

Pricing without martingale measure

Julien Baptiste, Laurence Carassus, Emmanuel Lépinette

▶ **To cite this version:**

Julien Baptiste, Laurence Carassus, Emmanuel Lépinette. Pricing without martingale measure. ESAIM Proceedings MAS 2022, 2023. hal-01774150v4

HAL Id: hal-01774150
https://hal.science/hal-01774150v4

Submitted on 12 Jul 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Pricing without martingale measure

Julien Baptiste,[1] **Laurence Carassus,**[2] **Emmanuel Lépinette,**[1,3]

[1] *Paris Dauphine university, PSL research university, Ceremade, CNRS, UMR, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France.*
Email: baptiste@ceremade.dauphine.fr, emmanuel.lepinette@ceremade.dauphine.fr

[2] *Léonard de Vinci Pôle Universitaire, Research Center, 92 916 Paris La Défense, France and Laboratoire de Mathématiques de Reims, UMR9008 CNRS et Université de Reims Champagne-Ardenne, France*
Email: laurence.carassus@devinci.fr

[3] *Gosaef, Faculté des Sciences de Tunis, 2092 Manar II-Tunis, Tunisia.*

Abstract: For several decades, the martingale measures have played a major role in the financial asset's pricing theory. Here, we follow an approach based on the conditional support of the asset price increments. We propose a numerical illustration on real data from the French CAC 40 index.

Keywords and phrases: Financial market models, CAC 40, Super-hedging prices, AIP, Conditional support, Essential supremum.
2000 MSC: 60G44, G11-G13.

1. Introduction

The problem of giving a fair price to a financial asset G is central in the economic and financial theory. A selling price should be an amount which is enough to initiate a hedging strategy for G, i.e., a strategy whose value at maturity is always above G. It seems also natural to ask for the infimum of such amount. This is the so called super-replication (or super-hedging) price and it has been introduced in the binomial setup for transaction costs by [3]. Characterizing and computing the super-replication price has become one of the central issue in the mathematical finance theory. The super-hedging theorem also called dual formulation show that the super-replication price can be computed using martingale measures (see, [9], [7] and the references therein). We argue that, in a discrete time model without frictions, the super-replication price can be computed using rather the conditional support of the distribution of the assets price increments. This is a well-known approach for the characterization of the condition of no arbitrage, see, [8]. Recently, [5] have shown that it is also fruitful for the computation of the super-replication price. In this last reference, the super-replication price are expressed using Fenchel conjugate and bi-conjugate without postulating any no-arbitrage condition. The finiteness of the super-hedging prices leads to a weak no-arbitrage condition called Absence of Instantaneous Profit (AIP). This condition is equivalent to the fact that 0 belongs to the convex hull of the conditional support of the assets price increments. Under AIP condition, the one-step infimum super-hedging cost is the concave envelop of the payoff relatively to the convex envelop of the conditional support.

In this paper, we propose in the multiple-period framework a recursive scheme for the computation of the super-hedging prices of an European option with convex payoff. Note that we obtain the same computation scheme as in [4] and [6], but under the weaker assumption of AIP and without using martingale measures.

So far, there has been little attention about the expermimental implementation of this algorithm. We calibrate historical data of the French index CAC 40 and implement our super-hedging strategy for a call option. Our procedure is, somehow, model free and based only on statistical estimations. We actually observe that the same strategy, based on the implied volatility rather than conditional support, provides worse results but at a lower price.

The paper is organized as follows. In Section 2, we present our algorithm and in Section 3, the numerical experiments. The proof of our algorithm is postponed to the appendix.

2. Theoritical algorithm

We present our framework and notations. Let $(\Omega, (\mathcal{F}_t)_{t\in\{0,\dots,T\}}\mathcal{F}_T, P)$ be a complete filtered probability space, where T is the time horizon. For any σ-algebra $\mathcal{H}$, we denote by $L^0(\mathbb{R}, \mathcal{H})$ the set of $\mathcal{H}$-measurable and $\mathbb{R}$-valued random variables. We consider a non-negative process $S := \{S_t,\ t \in \{0,\dots,T\}\}$ such that $S_t \in L^0(\mathbb{R}, \mathcal{F}_t)$ for all $t \in \{0,\dots,T\}$. The vector S_t represents the price at time t of the risky asset in the financial market of consideration. Trading strategies are given by processes $\theta := \{\theta_t, t \in \{0,\dots,T-1\}\}$ such that $\theta_t \in L^0(\mathbb{R}, \mathcal{F}_t)$ for all $t \in \{0,\dots,T-1\}$. The vector θ_t represents the investor's holding in the risky asset between times t and $t+1$. We assume that trading is self-financing and that the riskless asset's price is a constant equal to 1. The value at time t of a portfolio θ starting from initial capital $x \in \mathbb{R}$ is then given by

$$V_t^{x,\theta} = x + \sum_{u=1}^{t} \theta_{u-1}\Delta S_u,$$

where $\Delta S_u = S_u - S_{u-1}$ for $u \geq 1$.

We propose an algorithm for the computation of the super-replication price of a European contingent claim $h(S_T)$, where h is convex. This algorithm have already be obtained in Proposition 2.2 of [6], under the NA condition, using the dual representation of the superreplication price. This algorithm can be obtained under a weaker assumption, called Absence of Instantaneous Profit. This condition asserts that it is not possible to super-replicate the contingent claim 0 at a negative super-hedging price. This is also the minimal requirement for the super-hedging cost not being equal to $-\infty$ and so to get a financial market where pricing is possible. The AIP is very weak : If the initial information is trivial, a one period instantaneous profit is a strategy starting from 0 and leading to a terminal wealth larger than some strictly positive constant. We now define the super-hedging prices, the super-hedging cost and the AIP condition. For every $t \in \{0,\dots,T\}$, the set $\mathcal{R}_t^T$ of all claims that can be super-replicated from the zero

initial endowment at time t is defined by

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^{T} \theta_{u-1}\Delta S_u - \epsilon_T^+,\ \theta_{u-1} \in L^0(\mathbb{R}, \mathcal{F}_{u-1}),\ \epsilon_T^+ \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}. \tag{2.1}$$

The set of super-hedging prices and the infimum super-hedging cost of some contingent claim $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$ at time t are given for all $t \in \{0, \dots, T\}$, by

$$\begin{aligned} \mathcal{P}_{T,T}(g_T) &= \{g_T\} \text{ and } \pi_{T,T}(g_T) = g_T, \\ \mathcal{P}_{t,T}(g_T) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t),\ \exists R \in \mathcal{R}_t^T,\ x_t + R = g_T \text{ a.s.}\}, \\ \pi_{t,T}(g_T) &= \operatorname{ess\,inf} \mathcal{P}_{t,T}(g_T). \end{aligned} \tag{2.2}$$

The infimum super-hedging cost is not necessarily a price in the sense that $\pi_{t,T}(g_T) \notin \mathcal{P}_{t,T}(g_T)$, when $\mathcal{P}_{t,T}(g_T)$ is not closed.

Definition 2.1. *The AIP condition holds true if for all* $t \in \{0, \dots, T\}$

$$\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}.$$

The AIP condition can be characterized using the notion of conditional support and of conditional essential supremum (and infimum). Recall that if $X_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$, $\operatorname{supp}_{\mathcal{F}_t} X_{t+1}$, the conditional support of X_{t+1} with respect to $\mathcal{F}_t$, is a random set $\Omega \twoheadrightarrow \mathbb{R}$ defined by

$$\operatorname{supp}_{\mathcal{F}_t} X_{t+1}(\omega) := \bigcap \left\{ A \subset \mathbb{R}^d,\ \text{closed},\ P(X_{t+1} \in A | \mathcal{F}_t)(\omega) = 1 \right\}, \tag{2.3}$$

when $P(X_{t+1} \in A|\mathcal{F}_t)(\omega)$ is a regular version of the conditional law of X_{t+1} knowing $\mathcal{F}_t$. The conditional essential supremum allows to incorporate measurability in the definition of the essential supremum (see, [2] and [10]):

Proposition 2.2. *Let* $\mathcal{F}_t$ *and* $\mathcal{F}_{t+1}$ *be complete* σ*-algebras such that* $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ *and let* $\Gamma = (\gamma_i)_{i \in I}$ *be a family of real-valued* $\mathcal{F}_T$*-measurable random variables. There exists a unique* $\mathcal{F}_t$*-measurable random variable* $\gamma_t \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{F}_t)$ *denoted* $\operatorname{ess\,sup}_{\mathcal{F}_t} \Gamma$ *which satisfies the following properties*

1. *For every* $i \in I$, $\gamma_t \geq \gamma_i$ *a.s.*
2. *If* $\zeta_t \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{F}_t)$ *satisfies* $\zeta_t \geq \gamma_i$ *a.s.* $\forall i \in I$, *then* $\zeta_t \geq \gamma_t$ *a.s.*

The following proposition gives a characterization of AIP, see Proposition 3.4 in [5].

Proposition 2.3. *The following assertions are equivalent.*

1. *AIP holds true.*
2. $0 \in \operatorname{convsupp}_{\mathcal{F}_t} \Delta S_{t+1}$ *a.s. or* $S_t \in [\operatorname{ess\,inf}_{\mathcal{F}_t} S_{t+1}, \operatorname{ess\,sup}_{\mathcal{F}_t} S_{t+1}] \cap \mathbb{R}$ a.s. *for all* $t \in \{0, \dots, T-1\}$.
3. $\pi_{t,T}(0) = 0$ a.s. *for all* $t \in \{0, \dots, T-1\}$,

where $\operatorname{convsupp}_{\mathcal{F}_t} \Delta S_{t+1}$ *is the convex envelop of* $\operatorname{supp}_{\mathcal{F}_t} \Delta S_{t+1}$ *i.e., the smallest convex set that contains* $\operatorname{supp}_{\mathcal{F}_t} \Delta S_{t+1}$.

We are now in position to present our pricing algorithm.

Proposition 2.4. *Suppose that the model is defined by* $\operatorname{ess\,inf}_{\mathcal{F}_{t-1}} S_t = k^d_{t-1} S_{t-1}$ a.s. *and* $\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} S_t = k^u_{t-1} S_{t-1}$ a.s. *where* $(k^d_{t-1})_{t\in\{1,\dots,T\}}$, $(k^u_{t-1})_{t\in\{1,\dots,T\}}$ *and* S_0 *are deterministic non-negative numbers.*

The AIP condition is verified if and only if $k^d_{t-1} \in [0,1]$ *and* $k^u_{t-1} \in [1,+\infty]$ *for all* $t \in \{1,\dots,T\}$.

Suppose that the AIP condition holds. Let $h : \mathbb{R} \to \mathbb{R}$ *be a non-negative convex function with* $\operatorname{dom} h = \mathbb{R}$ *such that* $\lim_{z\to+\infty} \frac{h(z)}{z} \in [0,\infty)$. *Then, the infimum super-hedging cost of the European contingent claim* $h(S_T)$ *is a price and it is given by* $\pi_{t,T}(h) = h(t,S_t) \in \mathcal{P}_{t,T}(h(S_T))$ *a.s. where*

$$\begin{aligned} h(T,x) &= h(x) \\ h(t-1,x) &= \lambda_{t-1} h(t, k^d_{t-1} x) + (1-\lambda_{t-1}) h\left(t, k^u_{t-1} x\right), \end{aligned} \tag{2.4}$$

and $\lambda_{t-1} = \frac{k^u_{t-1}-1}{k^u_{t-1}-k^d_{t-1}} \in [0,1]$ *and* $1-\lambda_{t-1} = \frac{1-k^d_{t-1}}{k^u_{t-1}-k^d_{t-1}} \in [0,1]$, *with the following conventions. When* $k^d_{t-1} = k^u_{t-1} = 1$ *or* $S_{t-1} = 0$, $\lambda_{t-1} = \frac{0}{0} = 0$ *and* $1-\lambda_{t-1} = 1$ *and when,* $k^d_{t-1} < k^u_{t-1} = \infty$, $\lambda_{t-1} = \frac{\infty}{\infty} = 1$,

$$(1-\lambda_{t-1}) h(t,(+\infty)x) = (1-k^d_{t-1}) x \frac{h(t,(+\infty x))}{(+\infty x)} = (1-k^d_{t-1}) x \lim_{z\to+\infty} \frac{h(z)}{z}. \tag{2.5}$$

Moreover, for every $t \in \{1,\dots,T\}$, $\lim_{z\to+\infty} \frac{h(z)}{z} = \lim_{z\to+\infty} \frac{h(t,z)}{z}$ *and* $h(\cdot,x)$ *is non-increasing for all* $x \geq 0$.

The strategy associated to the infimum super-hedging price is given by:

$$\theta^*_t = \frac{h(t+1, k^u_t S_t) - h(t+1, k^d_t S_t)}{(k^u_t - k^d_t) S_t}. \tag{2.6}$$

The infimum super-hedging cost of the European contingent claim $h(S_T)$ in our model is a price, precisely the same than the price we get in a binomial model where S_t belongs to $\{k^d_{t-1} S_{t-1}, k^u_{t-1} S_{t-1}\}$ a.s., $t \in \{1,\dots,T\}$.

3. Numerical experiments

We suppose that the discrete dates are given by $t^n_i = \frac{iT}{n}$, $i \in \{0,\dots,n\}$ where $n \geq 1$ and that

$$\operatorname{ess\,inf}_{\mathcal{F}_{t^n_i}} S_{t^n_{i+1}} = k^d_{t^n_i} S_{t^n_i} \text{ a.s. and } \operatorname{ess\,sup}_{\mathcal{F}_{t^n_i}} S_{t^n_{i+1}} = k^u_{t^n_i} S_{t^n_i} \text{ a.s.}$$

Below we present different approach for the determination of $k^u_{t^n_i}$ and $k^d_{t^n_i}$. By Proposition 2.4, the infimum super-hedging cost of the European Call option $(S_T - K)^+$ is given by $h^n\left(t^n_i, S_{t^n_i}\right)$, where h^n is defined by (2.4) with the terminal condition $h^n(T,x) = g(x) = (x-K)^+$. The associated super-hedging strategies $(\theta^*_{t^n_i})_{i\in\{0,\dots,n-1\}}$ are given by (2.6). We denote by V_T the terminal value of our strategy starting from the super-hedging

cost $V_0 = \pi_{0,T} = h(0, S_0)$, i.e., $V_T = V_0 + \sum_{i=0}^{n-1} \theta^*_{t_i^n} \Delta S_{t_{i+1}^n}$. Then, the super-hedging error is the difference between the terminal value of our strategy and the call option, i.e., $\varepsilon_T = V_T - (S_T - K)^+$. If it is negative, our strategy is not conservative and does not allow to cover the position.

3.1. Implied volatility

In this subsection, we assume that for $i \in \{0, \dots, n-1\}$

$$k^u_{t_i^n} = 1 + \sigma_{t_i^n} \sqrt{\Delta t^n_{i+1}} \text{ and } k^d_{t_i^n} = 1 - \sigma_{t_i^n} \sqrt{\Delta t^n_{i+1}} \geq 0, \tag{3.7}$$

where $t \mapsto \sigma_t$ is a positive Lipschitz-continuous function on $[0, T]$. Our aim is to calibrate σ to the observed European call option prices. We extend the function h^n on $[0, T]$ in such a way that h^n is constant on each interval $[t^n_i, t^n_{i+1}[$, $i \in \{0, \dots, n\}$. Such a scheme is proposed by Milstein [12], where a convergence theorem is proved when the terminal condition, i.e., the payoff function is smooth. Precisely, the sequence of functions $(h^n(t, x))_{n \geq 1}$ converges uniformly to $h(t, x)$, solution of the Black and Scholes formula with time-dependent volatility:

$$\partial_t h(t, x) + \sigma_t^2 \frac{x^2}{2} \partial_{xx} h(t, x) = 0, \quad h(T, x) = g(x). \tag{3.8}$$

In [12], it is supposed that the successive derivatives of the solution of the P.D.E. solution h are uniformly bounded. This is not the case for the Call payoff function g. On the contrary, the successive derivatives of the solution of the P.D.E. explode at the horizon date, see [11]. In [1], it is proven that the uniform convergence still holds when the payoff function is not smooth provided that the successive derivatives of the solution of the P.D.E. do not explode *too much*.

Supposing that Δt^n_i is closed to 0, we can identify the observed prices of the Call option with the theoretical limit prices $h(t, S_t)$ at any instant t given by (3.8). For several strikes, matching the observed prices to the theoretical ones allows to deduce the associated *implied* volatility $t \mapsto \sigma_t$.

The data set is composed of historical values of the french index CAC 40 and European call option prices of maturity 3 months from the 23rd of October 2017 to the 19th of January 2018. The values of S are distributed as in Figure 1.

Now, we want to check the proportion of observations satisfying (3.9) which is the direct consequence of the assumptions made on the multipliers $k^u_{t^n_{i-1}}$ and $k^d_{t^n_{i-1}}$ in (3.7):

$$\left| \frac{S_{t^n_{i+1}}}{S^n_{t_i}} - 1 \right| \leq \sigma_{t_i^n} \sqrt{\Delta t^n_{i+1}}, \text{ a.s.} \tag{3.9}$$

The results are satisfactory for strikes lower that 5000, see Fig. 2. Note that, when the strikes are too large with respect to the current price S, price observations are less available for the calibration, see Figure 1. This could explain the degradation of our results.

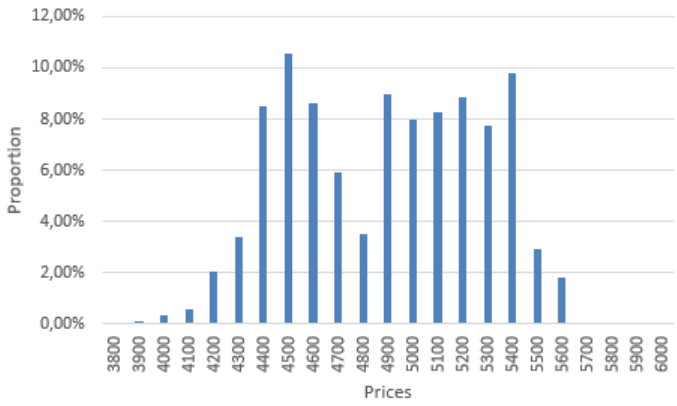


Fig 1: Distribution of the observed prices.

Strike	4800	4900	5000	5100	5200	5300
Ratio	96,7%	95,1%	95,1%	88,5%	86,9%	80,3%
Strike	5400	5500	5600	5700	5800	5900
Ratio	70,5%	78,7%	75,4%	77,0%	73,8%	75,4%

Fig 2

3.2. Direct calibration

We propose another approach where the coefficients k^u and k^d of Proposition 2.4 are calibrated directly on the value of the CAC 40 and the quality of the model is measured through the super-hedging error. This allows to obtain coefficients k^u and k^d which are not strike dependent as before and in this sense model free.

The data set is composed of historical daily closing values of the French index CAC 40 from the 5th of January 2015 to the 12th of March 2018. The chosen interval $[0, T]$ corresponds to one week of 5 working days and $n = 4$. We propose two estimators for $k^d_{t^4_{i-1}}$ and $k^u_{t^4_{i-1}}$. The first approach is called symmetric : $\sigma_{t^4_i}$ is estimated as an upper bound in (3.9) and then the $k^d_{t^4_i}$ and $k^u_{t^4_i}$ are given as in (3.7).

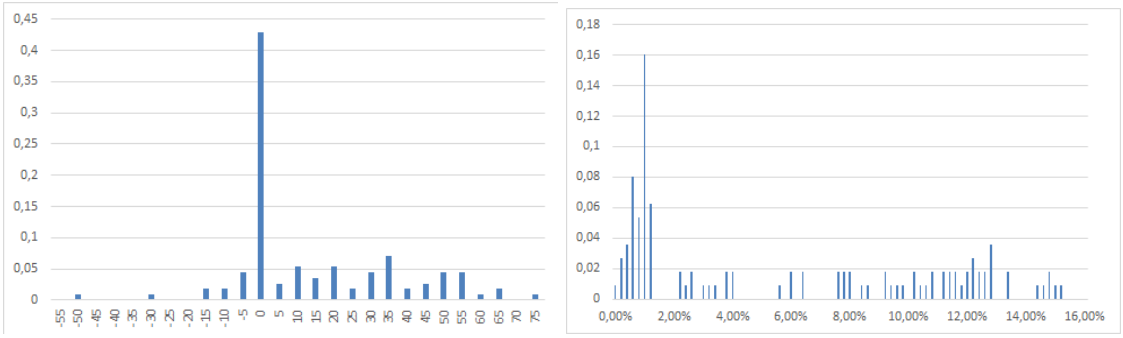
$$\begin{aligned} \sigma_{t^4_i} &= \overline{\max}\left(\left|\frac{S_{t^4_{i+1}}}{S^4_{t_i}} - 1\right| / \sqrt{\Delta t^4_{i+1}},\right) \\ k^u_{t^4_i} &= 1 + \sigma_{t^4_i}\sqrt{\Delta t^4_{i+1}} \text{ and } k^d_{t^4_i} = 1 - \sigma_{t^4_i}\sqrt{\Delta t^4_{i+1}}. \end{aligned} \tag{3.10}$$

The asymmetric approach is perhaps more intuitive as the empirical minimum and maximum are taken.

$$k^d_{t^4_i} = \overline{\min}\frac{S_{t^4_{i+1}}}{S_{t^4_i}} \text{ and } k^u_{t^4_i} = \overline{\max}\frac{S_{t^4_{i+1}}}{S_{t^4_i}}.$$

Note that $\overline{\max}$ (resp. $\overline{\min}$), the empirical maximum (resp. min), is taken over a one year sliding sample window of 52 weeks. We estimate the parameters on 52 weeks and we implement our hedging strategy on the fifty third one. We then repeat the procedure by sliding the window of one week.

We study below the super-hedging error $\varepsilon_T = V_T - (S_T - K)^+$ for different strikes. In the symmetric case, we present in Figures 3a and 3b the distribution of the super-hedging error ε_T and of V_0/S_0 for $K = 4700$.

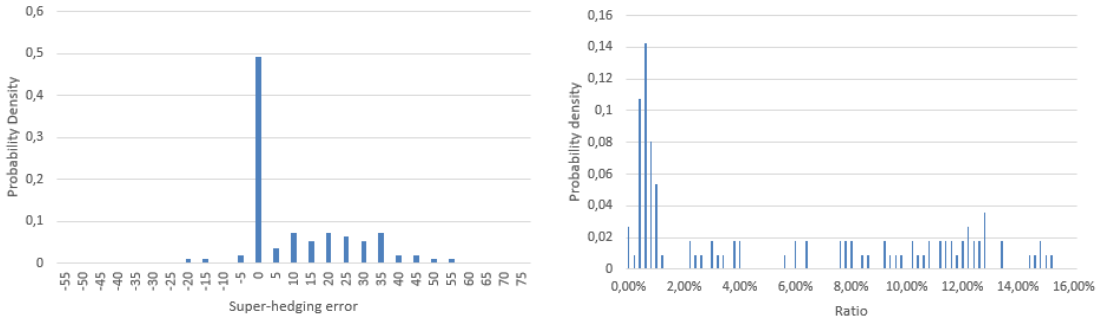


(a) Distribution of the super-hedging error $\varepsilon_T = V_T - (S_T - K)^+$ for $K = 4700$.

(b) Distribution of the ratio V_0/S_0 for $K = 4700$.

Fig 3: Symmetric case.

The asymmetric case is presented in Figures 6a and 6b still for $K = 4700$.



(a) Distribution of the super-hedging error $\varepsilon_T = V_T - (S_T - K)^+$ for $K = 4700$.

(b) Distribution of the ratio V_0/S_0 for $K = 4700$.

Fig 4: Asymmetric case.

The graphs in the Symmetric and Asymmetric cases look similar. So, we now compare both methods in the table below.

In the symmetric (resp. asymmetric) case the empirical average of the error ε_T is 12.76 (resp. 9.47) and its standard deviation is 21.65 (resp. 14.20). This result is rather satisfactory in comparison to the large value of the empirical mean of S_0 which is equal

	Mean of V_0/S_0	Variance of V_0/S_0	Mean of ε_T	Variance of ε_T	$P(\varepsilon_T < 0)$	VaR 95 %
Symmetric	5.61%	5.14 %	12.76	21.65	14.29%	-10.33
Asymmetric	5.52%	5.22%	9.47	14.20	8.04%	-1.81

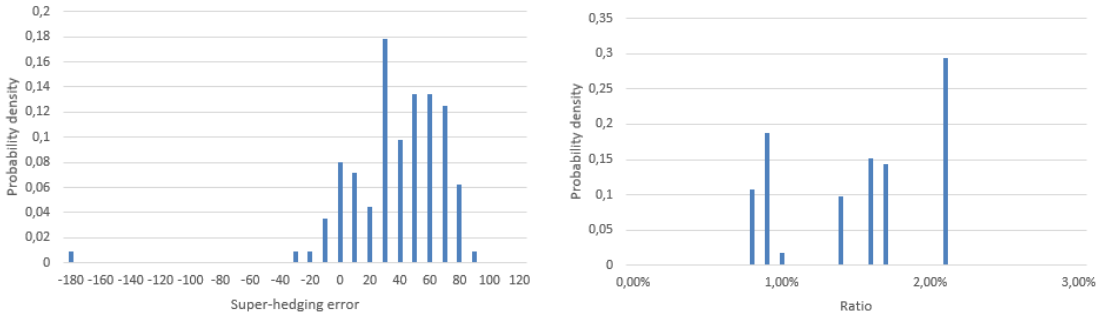
Fig 5: Comparison of the two methods of estimation for $K = 4700$. The mean of S_0 is 4844,93 and the mean of $(S_0 - K)^+$ is 278.73.

to 4844.93. Notice that we observe $E(S_T - K)^+ \simeq 278.73$. This empirically confirms the efficiency of our suggested method. The empirical probability of $\{\varepsilon_T < 0\}$ is equal to 14.29% (resp. 8.04%) but the Value at Risk at 95 % is -10.33 (resp. -1.81) which shows that our strategy is conservative.

Now, we compare the cost of our strategy to S_0, which is the theoretical super-hedging price in some incomplete markets (this is for example the case when $k^d = 0$ and $k^u = \infty$, in particular when the dynamics of S is modeled by a (discrete) geometric Brownian motion, see [4]). The empirical average of V_0/S_0 is 5.61% (resp. 5.52%) and its standard deviation is 5.14% (resp. 5.22%). This means that V_0 is much smaller than S_0.

Note that the huge loss (50 in the symmetric case) is linked to so-called *black friday* week that occurs the 24th of June 2016. Large falls of risky assets were observed in European markets, mainly explained by the Brexit vote. In particular, the CAC 40 felt from 4340 to 4106, with a loss of -8% on Friday.

We also present the "at the money" case $K = S_0$, see Fig. 6.



(a) Distribution of the super-hedging error $\varepsilon_T = V_T - (S_T - K)^+$ for $K = S_0$.

(b) Distribution of the ratio V_0/S_0 for $K = S_0$.

Fig 6: Asymmetric case.

We also provide the comparison between the symmetric and asymmetric cases.

	Mean of V_0/S_0	Variance of V_0/S_0	Mean of ε_T	Variance of ε_T	$P(\varepsilon_T < 0)$	VaR 95 %
Symmetric	1.51%	0.47 %	35.69	34.11	9.82 %	-11.41%
Asymmetric	1.47%	0.49%	33.37	32.78	12.50%	-9.29

Fig 7: Comparison of the two methods of estimation for $K = S_0$.

We see that the results are better than for $K = 4700$: V_0 is small with respect to S_0 and the probability of loss is small as well.

3.3. Comparison between implied volatility and direct calibration

We finish with the comparison of the results of Section 3.2 in the symmetric case, i.e., when σ_{t_i} is given by (3.10), to the ones when σ_{t_i} is the implied volatility. The comparison is made for $K = 5400$. The implementation of the classical approach through implied volatility gives worse results as we may expect. Indeed, as observed in Section 3.1, the implied volatility does not capture the coefficients as well as the empirical observations do. Therefore, from the implied volatility, we get larger coefficients k^d and smaller coefficients k^u. Thus, the minimal super-hedging prices are smaller, as observed in Figure 8, but the super-hedging error is negative in most of the cases, see Figure 9. Our approach is definitively more conservative.



Fig 8: Minimum Super-hedging prices $K = 5400$.

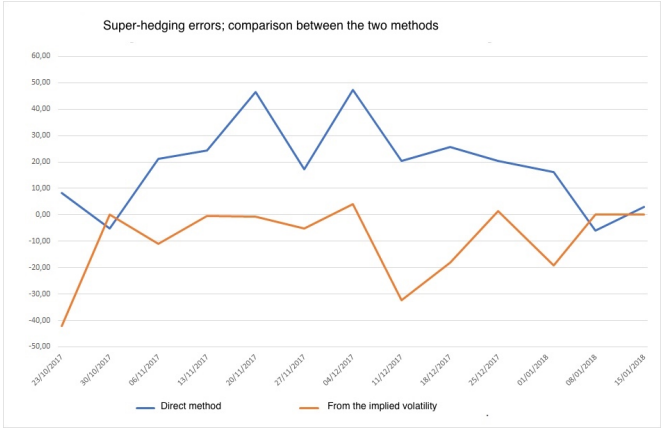


Fig 9: Super-heding errors for $K = 5400$.

4. Appendix

Proof of Proposition 2.4 The conditions $k^d_{t-1} \in [0,1]$ and $k^u_{t-1} \in [1,+\infty]$ for all $t \in \{1,\dots,T\}$ are equivalent to the AIP condition (see Proposition 2.3). Let $M = \frac{h(\infty)}{\infty}$ and $M_t = \lim_{z\to+\infty} \frac{h(t,z)}{z}$. We prove the second statement. Assume that AIP holds true. We establish (i) the recursive formulation $\pi_{t,T}(h(S_T)) = h(t,S_t)$ given by (2.4), (ii) $h(t,\cdot) \geq h(t+1,\cdot)$ and (iii) $M_t = M_{t+1}$. The case $t = T$ is immediate. As $h : \mathbb{R} \to \mathbb{R}$ is a convex function with $\operatorname{dom} h = \mathbb{R}$, h is clearly a $\mathcal{F}_{T-1}$-normal integrand, we can apply Corollary 2.23 in [5] and we get that a.s.

$$\begin{aligned}
\pi_{T-1,T}(h(S_T)) &= \inf\left\{\theta S_{T-1} + \beta,\ \theta \in \mathbb{R},\ \beta \in \mathbb{R},\ \theta x + \beta \geq h(x),\ \forall x \in \operatorname{supp}_{\mathcal{F}_{T-1}} S_T\right\} \\
&= \theta^*_{T-1} S_{T-1} + \beta^* = h(\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T) + \theta^*_{T-1}\left(S_{T-1} - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T\right) \\
&= h(k^d_{T-1} S_{T-1}) + \theta^*_{T-1}\left(S_{T-1} - k^d_{T-1} S_{T-1}\right), \\
\theta^*_{T-1} &= \frac{h(k^u_{T-1} S_{T-1}) - h(k^d_{T-1} S_{T-1})}{k^u_{T-1} S_{T-1} - k^d_{T-1} S_{T-1}},
\end{aligned}$$

with the conventions $\theta^*_{T-1} = \frac{0}{0} = 0$ if either $S_{T-1} = 0$ or $k^u_{T-1} = k^d_{T-1} = 1$ and $\theta^*_{T-1} = \frac{h(\infty)}{\infty} = M$ if $k^d_{T-1} < k^u_{T-1} = +\infty$. Moreover, we obtain that

$$\pi_{T-1,T}(h(S_T)) + \theta^*_{T-1}\Delta S_T \geq h(S_T) \text{ a.s., i.e., } \pi_{T-1,T}(h(S_T)) \in \mathcal{P}_{T-1,T}(h(S_T)).$$

For any $t \in \{0,\dots,T-1\}$ and any $g_{t+1} \in L^0(\mathbb{R},\mathcal{F}_{t+1})$, we introduce the one-step super-hedging prices and cost :

$$\begin{aligned}
\mathcal{P}_{t,t+1}(g_{t+1}) &= \left\{x_t \in L^0(\mathbb{R},\mathcal{F}_t), \exists\, \theta_t \in L^0(\mathbb{R},\mathcal{F}_t),\ x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.}\right\} \\
\pi_{t,t+1}(g_{t+1}) &= \operatorname{ess\,inf} \mathcal{P}_{t,t+1}(g_{t+1})
\end{aligned}$$

Then, using Lemma 3.1 of [5], we get that $\mathcal{P}_{T-2,T}(h(S_T)) = \mathcal{P}_{T-2,T-1}(\pi_{T-1,T}(h(S_T)))$,

$$\pi_{T-2,T}(h(S_T)) = \pi_{T-2,T-1}(\pi_{T-1,T}(h(S_T)))$$

and we may continue the recursion as soon as $\pi_{T-1,T}(h(S_T)) = h(T-1,S_{T-1})$ where $h(T-1,\cdot)$ satisfies (2.4), is convex with domain equal to $\mathbb{R}$, is such that $h(T-1,z) \geq 0$ for all $z \geq 0$ and $M_{T-1} = M \in [0,\infty)$. To see that, we distinguish three cases. If either $S_{T-1} = 0$ or $k^u_{T-1} = k^d_{T-1} = 1$, $\pi_{T-1,T}(h(S_T)) = h(S_{T-1})$ and $h(T-1,z) = h(z) = h(T,z)$ satisfies all the required conditions. If $k^d_{T-1} < k^u_{T-1} = +\infty$, $\theta^*_{T-1} = M$ and $\pi_{T-1,T}(h(S_T)) = h(T-1,S_{T-1})$, where

$$\begin{aligned}
h(T-1,z) &= h(k^d_{T-1} z) + Mz\left(1 - k^d_{T-1}\right) \\
&= \lim_{k^u\to+\infty} \left(\frac{k^u - 1}{k^u - k^d_{T-1}} h(k^d_{T-1} z) + \frac{1 - k^d_{T-1}}{k^u - k^d_{T-1}} h(k^u z)\right),
\end{aligned}$$

using (2.5). The term in the r.h.s. above is larger than $h(z) = h(T,z)$ by convexity. As $k^d_{T-1} \in [0,1]$ and $M \in [0,\infty)$, $h(T-1,z) \geq 0$ for all $z \geq 0$, we get that $h(T-1,\cdot)$

is convex function with domain equal to $\mathbb{R}$ since h is so. The function $h(T-1,\cdot)$ also satisfies (2.4) (see (2.5)). Finally

$$M_{T-1} = \lim_{z\to+\infty} k^d_{T-1}\frac{h(k^d_{T-1}z)}{k^d_{T-1}z} + M\left(1-k^d_{T-1}\right) = M.$$

The last case is when $S_{T-1} \neq 0$ and $k^u_{T-1} \neq k^d_{T-1}$ and $k^u_{T-1} < +\infty$. It is clear that (2.6) implies (2.4). Moreover, as $k^d_{T-1} \in [0,1]$ and $k^u_{T-1} \in [1,+\infty)$, $\lambda_{T-1} \in [0,1]$, $1-\lambda_{T-1} \in [0,1]$ and (2.4) implies that $h(T-1,z) \geq 0$ for all $z \geq 0$, $h(T-1,\cdot)$ is convex with domain equal to $\mathbb{R}$ since h is so. Finally, by convexity

$$M_{T-1} \;=\; \lambda_{T-1}k^d_{T-1}\lim_{z\to+\infty}\frac{h(k^d_{T-1}z)}{k^d_{T-1}z} + (1-\lambda_{T-1})k^u_{T-1}\lim_{z\to+\infty}\frac{h(k^u_{T-1}z)}{k^u_{T-1}z} = M. \;\square$$

Competing interests: The authors declare none.

References

[1] Baptiste J. and E. Lépinette . Diffusion equations: convergence of the functional scheme derived from the binomial tree with local volatility for non smooth payoff functions. *Applied Mathematical Finance*, 25, 511-532, 2018.

[2] Barron E. N., Cardaliaguet, P. and R. Jensen . Conditional Essential Suprema with Applications. *Appl Math Optim*, 48, 229-253, 2003.

[3] Bensaid, B., Lesne J.P., Pagès H. and J. Scheinkman. Derivative asset pricing with transaction costs . *Math.Fin.*, 2, 63-86, 1992.

[4] Carassus, L., Gobet, E. and E. Temam. A class of financial products and models where super-replication prices are explicit *"International Symposium, on Stochastic Processes and Mathematical Finance"* at Ritsumeikan University, Kusatsu, Japan, March 2006.

[5] Carassus L. and E. Lépinette. Pricing without no-arbitrage condition in discrete time. Journal of Mathematical Analysis and Applications, to appear, 125441, ISSN 0022-247X, https://doi.org/10.1016/j.jmaa.2021.125441, 2021.

[6] Carassus L. and T. Vargiolu. Super-replication price: it can be ok. *ESAIM: proceedings and surveys*, 64, 54-64, 2018.

[7] Föllmer H. and A. Schied. Stochastic Finance: An introduction in discrete time. 2nd. Ed., de Gruyter Studies in Mathematics, Walter de Gruyter, Berlin-New York, 2004.

[8] Jacod, J. and A. N. Shiryaev. Local martingales and the fundamental asset pricing theorems in the discrete-time case, *Finance Stochastic*, 2, 259-273, 1998.

[9] Karatzas I. and S. Shreve. *Methods of Mathematical Finance.* Springer-Verlag, 1988.

[10] Kabanov Y. and E. Lépinette. Essential supremum with respect to a random partial order. *Journal of Mathematical Economics*, 49 , 6, 478-487, 2013.

[11] Lépinette E. and T. Tran. Approximate hedging in a local volatility model with proportional transaction costs. *Applied Mathematical Finance*, 21, 4, 313-341, 2014.

[12] Milstein, G.N. The Probability Approach to Numerical Solution of Nonlinear Parabolic Equations. *Numerical methods for partial differential equations*, 18, 4, 490-522, 2002.