introduce functions Q^1 and Q^2.

We have obtained the following equations:

$$\Omega_1^1 := d\omega_1^1 - \frac{3}{2}\omega^2 \wedge \varphi^1 - \frac{3}{2}\omega^1 \wedge \varphi^2 = 0, \tag{6}$$

$$\Phi^1 := d\varphi^1 - \varphi^1 \wedge \omega_1^1 - \frac{1}{2}\omega^1 \wedge \psi - \frac{1}{2}\varphi \wedge \varphi^1 = Q^1\omega \wedge \omega^2, \tag{7}$$

$$\Phi^2 := d\varphi^2 + \varphi^2 \wedge \omega_1^1 - \frac{1}{2}\omega^2 \wedge \psi - \frac{1}{2}\varphi \wedge \varphi^2 = Q^2\omega \wedge \omega^1. \tag{8}$$

4.1.3

Equation $d(d\varphi^1) = 0$ obtained differentiating Φ^1 above is simplified to

$$0 = \frac{1}{2}\omega \wedge \omega^1 \wedge \nu - \omega \wedge \omega^2 \wedge \left(dQ^1 + 2Q^1\omega_1^1 - 2Q^1\varphi\right).$$

It implies that

$$\nu = U_1\omega^1 + U_2\omega^2$$

and

$$dQ^1 + 2Q^1\omega_1^1 - 2Q^1\varphi = S^1\omega - \frac{1}{2}U_2\omega^1 + T^1\omega^2,$$

where we introduced functions U_1, U_2, S^1 and T^1.

We obtain, substituting the expression for ν in equation 5, the following expression

$$\Psi := d\psi - 2\varphi^1 \wedge \varphi^2 - \varphi \wedge \psi + \omega^1 \wedge \nu^2 - \omega^2 \wedge \nu^1 = (U_1\omega^1 + U_2\omega^2) \wedge \omega. \tag{9}$$

Therefore, as $\nu^1 = Q^1\omega^1$ and $\nu^2 = Q^2\omega^2$ we have

$$\Psi := d\psi - 2\varphi^1 \wedge \varphi^2 - \varphi \wedge \psi = (U_1\omega^1 + U_2\omega^2) \wedge \omega. \tag{10}$$

4.1.4

Anagously, equation $d(d\varphi^2) = 0$ obtained differentiating Φ^2 above is simplified to

$$0 = \omega \wedge \omega^1 \wedge \left(dQ^2 - 2Q^2\omega_1^1 - 2Q^2\varphi + \frac{1}{2}U_1\omega^2\right).$$

It implies that

$$dQ^2 - 2Q^2\omega_1^1 - 2Q^2\varphi = S^2\omega - \frac{1}{2}U_1\omega^2 + T^2\omega^1,$$

where we introduced new functions S^2 and T^2.

4.1.5

Finally, equation $d(d\psi)$ obtained from 5 simplifies to

$$0 = \omega \wedge \omega^1 \left(dU_1 - \frac{5}{2}U_1\varphi - U_1\omega_1^1 + 2Q^2\varphi^1\right) + \omega \wedge \omega^2 \left(dU_2 - \frac{5}{2}U_2\varphi + U_2\omega_1^1 - 2Q^1\varphi^2\right)$$

which implies that

$$dU_1 - \frac{5}{2}U_1\varphi - U_1\omega_1^1 + 2Q^2\varphi^1 = A\omega + B\omega^1 + C\omega^2$$

and

$$dU_2 - \frac{5}{2}U_2\varphi + U_2\omega_1^1 - 2Q^1\varphi^2 = D\omega + C\omega^1 + E\omega^2.$$

4.2 The CR case

In order to make the construction of the bundle Y compatible with the CR bundle constructed in [C, CM] one verifies first that $\omega^2 = i\bar{\omega}^1$. We have indeed

$$d\omega = \omega \wedge \varphi + i\omega^1 \wedge \bar{\omega}^1.$$

The form ω can be taken to be real so φ is also real.

From equations 2 one has

$$\omega_1^1 + \bar{\omega}_1^1 = 0 \quad \varphi^2 = i\bar{\varphi}^1.$$

From equation 6 we obtain

$$d\omega_1^1 - \frac{3}{2}i\bar{\omega}^1 \wedge \varphi^1 - \frac{3}{2}\omega^1 \wedge \bar{\varphi}^1 = 0,$$

From equations 7,8

$$d\varphi^1 - \varphi^1 \wedge \omega_1^1 - \frac{1}{2}\omega^1 \wedge \psi - \frac{1}{2}\varphi \wedge \varphi^1 = Q^1 \omega \wedge \omega^2,$$

$$d\varphi^2 + \varphi^2 \wedge \omega_1^1 - \frac{1}{2}\omega^2 \wedge \psi - \frac{1}{2}\varphi \wedge \varphi^2 = Q^2 \omega \wedge \omega^1,$$

we have $\psi = \bar{\psi}$ and $Q^1 = \bar{Q}^2$. From equation 10 we obtain

$$d\psi - 2\varphi^1 \wedge \bar{\varphi}^1 - \varphi \wedge \psi - (U_1\omega^1 + U_2\bar{\omega}^1) \wedge \omega = 0.$$

From that equation we observe that $U_1 = -i\bar{U}_2$ (in the CR literature U_1 is denoted R_1).

5 The Cartan connection

We consider the bundle $Y \to M$ as a principal bundle with structure group $H \subset \mathbf{SL}(3, \mathbb{C})$, the Borel group of triangular matrices. Observe that, although M is a real manifold, each fiber is a complex space of dimension five which can be identified to the Borel subgroup H. The real dimension of Y is 13 which is $\dim_{\mathbb{R}} \mathbf{SL}(3, \mathbb{C}) - 3$. This dimension difference does not allow us to obtain a genuine Cartan connection but a slight generalization of the definition will be sufficient for our purposes.

Recall that $X^*(y) = \frac{d}{dt}_{t=0} ye^{tX}$ where e^{tX} is the one parameter group generated by X.

Definition 5.1 *A Cartan connection on Y is a 1-form $\pi : TY \to \mathfrak{sl}(3, \mathbb{C})$ satisfying:*
0. $\pi_p : T_pY \to \mathfrak{sl}(3, \mathbb{C})$ is injective.
1. If $X \in \mathfrak{h}$ and $X^* \in TY$ is the vertical vector field canonically associated to X then $\pi(X^*) = X$.
2. If $h \in H$ then $(R_h)^*\pi = Ad_{h^{-1}}\pi$

Note that contrary to the usual definition of Cartan connection we don't impose that $\pi_p : T_pY \to \mathfrak{sl}(3, \mathbb{C})$ be an isomorphism as the dimensions are different.

We can represent the structure equations 1, 2, 3, 4 and 5 as a matrix equation whose entries are differential forms. The forms are disposed in the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ as

$$\pi = \begin{pmatrix} -\frac{1}{2}\varphi - \frac{1}{3}\omega_1^1 & \varphi^2 & -\frac{1}{4}\psi \\ \omega^1 & \frac{2}{3}\omega_1^1 & \frac{1}{2}\varphi^1 \\ 2\omega & 2\omega^2 & \frac{1}{2}\varphi - \frac{1}{3}\omega_1^1 \end{pmatrix}$$

It is a simple verification to show that

$$d\pi + \pi \wedge \pi = \Pi \tag{11}$$

where

$$\Pi = \begin{pmatrix} 0 & -\Phi^2 & -\frac{1}{4}\Psi \\ 0 & 0 & \frac{1}{2}\Phi^1 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\Phi^1 = Q^1\omega \wedge \omega^2$, $\Phi^2 = Q^2\omega \wedge \omega^1$ and $\Psi = (U_1\omega^1 + U_2\omega^2) \wedge \omega$.

Theorem 5.2 *The form π is a Cartan connection on $Y \to M$.*

Proof. The action by H can be replaced by the action of the Borel group of upper triangular matrices as described in a previous section. The action on the right by an element $h \in H$ on a coframe $y \in Y$ is denoted by $R_h(y)$. As $y \in Y$ is a coframe of E, one may consider tautological lifts y^{taut} on Y defined by the elements of the coframe y. The action on Y lifts to an action on its tautological lifts as follows:

$$R_h{}^*(y^{taut}) = R_h(y)^{taut}.$$

We compute $Ad_{h^{-1}}\pi$ for an element

$$h = \begin{pmatrix} a & c & e \\ 0 & \frac{1}{ab} & d \\ 0 & 0 & b \end{pmatrix}$$

and verify that the tautological forms $\omega, \omega^1, \omega^2, \varphi$ on Y (which appear as certain components of the connection) change according to the right action above.

It remains to show that the other components change similarly. Now given $\omega, \omega^1, \omega^2, \varphi$ tautological forms on Y we defined unique forms $\omega_1^1, \varphi^1, \varphi^2, \psi$ such that the curvatures Φ^1, Φ^2 and Ψ had special properties.

We have

$$Ad_{h^{-1}}(d\pi - \pi \wedge \pi) = dAd_{h^{-1}}\pi - Ad_{h^{-1}}\pi \wedge Ad_{h^{-1}}\pi.$$

Writing

$$\tilde{\pi} = Ad_{h^{-1}}\pi,$$

we obtain $Ad_{h^{-1}}\Pi = \tilde{\Pi}$, where $\tilde{\Pi} = d\tilde{\pi} - \tilde{\pi} \wedge \tilde{\pi}$. A computation shows that

$$
\begin{aligned}
\tilde{\omega} &= \frac{a}{b}\,\omega \\
\tilde{\omega}^1 &= a^2 b\,\omega^1 - 2da^2\,\omega \\
\tilde{\omega}^2 &= \frac{1}{ab^2}\,\omega^2 + \frac{c}{b}\,\omega \\
\tilde{\varphi} &= \varphi + abc\,\omega^1 + 2\frac{d}{b}\,\omega^2 + (\frac{4e}{b} - 2dac)\,\omega \qquad (12)\\
\tilde{\omega}^1_1 &= \omega^1_1 + \frac{3}{2}abc\,\omega^1 - 3\frac{d}{b}\,\omega^2 - 3dac\,\omega \\
\tilde{\varphi}^1 &= b^2 a\,\varphi^1 + 2dab\,\omega^1_1 - bad\,\varphi + 2bae\,\omega^1 - 4d^2 a\,\omega^2 - 4dae\,\omega \\
\tilde{\varphi}^2 &= \frac{1}{ba^2}\,\varphi^2 + \frac{c}{a}\,\omega^1_1 + \frac{1}{2}ca\,\varphi + bc^2\,\omega^1 + (\frac{2e}{a^2b^2} - \frac{2cd}{ab})\,\omega^2 + (\frac{2ce}{ab} - 2dc^2)\,\omega \\
\tilde{\psi} &= \frac{b}{a}\,\psi + (\frac{4e}{a} - 2bc)\,\varphi + 4bce\,\omega^1 + (\frac{8de}{ab} - 8cd^2)\,\omega^2 + 2cb^2\,\varphi^1 + 4\frac{d}{a}\,\varphi^2 + 4dbc\,\omega^1_1 + (\frac{8e^2}{ab} - 8dce)\,\omega
\end{aligned}
$$

therefore it suffices to verify that the new curvature forms Φ'^1, Φ'^2 and Ψ' obtained from $Ad_{h^{-1}}\Pi$ verify the same properties. Indeed

$$
Ad_{h^{-1}}\Pi = \tilde{\Pi} = \begin{pmatrix} 0 & \frac{1}{a^2b}\Pi_{12} & \frac{b}{a}\Pi_{13} + \frac{d}{a}\Pi_{12} - cb^2\Pi_{23} \\ 0 & 0 & ab^2\Pi_{23} \\ 0 & 0 & 0 \end{pmatrix}
$$

and we see that the new curvatures satisfy the same properties. □

In particular, one obtains that

$$\tilde{\Phi}^1 = 2\tilde{\Pi}_{23} = 2ab^2\Pi_{23}\Pi_{23} = ab^2\Phi^1$$

and therefore

$$\tilde{Q}^1\tilde{\omega} \wedge \tilde{\omega}^2 = ab^2\,Q^1\omega \wedge \omega^2.$$

But $\tilde{Q}^1\tilde{\omega} \wedge \tilde{\omega}^2 = \tilde{Q}^1\frac{a}{b}\,\omega \wedge \frac{1}{ab^2}\omega^2$ and then

$$\tilde{Q}^1 = ab^5\,Q^1.$$

Analogously, from

$$\tilde{\Phi}^2 = -\tilde{\Pi}_{12} = -\frac{1}{a^2b}\Pi_{12}\Pi_{23} = \frac{1}{a^2b}\Phi^2$$

we obtain that

$$\tilde{Q}^2 = \frac{1}{a^5b}\,Q^2.$$

These transformation properties imply that we can define two tensors on Y which are invariant under H and will give rise to two functions on M. Indeed

$$Q^1\,\omega^2 \wedge \omega \otimes \omega \otimes e_1$$

and

$$Q^2\,\omega^1 \wedge \omega \otimes \omega \otimes e_2,$$

where e_1 and e_2 are duals to ω^1 and ω^2 in the dual frame of the coframe bundle of Y are easily seen to be H-invariant.

5.1 Null curvature models

A local characterization of null curvature path geometries is given in the following theorem. Recall definition 2.2 of a totally real embedding $\varphi : M \to \mathbf{SL}(3,\mathbb{C})/B$ and its associated flag structure which correspond to $\varphi_*(T^1) = \mathfrak{T}_1$ and $\varphi_*(T^2) = \mathfrak{T}_2$ on the flag space $\mathbf{SL}(3,\mathbb{C})/B$.

Theorem 5.3 *A totally real embedding $\varphi : N \to \mathbf{SL}(3,\mathbb{C})/B$ with induced flag structure on $TM^{\mathbb{C}}$ given by T^1 and T^2 as above is a contact path structure with adapted connection having null curvature. Conversely a contact path structure whose adapted connection has zero curvature is locally equivalent to a totally real embedding with induced path structure defined by T^1 and T^2 as above.*

Observe that null curvature does not define a unique flag structure on a real manifold but instead decides whether is can be embedded as a totally real submanifold in flag space.

Proof. The fact that N, equipped with the two sub-bundles T^1 and T^2 is a contact path structure with adapted connection having null curvature follows from the fact that the adapted principal bundle Y associated to N is identified to the restriction to N of the bundle $\mathbf{SL}(3,\mathbb{C}) \to \mathbf{SL}(3,\mathbb{C})/H$. The adapted connection on Y is then the Maurer-Cartan form of $\mathbf{SL}(3,\mathbb{C})$ restricted to this bundle and therefore has zero curvature.

Suppose now that M has a contact path structure defined by sub-bundles T^1 and T^2 in $TN \otimes \mathbb{C}$. Let $\pi : TY \to \mathfrak{sl}(3,\mathbb{C})$ be an adapted connection. Suppose that $d\pi + \pi \wedge \pi = \Pi = 0$ and let $\tilde{\omega}$ be the Maurer-Cartan form of the group $\mathbf{SL}(3,\mathbb{C})$. By Cartan's theorem (see theorem 1.6.10 in [IL]) every $y \in Y$ is contained in an open neighborhood $U \subset Y$ where an immersion $f : U \to \mathbf{SL}(3,\mathbb{C})$ is defined satisfying $\pi = f^*(\tilde{\omega})$. Moreover, any other immersion $\tilde{f}$ satisfying the same equation is related by a translation by an element $a \in \mathbf{SL}(3,\mathbb{C})$ in the group, that is, $\tilde{f} = af$.

If $X \in \mathfrak{h}$ then $X = \pi(X^*) = f^*(\tilde{\omega})(X^*) = \tilde{\omega}(f_*X^*)$. Therefore f_*X^* is tangent to the fibers of $\mathbf{SL}(3,\mathbb{C}) \to \mathbf{SL}(3,\mathbb{C})/H$. We conclude that $f : U \to \mathbf{SL}(3,\mathbb{C})$ projects to an immersion $\bar{f} : V \to \mathbf{SL}(3,\mathbb{C})/H$ where $V \subset M$. The subspaces $\bar{f}_*(T^1)$ and $\bar{f}_*(T^2)$ are precisely the subspaces $\mathfrak{T}_1$ and $\mathfrak{T}_2$ restricted to $\bar{f}(V)$.

□

The following theorem shows the rigidity of the real models in flag space. It remains the possibility that general CR structures or path structures might be deformed in higher dimensional flag spaces.

Theorem 5.4

- *Any local embedding of a CR structure into the flag space $\mathbf{SL}(3,\mathbb{C})/B$ coincides locally with $\varphi_{CR} : \mathbb{S}^3 \to \mathbf{SL}(3,\mathbb{C})/B$ up to a translation. In particular, only spherical CR structures can be embedded.*

- *Any local embedding of a path structure into the flag space $\mathbf{SL}(3,\mathbb{C})/B$ coincides locally with $\varphi_{\mathbb{R}} : F = \mathbf{SL}(3,\mathbb{R})/B_{\mathbb{R}} \to \mathbf{SL}(3,\mathbb{C})/B$ up to a translation. In particular, only flat path geometries can be embedded.*

Proof. By the previous theorem the only CR structures which can be embedded are the spherical ones. On the other hand, null curvature CR structures are known ([C]) to be locally equivalent to $\mathbb{S}^3$ equipped to its standard structure. By the theorem again, the null curvature structures admit embeddings which differ at most by a translation. The proof in the case of path geometry is similar. □

6 A global invariant

The second Chern class of the bundle Y with connection form π is given by

$$c_2(Y,\pi) = \frac{1}{8\pi^2}\mathrm{tr}\,(\Pi\wedge\Pi).$$

In the case of the connection form π we obtain

$$\begin{pmatrix} 0 & -\Phi^2 & -\frac{1}{4}\Psi \\ 0 & 0 & \frac{1}{2}\Phi^1 \\ 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\Phi^2 & -\frac{1}{4}\Psi \\ 0 & 0 & \frac{1}{2}\Phi^1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\Phi^1\wedge\Phi^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As $\Phi^1 = Q^1\omega\wedge\omega^2$ and $\Phi^2 = Q^2\omega\wedge\omega^1$ we have $\Pi\wedge\Pi = 0$ and

$$c_2(Y,\pi) = 0.$$

We include the proof of the next lemma although it is standard.

Lemma 6.1 *The form*

$$TC_2(\pi) = \frac{1}{8\pi^2}\left(\mathrm{tr}\,(\pi\wedge\Pi) + \frac{1}{3}\mathrm{tr}\,(\pi\wedge\pi\wedge\pi)\right) = \frac{1}{24\pi^2}\mathrm{tr}\,(\pi\wedge\pi\wedge\pi)$$

is closed.

Proof. Observe first that differentiating the curvature form we obtain $d\Pi = \Pi\wedge\pi - \pi\wedge\Pi$. Next we compute

$$d\,\mathrm{tr}\,(\Pi\wedge\pi) = \mathrm{tr}\,(d\Pi\wedge\pi + \Pi\wedge d\pi) = \mathrm{tr}\,((\Pi\wedge\pi - \pi\wedge\Pi)\wedge\pi + \Pi\wedge(\Pi - \pi\wedge\pi))$$
$$= -\mathrm{tr}\,(\pi\wedge\Pi\wedge\pi) = 0$$

because

$$\mathrm{tr}\,(\Pi\wedge\pi) = -\Phi^2\wedge\omega^1 + \Phi^1\wedge\omega^2 = 0.$$

Note that $\mathrm{tr}\,(\alpha\wedge\beta) = (-1)^{kl}\mathrm{tr}\,(\beta\wedge\alpha)$ if α and β are two matrices of forms of degree k and l respectively. Therefore, computing

$$\frac{1}{3}d\,\mathrm{tr}\,(\pi\wedge\pi\wedge\pi) = \mathrm{tr}\,(d\pi\wedge\pi\wedge\pi) = \mathrm{tr}\,((\Pi - \pi\wedge\pi)\wedge\pi\wedge\pi)$$
$$= -\mathrm{tr}\,(\pi\wedge\pi\wedge\pi\wedge\pi) = 0.$$

□

Remark that $0 = c_2(Y,\pi) = d\,TC_2(\pi)$.

Definition 6.1 *Suppose that the fiber bundle $Y \to M$ is trivial and let $s : M \to Y$ be a section, we define then*

$$\mu = \int_M s^* TC_2(\pi) = \frac{1}{24\pi^2} \int_M s^* \mathrm{tr}\,(\pi \wedge \pi \wedge \pi).$$

In principle that integral depends on the section but the following proposition shows that the integrand

$$s^* TC_2(\pi)$$

defines an element in the cohomology which does not depend on the section.

Proposition 6.2 *Suppose s and $\tilde{s}$ are two sections. Then*

$$\tilde{s}^* TC_2(\pi) - s^* TC_2(\pi) = -\frac{1}{8\pi^2} d\, s^* \mathrm{tr}\,(h^{-1}\pi \wedge d\,h).$$

Proof. Fix the section s. Than there exists a map $h : M \to H$ such that $\tilde{s} = R_h \circ s$. We have then

$$\tilde{s}^* TC_2(\pi) = \frac{1}{24\pi^2} s^* \mathrm{tr}\,(R_h^* \pi \wedge R_h^* \pi \wedge R_h^* \pi).$$

From the formula

$$R_h{}^* \pi = h^{-1} d\,h + Ad_{h^{-1}} \pi,$$

we obtain

$$\mathrm{tr}\,(R_h^* \pi \wedge R_h^* \pi \wedge R_h^* \pi) =$$

$$\mathrm{tr}\,\left(h^{-1} d\,h \wedge h^{-1} d\,h \wedge h^{-1} d\,h + 3h^{-1} d\,h \wedge h^{-1} \pi \wedge d\,h + 3h^{-1} \pi \wedge \pi \wedge d\,h + \pi \wedge \pi \wedge \pi\right)$$

$$= \mathrm{tr}\,\left(-h^{-1} d\,h \wedge d\,h^{-1} \wedge d\,h - 3d\,h^{-1} \wedge \pi \wedge d\,h + 3h^{-1} \pi \wedge \pi \wedge d\,h + \pi \wedge \pi \wedge \pi\right).$$

Lemma 6.2 $\mathrm{tr}\,(h^{-1} d\,h \wedge d\,h^{-1} \wedge d\,h) = 0.$

Proof. Observe that $d\,h^{-1} \wedge d\,h$ is upper triangular with null diagonal. Moreover $h^{-1} d\,h$ is upper triangular and therefore the Lie algebra valued form also has zero diagonal.

□

Lemma 6.3 $d\,\mathrm{tr}\,(h^{-1}\pi \wedge d\,h) = \mathrm{tr}\,(d\,h^{-1} \wedge \pi \wedge d\,h - h^{-1}\pi \wedge \pi \wedge d\,h).$

Proof. Compute $d\mathrm{tr}\,(h^{-1}\pi \wedge d\,h) = \mathrm{tr}\,(d\,h^{-1} \wedge \pi \wedge d\,h + h^{-1} d\pi \wedge d\,h)$

$$= \mathrm{tr}\,\left(d\,h^{-1} \wedge \pi \wedge d\,h + h^{-1}(\Pi - \pi \wedge \pi) \wedge d\,h\right)$$

$$= \mathrm{tr}\,\left(d\,h^{-1} \wedge \pi \wedge d\,h - h^{-1}\pi \wedge \pi \wedge d\,h\right)$$

because $\mathrm{tr}\,(h^{-1}\Pi \wedge d\,h) = 0$ as in the previous lemma.

□

The proposition follows from the two lemmas.

□

6.1 First variation

We obtain in this section a first variation formula for the invariant μ when the flag structure is deformed through a smooth path. Let $\mu(t)$ be the invariant defined as a function of the a parameter describing the deformation of the structure on a closed manifold M and define $\delta\mu = \frac{d}{dt}\mu(0)$.

Proposition 6.3 $\delta\mu = -\frac{1}{4\pi^2}\int_M s^*\mathrm{tr}\,(\dot\pi \wedge \Pi)$.

Proof. Differentiating $\mu(t) = \frac{1}{24\pi^2}\int_M s^*\mathrm{tr}\,(\pi_t \wedge \pi_t \wedge \pi_t)$ we have

$$\delta\mu = \frac{1}{8\pi^2}\int_M s^*\mathrm{tr}\,(\pi \wedge \pi \wedge \dot\pi).$$

Using the formula $\dot\Pi = d\dot\pi + \dot\pi \wedge \pi + \pi \wedge \dot\pi$ we write

$$\mathrm{tr}\,(\pi \wedge \dot\Pi) = \mathrm{tr}\,(\pi \wedge d\dot\pi + 2\pi \wedge \pi \wedge \dot\pi)$$

and therefore

$$\mathrm{tr}\,(\pi \wedge \pi \wedge \dot\pi) = \frac{1}{2}\mathrm{tr}\,(\pi \wedge \dot\Pi - \pi \wedge d\dot\pi) = \frac{1}{2}\mathrm{tr}\,(-\dot\pi \wedge \Pi - d\pi \wedge \dot\pi).$$

Where, in the last equality, we used that on a closed manifold $\int_M \mathrm{tr}\,(\pi \wedge d\dot\pi) = \int_M \mathrm{tr}\,(d\pi \wedge \dot\pi)$ and that, differentiating $\mathrm{tr}\,(\pi \wedge \Pi) = 0$, we have $\mathrm{tr}\,(\dot\pi \wedge \Pi + \pi \wedge d\dot\Pi) = 0$.

Substituting $\Pi - \pi \wedge \pi = d\pi$ we obtain

$$\mathrm{tr}\,(\pi \wedge \pi \wedge \dot\pi) = \frac{1}{2}\mathrm{tr}\,(-2\dot\pi \wedge \Pi - \pi \wedge \pi \wedge \dot\pi)$$

and therefore

$$\delta\mu = \frac{1}{8\pi^2}\int_M s^*\mathrm{tr}\,(\pi \wedge \pi \wedge \dot\pi) = -\frac{1}{4\pi^2}\int_M s^*\mathrm{tr}\,(\dot\pi \wedge \Pi).$$

□

Observe that an explicit computation gives

$$\mathrm{tr}\,(\dot\pi \wedge \Pi) = -\dot\omega^1 \wedge \Phi^2 + \dot\omega^2 \wedge \Phi^1 - \frac{1}{2}\dot\omega \wedge \Psi.$$

7 Pseudo flag geometry

In this section we fix a contact form and obtain a reduction of the structure group of a path geometry. We will obtain the relations between the invariants of the reduced structure to the original one. This is similar to the reduction of a CR structure to a pseudo hermitian structure.

We consider a form θ on $T^{\mathbb{C}}$ such that $\ker\theta = T^1 \oplus T^2$ is non-integrable. Define forms Z^1 and Z^2 on $T^{\mathbb{C}}$ satifying

$$Z^1(T^1) \neq 0 \ \text{ and } Z^2(T^2) \neq 0,$$

$$\ker \theta^1 \supset T^2 \quad \text{and} \quad \ker \theta^2 \supset T^1$$

and such that $d\theta = Z^1 \wedge Z^2$.

Fixing one choice, all others are given by $\theta^1 = aZ^1$ and $\theta^2 = \frac{1}{a}Z^2$, with $a \in \mathbb{C}^*$. We consider now the $\mathbb{C}^*$ coframe bundle X defined by the forms $\theta^1, \theta^2, \theta$. We have

$$d\theta = \theta^1 \wedge \theta^2. \tag{13}$$

Proposition 7.1 *There exist unique forms θ_1^1, τ^1 and τ^2 on X such that*

$$d\theta^1 = \theta^1 \wedge \theta_1^1 + \theta \wedge \tau^1 \tag{14}$$

$$d\theta^2 = -\theta^2 \wedge \theta_1^1 + \theta \wedge \tau^2 \tag{15}$$

with $\theta_1^1 = -\frac{da}{a} \mod \theta^1,\ \theta^2,\ \theta$ and $\tau^1 \wedge \theta^2 = \tau^2 \wedge \theta^1 = 0$.

Proof. Define functions z_{12}^i, z_{j0}^i by

$$dZ^i = z_{12}^i Z^1 \wedge Z^2 + z_{10}^i Z^1 \wedge \theta + z_{20}^i Z^2 \wedge \theta.$$

Then

$$d\theta^1 = \frac{da}{a} \wedge \theta^1 + a dZ^1 = \frac{da}{a} \wedge \theta^1 + a\left(z_{12}^1 Z^1 \wedge Z^2 + z_{10}^1 Z^1 \wedge \theta + z_{20}^1 Z^2 \wedge \theta\right).$$

which can be written as

$$d\theta^1 = \theta^1 \wedge \theta_1^1 + \theta \wedge \tau^1 \tag{16}$$

where $\theta_1^1 = -\frac{da}{a} + z_{12}^1 Z^2 - z_{12}^2 Z^1$ (where we added a term in Z^1 in order to have a compatibility with the formula for $d\theta^2$ bellow) and $\tau^1 = -z_{10}^1\theta^1 - z_{20}^1 a^2\theta^2$.

Anagously, from

$$d\theta^2 = -\frac{da}{a} \wedge \theta^2 + \frac{1}{a} dZ^2 = -\frac{da}{a} \wedge \theta^2 + \frac{1}{a}\left(z_{12}^2 Z^1 \wedge Z^2 + z_{10}^2 Z^1 \wedge \theta + z_{20}^2 Z^2 \wedge \theta\right)$$

we obtain

$$d\theta^2 = -\theta^2 \wedge \theta_1^1 + \theta \wedge \tau^2 \tag{17}$$

where $\theta_1^1 = -\frac{da}{a} + z_{12}^1 Z^2 - z_{12}^2 Z^1$ and $\tau^2 = -z_{10}^2 a^{-2}\theta^1 - z_{20}^2\theta^2$.

Observe also that, differentiating equation 13 and using 16 and 17, we obtain $\theta \wedge (\tau^1 \wedge \theta^2 - \tau^2 \wedge \theta^1) = 0$ which implies that, writing $\tau^i = \tau_1^i\theta^1 + \tau_2^i\theta^2$,

$$\tau_1^1 + \tau_2^2 = 0.$$

Now, if $\theta_1'^1$, τ'^1 and τ'^2 are other forms satisfying the equations, then from the above equations we obtain

$$\theta_1^1 - \theta_1'^1 = A\theta \quad \text{and} \quad \tau_1^1 - \tau_1'^1 = A.$$

Choosing an appropriate A we can therefore fix $\tau_1^1 = 0$ and the forms θ_1^1, τ^1 and τ^2 are uniquely determined as claimed. □

Differentiating equations 16 and 17 we obtain

$$\theta^1 \wedge d\theta_1^1 + \theta \wedge (d\tau^1 - \tau^1 \wedge \theta_1^1) = 0$$

$$-\theta^2 \wedge d\theta_1^1 + \theta \wedge (d\tau^2 + \tau^2 \wedge \theta_1^1) = 0$$

and therefore

$$d\theta_1^1 = R\theta^1 \wedge \theta^2 + W^1\theta^1 \wedge \theta + W^2\theta^2 \wedge \theta \tag{18}$$

$$d\tau^1 - \tau^1 \wedge \theta_1^1 = -W^2\theta^1 \wedge \theta^2 + S_1^1\theta \wedge \theta^1 + S_2^1\theta \wedge \theta^2 \tag{19}$$

$$d\tau^2 + \tau^2 \wedge \theta_1^1 = -W^1\theta^1 \wedge \theta^2 + S_1^2\theta \wedge \theta^1 + S_2^2\theta \wedge \theta^2 \tag{20}$$

Moreover, differentiating equation $\tau^1 \wedge \theta^2 = 0$ and $\tau^2 \wedge \theta^1 = 0$ we obtain

$$S_1^1 = S_2^2 = \tau_2^1\tau_1^2.$$

7.1 Curvature identities

Differentiating equation 18 one gets

$$dR \wedge \theta^1 \wedge \theta^2 + (dW^1 - W^1\theta_1^1) \wedge \theta^1 \wedge \theta(dW^2 + W^2\theta_1^1) \wedge \theta^2 \wedge \theta = 0.$$

Writing

$$dR = R_0\theta + R_1\theta^1 + R_2\theta^2,$$

$$dW^1 - W^1\theta_1^1 = W_0^1\theta + W_1^1\theta^1 + W_2^1\theta^2$$

and

$$dW^2 + W^2\theta_1^1 = W_0^2\theta + W_1^2\theta^1 + W_2^2\theta^2$$

Then

$$R_0 = W_2^1 - W_1^2.$$

Differentiating equation 20 and writing $dR_0 = R_{00}\theta + R_{01}\theta^1 + R_{02}\theta^2$, one gets

$$dR_1 - R_1\theta_1^1 + R_2\tau_1^2\theta - \frac{1}{2}R_0\theta^2 = R_{01}\theta + R_{11}\theta^1 + R_{12}\theta^2$$

and

$$dR_2 + R_2\theta_1^1 + R_1\tau_2^1\theta + \frac{1}{2}R_0\theta^1 = R_{02}\theta + R_{12}\theta^1 + R_{22}\theta^2$$

We also obtain differentiating 19 and 20

$$d\tau_2^1 + 2\tau_2^1\theta_1^1 = -W^2\theta^1 + S_2^1\theta \mod \theta^2$$

and

$$d\tau_1^2 - 2\tau_1^2\theta_1^1 = W^1\theta^2 + S_1^2\theta \mod \theta^1.$$

7.2 Embedding $X \to Y$

Recall that X is the coframe bundle of forms $(\theta, \theta^1, \theta^2)$ over M. We chose a section of this bundle. The forms over M will also be denoted by $(\theta, \theta^1, \theta^2)$. The goal now is to obtain an immersion $X \to Y$. Let $s : M \to Y$ such that $s^*\omega = \theta$ and write

$$s^*\varphi = A_1\theta^1 + A_2\theta^2 + A_0\theta.$$

for functions A_i on M. To choose a section we will impose that $s^*\varphi = 0$. For that sake we start with a particular section and move it using the action of the structure group H.

Consider

$$h = \begin{pmatrix} a & c & e \\ 0 & \frac{1}{ab} & f \\ 0 & 0 & b \end{pmatrix}$$

which gives

$$h^{-1}d\,h = \begin{pmatrix} a^{-1}d\,a & a^{-1}d\,c + c(\frac{d\,b}{ab} + \frac{d\,a}{a^2}) & a^{-1}(d\,e) - bc(d\,f) + (cf - \frac{e}{ab})d\,b \\ 0 & -\frac{d\,(ab)}{ab} & ab(d\,f) - af(d\,b) \\ 0 & 0 & \frac{d\,b}{b} \end{pmatrix}. \tag{21}$$

We will use the formula

$$R_h{}^*\pi = h^{-1}d\,h + Ad_{h^{-1}}\pi.$$

If $h : M \to H$ is given by

$$\begin{pmatrix} 1 & c & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$$

then, using formula 21 and formula 12 for $Ad_{h^{-1}}\pi$ in the expression of $R_h{}^*\pi$ we obtain

$$\tilde{\varphi} = \varphi + c\omega^1 + 2f\omega^2 + (4e - 2cf)\omega.$$

Observe that $a = b$ is imposed by the condition $s^*\omega = \theta$. Starting with any section defining functions A_1, A_2 and A_3 we obtain a new section by acting by a section of $h : M \to H$ given by $c = -A_1,\ f = -A_2/2,\ e = (A_1A_2 - A_0)/4$. In that case we have

$$s^*\tilde{\varphi} = 0.$$

The maps from E to the fiber group of $Y \to E$, such that $s^*R_h{}^*\tilde{\varphi} = 0$ are fixed, because $R_h{}^*\tilde{\varphi} = \tilde{\varphi} + c\,\omega^1 + 2f\,\omega^2 + (4e - 2cf)\,\omega$, so $c = f = e = 0$.

After fixing c, f, e, we allow maps $h : M \to H$ acting by R_h on $Y \to M$ by elements of the form

$$h = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & a \end{pmatrix}$$

so that the form θ be preserved. This gives the embedding of X into Y.

We may suppose that $s^*\varphi = 0$ and then obtain the following equations by pulling back to M the structure equations on Y:

$$d\theta = \theta^1 \wedge \theta^2 \tag{22}$$

$$d\theta^1 = \theta^1 \wedge \omega_1^1 + \theta \wedge \varphi^1 \tag{23}$$

$$d\theta^2 = -\theta^2 \wedge \omega_1^1 + \theta \wedge \varphi^2 \tag{24}$$

$$\theta^1 \wedge \varphi^2 - \theta^2 \wedge \varphi^1 + \theta \wedge \psi = 0 \tag{25}$$

$$d\omega_1^1 - \frac{3}{2}\theta^1 \wedge \varphi^2 - \frac{3}{2}\theta^2 \wedge \varphi^1 = 0 \tag{26}$$

$$d\varphi^1 - \varphi^1 \wedge \omega_1^1 - \frac{1}{2}\theta^1 \wedge \psi = Q^1\theta \wedge \theta^2 \tag{27}$$

$$d\varphi^2 + \varphi^2 \wedge \omega_1^1 - \frac{1}{2}\theta^2 \wedge \psi = Q^2\theta \wedge \theta^1 \tag{28}$$

$$d\psi - 2\varphi^1 \wedge \varphi^2 = U_1\theta^1 \wedge \theta + U_2\theta^2 \wedge \theta \tag{29}$$

In the formulae above we write the pull back of any form α defined on Y using the same notation α. It follows from equations 23 and 24, comparing with proposition 7.1 that

$$\omega_1^1 = \theta_1^1 + c\theta$$

and therefore

$$d\theta^1 = \theta^1 \wedge \theta_1^1 + \theta \wedge (\varphi^1 - c\theta^1) \tag{30}$$

$$d\theta^2 = -\theta^2 \wedge \theta_1^1 + \theta \wedge (\varphi^2 + c\theta^2) \tag{31}$$

which implies

$$\varphi^1 = c\theta^1 + E^1\theta + \tau^1,$$

$$\varphi^2 = -c\theta^2 + E^2\theta + \tau^2,$$

Substituting the formulae above in equation 25, and using again proposition 7.1 we obtain

$$\theta \wedge (\psi - E^2\theta^1 + E^1\theta^2) = 0$$

and therefore

$$\psi = E^2\theta^1 - E^1\theta^2 + G\theta.$$

Substituting the expressions of ω_1^1, φ^1 and φ^2 in equation 26 and using equation 18 we obtain

$$(R + 4c)\,\theta^1 \wedge \theta^2 + (W^1 - \frac{3}{2}E^2)\,\theta^1 \wedge \theta + (W^2 - \frac{3}{2}E^1)\,\theta^2 \wedge \theta + dc \wedge \theta = 0.$$

This implies

$$c = -\frac{R}{4}$$

writing, as before, $dR = R_0\theta + R_1\theta^1 + R_2\theta^2$ and substituting in the above expression we get

$$(W^1 - \frac{3}{2}E^2 - \frac{1}{4}R_1)\,\theta^1 \wedge \theta + (W^2 - \frac{3}{2}E^1 - \frac{1}{4}R_2)\,\theta^2 \wedge \theta = 0$$

which implies

$$E^2 = \frac{2}{3}(W^1 - \frac{1}{4}R_1)$$

and

$$E^1 = \frac{2}{3}(W^2 - \frac{1}{4}R_2).$$

We now write equation 27, substituting the expressions for φ^1 and ω^1_1 obtained above:

$$0 = d\tau^1 - \frac{1}{4}(R_0\theta + R_2\theta^2)\wedge\theta^1 - \frac{1}{4}R(\theta^1\wedge\theta^1_1 + \theta\wedge\tau^1) + dE^1\wedge\theta + E^1\theta^1\wedge\theta^2$$

$$-(\tau^1 - \frac{1}{4}\theta^1 + E^1\theta)\wedge(\theta^1_1 - \frac{1}{4}R\theta) - \frac{1}{2}\theta^1\wedge(E^2\theta^1 - E^1\theta^2 + G\theta) - Q^1\theta\wedge\theta^2$$

Using curvature identities in order to simplify the above expression we obtain after a computation

$$0 = \theta\wedge\theta^1(S^1_1 - \frac{1}{3}R_0 + \frac{1}{16}R^2 + \frac{1}{2}G - \frac{2}{3}W^2_1 + \frac{1}{6}R_{21}) + \theta\wedge\theta^2(S1_2 - \frac{1}{2}R\tau^1_2 - Q^1 - \frac{2}{3}W^2_2 + \frac{1}{6}R_{22})$$

Therefore

$$S^1_1 - \frac{1}{3}R_0 + \frac{1}{16}R^2 + \frac{1}{2}G - \frac{2}{3}W^2_1 + \frac{1}{6}R_{21} = 0$$

and

$$S^1_2 - \frac{1}{2}R\tau^1_2 - Q^1 - \frac{2}{3}W^2_2 + \frac{1}{6}R_{22} = 0.$$

Analogously, from equation 28, we obtain the identities

$$S^2_2 + \frac{1}{3}R_0 + \frac{1}{16}R^2 + \frac{1}{2}G - \frac{2}{3}W^1_2 + \frac{1}{6}R_{12} = 0$$

and

$$S^2_1 + \frac{1}{2}R\tau^2_1 - Q^2 - \frac{2}{3}W^1_1 + \frac{1}{6}R_{11} = 0.$$

7.2.1 The global invariant

A simple computation gives the formula

$$\mathrm{tr}\,(\pi\wedge\pi\wedge\pi) = \frac{3}{2}(\omega\wedge\varphi + \omega^1\wedge\omega^2)\wedge\psi + 3\omega\wedge\varphi^1\wedge\varphi^2 + 3\omega^1\wedge(\frac{1}{2}\varphi + \omega^1_1)\wedge\varphi^2 - 3\omega^2\wedge(\frac{1}{2}\varphi - \omega^1_1)\wedge\varphi^1.$$

Therefore using the embedding of the previous section we obtain by a computation:

Proposition 7.2 *For a section $s : M \to Y$ factoring through an embedding of X into Y such as $s^*\varphi = 0$ we have*

$$s^*TC_2(\pi) = \frac{1}{8\pi^2}\left(\theta\wedge\theta^1\wedge\theta^2(\frac{1}{2}G + \frac{1}{16}R^2 - \tau^1_2\tau^2_1) + \theta^1_1\wedge(E^2\theta\wedge\theta^1 + E^1\theta\wedge\theta^2 - \frac{R}{2}\theta^1\wedge\theta^2)\right).$$

8 Homogeneous flag structures on $\mathbf{SU}(2)$

Let α, β, γ be a basis of left invariant 1-forms defined on $\mathbf{SU}(2)$ with

$$d\alpha = -\beta \wedge \gamma, \quad d\beta = -\gamma \wedge \alpha, \quad d\gamma = -\alpha \wedge \beta$$

We define a pseudo flag structure choosing a map from $\mathbf{SU}(2)$ to $\mathbf{SL}(2, \mathbb{C})$:

$$\theta = \gamma, \quad Z^1 = r_1\beta + r_2\alpha, \quad Z^2 = s_1\beta + s_2\alpha,$$

with $r_1s_2 - r_2s_1 = 1$. Then

$$d\theta = Z^1 \wedge Z^2.$$

In the case the map $\mathbf{SU}(2) \to \mathbf{SL}(2, \mathbb{C})$ is constant, from $\beta = s_2Z^1 - r_2Z^2$ and $\alpha = -s_1Z^1 + r_1Z^2$, we obtain

$$dZ^1 = r_1 d\beta + r_2 d\alpha = \theta \wedge \left(xZ^1 + yZ^2\right)$$

and analogously,

$$dZ^2 = \theta \wedge \left(zZ^1 - xZ^2\right),$$

where

$$x = r_1s_1 + r_2s_2, \quad y = -(r_1^2 + r_2^2), \quad z = s_1^2 + s_2^2.$$

Observe that $x^2 + yz = -1$. Then for a pseudo flag structure with coframes obtained from the tautological forms $\theta^1 = aZ^1, \theta^2 = a^{-1}Z^2$

$$d\theta^1 = \theta^1 \wedge (-\frac{da}{a} - x\theta) + \theta \wedge (ya^2\theta^2)$$

$$d\theta^2 = -\theta^2 \wedge (-\frac{da}{a} - x\theta) + \theta \wedge (za^{-2}\theta^1)$$

From Proposition 7.1 we have

$$\theta_1^1 = -\frac{da}{a} - x\theta,$$

$$\tau^1 = ya^2\theta^2, \quad \tau^2 = za^{-2}\theta^1.$$

and therefore

$$d\theta_1^1 = -x\theta^1 \wedge \theta^2$$

so that $R = -x, W^1 = W^2 = 0$. In order to compute the curvature invariants from the pseudo flag structure we use the embedding in section 7.2. We compute first

$$d\tau^1 = ya^2\theta^2 \wedge \theta_1^1 + yz\theta \wedge \theta^1 - 2a^2xy\theta \wedge \theta^2$$

and

$$d\tau^2 = -za^{-2}\theta^1 \wedge \theta_1^1 + 2xza^{-2}\theta \wedge \theta^1 + yz\theta \wedge \theta^2.$$

Now, as $d\tau^1 - \tau^1 \wedge \theta_1^1 = -W^2\theta^1 \wedge \theta^2 + S_1^1\theta \wedge \theta^1 + S_2^1\theta \wedge \theta^2$ and $d\tau^2 + \tau^2 \wedge \theta_1^1 = -W^1\theta^1 \wedge \theta^2 + S_1^2\theta \wedge \theta^1 + S_2^2\theta \wedge \theta^2$ (cf. 19,20) we obtain

$$S_1^1 = yz, \quad S_2^1 = -2a^2xy, \quad S_1^2 = 2xza^{-2}, \quad S_2^2 = yz.$$

From the embedding equations we have

$$Q^1 = S^1_2 - \frac{1}{2}R\tau^1_2 - \frac{2}{3}W^2_2 + \frac{1}{6}R_{22} = -\frac{3}{2}xya^2$$

and, analogously

$$Q^2 = S^2_1 + \frac{1}{2}R\tau^2_1 - \frac{2}{3}W^1_1 + \frac{1}{6}R_{11} = \frac{3}{2}xza^{-2}.$$

We conclude that $Q^1 = Q^2 = 0$ if and only if $x = 0$ or $y = z = 0$.

8.0.1 The global invariant

We compute, using 7.2, the global invariant for the family of structures defined on $\mathbf{SU}(2)$ We have $R = -x$, $\tau^1_2 = a^2y$, $\tau^2_1 = a^{-2}z$, $E^1 = E^2 = 0$ and $G = -2yz - \frac{1}{8}x^2$. Therefore for a section $s : \mathbf{SU}(2) \to Y$ as above we obtain

$$s^*TC_2(\pi) = -\frac{1}{8\pi^2}\gamma \wedge \beta \wedge \alpha(2yz + \frac{1}{2}x^2).$$

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E. Falbel
Institut de Mathématiques
de Jussieu-Paris Rive Gauche
CNRS UMR 7586 and INRIA EPI-OURAGAN
Sorbonne Université, Faculté des Sciences
4, place Jussieu 75252 Paris Cedex 05, France
elisha.falbel@imj-prg.fr

J. M. Veloso
Faculdade de Matemática - ICEN
Universidade Federal do Pará
66059 - Belém- PA - Brazil
veloso@ufpa.br