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▶ **To cite this version:**

Nicolas Martin, Paolo Frasca, Carlos Canudas de Wit. Network reduction towards a scale-free structure preserving physical properties. Complex Networks 2017 - The 6th International Conference on Complex Networks and Their Applications, Nov 2017, Lyon, France. pp.294-296. hal-01769015v2

HAL Id: hal-01769015
https://hal.science/hal-01769015v2

Submitted on 19 Apr 2018

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Network reduction towards a scale-free structure preserving physical properties

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1 Introduction

The analysis and control of dynamics on large networks is often a challenging task. Reduction methods which can cut this complexity have intensively been developed and see for example these recent works [1, 2]. We propose here a new approach of reduction allowing to drive an original arbitrary large network towards a particular structure: a scale-free network. Scale-free networks are characterized by a large number of nodes with a small degree and some nodes with a very large degree: their degree distribution is a power law. The scale-freeness of a network implies interesting properties: robustness to failures, ultra-small world property [3], and interesting features for control design [4]. Then, it appears that, for some applications, the reduction method would be more interesting if it allows to obtain a reduced network with a scale-free structure. The objective of the method developed here is starting from an arbitrary large weighted and directed network and finding a reduced network with a scale-free structure, while preserving some physical properties of the initial graph for consistency.

2 Problem formulation

The reduction is made by partitioning the initial graph into groups of connected nodes and assigning each part as a new node in the reduced graph. By this way, the topological structure of the graph (interconnections) is preserved. Then the problem can be viewed as a minimization problem where we look for a partition of the initial graph G_0 such that the graph $\tilde{G}$ coming out of this partition minimizes the sum of two cost functions: $\mathrm{J}_{\mathrm{SF}_\alpha}(G) + \mathrm{J}_{\mathrm{sim}}(G, G_0)$ where $\mathrm{J}_{\mathrm{SF}_\alpha}(G)$ is the scale-free error and gives indication on the scale-freeness of the graph and $\mathrm{J}_{\mathrm{sim}}(G, G_0)$ is the similarity cost function between G and G_0 and gives an indication on the error between the graph and the initial graph. The minimization is done under a constraint: $\tilde{G}$ has to preserve some physical properties.
The general problem being stated, we give some specifications to solve a particular case:

- The scale-free error is the 2-norm difference between the degree distribution of the graph and the power law corresponding to the desired scale-free distribution.

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– The similarity cost function is written as follow:

$$\mathrm{J}_{\mathrm{sim}}(G_1, G_0) = \frac{\|x^\star_{G_1} - \mathrm{Proj}(x^\star_{G_0})\|_2}{\|\mathrm{Proj}(x^\star_{G_0})\|_2} \tag{1}$$

where $x^\star_G$ is an indicator of spectral centrality in the graph G (not detailed here), and $\mathrm{Proj}(x)$ is the vector of the sum of the element of x within each clusters. This cost function translates the closeness between the centrality in the reduced graph and the sum of the centrality within each cluster in the initial graph.

– The only physical property we want to preserve is the mass conservation: for every node the sum of the weights of incoming edges is equal to the sum of the weights of outgoing edges.

3 Results: Theorem and simulations

Definition: A merging is a particular partition where only two connected nodes are gathered. Precisely if the set of edges is $V = \{1, ..., n\}$, a merging is a partition with the form $\{\{1\}, ..., \{v-1\}, \{v+1\}, ..., \{w-1\}, \{w+1\}, ..., \{n\}, \{v, w\}\}$ where (v, w) is an edge of the graph.

Within the specifications given previously we have the following theorem.

Theorem 1. *Let G_0 be a weighted graph. For all merging M, the weights of the graph G_1 issued of the merging M of G_0 can be chosen such that G_1 preserves the mass conservation property and such that the similarity cost function is null.*

By exploiting this theorem, we have designed an iterative algorithm to solve the above optimisation problem: starting from an initial large graph, we look at each step for the best merging which is the merging whose resulting graph minimizes the scale-free error. We do not look at the value of the similarity cost function as it is null for all merging just by choosing the good weights. Instead of looking for the best edge within the whole set of edges, we only search within a small randomly selected set of edges. This choice allows to cut considerably the computation time and figure 2 shows that it has a minor influence on the efficiency of the algorithm. The complexity of the algorithm is $O(N_r N_v^2)$, where N_v is the number of nodes in the initial graph and N_r the size of the random set of edges tested.
Although the algorithm does not provide the exact solution of the problem, simulations show that the graph obtained is really close to the scale-free structure desired. Figure 1 shows the results of the simulation on the urban traffic network of Grenoble which is originally rather far from a scale-free structure.

Acknowledgements

The research leading to these results has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement N 694209)

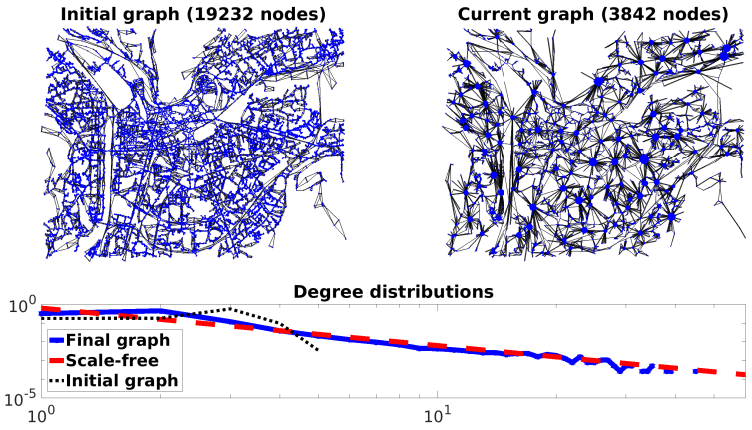

Fig. 1: Top left: the initial graph of Grenoble (double-way roads are thicker than one-way ones). Top right: result of the algorithm, a scale-free graph five times smaller than the initial one. The graph respects the mass conservation property and allows to reconstruct the centrality of each zone with a perfect consistency with the centrality in the initial graph. Bottom: the degree distributions show the ability of the algorithm to drive the initial graph to the desired structure.

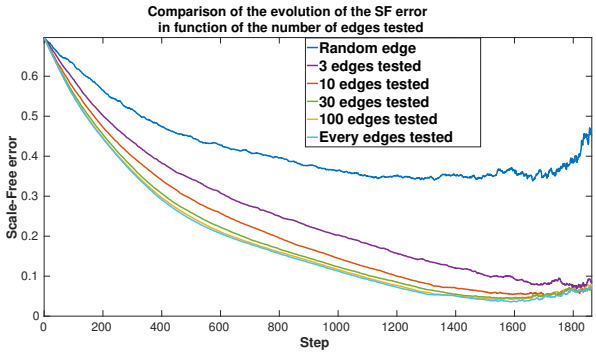

Fig. 2: Evolution of the scale-free error through the algorithm for different values of the size of the random set of edges: there is no significant advantage to explore a very large set of edges.

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