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Abstract. A family of self-similar and translation-invariant random sup-measures with long-range dependence are investigated. They are shown to arise as the limit of the empirical random sup-measure of a stationary heavy-tailed process, inspired by an infinite urn scheme, where same values are repeated at several random locations. The random sup-measure reflects the long-range dependence nature of the original process, and in particular characterizes how locations of extremes appear as long-range clusters represented by random closed sets. A limit theorem for the corresponding point-process convergence is established.

1. Introduction

There is a recently renewed interest in limit theorems for extreme values of stationary processes in the presence of long-range dependence [1, 27, 31]. Let $\{X_n\}_{n\in\mathbb{N}}$ be a stationary process. In extreme value theory, it is by now a classical problem to investigate the limit of the partial maxima $\{\max_{i=1,\dots,\lfloor nt\rfloor} X_i\}_{t\in[0,1]}$ as a process of $t \in [0,1]$, after appropriate normalization, as $n \to \infty$. It is further understood that such functional limit theorems are better illustrated in terms of convergence of point processes, in particular in the case when the dependence of the extremes of $\{X_n\}_{n\in\mathbb{N}}$ is weak. For a simple and yet representative example, take $\{X_n\}_{n\in\mathbb{N}}$ to be i.i.d. heavy-tailed random variables such that $\mathbb{P}(X_1 > x) \sim x^{-\alpha}$ as $x \to \infty$ with tail index $\alpha \in (0,\infty)$. It is well known that

$$\sum_{i=1}^{n} \delta_{(X_i/n^{1/\alpha},i/n)} \Rightarrow \sum_{\ell=1}^{\infty} \delta_{(\Gamma_\ell^{-1/\alpha},U_\ell)}, \tag{1.1}$$

where $\{(\Gamma_\ell, U_\ell)\}_{\ell\in\mathbb{N}}$ is a measurable enumeration of points from a Poisson point process on $\mathbb{R}_+ \times [0,1]$ with intensity $dxdu$. Such a point-process convergence provides a detailed description of the asymptotic behavior of extremes, by which we mean broadly the top order statistics instead of the largest one alone: the top order statistics normalized by $n^{1/\alpha}$ converge weakly to $\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \dots$, and their locations are asymptotically independent and uniformly distributed over $[0,1]$ [22]. Such a picture is representative for the general situation where $\{X_n\}_{n\in\mathbb{N}}$ have weak dependence. Classical references now include [7, 21, 29], among others.

The recent advances along this line, however, focus on the case when the stationary process $\{X_n\}_{n\in\mathbb{N}}$ has long-range dependence in the literature. The long-range dependence here, roughly speaking, means that with the same marginal law, the normalization of maxima is of a different order from $n^{1/\alpha}$ so that a non-degenerate limit arises [30, 31]. In the seminal work of O'Brien et al. [25], summarizing a series of developments in the 80s, it has been pointed out that all possible non-degenerate limits of extremes of a stationary sequence can be fit into the framework of convergence of *random sup-measures*. The framework could be

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viewed as a counterpart of the Lamperti's theorem [19] for extremes, in the sense that the limit random sup-measures are necessarily shift-invariant and self-similar. This framework of course includes the case (1.1), and the corresponding limit random sup-measure on $[0,1]$ can be represented as

$$\mathcal{M}_\alpha(\cdot) = \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{U_\ell \in \cdot\}}, \tag{1.2}$$

or more generally as a random sup-measure on $\mathbb{R}$ in the same notation with $\{(\Gamma_\ell, U_\ell)\}_{\ell\in\mathbb{N}}$ a Poisson point process on $\mathbb{R}_+\times\mathbb{R}$ with intensity $dxdu$. In this case, furthermore, the limit random sup-measure is *independently scattered* (a.k.a. completely random) and α-Fréchet, that is, its values over disjoint sets are independent and for every bounded open set A, $\mathcal{M}_\alpha(A)$ is α-Fréchet distributed with $\mathbb{P}(\mathcal{M}_\alpha(A)\le x) = \exp(-\mathrm{Leb}(A)x^{-\alpha})$, $x>0$. Independently scattered random sup-measures are fundamental in stochastic extremal integral representations of max-stable processes [34]. In general, the random sup-measure arising from a stationary sequence may not be independently scattered, or even Fréchet [32].

However, within the general framework of convergence of random sup-measures, to the best of our knowledge it is only very recently that other concrete non-trivial examples have been completely worked out. In a series of papers [18, 26, 32], the extremes of a well-known challenging example of heavy-tailed stationary processes with long-range dependence have been completely characterized in terms of limit theorems for random sup-measures. For this example, the limit random sup-measure obtained by Lacaux and Samorodnitsky [18] takes the form

$$\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}(\cdot) = \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\left\{\widetilde R_\ell^{(\beta)}\cap\,\cdot\,\neq\emptyset\right\}}, \tag{1.3}$$

where $\{\Gamma_\ell\}_{\ell\in\mathbb{N}}$ are as before, $\{\widetilde R_\ell^{(\beta)}\}_{\ell\in\mathbb{N}}$ are i.i.d. random closed sets of $[0,1]$, each consisting of a randomly shifted $(1-\beta)$-stable regenerative set (a stable regenerative set is the closure of a stable subordinator; see Example 6.1 below for a complete description of $\widetilde R_\ell^{(\beta)}$), and the two sequences are independent. We refer to this family of random sup-measures as *stable-regenerative random sup-measures* in this paper. More precisely, $\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}$ arises in limit theorems for a discrete model with parameters $\alpha>0$, $\beta\in(1/2,1)$ [18], and it can be naturally extended to all $\beta\in(0,1)$ (for the original problem in [18] with $\beta\in(0,1/2)$, a more complicated random sup-measure of non-Fréchet type is shown to arise in the limit in [32]; note also that a different parameterization $\widetilde\beta = 1-\beta$ was used in [32]).

One could draw a comparison between (1.2) and (1.3) by viewing each uniform random variable U_i in (1.2) as a random closed set consisting of a singleton point. From this point of view, for the stable-regenerative random sup-measures, the random closed sets $\{\widetilde R_\ell^{(\beta)}\}_{\ell\in\mathbb{N}}$ represent the limit law of positions of extremes, and in this case they reveal a much more intriguing structure: for example, each $\widetilde R_\ell^{(\beta)}$, as randomly shifted $(1-\beta)$-stable regenerative set, is uncountably infinite and with Hausdorff dimension $1-\beta$ almost surely. They reflect the picture that each top order statistic shows up at infinitely many different locations, even unbounded if $\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}$ is viewed as a random sup-measure on $\mathbb{R}$, in a sharp contrast to the situation of independently scattered random sup-measure (1.2) where each top order statistic occurs at a unique random location.

We refer to the phenomena that each top order statistic may show up at multiple and possibly infinitely many locations by *long-range clustering.* Clustering of extremes have been studied before, but in most examples clusters are *local* in the sense that, roughly speaking, each top order statistic is replaced by a cluster consisting of several correlated values at the same time point, due to certain local dependence structure of the original model (see e.g. [14, 21]).

In this paper, by examining another model of heavy-tailed stationary processes, we prove the convergence of empirical random sup-measures to a family of random sup-measures, exhibiting long-range clustering. We refer to this family as the *Karlin random sup-measures*, denoted by $\mathcal{M}_{\alpha,\beta}$ with $\alpha > 0$, $\beta \in (0,1)$ (see (3.5)). These random sup-measures are also in the form of (1.3): now each random closed set $\widetilde{R}_\ell^{(\beta)}$ is replaced by a new one consisting of a random number of independent uniform random variables, and hence its complexity is between the independently scattered random sup-measures (1.2) and stable-regenerative random sup-measures (1.3). In the literature, the Karlin random sup-measures have been considered recently by Molchanov and Strokorb [24], from the aspect of extremal capacity functionals.

The Karlin random sup-measures arise in our investigation on the so-called heavy-tailed Karlin model, a variation of an infinite urn scheme investigated by Karlin [17]. The model is a stationary heavy-tailed process where each top order statistic shows up at possibly multiple locations. It has been known to have long-range dependence, and functional central limit theorems for related partial sums have been recently investigated in [10, 11]. Here, for the extremes, we establish a limit theorem (Theorem 4.1) of point-process convergence encoding the values and corresponding locations of the stationary process as in (1.1), with now locations represented by random closed sets. In particular, the joint convergence describes the long-range clustering of the corresponding order statistics of the Karlin model, and as an immediate consequence the convergence of the empirical random sup-measure to the Karlin random sup-measure in the form of (1.3) follows (Theorem 4.2).

Another way to distinguish the Karlin random sup-measures from independently scattered and stable-regenerative random sup-measures is by noticing that they all have different ergodic properties. This can be understood by comparing the ergodic properties of the induced *max-increment processes* of each class. Each max-increment process of a max-stable random sup-measure is a stationary max-stable process. Ergodic properties of stationary max-stable processes have been recently investigated in the literature [9, 15, 16, 33]. In particular, it is known that the max-increment processes of independently scattered random sup-measures are mixing, those of stable-regenerative random sup-measures are ergodic but not mixing, and here we show that those of Karlin random sup-measures are not ergodic.

We also notice that the Karlin random sup-measures and stable-regenerative random sup-measures both have the same extremal process as a time-changed standard α-Fréchet extremal process, and this holds in a much greater generality. It is easy to see that the extremal process contains much less information than the corresponding random sup-measure. Here we elaborate the relation of the two by showing that for all self-similar Choquet random sup-measures (defined in Section 2), the associated extremal processes are time-changed standard extremal processes (Proposition A.1 in the appendix).

The paper is organized as follow. A general class of random sup-measures, the so-called Choquet random sup-measures, is presented in Section 2. In Section 3, we introduce the Karlin random sup-measures. In Section 4, we introduce the heavy-tailed Karlin model, and state our main results. The proofs are provided in Section 5. In Section 6 we discuss related random sup-measures having the same extremal process. The appendix is devoted to a general result concerning the relation between Choquet random sup-measures and their extremal processes. Some related background on random closed sets and random sup-measures are provided below.

Preliminary background. We start with spaces of closed sets. Our main reference is [23]. We shall consider the space of all closed subsets of a given metric space E, denoted by $\mathcal{F}(E)$, with only $E = [0,1]$, $\mathbb{R}$ or $\mathbb{R}_+ := [0,\infty)$ in this paper. The space $\mathcal{F} \equiv \mathcal{F}(E)$ is equipped with the Fell topology. That is, letting $\mathcal{G} \equiv \mathcal{G}(E)$ and $\mathcal{K} \equiv \mathcal{K}(E)$ denote the open and compact subsets of $\mathcal{F}$, respectively, the topology generated by the base of sets

$$\mathcal{F}_G := \{F \in \mathcal{F} : F \cap G \neq \emptyset\}, \quad G \in \mathcal{G}$$

and

$$\mathcal{F}^K := \{K \in \mathcal{K} : F \cap K = \emptyset\}, \quad K \in \mathcal{K}.$$

The Fell topology is also known as the hit-and-miss topology. With our choice of E (and more generally when it is locally compact and Hausdorff second countable), the Fell topology on $\mathcal{F}(E)$ is metrizable. Hence we define random closed sets as random elements in a metric space [3]. The law of a random closed set R is uniquely determined by

$$\varphi(K) := \mathbb{P}(R \cap K \neq \emptyset), \quad K \in \mathcal{K}(E),$$

where $\mathcal{K}(E)$ is the collection of all compact subsets of E, and φ is known as the *capacity functional* of R. Let $\{R_n\}_{n\in\mathbb{N}}$ and R be a collection of random closed sets in $\mathcal{F}$. A practical sufficient condition for the weak convergence $R_n \Rightarrow R$ in $\mathcal{F}(E)$ as $n \to \infty$ is that

$$\lim_{n\to\infty} \mathbb{P}(R_n \cap A \neq \emptyset) = \mathbb{P}(R \cap A \neq \emptyset),$$

for all $A \subset E$ which is a finite union of bounded open intervals such that $\mathbb{P}(R \cap \overline{A} \neq \emptyset) = \mathbb{P}(R \cap A \neq \emptyset)$ where $\overline{A}$ is the closure set of A [23, Corollary 1.6.9].

Next, we review basics on sup-measures on a metric space E. Our main references are [25, 35]. A sup-measure m on E is defined as a set function from $\mathcal{G} \equiv \mathcal{G}(E)$ to $\mathbb{R}_+$ (in general the sup-measure could take negative values, but not in the framework of this paper), and it can be uniquely extended to a set function from *all* subsets of E to $\mathbb{R}_+$. We start by recalling the definition of a sup-measure on $\mathcal{G}$. A set function $m : \mathcal{G} \to \mathbb{R}_+$ is a sup-measure, if $m(\emptyset) = 0$ and

$$m\left(\bigcup_\alpha G_\alpha\right) = \sup_\alpha m(G_\alpha)$$

for all arbitrary collection of $\{G_\alpha\}_\alpha \subset \mathcal{G}$. Let $\mathrm{SM}(E)$ denote the space of sup-measures from $\mathcal{G} \to \mathbb{R}_+$. The canonical extension of $m : \mathcal{G} \to \mathbb{R}_+$ to a sup-measure on all subsets of E is given by

$$m(A) := \inf_{G\in\mathcal{G}, A\subset G} m(G) \quad \text{for all } A \subset E, A \neq \emptyset.$$

The *sup-vague topology* on $\mathrm{SM}(E)$ is defined such that for $\{m_n\}_{n\in\mathbb{N}}$ and m elements of $\mathrm{SM}(E)$, $m_n \to m$ as $n \to \infty$ if the following two conditions hold

$$\limsup_{n\to\infty} m_n(K) \le m(K), \text{ for all } K \in \mathcal{K}(E),$$
$$\liminf_{n\to\infty} m_n(G) \ge m(G), \text{ for all } G \in \mathcal{G}(E).$$

This choice of topology makes $\mathrm{SM}(E)$ compact and metrizable. We then define random sup-measures again as random elements in a metric space. In particular, $M : \Omega \to \mathrm{SM}(E)$ is a random sup-measure, if and only if $M(A)$ is a $\mathbb{R}_+$-valued random variable for all open bounded intervals A or all compact intervals A, with rational end points. Examples of particular importance for us include *scaled indicator* random sup-measures in the form of

$$\zeta \mathbf{1}_{\{R\cap\,\cdot\,\neq\emptyset\}},$$

where ζ is a positive random variable and R a random closed set, the two not necessarily independent, and the maximum of a finite number of such scaled-indicators. A practical sufficient condition for weak convergence in $\mathrm{SM}(E)$ is the following. Let $\{M_n\}_{n\in\mathbb{N}}$ and M be random sup-measures on E. We have $M_n \Rightarrow M$ in $\mathrm{SM}(E)$, if

$$(M_n(A_1), \dots, M_n(A_d)) \Rightarrow (M(A_1), \dots, M(A_d)),$$

for all bounded open intervals $A_1, \dots, A_d$ of E such that $M(A_i) = M(\overline{A}_i)$ with probability one [25, Theorem 3.2].

Of particular importance among random sup-measures are *Fréchet (max-stable) random sup-measures*, which are random sup-measures with Fréchet finite-dimensional distributions. Recall that a random variable Z has an α-Fréchet distribution if $\mathbb{P}(Z \le z) = \exp(-\sigma z^{-\alpha})$, $z > 0$, for some constants $\sigma > 0$, $\alpha > 0$. A random vector $(Z_1, \dots, Z_d)$ has an α-Fréchet distribution if all its max-linear combinations $\max_{i=1,\dots,d} a_i Z_i$, for $a_1, \dots, a_d > 0$, have α-Fréchet distributions. Now, a random sup-measure is α-Fréchet if its joint law on finite sets is α-Fréchet. Equivalently, an α-Fréchet random sup-measure on E can be viewed as a set-indexed α-Fréchet max-stable process $\{M(A)\}_{A\subset E}$, that is, a stochastic process of which every finite-dimensional distribution is α-Fréchet. Fréchet random variables and Fréchet processes are fundamental objects in extreme value theory, as they arise in limit theorems for extremes of heavy-tailed models [6, 15, 29].

2. Choquet random sup-measures

A special family of Fréchet random sup-measures is the so-called *Choquet random sup-measures*, recently introduced by Molchanov and Strokorb [24]. It is known that every α-Fréchet random sup-measure M has the expression

$$\mathbb{P}(M(K) \le z) = \exp\left(-\frac{\theta(K)}{z^\alpha}\right), \quad K \in \mathcal{K}(E), \tag{2.1}$$

where $\theta(K)$ is referred to as the *extremal coefficient functional* of M. In general, different Fréchet random sup-measures may have the same extremal coefficient functional. Given an extremal coefficient functional θ, the so-called Choquet random sup-measure was introduced and investigated in [24], in the form of

$$\mathcal{M}(\cdot) \stackrel{d}{=} \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{R_\ell \cap \cdot \neq \emptyset\}}. \tag{2.2}$$

Here $\{(\Gamma_\ell, R_\ell)\}_{\ell\in\mathbb{N}}$ is a measurable enumeration of points from a Poisson point process on $(0,\infty) \times \mathcal{F}(E)$ with intensity $dxd\nu$, where ν is a locally finite measure on $\mathcal{F}(E)$ uniquely determined by

$$\nu(\mathcal{F}_K) \equiv \nu(\{F \in \mathcal{F}(E) : F \cap K \neq \emptyset\}) = \theta(K), \quad K \in \mathcal{K}(E).$$

The so-defined $\mathcal{M}$ in (2.2) turns out to be an α-Fréchet random sup-measure with extremal coefficient functional θ, and furthermore its law is uniquely determined by θ. It was demonstrated in [24] that this family of random sup-measures plays a crucial role among all Fréchet random sup-measures from several aspects, and the Choquet theorem plays a fundamental role in this framework, which explains the name.

In view of limit theorems, Choquet random sup-measures arise naturally in the investigation of extremes of a stationary sequence, including the independently scattered and stable-regenerative random sup-measures (see (1.2) and (1.3) respectively). In extreme value theory, many limit theorems are established in terms of extremal processes rather than random sup-measures. Given a general random sup-measure $\mathcal{M}$, let $\mathbb{M}(t) := \mathcal{M}([0,t])$, $t \ge 0$, denote its associated extremal process. It is well known that $\mathbb{M}$ contains much less information than $\mathcal{M}$ in general. This is particularly the case in the framework of self-similar Choquet random sup-measures, as their extremal processes are necessarily time-changed versions of a standard α-Fréchet extremal process. Recall that a random sup-measure $\mathcal{M}$ is H-self similar for some $H > 0$ if

$$\mathcal{M}(\lambda\,\cdot) \stackrel{d}{=} \lambda^H \mathcal{M}(\cdot), \text{ for all } \lambda > 0. \tag{2.3}$$

By *standard α-Fréchet extremal process*, we mean the extremal process determined by the independently scattered random sup-measure $\mathcal{M}_\alpha$, $\mathbb{M}_\alpha(t) := \mathcal{M}_\alpha([0,t])$. That is, using the same $\{(\Gamma_\ell, U_\ell)\}_{\ell\in\mathbb{N}}$ as in (1.2),

$$\mathbb{M}_\alpha(t) := \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{U_\ell \le t\}}, \quad t \ge 0. \tag{2.4}$$

Proposition 2.1. *For any H-self-similar Choquet α-Fréchet random sup-measure $\mathcal{M}$ with $H > 0$, the corresponding extremal process $\mathbb{M}$ satisfies*

$$\theta([0,1]) \{\mathbb{M}(t)\}_{t\ge 0} \stackrel{d}{=} \left\{\mathbb{M}_\alpha(t^{\alpha H})\right\}_{t\ge 0}.$$

To the best of our knowledge, this fact has not been noticed in the literature before. This proposition actually follows from a more general result on Choquet random sup-measures and the corresponding extremal processes, which is of independent interest and established in Proposition A.1 in the appendix. In the upcoming setting, this provides another justification that it is important to work with random sup-measures in the presence of long-range dependence, as the corresponding extremal processes capture much less information of the dependence. See also the discussion in Section 6.

3. Karlin random sup-measures

In this section we provide two representations of Karlin random sup-measures. They are Choquet random sup-measures with α-Fréchet marginals and they depend on a second parameter $\beta \in (0,1)$.

Let us denote by xA, for $x > 0$ and $A \subset \mathbb{R}$, the scaled set $\{xy : y \in A\}$. The Karlin random sup-measure $\mathcal{M}_{\alpha,\beta}$ on $\mathbb{R}$ is defined by the following representation

$$\mathcal{M}_{\alpha,\beta}(A) := \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\left\{\widetilde{\mathcal{N}}_\ell(x_\ell A)\neq 0\right\}}, \quad A \in \mathcal{G}(\mathbb{R}), \tag{3.1}$$

where $\{(\Gamma_\ell, x_\ell, \widetilde{\mathcal{N}}_\ell)\}_{\ell\in\mathbb{N}}$ is an enumeration of the points from a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathfrak{M}_+(\mathbb{R})$ with intensity measure $d\gamma \times \Gamma(1-\beta)^{-1}\beta x^{-\beta-1}dx \times d\widetilde{\mathbb{P}}$. Here $\mathfrak{M}_+(\mathbb{R})$ is the space of Radon point measures on $\mathbb{R}$ and $\widetilde{\mathbb{P}}$ is the probability measure on it induced by a standard Poisson random measure (with intensity dx). Equivalently, the Poisson point process $\{(\Gamma_\ell, x_\ell, \widetilde{\mathcal{N}}_\ell)\}_{\ell\in\mathbb{N}}$ can be viewed as the Poisson point process $\{(\Gamma_\ell, x_\ell)\}_{\ell\in\mathbb{N}}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $d\gamma \times \Gamma(1-\beta)^{-1}\beta x^{-\beta-1}dx$ and i.i.d. marks $\{\widetilde{\mathcal{N}}_\ell\}_{\ell\in\mathbb{N}}$ with law $\widetilde{\mathbb{P}}$.

To see that $\mathcal{M}_{\alpha,\beta}$ is a Choquet random sup-measure, we introduce the random closed set $\widetilde{\mathcal{R}}_\ell$ induced by $\widetilde{\mathcal{N}}_\ell$ as

$$\widetilde{\mathcal{R}}_\ell := \left\{t \in \mathbb{R} : \widetilde{\mathcal{N}}_\ell(\{t\}) = 1\right\},$$

and then write $\{\widetilde{\mathcal{N}}_\ell(x_\ell A) \neq 0\} = \{(\widetilde{\mathcal{R}}_\ell/x_\ell) \cap A \neq \emptyset\}$. So (3.1) now becomes

$$\mathcal{M}_{\alpha,\beta}(A) = \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\left\{(\widetilde{\mathcal{R}}_\ell/x_\ell)\cap A\neq\emptyset\right\}}, \quad A \in \mathcal{G}(\mathbb{R}),$$

as in (2.2), and then it can be extended to all $A \subset \mathbb{R}$ by the canonical extension of sup-measures.

Viewing $\{\mathcal{M}_{\alpha,\beta}(A)\}_{A\subset\mathbb{R}}$ as a set-indexed α-Fréchet max-stable process, we have the following joint distribution:

$$\begin{aligned}\mathbb{P}\left(\mathcal{M}_{\alpha,\beta}(A_1) \le z_1, \ldots, \mathcal{M}_{\alpha,\beta}(A_d) \le z_d\right) \\ = \exp\left(-\Gamma(1-\beta)^{-1}\int_0^\infty \beta x^{-\beta-1}\widetilde{\mathbb{E}}\left(\bigvee_{i=1}^d \frac{\mathbf{1}_{\left\{\widetilde{\mathcal{N}}(xA_i)\neq 0\right\}}}{z_i^\alpha}\right)dx\right),\end{aligned} \tag{3.2}$$

for all $d \in \mathbb{N}$, $z_1, \ldots, z_d > 0$, where $\widetilde{\mathbb{E}}$ is the expectation with respect to $\widetilde{\mathbb{P}}$. See [24, 34] for more details. It suffices to consider $A_1, \ldots, A_d$ as open (or compact) intervals in $\mathbb{R}$ (not necessarily disjoint) above to determine the law of $\mathcal{M}_{\alpha,\beta}$.

From the above presentation, in particular we compute for $d = 1$ and a compact set $K \subset \mathbb{R}$,

$$\mathbb{P}(\mathcal{M}_{\alpha,\beta}(K) \le z) = \exp\left(-\Gamma(1-\beta)^{-1}\int_0^\infty \beta x^{-\beta-1}\widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(xK) \ne 0\right)dx z^{-\alpha}\right).$$

Let Leb denote the Lebesgue measure on $\mathbb{R}$. We have

$$\begin{aligned}\int_0^\infty \beta x^{-\beta-1}\widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(xK) \ne 0\right)dx &= \int_0^\infty \beta x^{-\beta-1}\left(1-\exp(-x\mathrm{Leb}(K))\right)dx \\ &= \mathrm{Leb}(K)\int_0^\infty x^{-\beta}\exp(-x\mathrm{Leb}(K))dx \\ &= \Gamma(1-\beta)\mathrm{Leb}(K)^\beta.\end{aligned} \tag{3.3}$$

Therefore we arrive at, for all $z > 0$,

$$\mathbb{P}(\mathcal{M}_{\alpha,\beta}(K) \le z) = \exp\left(-\frac{\theta_\beta(K)}{z^\alpha}\right) \quad \text{with} \quad \theta_\beta(K) := \mathrm{Leb}(K)^\beta.$$

The function θ_β is the extremal coefficient functional of the random sup-measure $\mathcal{M}_{\alpha,\beta}$.

It is clear from the definition (3.2) that $\mathcal{M}_{\alpha,\beta}$ is β/α-self-similar in the sense of (2.3) and translation-invariant

$$\mathcal{M}_{\alpha,\beta}(\cdot) \stackrel{d}{=} \mathcal{M}_{\alpha,\beta}(x+\cdot), \text{ for all } x \in \mathbb{R}.$$

It is also remarkable that it is symmetric (or rearrangement invariant [24, Sect. 9]) in the sense that its law only depends on the Lebesgue measures of the sets. More precisely, for two collections of disjoint open intervals $\{A_1, \dots, A_d\}$ and $\{B_1, \dots, B_d\}$ such that $\mathrm{Leb}(A_i) = \mathrm{Leb}(B_i), i = 1, \dots, d$, we have

$$\left(\mathcal{M}_{\alpha,\beta}(A_1), \dots, \mathcal{M}_{\alpha,\beta}(A_d)\right) \stackrel{d}{=} \left(\mathcal{M}_{\alpha,\beta}(B_1), \dots, \mathcal{M}_{\alpha,\beta}(B_d)\right).$$

This is a stronger notion than the translation invariance, which has been known to hold true for all random sup-measures arising from stationary sequences [25].

By self-similarity essentially all properties of $\mathcal{M}_{\alpha,\beta}$ can be investigated by restricting to a bounded interval, in which case $\mathcal{M}_{\alpha,\beta}$ has a more convenient representation. We consider its restriction to $[0,1]$ here. In this case, θ_β determines the law of a random closed set $\mathcal{R}^{(\beta)}$ in $[0,1]$ by

$$\mathbb{P}(\mathcal{R}^{(\beta)} \cap K \ne \emptyset) = \frac{\theta_\beta(K)}{\theta_\beta([0,1])} = \mathrm{Leb}(K)^\beta, \text{ for all } K \subset [0,1] \text{ compact.} \tag{3.4}$$

Now, restricting to $[0,1]$, it follows that

$$\mathcal{M}_{\alpha,\beta}(\cdot) \stackrel{d}{=} \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}}\mathbf{1}_{\left\{\left(\mathcal{R}_\ell^{(\beta)}\cap\,\cdot\,\right)\ne\emptyset\right\}} \text{ on } [0,1], \tag{3.5}$$

where $\{\Gamma_\ell\}_{\ell\in\mathbb{N}}$ is the sequence of arrival times of a standard Poisson point process on $\mathbb{R}_+$, $\{\mathcal{R}_\ell^{(\beta)}\}_{\ell\in\mathbb{N}}$ are i.i.d. copies of $\mathcal{R}^{(\beta)}$, and the two sequences are independent. The fact that $\mathcal{M}_{\alpha,\beta}$ in (3.1) has the same presentation (in law) as in (3.5) when restricted to $[0,1]$, follows from either a straightforward computation of finite-dimensional distributions of random sup-measures based on (3.5), or from a more general property of Choquet random sup-measures [24, Corollary 4.5].

In addition, we have the following probabilistic representation of $\mathcal{R}^{(\beta)}$.

Lemma 3.1. *Suppose $\beta \in (0,1)$. Let Q_β be an $\mathbb{N}$-valued random variable with probability mass function*

$$\mathbb{P}(Q_\beta = k) = \frac{\beta(1-\beta)_{(k-1)\uparrow}}{k!} =: p_\beta(k), \quad k \in \mathbb{N},$$

with $(a)_{n\uparrow} = a(a+1)\cdots(a+n-1)$, $n \in \mathbb{N}$, $a \in \mathbb{R}$. *Let* $\{U_n\}_{n\in\mathbb{N}}$ *be i.i.d. random variables uniformly distributed over* $(0,1)$, *independent from* Q_β. *Then,*

$$\mathcal{R}^{(\beta)} \stackrel{d}{=} \bigcup_{i=1}^{Q_\beta}\{U_i\}.$$

Proof. It suffices to prove that $\bigcup_{i=1}^{Q_\beta}\{U_i\}$ has the same capacity functional as $\mathcal{R}^{(\beta)}$ in (3.4). We have, by first conditioning on Q_β,

$$\mathbb{P}\left(\left(\bigcup_{i=1}^{Q_\beta}\{U_i\}\right)\cap K \neq \emptyset\right) = \mathbb{E}\left[1-(1-\mathrm{Leb}(K))^{Q_\beta}\right].$$

One can show that the prescribed distribution of Q_β satisfies the property, for all $z \in (0,1)$,

$$1 - z^\beta = \mathbb{E}\left[(1-z)^{Q_\beta}\right].$$

See for example [28, Eq. (3.42)]. In view of (3.4), this completes the proof. □

Remark 3.2. The law of Q_β has been known to be related to the Karlin model defined in Section 4, and hence it is not a coincidence that it shows up in the limit random sup-measure. In fact, Q_β is a *size-biased sampling* from the asymptotic frequency $\{p_\beta(k)\}_{k\in\mathbb{N}}$ of blocks of size k of an infinite exchangeable random partition with β-diversity. See [28, Section 3.3] for more details and Remark 4.3 below.

Remark 3.3. The first representation of $\mathcal{M}_{\alpha,\beta}$ has been already considered by Molchanov and Strokorb [24]. Their description starts with and focuses on the extremal coefficient functional θ_β whereas we start from the underlying Poisson point process directly. This is suggested in [24, Remark 9.8], while more detailed discussions can be found in the first arXiv online version of the same paper. In particular, Example 9.5 therein provides the same representation as in (3.1). The interpretation of the set $\mathcal{R}^{(\beta)}$ in our Lemma 3.1 seems to be new.

The Karlin random sup-measures also interpolate between the independently scattered random sup-measures $\mathcal{M}_\alpha$ and the completely dependent one, defined as $\mathcal{M}^{\mathrm{c}}_\alpha(\cdot) = Z\mathbf{1}_{\{\,\cdot\,\neq\emptyset\}}$ for a standard α-Fréchet random variable Z (the random sup-measure taking the same value Z on any non-empty set).

Proposition 3.4. *For every* $\alpha > 0$, $\mathcal{M}_{\alpha,\beta} \Rightarrow \mathcal{M}_\alpha$ *as* $\beta \uparrow 1$, *and* $\mathcal{M}_{\alpha,\beta} \Rightarrow \mathcal{M}^{\mathrm{c}}_\alpha$ *as* $\beta \downarrow 0$.

Proof. It suffices to notice that by the capacity functional in (3.4), $\mathcal{R}^{(\beta)} \Rightarrow U$ as $\beta \uparrow 1$ where U is the random closed set induced by the uniform random variable on $(0,1)$, and $\mathcal{R}^{(\beta)} \Rightarrow [0,1]$, a deterministic set, as $\beta \downarrow 0$. □

We conclude this section by examining the ergodic properties of $\mathcal{M}_{\alpha,\beta}$. Every self-similar and translation invariant random sup-measure $\mathcal{M}$ naturally induces a stationary process, the so-called *max-increment process* defined as

$$\zeta(t) := \mathcal{M}((t-1,t]), \quad t \in \mathbb{R}. \tag{3.6}$$

Proposition 3.5. *The max-increment process* $\{\zeta_{\alpha,\beta}(t)\}_{t\in\mathbb{R}}$ *of* $\mathcal{M}_{\alpha,\beta}$ *is not ergodic.*

Proof. Introduce, for $z > 0$, $t \in \mathbb{R}$,

$$\tau_z(t) := \log \mathbb{P}(\zeta_{\alpha,\beta}(0) \le z, \zeta_{\alpha,\beta}(t) \le z) - 2\log\mathbb{P}(\zeta_{\alpha,\beta}(0) \le z).$$

A simple necessary and sufficient condition for ergodicity of a stationary α-Fréchet process is that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \tau_z(t)dt = 0 \text{ for all } z > 0,$$

see Kabluchko and Schlather [16]. Here we have, for $t > 1$,

$$\begin{aligned}
&-\log \mathbb{P}(\zeta_{\alpha,\beta}(0) \le z, \zeta_{\alpha,\beta}(t) \le z) \\
&= \frac{1}{z^\alpha}\Gamma(1-\beta)^{-1}\int_0^\infty \beta x^{-\beta-1}\widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(x(-1,0]) \ne 0, \widetilde{\mathcal{N}}(x(t-1,t]) \ne 0\right)dx \\
&= \frac{1}{z^\alpha}\Gamma(1-\beta)^{-1}\int_0^\infty \beta x^{-\beta-1}(1-e^{-x})^2 dx = (2-2^\beta)z^{-\alpha}.
\end{aligned}$$

In addition to (3.3), this implies for all $t > 1$, $z > 0$,

$$\tau_z(t) = \left[2\theta_\beta((-1,0]) - (2-2^\beta)\right] z^{-\alpha} = 2^\beta z^{-\alpha}.$$

The desired result hence follows. □

4. A heavy-tailed Karlin model

In this section, we introduce a discrete stationary process $\{X_n\}_{n\in\mathbb{N}}$ based on a model, originally studied by Karlin [17], which is essentially an infinite urn scheme. Here, we shall work with a heavy-tailed randomized version of the original model.

To start with, consider an $\mathbb{N}$-valued random variable Y with $\mathbb{P}(Y = k) = p_k$, $k \in \mathbb{N}$. We assume that $p_1 \ge p_2 \ge \cdots > 0$ and, for technical purpose, encode them into the measure

$$\nu := \sum_{\ell=1}^\infty \delta_{1/p_\ell}, \tag{4.1}$$

where δ_x is the unit point mass at x. The following regular variation assumption is made on the frequencies:

$$\nu((0,x]) = \max\{\ell \in \mathbb{N} : 1/p_\ell \le x\} = x^\beta L(x) \quad \text{with } \beta \in (0,1), \tag{4.2}$$

for some slowly varying function L at infinity.

The randomized Karlin model $\{X_n\}_{n\in\mathbb{N}}$ is defined through a two-layer construction. We imagine that there are infinitely many empty boxes indexed by $\mathbb{N}$. First, we independently associate a heavy-tailed random variable to each box. Second, at each round n, we throw a ball at random in one of the boxes (according to the law of Y) and we consider the corresponding heavy-tailed random variable as the value of our process at time n. Namely, let $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$ be i.i.d. random variables with common law such that

$$\mathbb{P}(\varepsilon_1 > y) \sim c_\alpha y^{-\alpha} \text{ as } y \to \infty \quad \text{with } \alpha > 0,\ c_\alpha \in (0,\infty), \tag{4.3}$$

each associated with the box with label $\ell \in \mathbb{N}$. Let $\{Y_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with common law as Y described above, independent of $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$. The stationary sequence $\{X_n\}_{n\in\mathbb{N}}$ is then obtained by setting

$$X_n := \varepsilon_{Y_n}, n \in \mathbb{N}.$$

Here, we are interested in the empirical random sup-measure of $\{X_n\}_{n\in\mathbb{N}}$ on $[0,1]$ introduced as

$$M_n(\cdot) := \max_{i/n \in \cdot} X_i,$$

and its limit as $n \to \infty$. Important quantities relying on the infinite urn scheme are,

$$K_{n,\ell} := \sum_{i=1}^n \mathbf{1}_{\{Y_i = \ell\}},\ \ell \ge 1, \quad \text{and} \quad K_n := \sum_{\ell=1}^\infty \mathbf{1}_{\{K_{n,\ell} \ne \emptyset\}},$$

the number of balls in the box ℓ and the number of non-empty boxes at time n, respectively. We know from [17] that, under (4.2), $K_n \sim \Gamma(1-\beta)n^\beta L(n)$ almost surely.

For a more detailed description of the model, we shall work within the framework of point-process convergence generalizing (1.1). For each $n \in \mathbb{N}$, introduce, for $\ell \ge 1$,

$$R_{n,\ell} = \{i \in \{1,\dots,n\} : Y_i = \ell\}.$$

The following point process ξ_n on $\mathbb{R}_+ \times \mathcal{F}([0,1])$ encode the information of our random model at time n:

$$\xi_n := \sum_{\ell \geq 1,\, K_{n,\ell} \neq 0} \delta_{(\varepsilon_\ell / b_n, R_{n,\ell}/n)}, \tag{4.4}$$

The first coordinate in the Dirac masses presents the value (normalized by b_n, given below) attached to the box ℓ and the second coordinate the possible multiple positions among $\{1,\dots,n\}$ (standardized by $1/n$) where this box has been chosen.

Our main results are the following. The first is a complete point-process convergence.

Theorem 4.1. *For the model above under assumptions* (4.2) *and* (4.3)*, with*

$$b_n := (c_\alpha \Gamma(1-\beta) n^\beta L(n))^{1/\alpha}, \tag{4.5}$$

we have

$$\xi_n \Rightarrow \xi := \sum_{\ell=1}^\infty \delta_{\left(\Gamma_\ell^{-1/\alpha}, \mathcal{R}_\ell^{(\beta)}\right)}, \text{ as } n \to \infty,$$

in $\mathfrak{M}_+((0,\infty) \times \mathcal{F}([0,1]))$*, where* $\{(\Gamma_\ell, \mathcal{R}_\ell^{(\beta)})\}_{\ell \in \mathbb{N}}$ *have the same law as in* (3.5)*.*

The second is the convergence of random sup-measures.

Theorem 4.2. *Under the assumption of Theorem 4.1, we have*

$$\frac{1}{b_n} M_n \Rightarrow \mathcal{M}_{\alpha,\beta}, \text{ as } n \to \infty,$$

in $\mathrm{SM}([0,1])$*.*

Theorem 4.1 is proved by analyzing the top order statistics of the model and their locations. Theorem 4.2 is a direct corollary of Theorem 4.1. Nevertheless, we will also give a second proof of it which is straightforward, without any analysis of the other top order statistics except the largest.

Remark 4.3. In the representation of the law of $\mathcal{R}^{(\beta)}$ in Lemma 3.1, the probability mass function $\{p_\beta(k)\}_{k\in\mathbb{N}}$ has an intrinsic connection to the Karlin model: each $p_\beta(k)$ is the asymptotic frequency of the number of boxes with exactly k balls, namely

$$\lim_{n\to\infty} \frac{1}{K_n} \sum_{\ell=1}^\infty \mathbf{1}_{\{K_{n,\ell}=k\}} = p_\beta(k) \text{ a.s.}$$

This has been known since Karlin [17].

Remark 4.4. For the sake of simplicity, we do not introduce a slowly varying function in (4.3) as in the common setup for heavy-tailed random variables. Replacing (4.3) by

$$\mathbb{P}(\varepsilon_1 > y) \sim y^{-\alpha} \ell(y) \text{ as } y \to \infty$$

with $\alpha > 0$ and ℓ a slowly varying function, the same limit arises while the correct normalization would involve the Bruijn conjugate (e.g. [4, Proposition 1.5.15]).

5. Proofs

In order to analyze the point process ξ_n, we introduce a description of it through the extreme values of the Karlin model. For each $n \in \mathbb{N}$, we consider the K_n random variables

$$\{\varepsilon_\ell : K_{n,\ell} \neq 0\}$$

and their order statistics denoted by

$$\varepsilon_{n,1} \geq \cdots \geq \varepsilon_{n,K_n}.$$

When there are no ties, we let $\widehat{\ell}_{n,k}$ denote the label of the box corresponding to the k-th order statistics, so that

$$\varepsilon_{n,k} = \varepsilon_{\widehat{\ell}_{n,k}}, \text{ for } k \leq K_n,$$

and set $\widehat{\ell}_{n,k} := 0$ for $k > K_n$. When there are ties among the order statistics, the aforementioned labeling is no longer unique, and we choose one at random among all possible ones in a uniform way. This procedure guarantees the independence between the values of the order statistics and the permutation that classifies them. That is, given K_n, the variables $\widehat{\ell}_{n,1}, \ldots, \widehat{\ell}_{n,K_n}$ are independent of the variables $\varepsilon_{n,1}, \ldots, \varepsilon_{n,K_n}$. Now, introduce the random closed sets

$$\widehat{R}_{n,k} := \left\{ i = 1, \ldots n : Y_i = \widehat{\ell}_{n,k} \right\}, \quad k = 1, \ldots, K_n,$$

and $\widehat{R}_{n,k} := \emptyset$ if $k > K_n$. The point processes ξ_n introduced in (4.4) can then be written as

$$\xi_n = \sum_{k=1}^{K_n} \delta_{\left(\varepsilon_{n,k}/b_n, \widehat{R}_{n,k}/n\right)}.$$

The key step in our proof is to investigate the following approximations of ξ_n, keeping only the top order statistics,

$$\xi_n^{(m)} := \sum_{k=1}^{m} \delta_{\left(\varepsilon_{n,k}/b_n, \widehat{R}_{n,k}/n\right)}, \quad m \in \mathbb{N}.$$

Here and below, we set $\varepsilon_{n,k} := 0$ if $k > K_n$.

Proposition 5.1. *For all $m \in \mathbb{N}$, we have*

$$\xi_n^{(m)} \Rightarrow \xi^{(m)} := \sum_{\ell=1}^{m} \delta_{\left(\Gamma_\ell^{-1/\alpha}, \mathcal{R}_\ell^{(\beta)}\right)}, \text{ as } n \to \infty,$$

in $\mathfrak{M}_+((0,\infty) \times \mathcal{F}([0,1]))$, where $\{(\Gamma_\ell, \mathcal{R}_\ell^{(\beta)})\}_{\ell \in \mathbb{N}}$ have the same law as in (3.5).

Proof. There is only a finite number of random points in both $\xi_n^{(m)}$ and $\xi^{(m)}$. Hence, it suffices to prove the joint convergence

$$\left(\frac{\varepsilon_{n,1}}{b_n}, \ldots, \frac{\varepsilon_{n,m}}{b_n}, \frac{\widehat{R}_{n,1}}{n}, \ldots, \frac{\widehat{R}_{n,m}}{n}\right) \Rightarrow \left(\Gamma_1^{-1/\alpha}, \ldots, \Gamma_m^{-1/\alpha}, \mathcal{R}_1^{(\beta)}, \ldots, \mathcal{R}_m^{(\beta)}\right) \tag{5.1}$$

in $\mathbb{R}_+^m \times \mathcal{F}([0,1])^m$, as $n \to \infty$. Under the heavy-tail assumption (4.3), the convergence of the first m coordinates, as the normalized m top order statistics of K_n i.i.d. random variables, is well known from [22] if K_n is a deterministic sequence increasing to infinity and the normalization (here b_n) is $c_\alpha^{1/\alpha} K_n^{1/\alpha}$. For the Karlin model, under the regular variation assumption (4.2), it has been shown that

$$\lim_{n\to\infty} \frac{K_n}{n^\beta L(n)} = \Gamma(1-\beta) \text{ a.s.},$$

see [13, Corollary 21]. Therefore the convergence of the first m coordinates follows. Further, on the left-hand side of (5.1), the first and last m coordinates are conditionally independent given the event $\{K_n \geq m\}$. Since $\mathbb{P}(K_n \geq m) \to 1$ as $n \to \infty$, it is sufficient to prove the convergence of the last m coordinates to conclude. The main difficulty in the analysis of the last m coordinates is due to their dependence. To overcome this difficulty, we first consider a coupled Poissonized version of the model. Namely, let $\{N(t)\}_{t\geq 0}$ denote a standard Poisson process on $\mathbb{R}_+$ independent of $\{Y_n\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n\}_{n\in\mathbb{N}}$, and let $0 < \tau_1 < \tau_2 < \cdots$ denote its consecutive arrival times. We consider the coupled model where we shift the fixed locations

$1, 2, \ldots, n$ of the original model to the random points corresponding to the consecutive random arrival times of N. The Poissonized process is then $\{X_{N(t)}\}_{t\geq 0}$. In this way, we set

$$\widetilde{K}_{n,\ell} := \sum_{i=1}^{\infty} \mathbf{1}_{\{Y_i=\ell,\, \tau_i \leq n\}} \quad \text{and} \quad \widetilde{K}_n := \sum_{\ell=1}^{\infty} \mathbf{1}_{\left\{\widetilde{K}_{n,\ell}\neq 0\right\}}. \tag{5.2}$$

It is important to keep in mind that, for this model, there are $\widetilde{K}_n$ different ε involved at time n, instead of K_n. Note that, thanks to the coupling, $\widetilde{K}_n = K_{N(n)}$. Thus, the order statistics of the set $\{\varepsilon_\ell : \widetilde{K}_{n,\ell} \neq 0\}$ are exactly $\varepsilon_{N(n),1} \geq \cdots \geq \varepsilon_{N(n),\widetilde{K}_n}$. Now, introduce $\widetilde{\ell}_{n,k}$ such that

$$\varepsilon_{\widetilde{\ell}_{n,k}} = \varepsilon_{N(n),k}, \quad k = 1, \ldots, \widetilde{K}_n,$$

and $\widetilde{\ell}_{n,k} := 0$ if $k > \widetilde{K}_n$. Again, in case of ties, we choose uniformly a random labeling as before. Then we define

$$\widetilde{R}_{n,k} := \left\{\tau_i : Y_i = \widetilde{\ell}_{n,k}\right\} \cap [0, n], \quad k = 1, \ldots, \widetilde{K}_n.$$

The key observation on the Poissonization procedure is that given that $\widetilde{\ell}_{n,1} = \ell_1, \ldots, \widetilde{\ell}_{n,m} = \ell_m$, with $\ell_1, \ldots, \ell_m > 0$, $\widetilde{R}_{n,1}, \ldots, \widetilde{R}_{n,m}$ are independent random closed sets; this is a consequence of the thinning property of Poisson processes. Moreover, the law of each $\widetilde{R}_{n,k}$ is the conditional law of the set of the arrival times of a Poisson process with intensity p_{ℓ_k} within $[0, n]$, given that it is not empty.

We first show that

$$\left(\frac{\widetilde{R}_{n,1}}{n}, \ldots, \frac{\widetilde{R}_{n,m}}{n}\right) \Rightarrow \left(\mathcal{R}_1^{(\beta)}, \ldots, \mathcal{R}_m^{(\beta)}\right). \tag{5.3}$$

Let $A_1, \ldots, A_m$ be m open intervals within $(0, 1)$. We first compute

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{k=1}^{m}\left\{\frac{1}{n}\widetilde{R}_{n,k} \cap A_k \neq \emptyset\right\}\right) \\ &\qquad = \sum_{\ell_1,\ldots,\ell_m\in\mathbb{N}} \mathbb{P}\left(\bigcap_{k=1}^{m}\left\{\frac{1}{n}\widetilde{R}_{n,k} \cap A_k \neq \emptyset\right\} \cap \left\{\widetilde{\ell}_{n,k} = \ell_k\right\}\right). \end{aligned} \tag{5.4}$$

For every choice of $\ell_1, \ldots, \ell_m \in \mathbb{N}$ that are mutually distinct (otherwise the probability above is zero), let N_k be a Poisson process with parameter p_{ℓ_k}, $k = 1, \ldots, m$, and $\widetilde{R}_k$ the corresponding random closed set induced by its arrival times in $[0, n]$. Given $\{\widetilde{K}_{n,\ell}\}_{\ell\in\mathbb{N}}$, the probability of the event $\{\widetilde{\ell}_{n,1} = \ell_1, \ldots, \widetilde{\ell}_{n,m} = \ell_m\}$ is

$$\mathbf{1}_{\{\widetilde{K}_{n,\ell_1}\neq 0,\ldots,\widetilde{K}_{n,\ell_m}\neq 0\}} \frac{(\widetilde{K}_n - m)!}{\widetilde{K}_n!}$$

as each non-empty box has equal probability to be the k-th largest (above $j!$ stands for the factorial of the non-negative integer j). Therefore by conditioning on the values of $\{\widetilde{\ell}_{n,k}\}_{k=1,\ldots,m}$ first, and then using the independence of the $\widetilde{K}_{n,\ell}$, we have, letting λ_k denote the Lebesgue measure of A_k,

$$
\begin{aligned}
&\mathbb{P}\left(\bigcap_{k=1}^m \left\{\frac{1}{n}\widetilde{R}_{n,k}\cap A_k \neq \emptyset\right\} \cap \left\{\widetilde{\ell}_{n,k} = \ell_k\right\}\right)\\
&= \mathbb{E}\left[\frac{(\widetilde{K}_n - m)!}{\widetilde{K}_n!}\prod_{k=1}^m \mathbf{1}_{\{\widetilde{K}_{n,\ell_k}\neq 0\}}\,\mathbb{P}\left(\widetilde{R}_k\cap nA_k \neq \emptyset \mid \widetilde{R}_k\cap[0,n]\neq\emptyset\right)\right]\\
&= \mathbb{E}\left[\frac{(\widetilde{K}_n - m)!}{\widetilde{K}_n!}\prod_{k=1}^m \mathbf{1}_{\{\widetilde{K}_{n,\ell_k}\neq 0\}}\left(\frac{1-e^{-\lambda_k n p_{\ell_k}}}{1-e^{-np_{\ell_k}}}\right)\right]\\
&= \mathbb{E}\left[\frac{\widetilde{K}_n^{(\ell_1,\dots,\ell_m)}!}{(m+\widetilde{K}_n^{(\ell_1,\dots,\ell_m)})!}\right]\prod_{k=1}^m(1-e^{-\lambda_k n p_{\ell_k}}),
\end{aligned}
$$

where

$$
\widetilde{K}_n^{(\ell_1,\dots,\ell_m)} = \sum_{\ell\geq 1,\,\ell\notin\{\ell_1,\dots,\ell_m\}} \mathbf{1}_{\{\widetilde{K}_{n,\ell}\neq 0\}}.
$$

We shall prove, in Lemma 5.2 below, that $\widetilde{\Phi}_n/((\widetilde{K}_n - m)\vee 1)\to 1$ and $\widetilde{\Phi}_n/(\widetilde{K}_n + m)\to 1$ in L^m, where $\widetilde{\Phi}_n := \mathbb{E}\widetilde{K}_n \sim \Gamma(1-\beta)n^\beta L(n)$ according to [13, Proposition 17 and Lemma 1]. Using that

$$
\frac{1}{(m+\widetilde{K}_n)^m} \leq \frac{\widetilde{K}_n^{(\ell_1,\dots,\ell_m)}!}{(m+\widetilde{K}_n^{(\ell_1,\dots,\ell_m)})!} \leq \frac{1}{((\widetilde{K}_n - m)\vee 1)^m},
$$

we infer that

$$
\frac{\widetilde{K}_n^{(\ell_1,\dots,\ell_m)}!}{(m+\widetilde{K}_n^{(\ell_1,\dots,\ell_m)})!} \sim \frac{1}{\widetilde{\Phi}_n^m} \text{ in } L^1, \text{ uniformly in } (\ell_1,\dots,\ell_m), \text{ as } n\to\infty.
$$

The right-hand side of (5.4) then becomes

$$
\begin{aligned}
&\sum_{\ell_1,\dots,\ell_m\in\mathbb{N},\neq} \mathbb{E}\left[\frac{\widetilde{K}_n^{(\ell_1,\dots,\ell_m)}!}{(m+\widetilde{K}_n^{(\ell_1,\dots,\ell_m)})!}\right]\prod_{k=1}^m(1-e^{-\lambda_k n p_{\ell_k}})\\
&\sim \frac{1}{(\Gamma(1-\beta)n^\beta L(n))^m}\sum_{\ell_1,\dots,\ell_m\in\mathbb{N},\neq}\prod_{k=1}^m\left(1-e^{-\lambda_k n p_{\ell_k}}\right), \text{ as } n\to\infty, \qquad (5.5)
\end{aligned}
$$

where in the summation, $\neq$ indicates that $\ell_1,\dots,\ell_m$ are mutually distinct. If we sum over all $\ell_1,\dots,\ell_m\in\mathbb{N}$ instead, recalling the definition of ν in (4.1), we have

$$
\sum_{\ell_1,\dots,\ell_m\in\mathbb{N}}\prod_{k=1}^m\left(1-e^{-\lambda_k n p_{\ell_k}}\right) = \prod_{k=1}^m\int_0^\infty(1-e^{-\lambda_k n/x})\nu(dx). \qquad (5.6)
$$

For the Karlin model, it is well known that the regular variation assumption (4.2) on ν leads to, after integration by parts and change of variables,

$$
\begin{aligned}
\int_0^\infty(1-e^{-\lambda n/x})\nu(dx) &= \int_0^\infty \frac{\lambda n}{x^2}e^{-\lambda n/x}\nu((0,x])dx\\
&\sim \nu((0,n])\lambda\int_0^\infty x^{\beta-2}e^{-\lambda/x}dx = \nu((0,n])\lambda^\beta\Gamma(1-\beta).
\end{aligned}
$$

This gives the asymptotic of (5.6), and also tells that the summations in (5.6) and (5.5) are asymptotically equivalent. Therefore, we have shown that

$$
\lim_{n\to\infty}\mathbb{P}\left(\bigcap_{k=1}^m\left\{\frac{1}{n}\widetilde{R}_{n,k}\cap A_k\neq\emptyset\right\}\right) = \prod_{k=1}^m\lambda_k^\beta.
$$

This established the claimed weak convergence in (5.3).

To complete the proof, it remains to show that $\widetilde{R}_{n,k}/n$ and $\widehat{R}_{n,k}/n$ can be made close with arbitrarily high probability by taking n large enough. To make this idea precise, we consider the Hausdorff metric $d_{\rm H}$ for non-empty compact sets defined as, for two non-empty compact sets F_1 and F_2,

$$d_{\rm H}(F_1,F_2) := \max\left\{\sup_{x\in F_1} d(x,F_2), \sup_{x\in F_2} d(x,F_1)\right\},$$

where d above is the distance between a point and a set induced in $\mathbb{R}$ by Euclidean metric: $d(x,A) := \inf_{y\in A}|x-y|$. It is known that $d_{\rm H}$ metricizes the Fell topology on $\mathcal{F}'([0,1]) := \mathcal{F}([0,1])\setminus\{\emptyset\}$. See for example [23, Appendix C]. For n large enough, consider the event

$$B_n^{(m)} := \{K_n \geq m\} \cap \{\widetilde{K}_n \geq m\},$$

so that, under $B_n^{(m)}$, $\widehat{R}_{n,k} \neq \emptyset$ and $\widetilde{R}_{n,k} \neq \emptyset$ for all $k = 1,\dots,m$. It is clear that $\lim_{n\to\infty}\mathbb{P}(B_n^{(m)}) = 1$. Therefore, (5.1) and hence the proposition shall follow from (5.3) and the fact that for all $\delta > 0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\left\{\max_{k=1,\dots,m} d_{\rm H}\left(\frac{\widehat{R}_{n,k}}{n}, \frac{\widetilde{R}_{n,k}}{n}\right) > \delta\right\} \cap B_n^{(m)}\right) = 0. \tag{5.7}$$

To prove (5.7), we first introduce the event

$$E_n^{(m)} = \left\{\widehat{\ell}_{n,1} = \widetilde{\ell}_{n,1},\dots,\widehat{\ell}_{n,m} = \widetilde{\ell}_{n,m}\right\},$$

and we shall prove that $\lim_{n\to\infty}\mathbb{P}(E_n^{(m)}) = 1$. Since the probability of the event

$$T_n^{(m)} := \left\{\text{no ties in the } m+1 \text{ top order statistics of } \{\varepsilon_\ell : K_{n,\ell} \neq 0 \text{ or } \widetilde{K}_{n,\ell} \neq 0\}\right\}$$

goes to 1 as $n\to\infty$, this will follow if one can show that

$$\lim_{n\to\infty}\mathbb{P}\left(E_n^{(m)} \cap B_n^{(m)} \cap T_n^{(m)}\right) = 1. \tag{5.8}$$

Assuming $B_n^{(m)}$ and $T_n^{(m)}$, the event $E_n^{(m)}$ holds if the m top order statistics from the set $\{\varepsilon_\ell : K_{n,\ell} \neq 0 \text{ or } \widetilde{K}_{n,\ell} \neq 0\}$ already appear in the subset $\{\varepsilon_\ell : K_{n,\ell} \neq 0 \text{ and } \widetilde{K}_{n,\ell} \neq 0\}$. Given $K_n^\wedge := K_n \wedge \widetilde{K}_n$ and $K_n^\vee := K_n \vee \widetilde{K}_n$, using the fact that the locations (labellings) of the order statistics among $\{1,\dots,K_n^\vee\}$ are uniformly distributed, the desired probability is the one that, when taking uniformly at random a permutation of $K_n^\vee$ elements, the m first elements of the permutation belong to a fixed subset of $K_n^\wedge$ elements. Thus, we infer that

$$\mathbb{P}\left(E_n^{(m)} \cap B_n^{(m)} \cap T_n^{(m)}\right) = \mathbb{E}\left[\frac{K_n^\wedge(K_n^\wedge-1)\cdots(K_n^\wedge-m+1)}{K_n^\vee(K_n^\vee-1)\cdots(K_n^\vee-m+1)}\mathbf{1}_{B_n^{(m)}}\mathbf{1}_{T_n^{(m)}}\right].$$

The quotient in the expectation converges to 1 almost surely and it is bounded by 1. Therefore, by the dominated convergence theorem, we obtain (5.8) and thus $\lim_{n\to\infty}\mathbb{P}(E_n^{(m)}) = 1$.

From now on, we assume that the events $E_n^{(m)}$ and $B_n^{(m)}$ hold. Let $k \in \{1,\dots,m\}$ be fixed and denote $\ell_k = \widehat{\ell}_{n,k} = \widetilde{\ell}_{n,k}$. Recall our definition of τ_i, the i-th arrival time of the Poisson process N in the Poissonization and set

$$\rho_n := \max_{i=1,\dots,n}|i-\tau_i|,$$

the maximal displacement of the positions $1,\dots,n$ by the Poissonization. Consider also the Poisson process N_k derived from N by keeping only the arrival times corresponding to the box ℓ_k ($N_k(t) := \sum_{i=1}^\infty \mathbf{1}_{\{\tau_i\leq t\}}\mathbf{1}_{\{Y_i=\ell_k\}}, t\geq 0$). Thus, N_k is a Poisson process of intensity p_{ℓ_k} and we denote by $\tau_1^{(k)} < \tau_2^{(k)} < \cdots$ its consecutive arrival times.

Consider $i \in \widehat{R}_{n,k}$ and first assume that i is such that $\tau_i \le n$ and hence $\tau_i \in \widetilde{R}_{n,k}$. In this case we have $d(i, \widetilde{R}_{n,k}) \le |i - \tau_i| \le \rho_n$. On the other hand, for $i \in \widehat{R}_{n,k}$ such that $\tau_i > n$, we have

$$d(i, \widetilde{R}_{n,k}) \le |i - \tau^{(k)}_{N_k(n)}| \le |i - n| \wedge |\tau^{(k)}_{N_k(n)} - n|.$$

Since in this case $N(n) < i < n$, we have $|i - n| \le |N(n) - n|$ and hence

$$\sup_{i \in \widehat{R}_{n,k}} d(i, \widetilde{R}_{n,k}) \le \max\left\{\rho_n, |N(n) - n|, |\tau^{(k)}_{N_k(n)} - n|\right\}. \tag{5.9}$$

Now, consider $\tau_i \in \widetilde{R}_{n,k}$. For such τ_i with $i \in \{1, \dots, n\}$, we have $d(\tau_i, \widehat{R}_{n,k}) \le |\tau_i - i| \le \rho_n$, whereas for $\tau_i \in \widetilde{R}_{n,k}$ with $i > n$, denoting by j_k the maximum of $\widehat{R}_{n,k}$ (non-empty by assumption), we have

$$d(\tau_i, \widehat{R}_{n,k}) \le |\tau_i - j_k| \le |\tau_i - \tau_{j_k}| + |\tau_{j_k} - j_k| \le |n - \tau_{j_k}| + |\tau_{j_k} - j_k|,$$

where we used that $\tau_{j_k} \le \tau_i \le n$ in the last inequality. Note that $\tau_{j_k} = \tau^{(k)}_{N_k(\tau_n)}$ and thus,

$$\sup_{i \colon \tau_i \in \widetilde{R}_{n,k}} d(\tau_i, \widehat{R}_{n,k}) \le \rho_n + |\tau^{(k)}_{N_k(\tau_n)} - n|.$$

Therefore, above and (5.9) yield

$$d_{\mathrm{H}}\left(\frac{\widehat{R}_{n,k}}{n}, \frac{\widetilde{R}_{n,k}}{n}\right) \le \max\left\{\frac{|N(n) - n|}{n}, \frac{|\tau^{(k)}_{N_k(n)} - n|}{n}, \frac{\rho_n}{n} + \frac{|\tau^{(k)}_{N_k(\tau_n)} - n|}{n}\right\}.$$

It is well known that $\lim_{n\to\infty} \rho_n/n = 0$ and $\lim_{n\to\infty} |N(n) - n|/n = 0$ almost surely. Furthermore,

$$\lim_{n\to\infty} \frac{\tau^{(k)}_{N_k(n)}}{n} = \lim_{n\to\infty} \frac{\tau^{(k)}_{N_k(n)}}{N_k(n)} \frac{N_k(n)}{n} = p_{\ell_k} \frac{1}{p_{\ell_k}} = 1 \text{ almost surely}$$

and hence $\lim_{n\to\infty} \tau^{(k)}_{N_k(\tau_n)}/n = 1$ almost surely. This established (5.7) and the proposition. $\square$

Lemma 5.2. *Let $\{\widetilde{K}_n\}_{n\ge 1}$ be the process defined in* (5.2) *and $\widetilde{\Phi}_n = \mathbb{E}\widetilde{K}_n$, $n \ge 1$. For any real constant c, we have*

$$\frac{\widetilde{\Phi}_n}{(\widetilde{K}_n + c) \vee 1} \to 1, \quad \text{as } n \to \infty,$$

almost surely and in L^p for all $p \ge 1$.

Proof. We know from [13] that $\widetilde{K}_n \sim \widetilde{\Phi}_n$ almost surely and thus the almost sure convergence above follows. Recalling that $\widetilde{K}_n$ is a sum of independent $\{0, 1\}$-valued random variables and that $\mathrm{Var}(\widetilde{K}_n) = \widetilde{\Phi}_{2n} - \widetilde{\Phi}_n \le \widetilde{\Phi}_n$, the Bernstein inequality (see e.g. [5]) gives

$$\mathbb{P}\left(\left|\frac{\widetilde{K}_n}{\widetilde{\Phi}_n} - 1\right| > \frac{1}{2}\right) \le 2\exp\left(-\frac{(\widetilde{\Phi}_n/2)^2}{2(\mathrm{Var}(\widetilde{K}_n) + \widetilde{\Phi}_n/6)}\right) \le 2\exp\left(-\frac{3}{28}\widetilde{\Phi}_n\right).$$

Let $p \ge 1$ and $q > p$ be fixed. Using the above inequality and the fact that $\widetilde{K}_n/((\widetilde{K}_n + c) \vee 1) \le 1 \vee (1 - c)$, we have

$$\begin{aligned}\mathbb{E}\left(\frac{\widetilde{\Phi}_n}{(\widetilde{K}_n + c) \vee 1}\right)^q &= \mathbb{E}\left(\left(\frac{\widetilde{\Phi}_n}{(\widetilde{K}_n + c) \vee 1}\right)^q \mathbf{1}_{\left\{\frac{\widetilde{K}_n}{\widetilde{\Phi}_n} \ge \frac{1}{2}\right\}}\right) + \mathbb{E}\left(\left(\frac{\widetilde{\Phi}_n}{(\widetilde{K}_n + c) \vee 1}\right)^q \mathbf{1}_{\left\{\frac{\widetilde{K}_n}{\widetilde{\Phi}_n} < \frac{1}{2}\right\}}\right)\\ &\le 2^q(1 \vee (1 - c))^q + 2\widetilde{\Phi}_n^q \exp\left(-\frac{3}{28}\widetilde{\Phi}_n\right).\end{aligned}$$

We infer that $\{\widetilde{\Phi}_n/((\widetilde{K}_n+c)\vee 1)\}_{n\geq 1}$ is bounded in L^q and then $\{[\widetilde{\Phi}_n/((\widetilde{K}_n+c)\vee 1)]^p\}_{n\geq 1}$ is uniformly integrable. The desired L^p convergence follows. $\square$

Proof of Theorem 4.1. To prove the convergence of the point processes of interest, we compute their Laplace transform:

$$\Psi_{\xi_n}(f) := \mathbb{E}\exp\left(-\xi_n(f)\right) = \mathbb{E}\exp\left(-\sum_{k=1}^{K_n} f\left(\varepsilon_{n,k}/b_n, \widehat{R}_{n,k}/n\right)\right),$$

for $f \in C_K^+((0,\infty)\times\mathcal{F}([0,1]))$, the space of non-negative continuous functions with compact support. Similarly,

$$\Psi_\xi(f) := \mathbb{E}\exp\left(-\sum_{\ell=1}^{\infty} f\left(\Gamma_\ell^{-1/\alpha}, \mathcal{R}_\ell^{(\beta)}\right)\right)$$

is the Laplace transform of ξ. Recall that the desired convergence follows if and only if

$$\lim_{n\to\infty}\Psi_{\xi_n}(f) = \Psi_\xi(f), \text{ for all } f\in C_K^+((0,\infty),\mathcal{F}([0,1])), \tag{5.10}$$

see for example [29, Proposition 3.19].

Now we prove (5.10). When investigating the weak convergence of point processes here, the topology on $(0,\infty)$ is such that all compact sets are bounded away from zero and $\mathcal{F}([0,1])$ is itself a compact metric space. So, for any $f\in C_K^+((0,\infty)\times\mathcal{F}([0,1]))$, there exists $\kappa=\kappa(f)>0$ so that $f(x,F)=0$ for all $x<\kappa$ and $F\in\mathcal{F}([0,1])$. Given f and thus $\kappa>0$ fixed, for all $\epsilon>0$, we can pick $m=m(\kappa,\epsilon)\in\mathbb{N}$ large enough, so that

$$\lim_{n\to\infty}\mathbb{P}\left(B_{\kappa,n}^{(m)}\right) = \mathbb{P}\left(\Gamma_m^{-1/\alpha}<\kappa\right) > 1-\epsilon \quad \text{ with } B_{\kappa,n}^{(m)} := \left\{\frac{\varepsilon_{n,m}}{b_n}<\kappa\right\}.$$

Now we express $\Psi_{\xi_n}(f)$ as

$$\Psi_{\xi_n}(f) = \mathbb{E}\left[\exp\left(-\xi_n(f)\right)\mathbf{1}_{B_{\kappa,n}^{(m)}}\right] + \mathbb{E}\left[\exp\left(-\xi_n(f)\right)\mathbf{1}_{(B_{\kappa,n}^{(m)})^c}\right].$$

The second term on the right-hand side above is then bounded by $1-\mathbb{P}(B_{\kappa,\epsilon,n}^{(m)})$. The first term equals

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{m} f\left(\varepsilon_{n,k}/b_n,\widehat{R}_{n,k}/n\right)\right)\mathbf{1}_{B_{\kappa,n}^{(m)}}\right]. \tag{5.11}$$

This is the expectation of a function from $\mathbb{R}_+^m\times\mathcal{F}([0,1])^m$ to $[0,1]$, continuous everywhere except at points from the set

$$\left\{(x_1,\dots,x_m,F_1,\dots,F_m)\in\mathbb{R}_+^m\times\mathcal{F}([0,1])^m : x_m=\kappa\right\}. \tag{5.12}$$

We have seen the convergence $(\varepsilon_{n,k}/b_n,\widehat{R}_{n,k}/n)_{k=1,\dots,m}\Rightarrow(\Gamma_k^{-1/\alpha},\mathcal{R}_k^{(\beta)})_{k=1,\dots,m}$ in Proposition 5.1, and we can notice that the set of discontinuity points (5.12) above is hit by $(\Gamma_1^{-1/\alpha},\dots,\Gamma_m^{-1/\alpha},\mathcal{R}_1^{(\beta)},\dots,\mathcal{R}_m^{(\beta)})$ with probability zero. Therefore, applying the continuous mapping theorem to (5.11), we have that

$$\begin{aligned}\limsup_{n\to\infty}\Psi_{\xi_n}(f) &\leq \mathbb{E}\left[\exp\left(-\sum_{k=1}^{m} f\left(\Gamma_k^{-1/\alpha},\mathcal{R}_k^{(\beta)}\right)\right)\mathbf{1}_{\left\{\Gamma_m^{-1/\alpha}<\kappa\right\}}\right]+\epsilon\\ &= \mathbb{E}\left[\exp\left(-\sum_{k=1}^{\infty} f\left(\Gamma_k^{-1/\alpha},\mathcal{R}_k^{(\beta)}\right)\right)\mathbf{1}_{\left\{\Gamma_m^{-1/\alpha}<\kappa\right\}}\right]+\epsilon\\ &\leq \Psi_\xi(f)+\epsilon.\end{aligned}$$

Similarly, one can show that

$$\liminf_{n\to\infty}\Psi_{\xi_n}(f) \geq \Psi_\xi(f) - \mathbb{P}\left(\Gamma_m^{-1/\alpha}\geq\kappa\right) \geq \Psi_\xi(f)-\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have thus proved (5.10) for every test function f, and hence the desired result. $\square$

Proof of Theorem 4.2. It suffices to prove, for all open intervals $A_1, \ldots, A_d$ in $[0,1]$ and positive reals $z_1, \ldots, z_d$, that

$$\begin{aligned} P_n &:= \mathbb{P}\left(\frac{M_n(A_1)}{b_n} > z_1, \ldots, \frac{M_n(A_d)}{b_n} > z_d\right) \\ &\longrightarrow \mathbb{P}(\mathcal{M}_{\alpha,\beta}(A_1) > z_1, \ldots, \mathcal{M}_{\alpha,\beta}(A_d) > z_d) := P, \text{ as } n \to \infty. \end{aligned}$$

This is a direct consequence of Theorem 4.1 since, denoting

$$\mathcal{F}_{A_i} = \{F \in \mathcal{F}([0,1]) : F \cap A_i \neq \emptyset\}, \ i = 1, \ldots, d,$$

we have

$$\begin{aligned} P_n &= \mathbb{P}\left(\xi_n\left((z_1,\infty) \times \mathcal{F}_{A_1}\right) \geq 1, \ldots, \xi_n\left((z_d,\infty) \times \mathcal{F}_{A_d}\right) \geq 1\right) \\ &\longrightarrow \mathbb{P}\left(\xi\left((z_1,\infty) \times \mathcal{F}_{A_1}\right) \geq 1, \ldots, \xi\left((z_d,\infty) \times \mathcal{F}_{A_d}\right) \geq 1\right) = P, \text{ as } n \to \infty. \end{aligned}$$

$\square$

Our proof of Theorem 4.2 is based on the presentation (3.5) of $\mathcal{M}_{\alpha,\beta}$, which we have shown at the beginning can be derived from the presentation (3.2). We conclude this section by giving a direct proof of Theorem 4.2 using the presentation (3.2) and also without using Proposition 5.1.

Second proof of Theorem 4.2. Fix $d \in \mathbb{N}$, open intervals $A_1, \ldots, A_d$ in $[0,1]$ and positive reals $z_1, \ldots, z_d$. We shall prove that

$$\mathbb{P}\left(\frac{M_n(A_k)}{b_n} \leq z_k, \ k = 1, \ldots, d\right) \to \mathbb{P}\left(\mathcal{M}_{\alpha,\beta}(A_k) \leq z_k, \ k = 1, \ldots, d\right),$$

as $n \to \infty$. For every $\ell \in \mathbb{N}$ and every $n \in \mathbb{N}$, we record whether $Y_i = \ell$ for some $i \in nA_k$, for each $k = 1, \ldots, d$, and count different types of boxes. More precisely, introduce $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d) \in \Lambda_d := \{0,1\}^d \setminus \{0, \ldots, 0\}$, and consider

$$\tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}(n) := \sum_{\ell=1}^{\infty} \prod_{\substack{k=1,\ldots,d \\ \delta_k = 1}} \mathbf{1}_{\{\exists i \in nA_k, Y_i = \ell\}} \prod_{\substack{k'=1,\ldots,d \\ \delta_{k'} = 0}} \mathbf{1}_{\{\forall i \in nA_{k'}, Y_i \neq \ell\}}.$$

For example, $\tau_{\boldsymbol{A}}^{1,\ldots,1}(n)$ is the number of box ℓ that has been sampled in some round $i_1 \in nA_1, i_2 \in nA_2, \ldots, i_d \in nA_d$, and $\tau_{\boldsymbol{A}}^{1,0,\ldots,0}(n)$ is the number of box ℓ that has been sampled in some round $i_1 \in nA_1$, but never in any round in $nA_2, \ldots, nA_d$. So all boxes that have been sampled during the first n rounds are divided into disjoint groups indexed by $\boldsymbol{\delta} \in \Lambda_d$.

Now we need the following limit theorem for $\tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}(n)$:

$$\lim_{n\to\infty} \frac{\tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}(n)}{n^\beta L(n)} = \tau_{\boldsymbol{A}}^{\boldsymbol{\delta}} := \int_0^\infty \beta x^{-\beta-1} \widetilde{\mathbb{P}}\left(\mathbf{1}_{\{\widetilde{\mathcal{N}}(xA_k) \neq 0\}} = \delta_k, k = 1, \ldots, d\right) dx \tag{5.13}$$

in probability. This follows from [10, Theorem 2] (which was also established by the Poissonization technique): the above identity therein was established for the corresponding Poisson random measures being even or odd, and we obtain the desired result here by applying the identity

$$\widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(A) \neq \emptyset\right) = \frac{1}{2}\widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(2A) \text{ odd}\right).$$

Then, conditioning on $\{Y_n\}_{n\in\mathbb{N}}$, we can write

$$\begin{aligned}
&\mathbb{P}\left(\frac{M_n(A_k)}{b_n} \le z_k, k=1,\dots,d\right)\\
&= \mathbb{E}\left[\prod_{\boldsymbol{\delta}\in\Lambda_d} \mathbb{P}_0\left(\frac{\varepsilon_0}{b_n} \le \min_{k=1,\dots,d,\delta_k=1} z_k\right)^{\tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}(n)}\right]\\
&= \mathbb{E}\exp\left\{\sum_{\boldsymbol{\delta}\in\Lambda_d} \tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}(n)\log\left[1-\mathbb{P}_0\left(\frac{\varepsilon_0}{b_n} > \min_{k=1,\dots,d,\delta_k=1} z_k\right)\right]\right\},
\end{aligned}$$

where ε_0, defined on another probability space $(\Omega_0,\mathcal{F}_0,\mathbb{P}_0)$, has the same distribution as ε_1. By (5.13) and heavy-tail assumption (4.3) on ε's distribution,

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{\boldsymbol{\delta}\in\Lambda_d} \tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}(n)\log\left[1-\mathbb{P}_0\left(\frac{\varepsilon_0}{b_n} > \min_{k=1,\dots,d,\delta_k=1} z_k\right)\right]\\
&\qquad = \Gamma(1-\beta)^{-1}\sum_{\boldsymbol{\delta}\in\Lambda_d} \tau_{\boldsymbol{A}}^{\boldsymbol{\delta}}\left(\min_{k=1,\dots,d,\delta_k=1} z_k\right)^{-\alpha} \text{ in probability.}
\end{aligned}$$

This last sum can be written as

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left(\sum_{\boldsymbol{\delta}\in\Lambda_d}\max_{k=1,\dots,d,\delta_k=1}\frac{1}{z_k^\alpha}\prod_{k=1}^d \mathbf{1}_{\left\{\mathbf{1}_{\{\widetilde{N}(xA_k)\neq0\}}=\delta_k\right\}}\right)\\
&= \widetilde{\mathbb{E}}\left(\sum_{\boldsymbol{\delta}\in\Lambda_d}\max_{k=1,\dots,d}\frac{\mathbf{1}_{\{\widetilde{N}(xA_k)\neq0\}}}{z_k^\alpha}\mathbf{1}_{\left\{\mathbf{1}_{\{\widetilde{N}(xA_k)\neq0\}}=\delta_k,k=1,\dots,d\right\}}\right)\\
&= \widetilde{\mathbb{E}}\left(\max_{k=1,\dots,d}\frac{\mathbf{1}_{\{\widetilde{N}(xA_k)\neq0\}}}{z_k^\alpha}\right).
\end{aligned}$$

Summing up, we have thus shown that

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{P}\left(\frac{M_n(A_k)}{b_n}\le z_k,\ k=1,\dots,d\right)\\
&\qquad = \exp\left(-\Gamma(1-\beta)^{-1}\int_0^\infty \beta x^{-\beta-1}\widetilde{\mathbb{E}}\left(\max_{k=1,\dots,d}\frac{\mathbf{1}_{\{\widetilde{N}(xA_k)\neq0\}}}{z_k^\alpha}\right)dx\right),
\end{aligned}$$

which is the desired finite-dimensional distribution as in (3.2). □

6. Random sup-measures and associated extremal processes

The extremal process associated to the Karlin random sup-measure $\mathcal{M}_{\alpha,\beta}$ appears to be a time-changed version of a standard α-Fréchet extremal process $\mathbb{M}_\alpha$, precisely

$$\left\{\mathbb{M}_\alpha(t^\beta)\right\}_{t\ge0}. \tag{6.1}$$

As noticed in Section 2, this is a consequence of the more general fact that the extremal process of any Choquet α-Fréchet random sup-measure is determined by the extremal coefficient functional evaluated on sets $\{[0,t]\}_{t>0}$ only. This is proved in Proposition A.1 in the appendix. The Karlin random sup-measure is of course not the only Choquet random sup-measure corresponding to the same extremal process (6.1). Another such family that arises naturally from limit theorems with long-range dependence are the stable-regenerative random sup-measures [18] recalled below.

Example 6.1. We recall the definition of stable-regenerative random sup-measures:

$$\mathcal{M}^{\rm sr}_{\alpha,\beta}(\cdot) := \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\left\{\left(V_\ell^{(\beta)}+R_\ell^{(\beta)}\right)\cap\,\cdot\,\neq\emptyset\right\}}, \tag{6.2}$$

where $\{(\Gamma_\ell, V_\ell^{(\beta)}, R_\ell^{(\beta)})\}_{\ell\in\mathbb{N}}$ is a Poisson point process on $\mathbb{R}_+\times\mathbb{R}_+\times\mathcal{F}(\mathbb{R}_+)$ with intensity $dx\beta v^{-(1-\beta)}dvdP_{1-\beta}$ where $P_{1-\beta}$ is the law of $(1-\beta)$-stable regenerative set (i.e., the closure of a $(1-\beta)$-stable subordinator [2]) on $\mathbb{R}_+$, and $\widetilde{R}_\ell^{(\beta)}$ in (1.3) is $V_\ell^{(\beta)} + R_\ell^{(\beta)}$ here. It was shown [18, 26] that

$$\left\{\mathcal{M}^{\rm sr}_{\alpha,\beta}([0,t])\right\}_{t\ge 0} \stackrel{d}{=} \left\{\mathbb{M}_\alpha(t^\beta)\right\}_{t\ge 0}.$$

(Strictly speaking only $\beta\in(1/2,1)$ was considered in [18], although the extension to $\beta\in(0,1)$ is straightforward.)

We now give an example of random sup-measure that is self-similar, *non-stationary*, and yet also has the same extremal process.

Example 6.2. For $\beta>0$, let T_β be the mapping between subsets of $\mathbb{R}_+$ induced by $t\mapsto t^\beta$. Then, $\mathcal{M}_\alpha\circ T_\beta$ is β/α-self-similar, but non-stationary, and the corresponding extremal process also has the form $\{\mathbb{M}_\alpha(t^\beta)\}_{t\ge 0}$.

In the special case $\beta\in(0,1)$, we provide another equivalent representation of $\mathcal{M}_\alpha\circ T_\beta$, which can also be connected to a variation of the Karlin model investigated in Section 4. Let $\widetilde{\mathcal{N}}$ be a Poisson random measure on $\mathbb{R}_+$, and view it as a Poisson process by letting $\widetilde{\mathcal{N}}(t) = \widetilde{\mathcal{N}}([0,t])\in\mathbb{N}_0 := \{0\}\cup\mathbb{N}$ denote the counting number of the Poisson process. We write

$$\widetilde{\mathcal{N}}[A] := \left\{\widetilde{\mathcal{N}}(t) : t\in A\right\}\subset\mathbb{N}_0,\ \text{for } A\subset\mathbb{R}_+.$$

We then introduce

$$\mathcal{M}^*_{\alpha,\beta}(\cdot) := \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\left\{\widetilde{N}_\ell[x_\ell\,\cdot\,]\ni 1\right\}} \text{ on } \mathbb{R}_+, \tag{6.3}$$

with $\{(\Gamma_\ell, \widetilde{\mathcal{N}}_\ell, x_\ell)\}_{\ell\in\mathbb{N}}$ defined as in (3.1). When restricted to $[0,1]$,

$$\mathcal{M}^*_{\alpha,\beta}(\cdot) \stackrel{d}{=} \sup_{\ell\in\mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\left\{\mathcal{R}_\ell^{(\beta)*}\cap\,\cdot\,\neq\emptyset\right\}},$$

with $\mathcal{R}^{(\beta)*}\stackrel{d}{=}\min\mathcal{R}^{(\beta)}$ (recall (3.4)). In fact, one could define $\mathcal{M}_{\alpha,\beta}$ and $\mathcal{M}^*_{\alpha,\beta}$ based on the same Poisson point process such that with probability one, $\mathcal{M}_{\alpha,\beta}(\cdot)\ge\mathcal{M}^*_{\alpha,\beta}(\cdot)$.

Proposition 6.3. *Let $\mathcal{M}_\alpha$ be defined as in* (1.2) *and T_β be the mapping between subsets of $\mathbb{R}_+$ induced by $t\mapsto t^\beta$ for some $\beta\in(0,1)$, then*

$$\mathcal{M}^*_{\alpha,\beta} \stackrel{d}{=} \mathcal{M}_\alpha\circ T_\beta \tag{6.4}$$

as random sup-measures on $\mathbb{R}_+$.

Proof. To show (6.4), by self-similarity it suffices to restrict to $[0,1]$ and compare the capacity functionals of the random closed sets in the Poisson point process presentation (6.3) and (1.2). We start by computing the extremal coefficient functional corresponding to (6.3): for an interval $A = (a, b]$,

$$\begin{aligned}
\Gamma(1-\beta)^{-1} \int_0^\infty & \beta x^{-\beta-1} \widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}[xA] \ni 1\right) dx \\
&= \Gamma(1-\beta)^{-1} \int_0^\infty \beta x^{-\beta-1} \widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(xa) = 0, \widetilde{\mathcal{N}}(xb) > 0\right) dx \\
&= \Gamma(1-\beta)^{-1} \int_0^\infty \beta x^{-\beta-1} \left[\widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(xa) = 0\right) - \widetilde{\mathbb{P}}\left(\widetilde{\mathcal{N}}(xb) = 0\right)\right] dx \\
&= (b^\beta - a^\beta).
\end{aligned}$$

This implies that the capacity functional for $\mathcal{R}^{(\beta)*}$ is

$$\theta([a,b])\mathbb{P}\left(\mathcal{R}^{(\beta)*} \cap [a,b] \neq \emptyset\right) = b^\beta - a^\beta = \mathrm{Leb}(T_\beta([a,b])),$$

whence

$$\mathcal{M}^*_{\alpha,\beta}(\cdot) \stackrel{d}{=} \sup_{\ell \in \mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{U_\ell \in T^\beta(\cdot)\}},$$

with $\{U_n\}_{n\in\mathbb{N}}$ being i.i.d. uniform random variables on $[0, 1]$. The desired result hence follows. □

The above representation of $\mathcal{M}_\alpha \circ T_\beta$ was discovered during our investigation on the limit of empirical random sup-measures for the following variation of the Karlin model

$$X^*_n := \varepsilon_{Y_n} \mathbf{1}_{\{K_{n,Y_n}=1\}}, \quad n \in \mathbb{N},$$

with $\{Y_n\}_{n\in\mathbb{N}}$, $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $K_{n,\ell}$, as in Section 4. In this variation, if a box ℓ is sampled ($Y_n = \ell$), then $X^*_n = \varepsilon_\ell$ only if this is the first time for the box ℓ, and $X^*_n = 0$ otherwise. For this model, one could establish a limit theorem for the empirical random sup-measure, and the limit is exactly the random sup-measure $\mathcal{M}^*_{\alpha,\beta}$. The sequence $\{X^*_n\}_{n\in\mathbb{N}}$ is not stationary, a drastically difference from $\{X_n\}_{n\in\mathbb{N}}$ considered in Section 4. Nevertheless, we see that partial maxima of both sequences are equal, explaining the equality of the corresponding extremal processes in the limit.

We conclude this section by the following remark comparing the aforementioned random sup-measures.

Remark 6.4. In summary, for $\beta \in (0, 1)$,

$$\mathcal{M}_{\alpha,\beta},\ \mathcal{M}^{\mathrm{sr}}_{\alpha,\beta},\ \text{and}\ \mathcal{M}_\alpha \circ T_\beta$$

all have the same extremal process as $\{\mathbb{M}_\alpha(t^\beta)\}_{t\geq 0}$. The independently scattered random sup-measure $\mathcal{M}_\alpha$, the stable-regenerative random sup-measure $\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}$, and the Karlin random sup-measure $\mathcal{M}_{\alpha,\beta}$ are all self-similar and shift-invariant. However, for the corresponding max-increment processes (3.6), $\mathcal{M}_\alpha$ is mixing, $\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}$ is ergodic but not mixing, and $\mathcal{M}_{\alpha,\beta}$ is not ergodic. The random sup-measure $\mathcal{M}_\alpha \circ T_\beta$ is self-similar but not shift-invariant.

Appendix A. Extremal processes of Choquet random sup-measures

As before, given a random sup-measure $\mathcal{M}$, we let $\mathbb{M}(t) := \mathcal{M}([0, t])$, $t \geq 0$, denote its associated extremal process. We denote by $\mathbb{M}_\alpha$ the standard α-Fréchet extremal process defined in (2.4). In the literature, $\mathbb{M}_\alpha$ was originally named *the* extremal process [12, 20]. The notion has become however more and more common to refer to various limits of partial-maxima processes. The same notion was also used for random sup-measures in [25].

Recall the definition Choquet random sup-measures (2.2) in Section 2. Proposition 2.1 therein is a special case of the following result.

Proposition A.1. *Let $\mathcal{M}$ be a Choquet α-Fréchet random sup-measure with extremal coefficient functional θ, and $\mathbb{M}$ its extremal process.*
(i) For $d \in \mathbb{N}, 0 < t_1 < \cdots < t_d$ and $x_1, \ldots, x_d \in \mathbb{R}_+$,

$$\mathbb{P}\left(\mathbb{M}(t_k) \le x_k, k = 1, \ldots, d\right) = \exp\left(-\sum_{k=1}^{d} a_k \theta([0, t_k])\right) \tag{A.1}$$

with

$$a_k := \frac{1}{\bigwedge_{j=k}^{d} x_j} - \frac{1}{\bigwedge_{j=k+1}^{d} x_j}, k = 1, \ldots, d-1,$$

and $a_d := 1/x_d$.
(ii) If in addition $\mathcal{M}$ is H-self-similar with $H > 0$, then,

$$\theta([0,1]) \left\{\mathbb{M}(t)\right\}_{t \ge 0} \stackrel{d}{=} \left\{\mathbb{M}_\alpha(t^{\alpha H})\right\}_{t \ge 0}.$$

Proof. We start by computing the finite-dimensional distribution of the associated extremal process. We write

$$\begin{aligned}\mathbb{P}(\mathbb{M}(t_k) \le x_k, k = 1, \ldots, d) &= \mathbb{P}\left(\mathcal{M}([0, t_k]) \le x_k, k = 1, \ldots, d\right) \\ &= \mathbb{P}\left(\int_{\mathbb{R}_+}^{\vee} \bigvee_{k=1}^{d} \frac{\mathbf{1}_{[0,t_k]}}{\bigwedge_{j=k}^{d} x_j} d\mathcal{M} \le 1\right).\end{aligned}$$

See [34] for background on stochastic extremal integrals $\int^\vee f d\mathcal{M}$. We then express the integrand as

$$f(t) := \bigvee_{k=1}^{d} \frac{\mathbf{1}_{[0,t_k]}(t)}{\bigwedge_{j=k}^{d} x_j} = \sum_{k=1}^{d} a_k \mathbf{1}_{[0,t_k]}(t).$$

In this way, we see that f is an upper-semi-continuous function expressed as the sum of d comonotonic functions. Let θ denote the extremal coefficient functional of $\mathcal{M}$. From [24], we know that $\mathbb{P}(\int^\vee g d\mathcal{M} \le t) = \exp(-\ell(g)/t)$, $t > 0$, where here and below, $\ell(g) := \int g d\theta$ (understood as a Choquet integral for upper-semi-continuous function g) is the *tail dependence functional* of $\mathcal{M}$, and $\ell(\mathbf{1}_K) = \theta(K)$. In particular we have

$$\begin{aligned}\mathbb{P}(\mathbb{M}(t_k) \le x_k, k = 1, \ldots, d) &= \mathbb{P}\left(\int^\vee f d\mathcal{M} \le 1\right) = \exp\left(-\ell(f)\right) \\ &= \exp\left(-\sum_{k=1}^{d} a_k \theta([0, t_k])\right),\end{aligned} \tag{A.2}$$

and in the last step we applied the comonotonic additivity of the tail dependence function ℓ for Choquet random sup-measures (i.e., for comonotonic functions g, h, $\int g + h d\theta = \int g d\theta + \int h d\theta$ [8, 24]). We have proved the first part of the proposition.

We also know that for an H-self-similar α-Fréchet random sup-measure, the extremal coefficient functional necessarily has the scaling property $\theta(\lambda[0,t]) = \lambda^{\alpha H}\theta([0,t])$ for all $\lambda > 0$ (see (2.1)). So for such a random sup-measure the conclusion of the first part becomes

$$\mathbb{P}\left(\mathbb{M}(t_k) \le x_k, k = 1, \ldots, n\right) = \exp\left(-\theta([0,1]) \sum_{k=1}^{\ell} a_k t_k^{\alpha H}\right).$$

Recall that for the independently scattered random sup-measure $\mathcal{M}_\alpha$, extremal coefficient functional is the Lebesgue measure. The second part of the proposition then follows. ☐

Remark A.2. We thank an anonymous referee for pointing out to us the following consequence: for a general Fréchet random sup-measure not of Choquet type, the statement (A.1) holds with '=' replaced by '$\ge$'. This is due to the stochastic dominance property of Choquet random sup-measures. Indeed, a general Fréchet random sup-measure $\widetilde{\mathcal{M}}$ can be coupled with a Choquet random sup-measure $\mathcal{M}$ *with the same extremal coefficient functional* θ. Let $\widetilde{\mathbb{M}}, \mathbb{M}$ and $\widetilde{\ell}, \ell$ be the extremal processes and the tail dependence functionals of the two random sup-measures, respectively. It is shown in [24, Corollary 5.4] that $\widetilde{\ell} \leq \ell$. Now, the aforementioned statement follows from the fact that the law of the extremal process is uniquely determined by the tail dependence functional (A.2).

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