Poisson statistics at the edge of Gaussian β-ensemble at high temperature

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Abstract

We study the asymptotic edge statistics of the Gaussian β-ensemble, a collection of n particles, as the inverse temperature β tends to zero as n tends to infinity. In a certain decay regime of β, the associated extreme point process is proved to converge in distribution to a Poisson point process as $n \to +\infty$. We also extend a well known result on Poisson limit for Gaussian extremes by showing the existence of an edge regime that we did not find in the literature.

Key words. Random matrices, Gaussian β-ensembles, Poisson statistics, Extreme value theory.
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1 Introduction

The study of spectral statistics in Random Matrix Theory has gathered a consequent volume of the research attention during the last decades. For several reasons, theses statistics are considered in the asymptotic regime : as the size of the matrix (and hence the number of eigenvalues) goes to infinity. One can inquiry about the behaviour of the whole spectrum (such as linear statistics), this is called *global* statistics (or regime). The main object to study in this context is the empirical spectral measure and the goal is to obtain a limiting distribution and give fluctuations around this limit. On the other hand, one can seek for more subtle, precise informations, like the spacing between two consecutive eigenvalues, or the nature of the largest eigenvalues; more generally, the joint distribution of eigenvalues in an interval of length $o(1)$. Such statistics are called *local*. In this particular regime, we differentiate between the *bulk* and the *edge* statistics. The bulk regime focuses on intervals inside the support of the limiting spectral measure while the edge regime at the boundary. In this article, we are mainly interested in the asymptotic local edge regime, which corresponds to the largest eigenvalues.

Among random matrix models, two matrix ensembles are distinguished : Wigner matrices and invariant ensembles. The first one indicates matrices with independent components while the second gathers matrices whose law is invariant by symmetry group action. Their intersection is known as the GOE, GUE and GSE. Their origin trace back to the pioneer Wigner. He wanted to model complex highly correlated systems with (or lacking) different kind of symmetries (see [12, 10]) and considered Hamiltonians as large random matrices. The name stems from the invariance under certain group actions. The joint density of the eigenvalues can be derived (see [2]) and is proportional to :

$$P(\lambda_1, ..., \lambda_n) \propto \exp\left(-\frac{1}{4}\sum_{i=1}^{n}\lambda_i^2\right)|\Delta_n(\lambda)|^\beta \prod_{i=1}^{n} d\lambda_i.$$

The Vandermonde determinant is noted $|\Delta_n(\lambda)|^\beta := \prod_{i<j}^{n} |\lambda_j - \lambda_i|^\beta$, and $\beta \in \{1, 2, 4\}$. The case $\beta = 1$ is the GOE which models Hamiltonians with time-reversal symmetry, $\beta = 2$ is the GUE which models Hamiltonians lacking time-reversal symmetry and $\beta = 4$ is the GSE which models Hamiltonians with time-reversal symmetry but no rotational symmetry (see [12]). Let us mention that when $\beta = 2$, the correlation functions, which will be our prime object, describe a determinantal process (see *Gaudin-Mehta formula*). The idea that

β taking different values gives rise to different models is known as the *Dyson's Threefold-Way* [8].

We can extend the model in two directions, allowing other values of β and other potentials, by writing for $\beta > 0$:

$$P_{n,\beta,V}(d\lambda_1, ..., d\lambda_n) := \frac{1}{Z_{n,\beta,V}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} V\left(\lambda_i\right)\right) |\Delta_n(\lambda)|^\beta \prod_{i=1}^{n} d\lambda_i.$$

We refer this as the *general β-ensemble*. If the potential is quadratic $V(x) := \frac{1}{2}x^2$, it reduces to the *Gaussian β-ensemble* which is the object of our work.

In this context, Dumitriu and Edelmann [5] made a major breakthrough by constructing a matrix model for such β-ensemble with any $\beta > 0$, hence extending the Dyson's Threefold-Way $\beta \in \{1, 2, 4\}$. It states that the Gaussian β-ensemble (viewed as a density probability function) is exactly the joint law of the spectrum of a certain matrix. The latter is obtained from successive Households transformations and has a symmetric tridiagonal form. This representation of the Gaussian β-ensemble by a matrix model [5] led the way for many progresses [9, 15, 13, 16] on the understanding of the asymptotic local eigenvalue statistics for general $\beta > 0$. In particular, the authors of [9], leaning on the symmetric tridiagonal structure of the Gaussian β-ensemble matrix model, gave multiple indications on how renormalized random matrices can be viewed as finite difference approximations to stochastic differential operators. This conjecture was investigated in [13] where the properly renormalized largest eigenvalues are shown to converge jointly in distribution to the low-lying eigenvalues of a one-dimensional Schrödinger operator, namely the stochastic Airy operator $\mathrm{SAO}_\beta := -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'_x,$: for $k \geq 1$ fixed, denoting $\lambda_1^\beta \geq \lambda_2^\beta \geq ... \geq \lambda_k^\beta$ the k largest eigenvalues of H_n^β and $\Lambda_0^\beta \leq \Lambda_1^\beta \leq ... \leq \Lambda_{k-1}^\beta$ the k smallest eigenvalues of SAO_β :

$$(n\beta)^{\frac{2}{3}} \left(2 - \frac{\lambda_i^\beta}{\sqrt{n\beta}}\right)_{1\leq i\leq k} \xrightarrow[n\infty]{\text{law}} \left(\Lambda_i^\beta\right)_{0\leq i\leq k-1}.$$

Since the minimal eigenvalue Λ_0 of SAO_β has distribution minus TW_β, this work thereby enlarges Tracy-Widom law to all $\beta > 0$, that is :

$$(n\beta)^{\frac{2}{3}} \left(\frac{\lambda_i^\beta}{\sqrt{n\beta}} - 2\right) \xrightarrow[n\infty]{\text{law}} \mathrm{TW}_\beta.$$

The Tracy-Widom law (with parameter β) is qualified as *universal*, in the sense that such local statistics hold for various matrix models (but also for objects outside of the random matrix field) and arises from highly correlated systems (such as modeled by some random matrices).

For finite dimension n, one can choose $\beta = 0$ in the joint law P of the Gaussian β-ensemble, which displays a lack of repulsion force as the Vandermonde factor vanishes, hence the correlation decreases, which means that randomness increases. In a Gibbs interpretation (which besides makes us refer to $Z_{n,\beta,V}$ and its counterparts as *partition functions*), it comes down to consider an infinite temperature in such log-gas (terminology due to Dyson [8]). Readily, the joint density for $\beta = 0$ is the density of n i.i.d. Gaussian random variables whose maximum is known [14] to converge weakly, as $n \to +\infty$, when properly renormalized, to the Gumbel distribution, one of the three *universal* distributions classes of the classical Extreme Value Theory. One deduces (see [4, Th 7.1]) Poisson limit for the Gaussian (ie: when $\beta = 0$) extreme point process as the number n of particles grows to infinity. This qualitative statistic is special to us since it is essentially our purpose in this article. It also carries more information and implies the limiting Gumbel distribution.

As the Gumbel law governs the typical fluctuations of the maximum of independent Gaussian variables, which corresponds to the case $\beta = 0$, and the Tracy-Widom law stems from complicated (highly dependent) systems such as the largest particles in the case $\beta > 0$ fixed and $n \to +\infty$, it is thus natural to ask for an interpolation between these two phases. The authors of [1] answer this question by proving that the properly renormalized Tracy-Widom$_\beta$ converges in distribution to the Gumbel law as $\beta \to 0$. They use the characterization of the distribution of the bottom eigenvalues of the stochastic Airy operator in terms of the explosion times process of its associated Riccati diffusion (see [13]). Regarding to our motivation, they could unfortunately not prove Poissonian statistics for the minimal eigenvalues $\left(\Lambda_i^\beta\right)$, distributed according to the Tracy-Widom$_\beta$ law, in the limit $\beta \to 0$. This procedure would exactly reverse the order of the limits $\beta \to 0, n \to +\infty$ considered previously. Nonetheless, the authors investigated the weak convergence of the top eigenvalues in the double limit $\beta := \beta_n \underset{n\infty}{\longrightarrow} 0$ by heuristic and numeric arguments. They alluded to the idea that one can achieve Poissonian statistics for β-ensemble using the same techniques as [13, 9], at high temperature within the regime $n\beta \underset{n\infty}{\longrightarrow} +\infty$. Concerning the bulk statistics, such work has been accomplished in the regime $\beta \sim n^{-1}$, that is Poisson convergence of the point process $\sum_{i=1}^{n} \delta_{n(\lambda_i - E)}$ with $E \in (-2, 2)$ an energy level in the *Wigner sea* (see [7, 6]).

The goal of this paper is to understand the behavior of the largest particles of the Gaussian β-ensemble as the inverse temperature β_n converges to 0 as n goes to infinity. To this purpose, we study the limiting process of the extremes of the Gaussian β-ensemble. Since β can decay with any arbitrary rate, we restrict ourselves to the regime $n\beta \underset{n\infty}{\longrightarrow} 0$. More precisely, our main result gives the convergence as $n \to +\infty$ of the extreme process toward a Poisson point process on $\mathbb{R}$, which can be inhomogeneous or not, according to the scaling sequences. Roughly speaking, the rescaled extreme eigenvalues approximate a Poisson point process which means that adjacent top particles are statistically independent. Our work also applies when β is set to 0 and *de facto* includes asymptotics $(n \to +\infty)$

of extremes of Gaussian variables ($\beta = 0$). While the outcomes are identical for both β cases, we want to stress out that the models are intrinsically distinct. We investigate this question in the subsequent Remark 1.2. Doing such simultaneous double scaling limit, we fulfill the corresponding task addressed by Allez and Dumaz in [1] within another regime mentioned in their work and by other means.

For $u = u_n$ and $v = v_n$ two sequences, we adopt the notation $u \ll v \iff \dfrac{u}{v} \underset{n\infty}{\longrightarrow} 0$ and state our main result :

Theorem 1.1. *Let $\beta = \beta_n$ be such that $0 \le \beta \ll \dfrac{1}{n\log(n)}$. Let $(\lambda_1, ..., \lambda_n)$ a family of random variables with joint law $P_{n,\beta}$:*

$$P_{n,\beta}(d\lambda_1, ..., d\lambda_n) := \frac{1}{Z_{n,\beta}} \exp\left(-\frac{1}{2}\sum_{i=1}^n \lambda_i^2\right) |\Delta_n(\lambda)|^\beta \prod_{i=1}^n d\lambda_i,$$

with normalization constant $Z_{n,\beta}$ and Vandermonde determinant $|\Delta_n(\lambda)|^\beta := \prod_{i<j}^n |\lambda_j - \lambda_i|^\beta$. Let (δ_n) a positive sequence and the modified Gaussian scaling :

$$b_n := \sqrt{2\log(n)} - \frac{1}{2}\frac{\log\log(n) + 2\log(\delta_n) + \log(4\pi)}{\sqrt{2\log(n)}}, \qquad a_n := \delta_n\sqrt{2\log(n)}.$$

- *Assume $\delta_n \underset{n\infty}{\longrightarrow} \delta > 0$. Then the random point process $\sum_{i=1}^n \delta_{a_n(\lambda_i - b_n)}$ converges in distribution to an inhomogeneous Poisson point process with intensity $e^{-\frac{x}{\delta}}dx$.*
- *Assume $\delta_n \gg 1$ such that $\log(\delta_n) \ll \sqrt{\log(n)}$. Then the random point process $\sum_{i=1}^n \delta_{a_n(\lambda_i - b_n)}$ converges in distribution to a homogeneous Poisson point process with intensity 1.*
- *When $\beta = 0$, the condition on (δ_n) is weakened to : $\log(\delta_n) \ll \log(n)$.*

Let us first discuss the assumptions and conclusions of the theorem. We prove convergence of extreme point processes

$$\mathcal{P}_n := \sum_{i=1}^n \delta_{a_n(\lambda_i - b_n)}$$

toward a Poisson point process on $\mathbb{R}$ with intensity $d\mu$ as $n \to +\infty$ for suitably chosen scaling sequences $(a_n), (b_n)$ and intensity μ. This convergence occurs regardless to $\beta > 0$ or $\beta = 0$ although this gives rise to two different models. The scaling sequences are exactly the same in both cases and are derived from the classical Gaussian scaling (see [14]), except that we increase the scale a_n by a multiplicative term δ_n and lower down the center b_n by an additive term involving δ_n. We then observe two regimes : first, when

$$\delta_n \underset{n\infty}{\longrightarrow} \delta > 0,$$

the limiting process is an inhomogeneous Poisson process with intensity $e^{-\frac{x}{\delta}}dx$ (which is a classical result in the purely Gaussian setting, when $\beta = 0$). When $\delta_n \gg 1$, even in the purely Gaussian setting ($\beta = 0$), we obtain a result that we did not find in the literature [4, 11, 14] : in this case, even though the interval considered (centered at b_n and with width of order a_n) goes to $+\infty$, the limiting process is a homogeneous Poisson process. An illustration of these phenomena is given in Figure 1 below.

Remark 1.2. As previously mentionned, the Poissonian description of the extreme process, along with the normalizing constants $(a_n), (b_n)$ which display no dependence on β, is valid for both cases $\beta_n = 0$ and $\beta_n > 0$. The question of how close both models are is then raised. Therefore, we need to measure the impact of the decay rate of β_n upon the model. In this direction, one can compare the normalization constants $Z_{n,\beta}$ between different β_n regimes. This idea emerges from equilibrium statistical mechanics where the $Z_{n,\beta}$ is seen as the *partition function* in the Gibbs interpretation. The computations show a transition : when $\beta \ll n^{-2}$, both models are equivalent. As soon as $\beta \gg n^{-2}$, the repulsion is significant. We state this result in the forthcoming Lemma 1.3 whose proof is postponed to Section 2.2. It indicates that our main theorem gains value when compelling

$$n^{-2} \ll \beta \ll (n\log(n))^{-1},$$

which corresponds to the regime where both models are truly distinct. The critical role of n^2 in this description is consistent with the fact that one can write

$$\log|\Delta_n(\lambda)|^\beta = \exp\left(\beta \sum_{i<j}^{n} \log|\lambda_j - \lambda_i|\right)$$

with the sum having $n^2(1 + o(1))$ terms.

Lemma 1.3. *Let $\beta \geq 0$ and $\beta' > 0$.*

- *Assume $0 \leq \beta \ll \beta' \ll \dfrac{1}{n^2}$, then $Z_{n,\beta'} \sim Z_{n,\beta}$.*
- *Assume $0 \leq \beta \ll \beta' \ll \dfrac{1}{n}$ and $\beta' \gg \dfrac{1}{n^2}$, then $Z_{n,\beta'} \ll Z_{n,\beta}$.*

The convergence toward a Poisson process for the extreme process is a much stronger information than the limiting distribution of the maximum. Indeed, one can deduce the limiting distribution as follows, but we postpone the proof to Section 2. Also, one can derive the limiting distribution for the k^{th} largest eigenvalue for fixed $k \geq 1$.

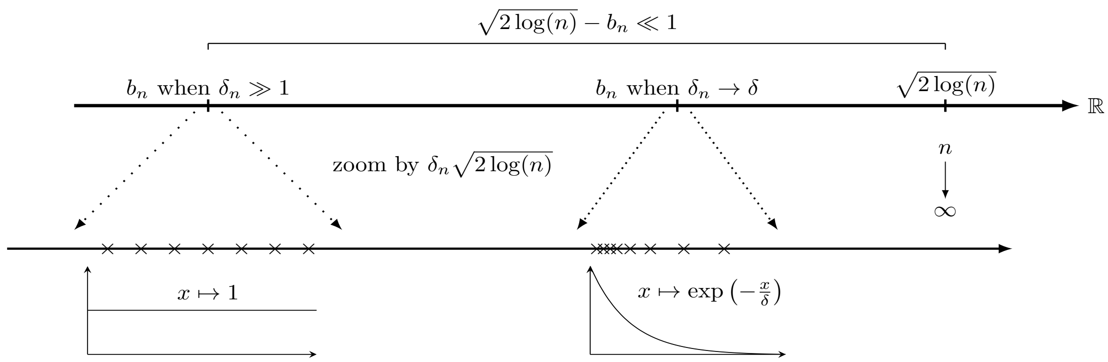

Figure 1: The centering at b_n for both cases $\delta_n \to \delta > 0$ and $\delta_n \to \infty$ are represented on the main line. We zoom in around each b_n by a factor $\delta_n\sqrt{2\log(n)}$ and let n go to ∞. For $\delta_n \gg 1$, the limiting object is a Poisson point process with intensity 1. For $\delta_n \xrightarrow[n\infty]{} \delta$, it leads to a Poisson point process with intensity $e^{-\frac{x}{\delta}}$.

Corollary 1.4. *Let $\beta = \beta_n$ be such that $0 \le \beta \ll \dfrac{1}{n\log(n)}$. Let $(\lambda_1, ..., \lambda_n)$ with joint law $P_{n,\beta}$. Let $(a_n), (b_n)$ from Theorem 1.1 for $\delta = 1$. Then,*

$$P_{n,\beta}\left(a_n\left(\lambda_{\max} - b_n\right) \le x\right) \xrightarrow[n\infty]{} \exp\left(-\exp\left(-x\right)\right).$$

Remark 1.5. This result shows that we recover the Gumbel law as limiting distribution of the largest particle from the Poisson limit, so that we retrieve the result of [1] corresponding to our setup. Besides, in view of (1) in the next section, we know that the largest eigenvalue is unbounded when n goes to infinity since the Gaussian distribution has unbounded support. In addition to this observation, our main result provides the explicit order and Gumbel fluctuations for the maximum eigenvalue.

The paper is organized as follows : first, we introduce and comment our model. To derive Poisson statistics, our method is the study of the correlation functions associated to the extreme point process. We refer to this as our *main tool* and explain how it is exploited. Since the computations involve various estimates and quantities, we exhibit them as independent claims outside the main proof. The other sections are devoted to the precise proof of our result. We treat the inhomogeneous case first as it plainly describes the method used. It naturally includes the case $\beta = 0$ as the computations are similar. In a second time, we transpose our work to the homogeneous case and give a peculiar proof of the statement when $\beta = 0$. This is done by other means and displays a wider asymptotic regime for the perturbation (δ_n), so we present it as an independent result.

Remark 1.6. We consider two cases : $\delta_n = O(1)$ and $\delta_n \gg 1$. For the second case, the assumption required is (δ_n) such that $\log(\delta_n) \ll \sqrt{\log(n)}$. Nonetheless, most of our results remain valid under both regimes and with a weaker growth restriction. For this reason, in this text, the reader will encounter a less restrictive hypothesis on (δ_n), namely $\log(\delta_n) \ll \log(n)$. It ensures that b_n is equivalent to $\sqrt{2\log(n)}$ for any such (δ_n) as n goes to infinity.

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2 General model of the Gaussian β-ensemble for $\beta \ll 1$ and $\alpha > 0$

For any $\alpha > 0$, $\beta \geq 0$, and $n \geq 1$, we define :

$$Z_{n,\alpha,\beta} := \int_{\mathbb{R}^n} \exp\left(-\frac{\alpha}{2}\sum_{i=1}^{n} \lambda_i^2\right) |\Delta_n(\lambda)|^{\beta} \prod_{i=1}^{n} d\lambda_i$$

with the Vandermonde determinant factor :

$$|\Delta_n(\lambda)|^{\beta} := \prod_{i<j}^{n} |x_i - x_j|^{\beta},$$

and consider an exchangeable family $(\lambda_1, ..., \lambda_n)$ of random variables with joint law

$$P_{n,\alpha,\beta}(d\lambda_1, ..., d\lambda_n) := \frac{1}{Z_{n,\alpha,\beta}} \exp\left(-\frac{\alpha}{2}\sum_{i=1}^{n} \lambda_i^2\right) |\Delta_n(\lambda)|^{\beta} \prod_{i=1}^{n} d\lambda_i.$$

When $\alpha = 1$, we adopt the following notation :

$$Z_{n,\beta} := \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2\right) |\Delta_n(\lambda)|^{\beta} \prod_{i=1}^{n} d\lambda_i$$

$$P_{n,\beta}(d\lambda_1, ..., d\lambda_n) := \frac{1}{Z_{n,\beta}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2\right) |\Delta_n(\lambda)|^{\beta} \prod_{i=1}^{n} d\lambda_i.$$

In the sequel, the parameter α is always assumed to be 1 except in some specific cases which will be mentionned. The reason of this choice shall be clear after incoming explanations.

Remark 2.1. For $\beta = 0$, we retrieve the density of n i.i.d. Gaussian random variables, which form a system of uncorrelated particles. The partition function in this case is just $Z_{n,\beta=0} = (2\pi)^{\frac{n}{2}}$. Allowing $\beta > 0$, the Vandermonde factor vanishes when $\lambda_i = \lambda_j$ and acts as a repulsion (long range) force between the particles, which thereby constitutes a correlated system. The smaller β is, the weaker repulsion operates.

From the crucial matrix model [5], we endow the Gaussian β-ensemble with a matrix structure. Recall that $\chi(k) = \sqrt{\Gamma\left(\frac{k}{2}, \frac{1}{2}\right)}$ where a $\Gamma(a,b)$-distributed random variable has density $\frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)}$ on $(0, +\infty)$. We state the corresponding result for our setup :

Theorem 2.2. *Let $H := H_{n,\alpha,\beta}$ the tridiagonal symmetric random matrix defined as :*

$$\frac{1}{\sqrt{\alpha}} \begin{pmatrix} g_1 & \frac{1}{\sqrt{2}} X_{n-1} & & & & \\ \frac{1}{\sqrt{2}} X_{n-1} & g_2 & \frac{1}{\sqrt{2}} X_{n-2} & & & \\ & \frac{1}{\sqrt{2}} X_{n-2} & g_3 & \frac{1}{\sqrt{2}} X_{n-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \frac{1}{\sqrt{2}} X_1 \\ & & & & \frac{1}{\sqrt{2}} X_1 & g_n \end{pmatrix},$$

with $(g_i)_{1 \le i \le n} \sim \mathcal{N}(0,1)$ i.i.d. sequence, $(X_i)_{1 \le i \le n-1}$ an independent sequence such that $X_i \sim \chi(i\beta)$ and independent overall entries up to symmetry.

For any $\alpha > 0$, $\beta \ge 0$, the joint law of the eigenvalues $(\lambda_1, ..., \lambda_n)$ of H is $P_{n,\alpha,\beta}$.

It makes the connection between the particles of law $P_{n,\alpha,\beta}$ and the spectrum of H. By trace invariance, we can easily access to further information : when $\alpha \sim 1 + \frac{n\beta}{2}$, the empirical spectral distribution $L_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\{\lambda_i\}}$ of $H_{n,\alpha,\beta}$ has asymptotic first moment 0 and second moment 1. In our setting $\beta \ll \frac{1}{n}$, it reduces to consider $\alpha \sim 1$.

In [3], with the choice $\alpha \sim 1 + \frac{n\beta}{2}$, the authors proved under the assumption of simultaneous limit $n\beta_n \to 2\gamma$ as $n \to +\infty$, a continuous asymptotic interpolation for the empirical spectral measure between the Wigner semicircle law ($\gamma \to +\infty$) and the Gaussian distribution ($\gamma = 0$). The latter case is of our interest and particularly to the setting $\beta \ll \frac{1}{n}$, they proved that :

$$\frac{1}{n} \sum_{i=1}^{n} f(\lambda_i) \xrightarrow[n\infty]{\mathbb{P}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} f(x) e^{-\frac{x^2}{2}} dx, \qquad \forall f \in \mathcal{C}_b(\mathbb{R}). \tag{1}$$

It also justifies the choice $\alpha = 1$ in our model for $\beta \ll \frac{1}{n}$.

We now give the proof of Corollary 1.4 mentionned earlier.

Proof of Corollary 1.4. Let $\xi_n := \sum_{i=1}^{n} \delta_{a_n(\lambda_i - b_n)}$ the extreme process associated to $(\lambda_i)_{1\le i\le n}$ and ξ a Poisson point process with intensity e^{-x}. By Theorem 1.1, the process ξ_n converges weakly to ξ and hence,

$$P_{n,\beta}\left(a_n\left(\lambda_{\max} - b_n\right) \le x\right) = P_{n,\beta}\left(\xi_n\left([x, +\infty[\right) = 0\right) \underset{n\infty}{\longrightarrow} \mathbb{P}\left(\xi\left([x, +\infty[\right) = 0\right).$$

The number of points of a Poisson process with intensity λ in an interval (a, b) is a Poisson distributed random variable with mean $\int_a^b \lambda(t)dt$. It follows that :

$$\mathbb{P}\left(\xi\left([x, +\infty[\right) = 0\right) = \mathbb{P}\left(\mathcal{P}\left(\int_x^{+\infty} e^{-t}dt\right) = 0\right) = \exp\left(-\exp\left(-x\right)\right).$$

$\square$

2.1 Main tool

The theorem we intend to prove will stem from the following result, which thereby makes it the cornerstone of our demonstration. It ensures that under pointwise convergence of the correlation functions and some uniform bound on it, the initial point process converges to a Poisson process.

Lemma 2.3. *Let X be a locally compact Polish space and μ a Radon measure on X. Let $(\lambda_1, ..., \lambda_n)$ be an exchangeable random vector taking values in X with density ρ_n with respect to $\mu^{\otimes n}$. For $1 \le k \le n$, we define the k-th correlation function on X^k :*

$$R_k^n(x_1, ..., x_k) := \frac{n!}{(n-k)!} \int_{(x_{k+1},...,x_n)\in X^{n-k}} \rho_n(x_1, ..., x_n) d\mu^{\otimes(n-k)}(x_{k+1}, ..., x_n).$$

Suppose there exists $\theta \ge 0$ independent of n such that :

- *For $k \ge 1$, on X^k, we have the pointwise convergence :*

$$R_k^n(x_1, ..., x_n) \underset{n\infty}{\longrightarrow} \theta^k.$$

- *For each compact $K \subset X$, there exists θ_K such that for all n, k, on K^k, we have :*

$$1_{\{k\le n\}} R_k^n(x_1, ..., x_k) \le \theta_K^k.$$

Then, the point process $\mathcal{P}_n := \sum_{i=1}^{n} \delta_{\lambda_i}$ converges in distribution to a Poisson point process with intensity $\theta d\mu$ as $n \to +\infty$.

The proof can be found in [3, Prop.5.6]. We announce how we use our main tool.

Remark 2.4. We consider the point process $\mathcal{P}_n = \sum_{i=1}^{n} \delta_{a_n(\lambda_i - b_n)}$ with $(\lambda_i)_{i\leq n} \sim P_{n,\alpha,\beta}$, $\beta := \beta_n \ll \frac{1}{n\log(n)}$ and $\alpha = 1$.

- For $\delta_n = \delta + o(1)$, we will prove Poisson convergence according to the lemma with $\mu = e^{-\frac{x}{\delta}}dx$ and $(\lambda_1, ..., \lambda_n)$ with law
$$\rho_n d\mu^{\otimes n}(\lambda_1, ..., \lambda_n) = e^{-\frac{\alpha}{2}\sum_{i=1}^{n}\lambda_i^2} |\Delta_n(\lambda)|^\beta e^{\frac{1}{\delta}\sum_{i=1}^{n}\lambda_i} \prod_{i=1}^{n} d\lambda_i.$$
- For $\delta_n \gg 1$, ρ_n is just $P_{n,\alpha,\beta}$ and the intensity is the Lebesgue measure $\mu = d\lambda$.

2.2 Partition functions

In this section, we list some identities, bounds and asymptotics involving partition fonctions. They will be used from time to time in the sequel of the text.

First, we give the main formula for the partition functions. From this, we will be able to compute several asymptotics of partition functions ratio.

Lemma 2.5. *For any $\alpha, \beta > 0$ and $n \geq 1$, the following identity holds :*
$$Z_{n,\alpha,\beta} = (2\pi)^{\frac{n}{2}}(n!)\alpha^{-\beta\frac{n(n-1)}{4} - \frac{n}{2}} \prod_{i=0}^{n-1} \frac{\Gamma\left((i+1)\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}.$$
If $\beta \geq 0$, one has also :
$$Z_{n,\alpha,\beta} = (2\pi)^{\frac{n}{2}}\alpha^{-\beta\frac{n(n-1)}{4} - \frac{n}{2}} \prod_{i=1}^{n} \frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)}.$$

Proof. Let $\beta > 0$. By the Selberg integral theorem in [2], we have :
$$\int_{\mathbb{R}^n} |\Delta_k(x)|^\beta e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2} dx_1...dx_n = (n!)(2\pi)^{\frac{n}{2}} \prod_{i=0}^{n-1} \frac{\Gamma\left((i+1)\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}.$$
By the change of variable $x_i = y_i\sqrt{\alpha}$, we get the fundamental identity on partition functions
$$\begin{aligned} Z_{n,\alpha,\beta} &= (2\pi)^{\frac{n}{2}}(n!)\alpha^{-\beta\frac{n(n-1)}{4} - \frac{n}{2}} \prod_{i=0}^{n-1} \frac{\Gamma\left((i+1)\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} \\ &= (2\pi)^{\frac{n}{2}}\alpha^{-\beta\frac{n(n-1)}{4} - \frac{n}{2}} \prod_{i=1}^{n} \frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)}. \end{aligned}$$
The case $\beta = 0$ is easily treated. $\square$

We are now ready to prove several results needed later.

Lemma 2.6. *Assume $n\beta \ll 1$. Then, for k fixed, $\alpha > 0$ and $\beta \geq 0$,*

$$\frac{Z_{n-k,\alpha,\beta}}{Z_{n,\alpha,\beta}} = (1+o(1))\,(2\pi)^{-\frac{k}{2}}\,\alpha^{\frac{k}{2}}.$$

Proof. For $u \ll 1$, recall the equivalence of the Gamma function near the origin :

$$\Gamma(u) = \frac{1}{u}\,(1+o(1)) \gg 1.$$

By Lemma 2.5, we compute the ratio for $\beta > 0$:

$$\begin{aligned}
\frac{Z_{n-k,\alpha,\beta}}{Z_{n,\alpha,\beta}} &= (2\pi)^{-\frac{k}{2}}\,\frac{(n-k)!}{n!}\alpha^{\frac{k}{2}+\frac{\beta}{4}(2nk-k(k+1))}\prod_{i=n-k}^{n-1}\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left((i+1)\,\frac{\beta}{2}\right)}\\
&= (1+o(1))\,(2\pi)^{-\frac{k}{2}}\,n^{-k}\alpha^{\frac{k}{2}}\prod_{i=n-k}^{n-1}\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left((i+1)\,\frac{\beta}{2}\right)}\\
&= (1+o(1))\,(2\pi)^{-\frac{k}{2}}\,n^{-k}\alpha^{\frac{k}{2}}\left(\frac{\beta}{2}\right)^{-k}\prod_{i=n-k}^{n-1}\frac{1}{\Gamma\left((i+1)\,\frac{\beta}{2}\right)}\\
&= (1+o(1))\,(2\pi)^{-\frac{k}{2}}\,n^{-k}\alpha^{\frac{k}{2}}\left(\frac{\beta}{2}\right)^{-k}\left(\frac{n\beta}{2}\right)^{k}\\
&= (1+o(1))\,(2\pi)^{-\frac{k}{2}}\,\alpha^{\frac{k}{2}}.
\end{aligned}$$

If $\beta = 0$, the identity claimed is readily computed from the main formula on partition functions. $\square$

Lemma 2.7. *Let any (δ_n) positive real sequence such that $\log(\delta_n) \ll \log(n)$. Let $\beta \in (0,2)$, $\alpha > 0$ and $n \geq 2$ such that $\alpha b_n^2 - \dfrac{\beta}{4} > 0$, then :*

$$\frac{Z_{n-1,\alpha b_n^2-\frac{\beta}{4},\beta}}{Z_{n,\beta}} \leq \frac{1}{\sqrt{2\pi}}\left(\sqrt{\alpha}b_n\right)^{-\beta\frac{(n-1)(n-2)}{2}-n+1}.$$

Proof. From the identity :

$$Z_{n,\alpha,\beta} = (2\pi)^{\frac{n}{2}}\alpha^{-\beta\frac{n(n-1)}{4}-\frac{n}{2}}\prod_{i=1}^{n}\frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)},$$

we have :

$$Z_{n-1,\alpha b_n^2-\frac{\beta}{4},\beta} = (2\pi)^{\frac{n-1}{2}}\left(\alpha b_n^2-\frac{\beta}{4}\right)^{-\beta\frac{(n-1)(n-2)}{4}-\frac{n-1}{2}}\prod_{i=1}^{n-1}\frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)}$$

and we can compute the ratio :

$$\frac{Z_{n-1,\alpha b_n^2-\frac{\beta}{4},\beta}}{Z_{n,\beta}} = \frac{(2\pi)^{\frac{n-1}{2}}\left(\alpha b_n^2-\frac{\beta}{4}\right)^{-\beta\frac{(n-1)(n-2)}{4}-\frac{n-1}{2}}\prod_{i=1}^{n-1}\frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)}}{(2\pi)^{\frac{n}{2}}\prod_{i=1}^{n}\frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)}}$$
$$= \frac{1}{\sqrt{2\pi}}\left(\alpha b_n^2-\frac{\beta}{4}\right)^{-\beta\frac{(n-1)(n-2)}{4}-\frac{n-1}{2}}\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{n\beta}{2}\right)}$$
$$= \frac{1}{\sqrt{2\pi}}\left(\sqrt{\alpha}b_n\right)^{-\beta\frac{(n-1)(n-2)}{2}-n+1}\left(1-\frac{\beta}{4\alpha b_n^2}\right)^{-\beta\frac{(n-1)(n-2)}{4}-\frac{n-1}{2}}\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{n\beta}{2}\right)}.$$

We apply the following inequality :

$$\frac{1}{1-x}\leq 4^x, \qquad x\in[0,\frac{1}{2}],$$

with $x=\dfrac{\beta}{4\alpha b_n^2}\leq\dfrac{1}{2}\iff\dfrac{\beta}{2}\leq\alpha b_n^2$ (which is obvious for $\beta\ll 1$). Thus,

$$\frac{Z_{n-1,b_n^2-\frac{\beta}{4},\beta}}{Z_{n,\beta}}\leq\left(\frac{1}{4}\right)^{\frac{\beta^2(n-1)(n-2)}{16\alpha b_n^2}+\frac{\beta(n-1)}{8\alpha b_n^2}}\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\sqrt{2\pi}\Gamma\left(1+\frac{n\beta}{2}\right)}\left(\sqrt{\alpha}b_n\right)^{-\beta\frac{(n-1)(n-2)}{2}-n+1}$$
$$\leq\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\sqrt{2\pi}\Gamma\left(1+\frac{n\beta}{2}\right)}\left(\sqrt{\alpha}b_n\right)^{-\beta\frac{(n-1)(n-2)}{2}-n+1}.$$

Gamma function has local minimum at ~ 0.8 with value ≈ 1.44, it follows that for $\beta\ll 1$,

$$\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\sqrt{2\pi}\Gamma\left(1+\frac{n\beta}{2}\right)}\leq\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\sqrt{2\pi}}\leq\frac{\Gamma(2)}{\sqrt{2\pi}}=\frac{1}{\sqrt{2\pi}}.$$

□

Lemma 2.8. *Let any* (δ_n) *positive real sequence such that* $\dfrac{\log(\delta_n)}{\log(n)}\ll 1$*. Assume* $\beta\ll\dfrac{1}{n}$*. Fix* $\alpha>0$ *and* $k\leq n$*. Then,*

$$\frac{Z_{n-k,\alpha-\frac{k\beta}{4b_n^2},\beta}}{Z_{n-k,\alpha,\beta}}=1+o(1).$$

Proof. For $\alpha>0$, we have :

$$Z_{n-k,\alpha-\frac{k\beta}{4b_n^2},\beta}=(2\pi)^{\frac{n-k}{2}}(n-k!)\alpha^{-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}}\left(1-\frac{k\beta}{4\alpha b_n^2}\right)^{-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}}\prod_{i=0}^{n-k-1}\frac{\Gamma\left((i+1)\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}.$$

Thus by a Taylor expansion of $x \mapsto \log(1-x)$ around 0 :

$$\begin{aligned}
\frac{Z_{n-k,\alpha-\frac{k\beta}{4b_n^2},\beta}}{Z_{n-k,\alpha,\beta}} &= \left(1-\frac{k\beta}{4\alpha b_n^2}\right)^{-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}} \\
&= \exp\left(\left(-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}\right)\log\left(1-\frac{k\beta}{4\alpha b_n^2}\right)\right) \\
&= \exp\left(\left(-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}\right)\left(-\frac{k\beta}{4\alpha b_n^2}+O\left(-\frac{k\beta}{4\alpha b_n^2}\right)^2\right)\right).
\end{aligned}$$

The last term converges to 1 under our hypothesis. $\square$

Lemma 2.9. *Let $\beta \geq 0$ and (δ_n) positive real sequence such that $\log(\delta_n) \ll \log(n)$. Assume $\beta \ll \frac{1}{n}$. Fix $k \leq n$. Then,*

$$\frac{Z_{n-k,1-\frac{k\beta}{4b_n^2},\beta}}{Z_{n-k,\beta}} \leq 4^k$$

$$\frac{Z_{n-k,\beta}}{Z_{n,\beta}} \leq \left(\sqrt{\frac{2}{\pi}}\right)^k.$$

Proof. Since the case $\beta = 0$ can be easily treated, we only consider $\beta > 0$. From our hypothesis, $k\beta$ is less than 1 and :

$$\frac{k\beta}{4b_n^2} \leq \frac{1}{2} \iff k\beta \leq 2b_n^2 \qquad \text{which is true.}$$

$$\frac{k\beta}{4b_n^2}\left(\beta\frac{(n-k)(n-k-1)}{4}+\frac{n-k}{2}\right) \leq \frac{(n\beta)^2}{16b_n^2}+\frac{n\beta}{8b_n^2} \leq 1.$$

So by applying the inequality $(1-x)^{-1} \leq 4^x$ on $[0,\frac{1}{2}]$, we compute :

$$\begin{aligned}
\frac{Z_{n-k,1-\frac{k\beta}{4b_n^2},\beta}}{Z_{n-k,\beta}} &= \frac{(2\pi)^{\frac{n-k}{2}}(n-k!)\left(1-\frac{k\beta}{4b_n^2}\right)^{-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}}\prod_{i=0}^{n-k-1}\frac{\Gamma\left((i+1)\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}}{(2\pi)^{\frac{n-k}{2}}(n-k!)\prod_{i=0}^{n-k-1}\frac{\Gamma\left((i+1)\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}} \\
&= \left(1-\frac{k\beta}{4b_n^2}\right)^{-\beta\frac{(n-k)(n-k-1)}{4}-\frac{n-k}{2}} \\
&\leq \exp\left(\frac{k\beta}{4b_n^2}\left(\beta\frac{(n-k)(n-k-1)}{4}+\frac{n-k}{2}\right)\log(4)\right) \\
&\leq 4^k.
\end{aligned}$$

We prove the second statement. Since

$$Z_{n,\beta} = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{i\beta}{2})},$$

then,

$$\frac{Z_{n-k,\beta}}{Z_{n,\alpha,\beta}} = (2\pi)^{-\frac{k}{2}} \prod_{i=n-k+1}^{n} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{i\beta}{2})}.$$

The Gamma function has local minimum at ≈ 1.46 with value ≈ 0.8, it follows that for any $i \leq n$, since $\beta \ll 1$,

$$\frac{1}{2} \leq \Gamma\left(1+\frac{i\beta}{2}\right) \leq \Gamma\left(1+\frac{\beta}{2}\right) \leq 1.$$

Hence,

$$\prod_{i=n-k+1}^{n} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{i\beta}{2})} \leq 2^k.$$

□

At last, we prove the previously stated lemma which compares the partition functions between different regime of β :

Proof of Lemma 1.3. Denoting γ the Euler constant, recall that for $x \ll 1$:

$$\log\Gamma\left(1+x\right) = -\gamma x + \frac{\pi^2}{12}x^2 + o(x^3).$$

Remark that for any $k \geq 1$, one has :

$$n^k \beta^{k-1} \gg n^{k+1}\beta^k.$$

We compute the ratios :

$$\begin{aligned}\frac{Z_{n,\beta}}{Z_{n,0}} &= \prod_{i=1}^{n} \frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{\beta}{2}\right)} \\ &= \exp\left(-n\log\Gamma\left(1+\frac{\beta}{2}\right) + \sum_{i=1}^{n}\log\Gamma\left(1+\frac{i\beta}{2}\right)\right)\end{aligned}$$

and,

$$\begin{aligned}\frac{Z_{n,\beta}}{Z_{n,\beta'}} &= \prod_{i=1}^{n} \frac{\Gamma\left(1+\frac{i\beta}{2}\right)}{\Gamma\left(1+\frac{i\beta'}{2}\right)} \\ &= \exp\left(-\sum_{i=1}^{n}\log\Gamma\left(1+\frac{i\beta'}{2}\right) + \sum_{i=1}^{n}\log\Gamma\left(1+\frac{i\beta}{2}\right)\right).\end{aligned}$$

Since the quantity $i\beta$ converges to 0 uniformly in $i \leq n$, we deduce that for some $\varepsilon_{i,n} = o\left((i\beta)^3\right)$, thus verifying $\varepsilon_{i,n} = O\left((n\beta)^3\right)$ for any $i \leq n$,

$$\begin{aligned}
\sum_{i=1}^{n} \log\Gamma\left(1+\frac{i\beta}{2}\right) &= -\frac{\gamma\beta}{2}\sum_{i=1}^{n} i + \frac{\pi^2}{48}\beta^2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\varepsilon_{i,n} \\
&= -\frac{\gamma\beta}{8}(n^2+n) + \frac{\pi^2}{48}\beta^2\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) + \sum_{i=1}^{n}\varepsilon_{i,n} \\
&= -\frac{\gamma\beta}{8}(n^2+n) + \frac{\pi^2}{48}\beta^2\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) + nO\left((n\beta)^3\right) \\
&= -\frac{\gamma}{8}n^2\beta\left(1+o(1)\right).
\end{aligned}$$

Indeed, if $\phi = nO\left((n\beta)^3\right)$, then $\dfrac{\phi}{n} = O\left((n\beta)^3\right)$, hence $\phi = o\left(n^4\beta^3\right)$ and by a previous remark, $\phi \ll n^2\beta$.

We deduce that :

$$\frac{Z_{n,\beta}}{Z_{n,0}} = \exp\left(-\frac{\gamma}{8}n^2\beta\left(1+o(1)\right)\right)$$

and,

$$\frac{Z_{n,\beta}}{Z_{n,\beta'}} = \exp\left(\frac{\gamma}{8}n^2\left(\beta'-\beta\right)\left(1+o(1)\right)\right).$$

The claims readily follow. $\square$

2.3 Estimates : bulk and largest eigenvalues

In the section, we establish some estimates on the eigenvalues $(\lambda_i)_{1\leq i\leq n}$ of $H_{n,\alpha,\beta}$, which are $P_{n,\alpha,\beta}$-distributed. Since the particles are exchangeable, every estimate will concern λ_1.

We give exponential type bound on the probability of a scaled eigenvalue to be larger than any arbitrary value. Same-wise, an exponential estimate for the probability of λ_1 to be as close as we want to any value is given.

These estimates will be crucial for the analysis of the integral term $\tilde{R}_k^n$, which presents itself as the expectation of some functional of (λ_i). The link is made through to the identity

$$\mathbb{E}\left|X\right| = \int_0^{+\infty}\mathbb{P}\left(|X|\geq t\right)dt.$$

We begin with a technical but fundamental lemma.

Lemma 2.10. *For any $a, b \in \mathbb{R}$ and $\beta > 0$, one has :*

$$|a+b|^\beta \leq 2^\beta e^{\beta \frac{a^2+b^2}{8}}.$$

Proof. First recall two inequalities :

$$|x| \leq 2e^{\frac{x^2}{16}}.$$

$$(x+y)^2 \leq 2x^2 + 2y^2.$$

Applying the first inequality with $x = a+b$, then using the second one give :

$$\begin{aligned}|a+b|^\beta &\leq \left(2e^{\frac{(a+b)^2}{16}}\right)^\beta \\ &\leq \left(2e^{\frac{a^2+b^2}{8}}\right)^\beta.\end{aligned}$$

□

This inequality is of interest because it roughly allows to gain quadratic sum bound a^2+b^2 from a quantity of type $\log|a+b|$. It provides an useful algebraic mean to upper-bound the integral term $\tilde{R}_k^n$ with a ratio of partition functions.

We show the following estimate on the scaled top eigenvalue :

Lemma 2.11. *Let $M > 0$ such that $\alpha \vee n\beta \leq M$. There exists a constant $C_M > 0$ such that for any $n \geq 1$, $\alpha, \beta, t > 0$ and $u \in \mathbb{R}$,*

$$P_{n,\alpha,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \geq t\right) \leq C_M b_n^{\beta(n-1)-4} \frac{\exp\left(-\frac{\alpha}{2}\left(b_n^2 - \frac{n\beta}{4\alpha}\right)^2\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)^2\right)}{\alpha^{1+\frac{\beta(n-1)(n-2)}{4}+\frac{n}{2}}\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)}.$$

Proof. Let $x \in \mathbb{R}$ fixed and set $u := 1 + \dfrac{x}{a_n b_n}$. Let $(\lambda_1, ..., \lambda_n)$ an exchangeable family of random variables distributed according to $P_{n,\alpha,\beta}$. Then, the family $\left(\tilde{\lambda}_i\right)_{1\leq i\leq n} := \left(\dfrac{\lambda_i}{b_n} - u\right)_{1\leq i\leq n}$ has law :

$$\frac{b_n^{n+\beta\frac{n(n-1)}{2}}}{Z_{n,\alpha,\beta}} |\Delta_n(z)|^\beta e^{-\frac{\alpha}{2}b_n^2\sum_{i=1}^n (z_i+u)^2} dz_1...dz_n.$$

Now for $t > 0$,

$$\begin{aligned}
\Lambda_{n,t,u} &:= P_{n,\alpha,\beta}\left(\left|1+\frac{x}{a_n b_n}-\frac{\lambda_1}{b_n}\right| \geq t\right) \\
&= P_{n,\alpha,\beta}\left(\left|\tilde{\lambda}_1\right| \geq a\right) \\
&= \frac{b_n^{n+\beta\frac{n(n-1)}{2}}}{Z_{n,\alpha,\beta}} \int_{|z_1|\geq t} \int_{(z_2,\ldots,z_n)\in\mathbb{R}^{n-1}} |\Delta_n(z)|^\beta \exp\left(-\frac{\alpha b_n^2}{2}\sum_{i=1}^n (z_i+u)^2\right) dz_1 \ldots dz_n \\
&= \frac{b_n^{n+\beta\frac{n(n-1)}{2}}}{Z_{n,\alpha,\beta}} \int_{|z_1|\geq t} \prod_{j=2}^n |z_1-z_j|^\beta e^{-\frac{\alpha b_n^2}{2}(z_1+u)^2} dz_1 \times \cdots \\
&\qquad \cdots \times \int_{\mathbb{R}^{n-1}} |\Delta_{n-1}(z_2,\ldots,z_n)|^\beta e^{-\frac{\alpha}{2}b_n^2\sum_{i=1}^n (z_i+u)^2} dz_2 \ldots dz_n.
\end{aligned}$$

The product term in the first integral involves every variables. We split this overlapping term thanks to the fundamental inequality of Lemma 2.10 :

$$|a+b|^\beta \leq 2^\beta e^{\beta\frac{a^2+b^2}{8}}.$$

It leads to :

$$\begin{aligned}
\Lambda_{n,t,u} \leq \frac{b_n^{n+\beta\frac{n(n-1)}{2}} 2^{n\beta}}{Z_{n,\alpha,\beta}} \int_{|z_1|\geq t} \exp\left(\frac{n\beta}{8}z_1^2 - \alpha\frac{b_n^2}{2}(z_1+u)^2\right) dz_1 \times \cdots \\
\cdots \times \int_{(z_2,\ldots,z_n)\in\mathbb{R}^{n-1}} |\Delta_{n-1}(z_2,\ldots,z_n)|^\beta \exp\left(\frac{\beta}{8}\sum_{i=2}^n z_i^2 - \alpha\frac{b_n^2}{2}\sum_{i=2}^n (z_i+u)^2\right) dz_2 \ldots dz_n.
\end{aligned}$$

The first integral term will be linked to a Gaussian tail and the second to a partition function. For this, we need to complete the square.

Using the two following algebraic identities :

$$\begin{aligned}
\frac{\beta}{8}\sum_{i=2}^n z_i^2 - \alpha\frac{b_n^2}{2}\sum_{i=2}^n (z_i+u)^2 = -\frac{\alpha}{2}\left(b_n^2-\frac{\beta}{4\alpha}\right)\sum_{i=2}^n\left(z_i+\frac{b_n^2 u}{b_n^2-\frac{\beta}{4\alpha}}\right)^2 \\
+\alpha\frac{b_n^4 u^2}{2\left(b_n^2-\frac{\beta}{4\alpha}\right)}(n-1) - \alpha\frac{b_n^2 u^2}{2}(n-1)
\end{aligned}$$

$$\frac{n\beta}{8}z_1^2 - \alpha\frac{b_n^2}{2}(z_1+u)^2 = -\frac{\alpha}{2}\left(b_n^2-\frac{n\beta}{4\alpha}\right)\left(z_1+\frac{b_n^2 u}{b_n^2-\frac{n\beta}{4\alpha}}\right)^2 + \alpha\frac{b^4u^2}{2\left(b_n^2-\frac{n\beta}{4\alpha}\right)} - \alpha\frac{b^2u^2}{2},$$

we can write :

$$\Lambda_{n,t,u} \leq \frac{b_n^{n+\beta\frac{n(n-1)}{2}} 2^{n\beta}}{Z_{n,\alpha,\beta}} e^{\alpha\frac{b^4u^2}{2\left(b_n^2-\frac{n\beta}{4\alpha}\right)}-\alpha\frac{b^2u^2}{2}} G(t)Z.$$

where

$$G(t) := \int_{|z_1|\geq t} \exp\left(-\frac{\alpha}{2}\left(b_n^2 - \frac{n\beta}{4\alpha}\right)\left(z_1 + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)^2\right) dz_1$$

$$Z := e^{\alpha \frac{b_n^4 u^2 (n-1)}{2\left(b_n^2 - \frac{\beta}{4\alpha}\right)} - \alpha \frac{b_n^2 u^2 (n-1)}{2}} \int_{\mathbb{R}^{n-1}} |\Delta_{n-1}(\lambda)|^\beta \exp\left(-\frac{\alpha}{2}\left(b_n^2 - \frac{\beta}{4\alpha}\right)\sum_{i=2}^n \left(\lambda_i + \frac{b_n^2 u}{b_n^2 - \frac{\beta}{4\alpha}}\right)^2\right) d\lambda_2 ... d\lambda_{n-1}$$

which is just :

$$Z = e^{\alpha \frac{b_n^4 u^2}{2\left(b_n^2 - \frac{\beta}{4\alpha}\right)}(n-1) - \alpha \frac{b_n^2 u^2}{2}(n-1)} Z_{n-1, \alpha b_n^2 - \frac{\beta}{4}, \beta}.$$

We treat the Gaussian integral term $G(t)$ in the RHS of :

$$\Lambda_{n,t,u} \leq \frac{b_n^{n+\beta\frac{n(n-1)}{2}} 2^{n\beta}}{Z_{n,\alpha,\beta}} e^{\alpha \frac{b_n^4 u^2}{2\left(b_n^2 - \frac{\beta}{4}\right)}(n-1) + \alpha \frac{b_n^4 u^2}{2\left(b_n^2 - \frac{n\beta}{4}\right)} - \alpha n \frac{b_n^2 u^2}{2}} Z_{n-1, \alpha b_n^2 - \frac{\beta}{4}, \beta} G(t).$$

By two successive change of variable,

$$\begin{aligned}
G(t) &= \int_{\left|z - \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right| \geq t} \exp\left(-\frac{\alpha}{2}\left(b_n^2 - \frac{n\beta}{4\alpha}\right) z^2\right) dz \\
&= \frac{1}{\sqrt{\alpha}\sqrt{b_n^2 - \frac{n\beta}{4}}} \int_{\left|\frac{z}{\alpha\left(b_n^2 - \frac{n\beta}{4\alpha}\right)} - \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right| \geq t} \exp\left(-\frac{z^2}{2}\right) dz \\
&= \frac{2}{\sqrt{\alpha}\sqrt{b_n^2 - \frac{n\beta}{4}}} \int_{\frac{z}{\alpha\left(b_n^2 - \frac{n\beta}{4\alpha}\right)} - \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}} \geq t} \exp\left(-\frac{z^2}{2}\right) dz \qquad \text{by symmetry} \\
&= \frac{2}{\sqrt{\alpha}\sqrt{b_n^2 - \frac{n\beta}{4\alpha}}} \int_{z \geq \alpha\left(b_n^2 - \frac{n\beta}{4\alpha}\right)\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)} \exp\left(-\frac{z^2}{2}\right) dz \\
&\leq \frac{2}{\alpha\sqrt{\alpha}} \frac{e^{-\frac{\alpha}{2}\left(b_n^2 - \frac{n\beta}{4\alpha}\right)^2\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)^2}}{\left(b_n^2 - \frac{n\beta}{4\alpha}\right)^{\frac{3}{2}}\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)}.
\end{aligned}$$

We used the classical Gaussian bound in the last line :

$$\int_y^{+\infty} e^{-\frac{y^2}{2}} dy \leq \frac{e^{-\frac{y^2}{2}}}{y}, \qquad y > 0.$$

Finally,

$$\Lambda_{n,t,u} \leq b_n^{n+\beta\frac{n(n-1)}{2}} 2^{n\beta+1} e^{\alpha \frac{b_n^4 u^2}{2\left(b_n^2 - \frac{\beta}{4\alpha}\right)}(n-1) + \alpha \frac{b_n^4 u^2}{2\left(b_n^2 - \frac{n\beta}{4\alpha}\right)} - \alpha n \frac{b_n^2 u^2}{2}} \frac{Z_{n-1, \alpha b_n^2 - \frac{\beta}{4}, \beta}}{Z_{n,\alpha,\beta}} \frac{e^{-\frac{\alpha}{2}\left(b_n^2 - \frac{n\beta}{4\alpha}\right)^2\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)^2}}{\alpha^{\frac{3}{2}}\left(b_n^2 - \frac{n\beta}{4\alpha}\right)^{\frac{3}{2}}\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4\alpha}}\right)}.$$

We deal with the ratio of partition functions by Lemma 2.7, so that the upper bound on $P_{n,\alpha,\beta}\left(\left|u-\frac{\lambda_1}{b_n}\right|\geq t\right)$ becomes :

$$\Lambda_{n,t,u}\leq \frac{b_n^{\beta(n-1)-1}2^{n\beta+1}}{\sqrt{2\pi}\left(b_n^2-\frac{n\beta}{4\alpha}\right)^{\frac{3}{2}}}e^{\alpha\frac{b_n^4u^2}{2\left(b_n^2-\frac{\beta}{4\alpha}\right)}(n-1)+\alpha\frac{b_n^4u^2}{2\left(b_n^2-\frac{n\beta}{4\alpha}\right)}-\alpha n\frac{b_n^2u^2}{2}}\frac{e^{-\frac{\alpha}{2}\left(b_n^2-\frac{n\beta}{4\alpha}\right)^2\left(t+\frac{b_n^2u}{b_n^2-\frac{n\beta}{4\alpha}}\right)^2}}{\alpha\left(t+\frac{b_n^2u}{b_n^2-\frac{n\beta}{4\alpha}}\right)}\alpha^{-\beta\frac{(n-1)(n-2)}{4}-\frac{n}{2}}.$$

Finally, for M such that $n\beta\wedge\alpha\leq M$:

$$\Lambda_{n,t,u}\leq C_Mb_n^{\beta(n-1)-4}\frac{\exp\left(-\frac{\alpha}{2}\left(b_n^2-\frac{n\beta}{4\alpha}\right)^2\left(t+\frac{b_n^2u}{b_n^2-\frac{n\beta}{4\alpha}}\right)^2\right)}{\alpha^{1+\frac{\beta(n-1)(n-2)}{4}+\frac{n}{2}}\left(t+\frac{b_n^2u}{b_n^2-\frac{n\beta}{4\alpha}}\right)}.$$

□

In the very same way, we show the following estimate for bulk eigenvalues :

Lemma 2.12. *Let $M>0$. There exists a constant $C_M>0$ such that for any $n\geq 1,\alpha>0,\beta>0,a\in\mathbb{R},\varepsilon\in(0,1)$ such that $\alpha\vee n\beta\leq M$,*

$$P_{n,\alpha,\beta}\left(|\lambda_1-a|\leq\varepsilon\right)\leq C_M\varepsilon\exp\left(\frac{n\alpha\beta}{2(4\alpha-\beta)}a^2\right).$$

Proof. We proceed as for the previous estimate. □

3 Inhomogeneous Poisson limit for $n\beta\ll 1$ and $\alpha=1$

This section is devoted to the proof of the first statement in our main result when $\beta>0$, namely, we consider $(\lambda_1,...,\lambda_n)\sim P_{n,\beta}$ and assume $\delta_n\underset{n\infty}{\longrightarrow}\delta>0$, then, for an appropriate choice of $(a_n),(b_n)$, the random point process $\sum_{i=1}^n\delta_{a_n(\lambda_i-b_n)}$ converges in distribution to a Poisson point process with intensity $e^{-\frac{x}{\delta}}dx$.

The plan, according to the lemma on Poisson convergence, is to first reformulate the correlation functions, and then establish the pointwise convergence to 1 of the correlation functions. The last step is to give an uniform upper bound which will end the proof of the theorem.

The case $\beta=0$ is much simpler. Following the same steps, it does not however involve the machinery of partition functions and tail bounds. So we will omit in the upcoming subsections.

3.1 Correlation functions

The first step is to give a satisfying formulation of the correlation function R_k^n. From its definition, we transpose it as product of multiple terms including an integral term $\tilde{R}_k^n$. Unlike the others, this quantity is more complicated and needs careful analysis.

Lemma 3.1. *Fix $\delta > 0$. Let $\alpha > 0$, $\beta \geq 0$ and $(\lambda_1, ..., \lambda_n)$ distributed according to $P_{n,\alpha,\beta}$. The k-th correlation function of the point process $\sum_{i=1}^{n} \delta_{a_n(\lambda_i - b_n)}$ is :*

$$R_k^n(x_1, ..., x_k) = \frac{n!}{(n-k)!} a_n^{-k-\frac{\beta}{2}k(k-1)} \left|\Delta_k(x)\right|^\beta \frac{Z_{n-k,\alpha,\beta}}{Z_{n,\alpha,\beta}} \times \cdots$$
$$\cdots \times \exp\left(-\frac{\alpha}{2}\sum_{i=1}^{k}\left(\frac{x_i}{a_n} + b_n\right)^2 + \frac{1}{\delta}\sum_{i=1}^{k} x_i + k\beta(n-k)\log(b_n)\right)\tilde{R}_k^n$$

with

$$\tilde{R}_k^n := \tilde{R}_k^n(x_1, ..., x_k) = \int_{\mathbb{R}^{n-k}} \exp\left(\beta \sum_{i=1}^{k}\sum_{j=1}^{n-k} \log\left|1 + \frac{x_i}{a_n b_n} - \frac{z_j}{b_n}\right|\right) dP_{n-k,\alpha,\beta}(z_1, ..., z_{n-k}).$$

Proof. Let $\mu = e^{-\frac{x}{\delta}}dx$, ie: $d\mu^{\otimes n}(x_1, ..., x_n) = e^{-\frac{1}{\delta}\sum_{i=1}^n x_i} dx_1 ... dx_n$.
Let $(\lambda_1, ..., \lambda_n)$ distributed according to $P_{n,\alpha,\beta}$:

$$P_{n,\alpha,\beta}(\lambda_1, ..., \lambda_n) = \frac{1}{Z_{n,\alpha,\beta}} \left|\Delta_n(\lambda)\right|^\beta \exp\left(-\frac{\alpha}{2}\sum_{i=1}^{n}\lambda_i^2\right) d\lambda_1 ... d\lambda_n.$$

Set $\left(\tilde{\lambda}_1, ..., \tilde{\lambda}_n\right)_{i\leq n} := \left(a_n\left(\lambda_i - b_n\right)\right)_{i\leq n}$. By a change of variable in $P_{n,\alpha,\beta}$, we get the joint density of $\left(\tilde{\lambda}_1, ..., \tilde{\lambda}_n\right)_{i\leq n}$:

$$\begin{aligned}\tilde{P}_{n,\alpha,\beta}\left(\tilde{\lambda}_1, ..., \tilde{\lambda}_n\right) &= \frac{a_n^{-\frac{n(n-1)}{2}\beta - n}}{Z_{n,\alpha,\beta}} \left|\Delta_n(\lambda)\right|^\beta e^{-\frac{\alpha}{2}\sum_{i=1}^n\left(\frac{\lambda_i}{a_n}+b_n\right)^2} d\lambda_1 ... d\lambda_n \\ &= \frac{a_n^{-\frac{n(n-1)}{2}\beta - n}}{Z_{n,\alpha,\beta}} \left|\Delta_n(\lambda)\right|^\beta e^{-\frac{\alpha}{2}\sum_{i=1}^n\left(\frac{\lambda_i}{a_n}+b_n\right)^2} e^{\frac{1}{\delta}\sum_{i=1}^n \lambda_i} d\mu^{\otimes n}\left(\lambda_1, ..., \lambda_n\right).\end{aligned}$$

Hence, we get the k-th correlation function :

$$R_k^n(x_1, ..., x_k) = \frac{n!}{(n-k)!} \frac{a_n^{-n-\beta\frac{n(n-1)}{2}}}{Z_{n,\alpha,\beta}} e^{-\frac{\alpha}{2}\sum_{i=1}^k\left(\frac{x_i}{a_n}+b_n\right)^2 + \frac{1}{\delta}\sum_{i=1}^k x_i} \times \cdots$$
$$\cdots \times \int_{\mathbb{R}^{n-k}} e^{-\frac{\alpha}{2}\sum_{i=k+1}^n\left(\frac{x_i}{a_n}+b_n\right)^2} \left|\Delta_n(x)\right|^\beta e^{\frac{1}{\delta}\sum_{i=k+1}^k x_i} d\mu^{\otimes(n-k)}(x_{k+1}, ..., x_n).$$

The goal is to extricate the $(x_1, ..., x_k)$ from the $(x_{k+1}, ..., x_n)$, and extract all leading order terms.

To this end, we begin with splitting the Vandermonde term :

$$\prod_{i<j}^{n} |x_i - x_j|^\beta = \left(\prod_{1 \le i<j \le k} |x_i - x_j|^\beta \right) \left(\prod_{k+1 \le i<j \le n} |x_i - x_j|^\beta \right) \left(\prod_{i=1}^{k} \prod_{j=k+1}^{n} |x_i - x_j|^\beta \right).$$

Note that in the RHS, the first term has $\frac{k(k-1)}{2}$ elements, the 2nd term has $\frac{(n-k)(n-k-1)}{2}$ elements and the last term has $k(n-k)$ elements.

Therefore,

$$R_k^n(x_1, ..., x_k) = \frac{n!}{(n-k)!} |\Delta_k(x)|^\beta \frac{a_n^{-n-\beta\frac{n(n-1)}{2}}}{Z_{n,\alpha,\beta}} \exp\left(-\frac{\alpha}{2} \sum_{i=1}^{k} \left(\frac{x_i}{a_n} + b_n \right)^2 + \frac{1}{\delta} \sum_{i=1}^{k} x_i \right) \Lambda.$$

where :

$$\Lambda := \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=k+1}^{n} \log|x_i - y_j|} e^{-\frac{\alpha}{2} \sum_{i=1}^{n-k} \left(\frac{y_i}{a_n} + b_n \right)^2} |\Delta_{n-k}(y)|^\beta \, e^{\frac{1}{\delta} \sum_{i=1}^{n-k} y_i} d\mu^{\otimes(n-k)}(y_1, ..., y_{n-k}).$$

We introduce the law $P_{n,\alpha,\beta}$ in the latter quantity. The change of variable $y = a_n(z - b_n)$ and little computation give :

$$\begin{aligned}
\Lambda &= \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log|x_i - y_j|} e^{-\frac{\alpha}{2} \sum_{i=1}^{n-k} \left(\frac{y_i}{a_n} + b_n \right)^2} \prod_{1 \le i<j \le n-k} |y_i - y_j|^\beta \prod_{i=1}^{n-k} dy_i \\
&= a_n^{n-k+\beta\frac{(n-k-1)(n-k)}{2}} \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log|x_i - a_n(z_j - b_n)|} e^{-\frac{\alpha}{2} \sum_{i=1}^{n-k} z_j^2} \prod_{1 \le i<j \le n-k} |z_i - z_j|^\beta \prod_{i=1}^{n-k} dz_i \\
&= a_n^{n-k+\beta\frac{(n-k-1)(n-k)}{2}} Z_{n-k,\alpha,\beta} \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log|x_i - a_n(z_j - b_n)|} dP_{n-k,\alpha,\beta}(z_1, ..., z_{n-k}) \\
&= a_n^{n-k+\beta\frac{(n-k-1)(n-k)}{2}} Z_{n-k,\alpha,\beta} \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log|x_i - a_n z_j + a_n b_n|} dP_{n-k,\alpha,\beta}(z_1, ..., z_{n-k}) \\
&= a_n^{n-k+\beta\frac{(n-k-1)(n-k)}{2}} Z_{n-k,\alpha,\beta} e^{k\beta(n-k)\log(a_n b_n)} \times \cdots \\
&\qquad \cdots \times \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log\left|1 + \frac{x_i}{a_n b_n} - \frac{z_j}{b_n}\right|} dP_{n-k,\alpha,\beta}(z_1, ..., z_{n-k}) \\
&= a_n^{n-k+\beta\frac{(n-k-1)(n-k)}{2} + k\beta(n-k)} Z_{n-k,\alpha,\beta} e^{k\beta(n-k)\log(b_n)} \times \cdots \\
&\qquad \cdots \times \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log\left|1 + \frac{x_i}{a_n b_n} - \frac{z_j}{b_n}\right|} dP_{n-k,\alpha,\beta}(z_1, ., z_{n-k}).
\end{aligned}$$

Thus the claim follows. □

3.2 Pointwise convergence of the correlation functions

The goal of this section is to establish the pointwise convergence $R_k^n(x_1, ..., x_k) \underset{n\infty}{\longrightarrow} 1$ for any fixed $\delta > 0$, $k \geq 1$ and $(x_1, ..., x_k) \in \mathbb{R}^k$ under the following hypothesis :

$$\beta \ll \frac{1}{n\log(n)}, \qquad \alpha = 1, \qquad \delta_n = \delta + o(1).$$

We have already shown the ratio of partition functions converges to $(2\pi)^{-\frac{k}{2}}$ in Lemma 2.6. The other terms are easily handable, so we begin by proving that the term $\tilde{R}_k^n$ converges to 1. To this end, we proceed by double inequality.

Let us show that :

$$\limsup_{n\infty} \tilde{R}_k^n(x_1, ..., x_k) \leq 1.$$

Lemma 3.2. *Let any (δ_n) positive real sequence such that $\log(\delta_n) \ll \log(n)$. Assume $\beta \ll \frac{1}{n}$. For $(x_1, ..., x_k) \in \mathbb{R}^k$ with k fixed, let*

$$\begin{aligned} \tilde{R}_k^n := \tilde{R}_k^n(x_1, ..., x_k) &= \int_{\mathbb{R}^{n-k}} \exp\left(\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log\left|1 + \frac{x_i}{a_n b_n} - \frac{z_j}{b_n}\right|\right) dP_{n-k,\beta}(z_1, ..., z_{n-k}) \\ &= \mathbb{E}_{P_{n-k,\beta}}\left(\exp\left(\beta \sum_{i=1}^{k} \sum_{j=1}^{n-k} \log\left|1 + \frac{x_i}{a_n b_n} - \frac{\lambda_j}{b_n}\right|\right)\right). \end{aligned}$$

Then,

$$\limsup_{n\infty} \tilde{R}_k^n(x_1, ..., x_k) \leq 1.$$

Proof. Applying the bound $|a+b|^\beta \leq 2^\beta e^{\beta\frac{a^2+b^2}{8}}$ of Lemma 2.5, we get :

$$\begin{aligned} \tilde{R}_k^n &= \int_{\mathbb{R}^{n-k}} e^{\sum_{i=1}^k \sum_{j=1}^{n-k} \log\left|1+\frac{x_i}{a_n b_n} - \frac{z_j}{b_n}\right|^\beta} dP_{n-k,\beta}(z_1, ..., z_{n-k}) \\ &\leq \int_{\mathbb{R}^{n-k}} e^{\sum_{i=1}^k \sum_{j=1}^{n-k} \beta\log 2 + \frac{\beta}{8}\left|1+\frac{x_i}{a_n b_n}\right|^2 + \frac{\beta}{8}\left|\frac{z_j}{b_n}\right|^2} dP_{n-k,\alpha,\beta}(z_1, ..., z_{n-k}) \\ &= \int_{\mathbb{R}^{n-k}} e^{\beta k(n-k)\beta\log 2 + \frac{\beta(n-k)}{8}\sum_{i=1}^k \left|1+\frac{x_i}{a_n b_n}\right|^2 + \frac{k\beta}{8}\sum_{j=1}^{n-k}\left|\frac{z_j}{b_n}\right|^2} dP_{n-k,\beta}(z_1, ..., z_{n-k}) \\ &\leq 2^{kn\beta} e^{\frac{n\beta}{8}\sum_{i=1}^k \left|1+\frac{x_i}{a_n b_n}\right|^2} \int_{\mathbb{R}^{n-k}} e^{\frac{k\beta}{8}\sum_{j=1}^{n-k}\left|\frac{z_j}{b_n}\right|^2} dP_{n-k,\beta}(z_1, ..., z_{n-k}) \\ &= 2^{kn\beta} e^{\frac{n\beta}{8}\sum_{i=1}^k \left|1+\frac{x_i}{a_n b_n}\right|^2} \frac{Z_{n-k,1-\frac{k\beta}{4b_n^2},\beta}}{Z_{n-k,\beta}}. \end{aligned}$$

It is now enough to show this ratio of partition functions converges to 1, which is provided by Lemma 2.8. ∎

We finish the proof of $\tilde{R}_k^n(x_1,...x,_k) \underset{n\infty}{\longrightarrow} 1$:

Lemma 3.3. *Assume* $\beta \ll \dfrac{1}{n\log(n)}$. *For* $(x_1,...,x_k) \in \mathbb{R}^k$ *with* k *fixed, let*

$$\begin{aligned}\tilde{R}_k^n(x_1,...,x_k) &:= \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^k \sum_{j=1}^{n-k} \log\left|1+\frac{x_i}{a_n b_n} - \frac{z_j}{b_n}\right|} dP_{n-k,\beta}(z_1,...,z_{n-k}) \\ &= \mathbb{E}_{P_{n-k,\beta}}\left(e^{\beta \sum_{i=1}^k \sum_{j=1}^{n-k} \log\left|1+\frac{x_i}{a_n b_n} - \frac{\lambda_j}{b_n}\right|}\right).\end{aligned}$$

Then

$$\liminf_{n\infty} \tilde{R}_k^n(x_1,...,x_k) \geq 1.$$

Proof. Since exp is convexe, by Jensen inequality, and exchangeability, it is enough to show that for any $x \in \mathbb{R}$ fixed :

$$\beta(n-k)\,\mathbb{E}_{P_{n-k,\beta}}\left|\log\left|1+\frac{x}{a_n b_n} - \frac{\lambda_1}{b_n}\right|\right| \underset{n\infty}{\longrightarrow} 0.$$

Since $k \geq 1$ is also fixed, it is enough to show that for $x \in \mathbb{R}$ fixed :

$$n\beta\,\mathbb{E}_{P_{n,\beta}}\left|\log\left|1+\frac{x}{a_n b_n} - \frac{\lambda_1}{b_n}\right|\right| \underset{n\infty}{\longrightarrow} 0.$$

We next prove the following result :

Lemma 3.4. *Let any* (δ_n) *positive real sequence such that* $\log(\delta_n) \ll \log(n)$. *Assume* $\beta \ll \dfrac{1}{n\log(n)}$. *Fix* $x \in \mathbb{R}$, *then :*

$$n\beta\,\mathbb{E}_{P_{n,\beta}}\left|\log\left|1+\frac{x}{a_n b_n} - \frac{\lambda_1}{b_n}\right|\right| \underset{n\infty}{\longrightarrow} 0.$$

Proof of Lemma 3.4. From the identity

$$\mathbb{E}\,|X| = \int_0^{+\infty} \mathbb{P}\left(|X| \geq t\right) dt,$$

and setting $u := 1 + \dfrac{x}{a_n b_n}$, we have :

$$\begin{aligned}\mathbb{E}_{P_{n,\beta}}\left|\log\left|1+\frac{x}{a_n b_n} - \frac{\lambda_1}{b_n}\right|\right| &= \int_0^{+\infty} P_{n-k,\beta}\left(\left|\log\left|u - \frac{\lambda_1}{b_n}\right|\right| \geq t\right) dt \\ &= \int_0^{+\infty} P_{n,\alpha,\beta}\left(\log\left|u - \frac{\lambda_1}{b_n}\right| \geq t\right) dt + \int_0^{+\infty} P_{n,\beta}\left(\log\left|u - \frac{\lambda_1}{b_n}\right| \leq -t\right) dt \\ &= \int_0^{+\infty} P_{n,\alpha,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \geq e^t\right) dt + \int_0^{+\infty} P_{n,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \leq e^{-t}\right) dt \\ &= \int_1^{+\infty} \frac{1}{y} P_{n,\alpha,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \geq y\right) dy + \int_0^1 \frac{1}{y} P_{n,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \leq y\right) dy.\end{aligned}$$

Next, we show that both integrals converge to 0. We set :

$$\Lambda_1 := \int_1^{+\infty} \frac{1}{y} P_{n,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \geq y\right) dy$$

$$\Lambda_2 := \int_0^1 \frac{1}{y} P_{n,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \leq y\right) dy.$$

Let's treat the term Λ_2.

Since $\alpha = 1$ and $n\beta \ll 1$, we fix $M = 1$, which gives a constant $C > 0$ independent of n, k, a, ε such that, we have the bulk estimate following from Lemma 2.12 :

$$P_{n,\beta}\left(|\lambda_1 - a| \leq \varepsilon\right) \leq C \exp\left(\log(\varepsilon) + \frac{\alpha n\beta}{8\left(1 - \frac{\beta}{4}\right)} a^2\right).$$

Then with $a = b_n + \dfrac{x_i}{a_n} = b_n u$,

$$\begin{aligned} P_{n,\beta}\left(\left|u - \frac{\lambda_j}{b_n}\right| \leq \varepsilon\right) &= P_{n-k,\beta}\left(|b_n u - \lambda_j| \leq b_n\varepsilon\right) \\ &\leq C b_n \varepsilon \exp\left(\frac{\alpha n\beta}{8\left(1 - \frac{\beta}{4}\right)} b_n^2 u^2\right) \end{aligned}$$

Hence,

$$\begin{aligned} 0 \leq (n-k)\beta\Lambda_2 &\leq Cn\beta b_n \exp\left(\frac{n\beta}{2(4-\beta)} b_n^2 u^2\right) \\ &\leq Cn\beta b_n \exp\left(\frac{n\beta}{8} b_n^2 + o(1)\right). \end{aligned}$$

The latter term goes to 0 if and only if $\beta \ll \dfrac{1}{n\log(n)}$. Observe that it is the only time we need to strengthen the restriction on $\beta \ll \dfrac{1}{n}$.

Now we treat the term Λ_1.

By top eigenvalue estimate Lemma 2.11,

$$P_{n,\beta}\left(\left|u - \frac{\lambda_1}{b_n}\right| \geq t\right) \leq C_M b_n^{\beta(n-1)-4} \frac{e^{-\frac{1}{2}\left(b_n^2 - \frac{n\beta}{4}\right)^2 \left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4}}\right)^2}}{\left(t + \frac{b_n^2 u}{b_n^2 - \frac{n\beta}{4}}\right)}.$$

Thus, one has :

$$\Lambda_1 \leq C_M b_n^{\beta(n-1)-4} \int_1^{+\infty} \frac{e^{-\frac{1}{2}\left(b_n^2-\frac{n\beta}{4}\right)^2\left(t+\frac{b_n^2 u}{b_n^2-\frac{n\beta}{4}}\right)^2}}{\left(t+\frac{b_n^2 u}{b_n^2-\frac{n\beta}{4}}\right)} dt$$

For $\alpha \underset{n\infty}{\longrightarrow} c \in]0,+\infty[$, $n\beta = O(1)$, the Lebesgue's dominated convergence theorem implies that integral term converges to 0 which leads to $n\beta\Lambda_1 \ll 1$. $\square$

$\square$

We are ready to achieve the goal of this section :

Lemma 3.5. *Let (δ_n) a positive real sequence such that $\delta_n \underset{n\infty}{\longrightarrow} \delta > 0$. Assume $\alpha = 1$, $\beta \ll \frac{1}{n\log(n)}$. Let $(a_n),(b_n)$ (modified Gaussian scaling) :*

$$b_n := \sqrt{2\log(n)} - \frac{\log\log(n) + 2\log(\delta_n) + \log(4\pi)}{2\sqrt{2\log(n)}}, \qquad a_n := \delta_n\sqrt{2\log(n)}.$$

Fix $(x_1,...,x_k) \in \mathbb{R}^k$, $k \geq 1$. Then the following pointwise convergence holds :

$$R_k^n(x_1,...,x_k) \underset{n\infty}{\longrightarrow} 1.$$

Proof. When $\alpha = 1$, by Lemma 3.1, one has :

$$R_k^n(x_1,...,x_k) = \frac{n!}{(n-k)!} a_n^{-k-\frac{\beta}{2}k(k-1)} |\Delta_k(x)|^\beta \frac{Z_{n-k,\beta}}{Z_{n,\beta}} \times \cdots$$
$$\cdots \times \exp\left(-\frac{1}{2}\sum_{i=1}^k\left(\frac{x_i}{a_n}+b_n\right)^2 + \frac{1}{\delta}\sum_{i=1}^k x_i + k\beta(n-k)\log(b_n)\right)\tilde{R}_k^n$$

with

$$\tilde{R}_k^n := \tilde{R}_k^n(x_1,...,x_k) = \int_{\mathbb{R}^{n-k}} \exp\left(\beta\sum_{i=1}^k\sum_{j=1}^{n-k}\log\left|1+\frac{x_i}{a_n b_n}-\frac{z_j}{b_n}\right|\right) dP_{n-k,\beta}(z_1,...,z_{n-k}).$$

We already proved that $\tilde{R}_k^n(x_1,...,x_k) \underset{n\infty}{\longrightarrow} 1$ by Lemma 3.2 and 3.3. Hence, we are reduced to show :

$$\frac{n!}{(n-k)!} a_n^{-k-\frac{\beta}{2}k(k-1)} |\Delta_k(x)|^\beta \frac{Z_{n-k,\beta}}{Z_{n,\beta}} e^{-\frac{1}{2}\sum_{i=1}^k\left(\frac{x_i}{a_n}+b_n\right)^2+\sum_{i=1}^n x_i} e^{k\beta(n-k)\log(b_n)} \underset{n\infty}{\longrightarrow} 1.$$

We have the following asymptotics for $k \geq 1$ fixed :

$$\frac{Z_{n-k,\beta}}{Z_{n,\beta}} = (2\pi)^{-\frac{k}{2}} + o(1).$$

$$\log a_n = \frac{1}{2}\log 2 + \frac{1}{2}\log\log n$$

$$\frac{n!}{(n-k)!} = (1+o(1))n^k$$

$$\begin{aligned} a_n^{-\beta\frac{k(k-1)}{2}-k} &= \exp\left(-k\log a_n\right)(1+o(1)) \\ &= \exp\left(-\frac{k}{2}\log 2 - \frac{k}{2}\log\log n\right)(1+o(1)) \end{aligned}$$

$$e^{k\beta(n-k)\log b_n} = e^{\frac{k}{2}n\beta\log\log(n)}\left(1+o(1)\right)$$

$$\Delta_k(x_1,...,x_k)^\beta = \prod_{i<j}^k |x_i - x_j|^\beta = \exp\left(\beta\sum_{i<j}^k \log|x_i - x_j|\right) = 1 + o(1).$$

Besides, for $\delta > 0$ fixed,

$$\begin{aligned} \exp\left(-\frac{1}{2}\sum_{i=1}^k\left(\frac{x_i}{a_n} + b_n\right)^2\right) &= \exp\left(-\frac{kb_n^2}{2} - \frac{\alpha b_n}{a_n}\sum_{i=1}^k x_i - \frac{1}{2a_n^2}\sum_{i=1}^n x_i^2\right) \\ &= \exp\left(-\frac{kb_n^2}{2} - \frac{1}{\delta}\sum_{i=1}^k x_i + o(1)\right). \end{aligned}$$

Note also that for $\delta_n \gg 1$,

$$\begin{aligned} \exp\left(-\frac{1}{2}\sum_{i=1}^k\left(\frac{x_i}{a_n} + b_n\right)^2\right) &= \exp\left(-\frac{kb_n^2}{2} - \frac{b_n}{a_n}\sum_{i=1}^k x_i - \frac{1}{2a_n^2}\sum_{i=1}^n x_i^2\right) \\ &= \exp\left(-\frac{kb_n^2}{2} + o(1)\right). \end{aligned}$$

So putting everything together :

$$\begin{aligned} R_k^n(x_1,...,x_k) &= \exp\left(k\left(\log(n) - \log(a_n) - \frac{1}{2}b_n^2 + n\beta\log(b_n) - \frac{1}{2}\log 2\pi\right)\right)(1+o(1)) \\ &:= e^{k\Lambda_n}\left(1+o(1)\right). \end{aligned}$$

It only remains to show that $\Lambda_n \ll 1$. Using the following asymptotics :

$$\log(a_n) = \frac{1}{2}\log(2) + \frac{1}{2}\log\log(n) + \log(\delta) + o(1)$$

$$b_n^2 = 2\log(n) - \log\log(n) - 2\log(\delta) - \log(4\pi) + o(1)$$
$$n\beta\log(b_n) = o(1),$$

we can compute and check the cancelation :

$$\begin{aligned}\Lambda_n &:= \log(n) - \log(a_n) - \frac{1}{2}b_n^2 - \frac{1}{2}\log(2\pi)\\ &= \log(n) - \frac{1}{2}\log(2) - \frac{1}{2}\log\log(n) - \log(\delta) - \log(n) + \frac{1}{2}\log\log(n) + \log(\delta)\\ &+ \frac{1}{2}\log(4\pi) - \frac{1}{2}\log(2\pi) + o(1)\\ &= o(1).\end{aligned}$$

□

3.3 Uniform upper-bound on the correlation functions

The goal of this section is to provide an uniform upper bound for the correlation functions. It constitutes the second hypothesis in the main tool required to show Poisson convergence.

Lemma 3.6. *Let $K \subset \mathbb{R}$ compact and $M := \sup K$. Set $\alpha = 1$. There exists a constant Θ_K independent of n, k such that for all $n \geq 1$, $k \leq n$ and $x_1, ..., x_k \in K$,*

$$R_k^n(x_1, ..., x_k) \leq \Theta_K^k.$$

Proof. Let $K \subset \mathbb{R}$ compact, $M := \sup K > 0$, $n \geq 1$, $k \leq n$, $x_1, ..., x_k \in K$ and $\alpha = 1$. Note that (δ_n) converges to $\delta > 0$, hence it is bounded.

$$\begin{aligned}R_k^n(x_1, ..., x_k) = \frac{n!}{(n-k)!} a_n^{-k-\frac{\beta}{2}k(k-1)} \left|\Delta_k(x)\right|^\beta \frac{Z_{n-k,\beta}}{Z_{n,\beta}} \times \cdots\\ \cdots \times \exp\left(-\frac{1}{2}\sum_{i=1}^k \left(\frac{x_i}{a_n} + b_n\right)^2 + \frac{1}{\delta}\sum_{i=1}^k x_i + k\beta(n-k)\log(b_n)\right)\tilde{R}_k^n.\end{aligned}$$

Note that the ratio of partition functions is bounded by $\left(\frac{2}{\pi}\right)^{\frac{k}{2}}$ according to Lemma 2.9.

First, we bound by elementary means the simple terms. The leading order terms will cancel each other in the computation. Then, we tackle the integral term $\tilde{R}_k^n$ by comparing it to a ratio of partition functions.

We begin with the bound of the Vandermonde determinant : since $\dfrac{\beta(k-1)}{2} \leq 1$,

$$|\Delta_k(x)|^\beta = \prod_{i<j}^{k} |x_i - x_j|^\beta \leq (2M)^{\beta \frac{k(k-1)}{2}} = \left((2M)^{\frac{\beta(k-1)}{2}}\right)^k \leq (2M)^k .$$

We bound the two first terms :

Note that :

$$a_n^{-k-\frac{\beta}{2}k(k-1)} \leq a_n^{-k} = \exp\left(-k\left(\log(\delta_n) + \frac{1}{2}\log\log(n) + \frac{1}{2}\log(2)\right)\right).$$

Also, a classical combinaison inequality, for $k \leq n$,

$$(n)_k := (n-k+1)...n = \frac{n!}{(n-k)!} \leq n^k \exp\left(-\frac{k(k-1)}{2n}\right).$$

Indeed, since $\forall x \in \mathbb{R}, 1 + x \leq e^x$, we have :

$$\frac{(n)_k}{n^k} = \prod_{i=0}^{k-1}\left(1 - \frac{i}{n}\right) \leq \prod_{i=0}^{k-1} \exp\left(-\frac{i}{n}\right) = \exp\left(\sum_{i=0}^{k-1} -\frac{i}{n}\right) = \exp\left(-\frac{(k-1)k}{2n}\right).$$

Let's now study the exponential terms :

$$\begin{aligned}
\exp\left(-\frac{\alpha}{2}\sum_{i=1}^{k}\left(\frac{x_i}{a_n} + b_n\right)^2\right) &= \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\left(\frac{x_i}{a_n} + b_n\right)^2\right) \\
&= \exp\left(-\frac{k}{2}b_n^2 - \frac{1}{2a_n^2}\sum_{i=1}^{k} x_i^2 - \frac{b_n}{a_n}\sum_{i=1}^{k} x_i\right) \\
&\leq \exp\left(-\frac{k}{2}b_n^2 - \frac{b_n}{a_n}\sum_{i=1}^{k} x_i\right) \\
&\leq \exp\left(-\frac{k}{2}b_n^2 + kc_\delta M\right) \\
&= \exp\left(-k\log(n) - \frac{k}{2}n\beta\log\log(n) + \frac{k}{2}\log\log(n) + \frac{k}{2}\log(4\pi) + kc_\delta M\right).
\end{aligned}$$

We used the fact that, since (δ_n) is bounded, there exists $c_\delta > 0$ such that :

$$\frac{b_n}{a_n} \leq \frac{1}{\delta_n} \leq c_\delta.$$

Also,

$$\begin{aligned}
k\beta(n-k)\log(b_n) &\leq kn\beta\log(b_n) \\
&\leq \frac{k}{2}n\beta\log\log(n) + \frac{k}{2}n\beta\log(2) \qquad \text{since } b_n \leq \sqrt{2\log(n)} \\
&\leq \frac{k}{2}n\beta\log\log(n) + k
\end{aligned}$$

Hence, the leading order terms cancel :

$$
\begin{aligned}
R_k^n(x_1, ..., x_k) &\leq e^{-\frac{k(k-1)}{2n}} \, (2M)^k \left(\frac{2}{\pi}\right)^{\frac{k}{2}} e^{k+\frac{k}{2}\log(4\pi)+kc_\delta M} \, \tilde{R}_k^n(x_1, ..., x_k) \\
&\leq (2M)^k \, e^{k+3k\log(2)+kc_\delta M} \, \tilde{R}_k^n(x_1, ..., x_k).
\end{aligned}
$$

It remains to bound the term $\tilde{R}_k^n$.

Applying the bound $|a+b|^\beta \leq 2^\beta e^{\beta\frac{a^2+b^2}{8}}$ of Lemma 2.10, we get :

$$
\tilde{R}_k^n(x_1, ..., x_k) \leq 2^{kn\beta} e^{\frac{n\beta}{8}\sum_{i=1}^k \left|1+\frac{x_i}{a_n b_n}\right|^2} \frac{Z_{n-k, 1-\frac{k\beta}{4b_n^2}, \beta}}{Z_{n-k,\beta}}.
$$

The ratio of partition functions is bounded by 4^k as we proved it in Lemma 2.9.

Besides,

$$
\exp\left(\frac{n\beta}{8}\sum_{i=1}^k \left|1+\frac{x_i}{a_n b_n}\right|^2\right) \leq \exp\left(\frac{1}{8}\sum_{i=1}^k \left(1+\frac{x_i^2}{a_n^2 b_n^2}+\frac{2x_i}{a_n b_n}\right)\right) \leq \exp\left(\frac{k}{8}+\frac{k}{8}M^2+\frac{k}{4}c_\delta M\right).
$$

Thus,

$$
\tilde{R}_k^n(x_1, ..., x_k) \leq 2^k e^{\frac{k}{8}+\frac{k}{8}M^2+\frac{k}{4}M} 4^k = 2^{3k} e^{\frac{k}{8}+\frac{k}{8}M^2+\frac{k}{4}c_\delta M}.
$$

And finally,

$$
R_k^n(x_1, ..., x_k) \leq (2M)^k \, e^{k+3k\log(2)+kc_\delta M} 2^{3k} e^{\frac{k}{8}+\frac{k}{8}M^2+\frac{k}{4}M} = (\Theta_K)^k.
$$

Where we have set :

$$
\Theta_K := e^{1+6\log(2)+c_\delta M+\frac{1}{8}\left(1+M^2+2c_\delta M\right)+\log(2M)}.
$$

□

4 Homogeneous Poisson limit for $n\beta \ll 1$ and $\alpha = 1$

In the sequel, we assume that $0 < \beta \ll \dfrac{1}{n}$, $\alpha = 1$ and above all $\delta_n \gg 1$.

We shall prove that the random process $\displaystyle\sum_{i=1}^n \delta_{a_n(\lambda_i - b_n)}$ converges in distribution to a homogeneous Poisson point process with intensity 1, for the same choices of the Gaussian modified scaling sequences (a_n), (b_n), under a certain decay rate of β and a restriction on (δ_n) which is discussed in the following remark :

Remark 4.1. The assumptions $\log(\delta_n) \ll \sqrt{\log(n)}$ and $\delta_n \gg 1$ mean that $\delta_n = e^{\varepsilon_n \sqrt{2\log(n)}}$ with $\dfrac{1}{\sqrt{\log(n)}} \ll \varepsilon_n \ll 1$. The perturbation by δ_n corresponds to an increase of the zoom around the same Gaussian center minus a negligible factor.

4.1 Proof of the theorem

We follow the same scheme of proof but with $\mu = d\lambda$ the Lebesgue measure.

So we compute first :

$$R_k^n(x_1, ..., x_k) = \frac{n!}{(n-k)!} a_n^{-k-\frac{\beta}{2}k(k-1)} |\Delta_k(x)|^\beta \frac{Z_{n-k,\beta}}{Z_{n,\beta}} e^{-\frac{1}{2}\sum_{i=1}^k \left(\frac{x_i}{a_n}+b_n\right)^2} e^{k\beta(n-k)\log(b_n)} \tilde{R}_k^n(x_1, ..., x_k)$$

with

$$\tilde{R}_k^n(x_1, ..., x_k) := \int_{\mathbb{R}^{n-k}} e^{\beta \sum_{i=1}^k \sum_{j=1}^{n-k} \log\left|1+\frac{x_i}{a_n b_n}-\frac{z_j}{b_n}\right|} dP_{n-k,\beta}(z_1, ..., z_{n-k}).$$

We already proved that $\tilde{R}_k^n(x_1, ..., x_k) \underset{n\infty}{\longrightarrow} 1$ in the last section, so that we have to show :

$$\frac{n!}{(n-k)!} a_n^{-k-\frac{\beta}{2}k(k-1)} |\Delta_k(x)|^\beta \exp\left(-\frac{1}{2}\sum_{i=1}^k \left(\frac{x_i}{a_n}+b_n\right)^2 + k\beta(n-k)\log(b_n)\right) \underset{n\infty}{\longrightarrow} 1.$$

We have the following asymptotics :

$$\frac{Z_{n-k,\beta}}{Z_{n,\beta}} = (2\pi)^{-\frac{k}{2}} + o(1).$$

$$\log a_n = \frac{1}{2}\log 2 + \frac{1}{2}\log\log n + \log(d_n)$$

$$\frac{n!}{(n-k)!} = n^k(1+o(1))$$

$$\begin{aligned} a_n^{-\beta\frac{k(k-1)}{2}-k} &= \exp\left(-k\log(a_n)\right)(1+o(1)) \\ &= \exp\left(-\frac{k}{2}\log(2) - \frac{k}{2}\log\log(n) - k\log(\delta_n)\right)(1+o(1)) \end{aligned}$$

$$\exp\left(k\beta(n-k)\log b_n\right) = \exp\left(\frac{k}{2}n\beta\log\log(n)\right)(1+o(1))$$

$$\Delta_k(x_1, ..., x_k)^\beta = \prod_{i<j}^k |x_i - x_j|^\beta = \exp\left(\beta\sum_{i<j}^k \log|x_i - x_j|\right) = 1 + o(1).$$

The main difference with the setup $a_n \sim b_n$ and $\mu = e^{-x}dx$ is the following asymptotic for $a_n \gg b_n$:

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{k}\left(\frac{x_i}{a_n}+b_n\right)^2\right) = \exp\left(-\frac{kb_n^2}{2} - \frac{b_n}{2a_n}\sum_{i=1}^{k}x_i - \frac{1}{2a_n^2}\sum_{i=1}^{n}x_i^2\right) = \exp\left(-\frac{kb_n^2}{2}+o(1)\right).$$

Hence,

$$\begin{aligned}
R_k^n(x_1,...,x_k) &= n^k a_n^{-k}(2\pi)^{-\frac{k}{2}}\exp\left(-\frac{k}{2}b_n^2 + kn\beta\log(b_n)\right)(1+o(1)) \\
&= \exp\left(k\left(\log(n)-\log(a_n)-\frac{1}{2}b_n^2+n\beta\log(b_n)-\frac{1}{2}\log 2\pi\right)\right)(1+o(1)) \\
&= e^{k\Lambda_n}\left(1+o(1)\right).
\end{aligned}$$

It only remains to show that $\Lambda_n \ll 1$. We compute the following asymptotics :

$$n\beta\log(b_n) = o(1) \qquad \text{since } \beta \ll \frac{1}{n\log(n)}$$

$$\log(a_n) = \frac{1}{2}\log(2)+\frac{1}{2}\log\log(n)+\log(\delta_n),$$

$$b_n^2 = 2\log(n)+\frac{\log^2(\delta_n)}{2\log(n)} - \log\log(n) - 2\log(\delta_n) - \log(4\pi) + o(1).$$

So that, we have :

$$\begin{aligned}
\Lambda_n &:= \log(n)-\log(a_n)-\frac{1}{2}b_n^2-\frac{1}{2}\log(2\pi) \\
&= -\frac{\log^2(\delta_n)}{2\log(n)}+o(1).
\end{aligned}$$

The latter quantity converges to 0 under our growth hypothesis on (δ_n).

To conclude the proof, we have to show the uniform upper-bound analog to Lemma 3.6, which is exactly the same work as done before.

4.2 The Gaussian case : $\beta = 0$

In this last subsection, we derive our result on homogeneous limiting Poisson process in the purely Gaussian case $\beta = 0$. Although the correlation functions method also applies (as we did for the inhomogeneous case when $\beta = 0$), it turns out that the classical method from EVT provides a better regime for the perturbation (δ_n). We formulate the result and prove it.

Lemma 4.2. *Let $(\lambda_i)_{i\leq n}$ an i.i.d. sequence of $\mathcal{N}(0,1)$. Let $\delta_n \gg 1$ such that $\log(\delta_n) \ll \log(n)$, and :*

$$a_n = \delta_n\sqrt{2\log(n)}$$

$$b_n = \sqrt{2\log(n)} - \frac{1}{2}\frac{\log\log(n) + 2\log(\delta_n) + \log(4\pi)}{\sqrt{2\log(n)}}.$$

Then, the point process $\sum_{i=1}^{n} \delta_{a_n(\lambda_i - b_n)}$ converges to a Poisson point process on $\mathbb{R}$ with intensity 1.

Proof. We set $\phi_n(x) = \dfrac{x}{a_n} + b_n$. Since we consider a collection of n i.i.d. random variables and a homogeneous limiting Poisson process, that is with intensity proportional to $d\lambda$ where λ is the Lesbegue measure on $\mathbb{R}$, it is enough [4, Th 7.1] to show that for any $x < y$,

$$\Lambda := n\left(\mathbb{P}\left(\lambda_1 \geq \phi_n(x)\right) - \mathbb{P}\left(\lambda_1 \geq \phi_n(y)\right)\right) \xrightarrow[n\infty]{} y - x.$$

By Mill's ratio, we know that for any $u \gg 1$,

$$\mathbb{P}\left(\lambda_1 \geq u\right) = \frac{\exp\left(-\frac{u^2}{2}\right)}{u\sqrt{2\pi}}\left(1 + o(1)\right).$$

Under the hypothesis $\log(\delta_n) \ll \log(n)$, one has $b_n \sim \sqrt{2\log(n)}$, hence $\phi_n(x) \sim \sqrt{2\log(n)}$.

We get :

$$\begin{aligned}
\Lambda &= \frac{n}{\sqrt{2\pi}}\left(\frac{e^{-\frac{\phi_n(x)^2}{2}}}{\phi_n(x)} - \frac{e^{-\frac{\phi_n(y)^2}{2}}}{\phi_n(y)}\right)(1 + o(1)) \\
&= \frac{n}{\phi_n(x)\phi_n(y)\sqrt{2\pi}}\left(\phi_n(y)e^{-\frac{\phi_n(x)^2}{2}} - \phi_n(x)e^{-\frac{\phi_n(x)^2}{2}}\right)(1 + o(1)) \\
&= \frac{ne^{-\frac{\phi_n(x)^2}{2}}}{\sqrt{2\log(n)}\sqrt{2\pi}}\left(1 - e^{\frac{\phi_n(x)^2 - \phi_n(y)^2}{2}}\right)(1 + o(1)).
\end{aligned}$$

A little computation gives :

$$\frac{\phi_n(x)^2 - \phi_n(y)^2}{2} = \frac{x^2 - y^2}{4\delta_n^2\log(n)} + \frac{x - y}{\delta_n} - \frac{(x-y)\log\log(n)}{2\delta_n\log(n)} - \frac{(x-y)\log(\delta_n)}{\delta_n\log(n)} - \frac{(x-y)\log(4\pi)}{2\delta_n\log(n)}.$$

The highest order term is $\dfrac{x-y}{\delta_n}$. Indeed,

$$\frac{\log(\delta_n)}{\delta_n\log(n)} \ll \frac{1}{\delta_n} \iff \log(\delta_n) \ll \log(n) \quad \text{which is true.}$$

We deduce that :

$$\Lambda = \frac{ne^{-\frac{\phi_n(x)^2}{2}}}{\sqrt{2\log(n)}\sqrt{2\pi}}\left(1 - e^{\frac{x-y}{\delta_n}(1+o(1))}\right)(1+o(1))$$
$$= \frac{ne^{-\frac{\phi_n(x)^2}{2}}}{\sqrt{2\log(n)}\sqrt{2\pi}}\left(\frac{y-x}{\delta_n}\right)(1+o(1))\,.$$

To conclude, we compute :

$$\frac{ne^{-\frac{\phi_n(x)^2}{2}}}{\sqrt{2\log(n)}\sqrt{2\pi}} = \delta_n\,(1+o(1))\,.$$

□

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