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DECAY ESTIMATES FOR LARGE VELOCITIES IN THE BOLTZMANN EQUATION WITHOUT CUT-OFF

CYRIL IMBERT, CLÉMENT MOUHOT, AND LUIS SILVESTRE

Abstract. We establish pointwise large velocity decay rates for the solution of the inhomogeneous Boltzmann equation without cutoff, under the assumption that three hydrodynamic quantities (mass, energy, entropy densities) stay under control.

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1. Introduction

1.1. **The Boltzmann equation.** The Boltzmann equation models the evolution of particle densities of gases. It sits at a mesoscopic scale between the hydrodynamic equations (e.g. the compressible Euler or Navier Stokes equations) describing the evolution of hydrodynamic quantities at a large scale, and the complicated dynamical system describing the movement of a huge number of particles. Fluctuations around steady state, at large scale, describe incompressible Navier-Stokes equations.

This density of particles is modelled by a function $f(t,x,v) : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ solving the equation

$$f_t + v \cdot \nabla_x f = Q(f,f) \quad \text{for } t \in (-1,0],\ x \in B_1,\ v \in B_1, \quad B_1 := \{z \in \mathbb{R}^d \ :\ |z| \le 1\}. \tag{1.1}$$

We recall that Boltzmann's collision operator $Q(f,f)$ is defined as follows

$$Q(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f(v'_*)f(v') - f(v_*)f(v)) B(|v-v_*|, \cos\theta)\, \mathrm{d}v_*\, \mathrm{d}\sigma$$

where the pre-collisional velocities v'_* and v' are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \quad \text{and} \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

and $\cos\theta$ (and $\sin(\theta/2)$) is *defined* as

$$\cos\theta := \frac{v-v_*}{|v-v_*|} \cdot \sigma \qquad \left(\text{and} \quad \sin(\theta/2) := \frac{v'-v}{|v'-v|} \cdot \sigma \right).$$

There are different modelling choices for the cross-section B. We are interested in the usual *non-cutoff* assumption. We consider a function B of the form

$$B(r, \cos\theta) = r^\gamma b(\cos\theta) \quad \text{with} \quad b(\cos\theta) \approx |\sin(\theta/2)|^{-(d-1)-2s} \tag{1.2}$$

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with $\gamma > -d$ and $s \in (0,1)$. In dimension $d = 3$, when the long-range interaction derives from a power-law repulsive force $F(r) = Cr^{-\alpha}$ with $\alpha \in (2, +\infty)$, then the exponents are given by: $\gamma = \frac{\alpha-5}{\alpha-1}$ and $s = \frac{1}{\alpha-1}$ (see [16] and [45, Chapter 1]). The case $\alpha = 5$, $\gamma = 0$ and $2s = 1/2$ is called *Maxwell molecules* [36], the case $\alpha \in (5, +\infty)$, $\gamma > 0$ and $2s \in (0, 1/2)$ is called *hard potentials (without cutoff)*, the case $\alpha \in [3, 5)$, $\gamma \in [-1, 0)$, $2s \in (1/2, 1]$ is called *moderately soft potentials (without cutoff)*, and finally the case $\alpha \in (2, 3)$, $\gamma \in (-3, -1)$, $2s \in (1, 2)$ is called *very soft potentials (without cutoff)*. The formal limit $s \to 1$ is called the *grazing collision limit*, and in this limit the Boltzmann collision operator converges to the Landau-Coulomb collision operator. It turns out that the limit between moderately and very soft potentials corresponds to $\gamma + 2s = 0$. We shall therefore call moderately soft potentials, in any dimension, the case $\gamma + 2s \in (0, 1)$.

The classical well posedness of the inhomogeneous Boltzmann equation is an outstanding open problem. In some sense it is a more detailed model than the Euler and Navier-Stokes equations, and therefore one can expect that it will share some of the (currently intractable) difficulties of these hydrodynamic models. Even in the spatially homegeneous case, the Cauchy problem is shown to be well-posed without perturbative assumption only in the case of moderately soft potentials [20]. At this time it seems out of reach to fully resolve the well-posedness problem in general. Our more realistic program is to show that for suitable initial data $f(0, x, v) = f_0(x, v)$, the equation (1.1) has a unique smooth solution for as long as its associated hydrodynamic quantities stay under control. Morally, our project neglects the hydrodynamic difficulties of the model and concentrates on the difficulties that are intrinsic to the kinetic representation of the fluid.

Let us state the longer-term conjecture. Consider the following hydrodynamic quantities

$$\text{(mass density)} \qquad M(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\, \mathrm{d}v,$$

$$\text{(energy density)} \qquad E(t,x) := \int_{\mathbb{R}^d} f(t,x,v)|v|^2\, \mathrm{d}v,$$

$$\text{(entropy density)} \qquad H(t,x) := \int_{\mathbb{R}^d} f \ln f(t,x,v)\, \mathrm{d}v.$$

Conditional regularity. Consider any solution to (1.1) on a time interval $[0, T]$ such that its hydrodynamical fields remain controlled on this time interval: more precisely assume that for all points (t, x), the mass density is bounded below and above $0 < m_0 \le M(t,x) \le M_0$, the energy density is bounded above $E(t,x) \le E_0$ and the entropy density is bounded above $H(t,x) \le H_0$. Then the solution is bounded and smooth, and such solution is unique.

Observe that contraposing this statement means that any finite-time blow-up in solutions to the Boltzmann equation with long-range interactions must include a blow-up in the hydrodynamical quantity or the creation of vaccum, i.e. one the bounds above has to degenerate as $t \to T$.

There are two natural ways in which this conjecture can be strenghtened or weakened:

(1) First, the blow-up scenario through the creation of vaccum is likely to be ruled out by further work, which means that the lower bound assumption on the mass could be removed. Mixing in velocity through collisions combined with transport effects indeed generate lower bounds in many settings, see [40, 24, 12, 13]; moreover it was indeed proved for the related Landau equation with moderately soft potentials in [30].

(2) Second, more regularity or decay could be assumed on the initial data, as long as it is propagated conditionally to the hydrodynamical bounds above. This would slightly weaken the conjecture but the contraposed conclusion would remain unchanged: any blow-up must occur at the level of the hydrodynamical quantitites.

In order to complete this program, we need to put together the following pieces of the puzzle.

- A short time existence result.
- An estimate in L^∞.
- A decay estimate for large velocities.
- An initial regularization estimate in Hölder spaces.
- Estimates of Schauder type to obtain higher regularity by bootstrap.

The short time existence of (1.1) was obtained in [6] for sufficiently regular initial data f_0. An estimate in L^∞ for $t > 0$ was obtained in [42]. The main result in this paper is the decay estimate for large velocities.

The initial regularization in C^α is obtained in [33]. The bootstrap mechanism to obtain higher regularity is the piece of the puzzle that currently remains unsolved.

In [6], the authors prove that if the solution f has five derivatives in L^2, with respect to all variables t, x and v, weighted by $(1+|v|)^q$ for arbitrarily large powers q, and in addition the mass density is bounded below, then the solution f is C^∞. This is the best conditional regularity result currently known that gives us C^∞ regularity. However, its hypotheses are too strong to apply with our current knowledge of the solution. Note also the previous partial result [21] and the subsequent follow-up papers [4, 31, 2, 5, 39] in the spatially homogeneous case, with less assumption on the initial data.

Remark that our program of "conditional regularity" has already been carried out for the inhomogeneous Landau equation with moderately soft potentials, which corresponds to the formal limit of the Boltzmann equation as $s \to 1$, when furthermore $\gamma + 2s \in [0, 1)$. The L^∞ estimate, as well as Gaussian upper bounds, were obtained in [15]. The regularization estimate in Hölder spaces was obtained in [27]. The bootstrap mechanism was developed in [29] in the form of Schauder estimates for kinetic parabolic equations. The regularity of solutions of the Landau equation is iteratively improved using Schauder estimates up to C^∞ regularity. In the physical case of the Landau-Coulomb equation ($\alpha = 2$, $\gamma = -3$, $s = 1$ in dimension 3), the conjecture is still open: the L^∞ bound is missing (see however partial results in this direction in [43]), and the Schauder estimates [29] do not cover this case even though this last point is probably only a milder technical issue.

Note that, drawing inspiration from the case of the Landau equation, in order for the iterative gain of regularity in [29] to work, it is necessary to start with a solution that decays, as $|v| \to \infty$, faster than any algebraic power rate $|v|^{-q}$. We expect the same general principle to apply to the Boltzmann equation. We do not yet have appropriate Schauder type estimates for kinetic integro-differential equations to carry out an iterative gain in regularity. That will be a future important step of our program.

1.2. **Main result.** It is proved in [42] that when $M(t,x)$, $E(t,x)$ and $H(t,x)$ are uniformly bounded above, and in addition $M(t,x)$ is bounded below by a positive constant, then the solution f satisfies an L^∞ a priori estimate depending on those bounds only (See Theorem 2.4 below). The result in this paper goes a step further by establishing a large velocity decay bound for such solutions.

We work with a strong notion of solution for the Boltzmann equation. We make this precise in the following definition. For convenience, we work in the periodic setting $x \in \mathbb{T}^d$ in space.

Definition 1.1 (Solutions for the Boltzmann equation with rapid decay)**.** *We say that a function $f : [0,+\infty) \times \mathbb{T}^d \times \mathbb{R}^d \to [0,+\infty)$ is a solution to the Boltzmann equation (1.1) if*

- *the function f is differentiable in t and x and twice differentiable in v everywhere; the equation (1.1) holds classically at every point;*
- *for any $q > 0$, the limit $\lim_{|v|\to\infty} |v|^q f(t,x,v) = 0$ holds uniformly in t and x.*

Let us comment briefly on the choice of solutions considered:

We work with smooth solutions. Indeed the only reasonable notion of weak solutions available in the case of long-range interactions is the so-called *renormalized solution with defect measure* [3] (extending the notion of renormalized solutions of [22]), and these solutions are too weak to be handled by the methods of this paper.

We also impose rapid decay for $|v| \to \infty$. This is a qualitative assumption that we make for technical reasons. None of the estimates in our theorems depends on the decay rate as $|v| \to \infty$ that is initially assumed for the solution (otherwise, they would be vacuous results). In Section 6, we explore what parts of out main theorem we can recover without this decay assumption.

We suppose periodicity in x: this is made for convenience so that we easily find a first contact point in the breakthough scenario described in Section 3.

Theorem 1.2 (Decay upper bound)**.** *Let $\gamma \in (-2,2)$ and $s \in (0,1)$ satisfy $\gamma + 2s \in [0,2]$ and f be a solution of the Boltzmann equation (1.1) as in Definition 1.1 such that*

$$\forall (t,x), \in (0,+\infty) \times \mathbb{T}^d, \quad 0 < m_0 \le M(t,x) \le M_0,\ E(t,x) \le E_0 \text{ and } H(t,x) \le H_0 \tag{1.3}$$

holds true for some positive constants m_0, M_0, E_0, H_0. Then

(1) *If $\gamma \le 0$, then*

$$f(t,x,v) \le N(1 + t^{-\frac{d}{2s}}) \min\left(1, |v|^{-(d+1+\frac{\gamma d}{2s})}\right)$$

where $N > 0$ depends on $m_0, M_0, E_0, H_0, d, \gamma$ and s.

(2) *If $\gamma > 0$ then, for any $q > 0$ there exists a constant N and a power $\beta > 0$ such that*

$$f(t,x,v) \leq N(1+t^{-\beta}) \min(1, |v|^{-q})$$

where $N > 0$ depends on $m_0, M_0, E_0, H_0, q, d, \gamma$ and s.

(3) *If $\gamma \leq 0$, there exists q_0 such that for all $q \geq q_0$ and $f_{in} \leq C \min(1, |v|^{-q})$ then*

$$f(t,x,v) \leq N \min(1, |v|^{-q})$$

where q_0 depends on dimension, s, γ, m_0, M_0, E_0, H_0 and N depends on $C, m_0, M_0, E_0, H_0, q, d, \gamma$ and s.

Remark 1.3. In the proof of Theorem 1.2-(2), we get $\beta = d/(2s) + q/\gamma$ if q is large. We have not tried to obtain the optimal exponent β and the one that we see in the proof probably is not.

Remark 1.4. The assumptions cover, in dimension 3, the case of *moderately soft potentials*, see the numerology above.

1.3. **Moment estimates in the kinetic theory.** The study of large velocity decay has appeared in the past mostly in the form of estimates for higher order moments. This is a very classical and important theme in kinetic equations. In particular, it is necessary to prove that appropriate weak solutions satisfy the conservation (or the decrease) of the kinetic energy, corresponding to the second moment.

As recalled in [45, Chapter 2.2], moment estimates are the first step in the theory of Boltzmann, since the velocity space is unbounded and the collision operator averages over all velocities. Their study is initiated for Maxwellian potentials ($\gamma = 0$) in the spatially homogeneous case [32, 44], where an infinite collection of exact differential equations is derived for the polynomial moments and their propagation is shown, without any possibility of appearance. In the case of hard potentials ($\gamma > 0$) with angular cutoff and spatial homogeneity, Povzner identities [41, 23, 46] play an important role:

- Elmroth [23] used them to prove that if any moment $q > 2$ exists initially, then they remain bounded for all times;
- Desvillettes [18] then made the crucial remark that all moments are generated as soon as one moment of order $q > 2$ exists initially;
- Finally [38, 34] proved that even the condition on one moment of order $s > 2$ can be dispensed with and only the conservation of the energy is required (and it was extended to the spatially homogeneous hard potentials without cutoff in [46]).

These polynomial moment estimates were used by Bobylev [9], through some clever refinement of the Povzner inequality and infinite summation, in order to prove, for spatially homogeneous hard potentials with cutoff, the propagation of (integral) exponential tail estimates $e^{-C|v|^b}$ with $b \in (0, 2]$ and C small enough if $b = 2$. This result was extended in [10] to more general collision kernels (still hard potentials with cutoff). The argument of Bobylev was improved to obtain *generation* of (integral) exponential tail estimates $e^{-C|v|^b}$ with $b \in (0, \gamma]$ in [37, 7].

In the case of spatially homogeneous soft potentials, Desvillettes [18] proved for $\gamma \in (-1, 0)$ that initially bounded moments grow at most linearly with time and it is explained in [45] that the method applies to $\gamma \in [-2, 0)$. This was later improved [19] into bounds uniform in time thanks to the convergence to equilibrium. The case of measure-valued solutions in the spatially homogeneous hard potentials with cutoff is considered in [35].

Moment estimates play a role in many nonlinear kinetic models, not only in the (elastic) Boltzmann equation. We previously mentioned the Landau equation, this is also the case for instance in the study of the inelastic Boltzmann equation [37] or in the study of the Boltzmann-Nordheim equation for bosons [14].

An extension of the (integral) polynomial and exponential tail bounds to the spatially inhomogeneous periodic case for hard potentials with cutoff is presented in [28], it relies on a priori bounds on the hydrodynamical quantities similar to what is assumed in this paper.

1.4. **Moment estimates VS pointwise estimates.** Most of the works mentioned in the previous subsection are limited to the space homogeneous case. It is mentioned in [45] that "*[i]n the case of the full, spatially inhomogeneous Boltzmann, there is absolutly no clue of how to get such [moment] estimates. This would be a major breakthrough in the theory.*" However the remark from [45] refers to unconditional moment bounds,

and, as the partial result in [28] mentioned above shows, it is expected that most of these estimates can be extended to the space inhomogeneous case under the assumptions that the hydrodynamic quantities stay under control. However, in the most optimistic scenario, moment estimates obtained using classical methods would always be integrated both in v and x. Pointwise upper bounds are certainly a stronger result.

Nonetheless, the first pointwise upper bound for the Boltzmann equation was obtained in [25] by first proving an integral estimate on an exponential moment. The proof in that paper uses that a form of the comparison principle holds for the linear Boltzmann equation. The authors obtain pointwise Gaussian upper and lower bounds for the space homogeneous cutoff Boltzmann equation. The method was extended in [11]. An exponential (but not exactly Gaussian) upper bound for the space homogeneous non-cutoff Boltzmann equation is obtained in [26]. In [26], they make use of the exponential moment estimate obtained in [8]. In order to apply methods similar to [25, 11, 26] to the space inhomogeneous case, we would first need strong (probably exponential) $L^\infty_{t,x}L^1_v(\omega)$ moments for some weight ω. The case of integral $L^1_{x,v}$ polynomial moments however can be handled with simpler Povzner estimates (see the appendix). We do not know any direct way to prove such moment density estimates other than as a consequence of our pointwise decay results. See for example [26, Theorem 4] where the authors address the space inhomogeneous case but they have to introduce a strong artificial assumption in terms of exponential moment densities.

Our results in this paper provide a priori upper bounds for large velocities for the inhomogeneous non cutoff model. Integral moment estimates (beyond mass and energy densities) play no role in the proof. We prove that some pointwise decay is *generated* in the case $\gamma \leq 0$; precisely, it decays at faster than $|v|^{-q_0}$ with $q_0 = d + 1 + \frac{\gamma d}{2s} = 1 + (\gamma + 2s)\frac{d}{2s} \geq 1$ (recall $\gamma + 2s \geq 0$). Of course, such a slow (pointwise algebraic) decay does not imply any control on moments. In the case where $\gamma > 0$, any algebraic (pointwise) decay is generated, which implies in turn that moments of any order are generated as well. In the case where $\gamma \leq 0$, any algebraic (pointwise) decay are propagated, which is arguably stronger than the classical result of propagation of moments of any order.

1.5. **Open questions.** There are some natural questions that remain unanswered and are natural problems to investigate in the future. Our result says, for some range of parameters, that the rate of decay of the solution $f(t, x, v)$ is faster than any power function $|v|^{-q}$ as $|v| \to \infty$. The most desirable result would be to obtain a Gaussian upper bound as in [25] or [15]. At the moment we have not obtained even an exponential upper bound as in [26]. Another open problem regards the range of parameters γ, s for which the bounds hold. In this paper, we assume $0 \leq \gamma + 2s \leq 2$, which restricts, in dimension 3, to moderately soft potentials. The case $\gamma + 2s \in (-1, 0)$ (very soft potentials) seems to be very difficult with current techniques since we cannot currently obtain the L^∞ estimate as in Theorem 2.4. In the case $\gamma + 2s > 2$, we would expect some form of Theorem 1.2 (2) to hold, but there is a difficulty in that the energy estimate $E(t, x) \leq E_0$ is not enough to obtain any meaningful estimate on the kernel K_f defined in (2.1). We have not been able, yet, to work around this difficulty.

For technical reasons, it may also be interesting to reproduce the upper bounds in Theorem 1.2 for weaker notions of solutions. We explore in Section 6 the case in which we drop the rapid decay assumption from Definition 1.1. In the case $\gamma \leq 0$, we recover essentially the same result as in Theorem 1.2 without assuming the rapid decay at infinity of solutions provided that $\gamma + 2s < 1$. In the case $\gamma > 0$, we can always generate decay of the form $f \leq N|v|^{-d-1}$. However, in order to obtain an upper bound that decays with a higher power, we need to make the qualitative assumption that $\lim_{|v|\to\infty} |v|^{q_0} f(t, x, v) = 0$ for some power q_0 that depends on all the other parameters. We would naturally expect the estimates in Theorem 1.2 to hold for any reasonable notion of solution f.

We make the assumptions (1.3) on the hydrodynamic quantities. As we explained before, estimates for the inhomogeneous Boltzmann equation without any assumption on the hydrodynamic quantities seem completely out of reach with the current techniques. It is conceivable, however, that one might be able to weaken the assumptions (1.3). Because of the recent result in [30], we expect that the lower bound $m_0 \leq M(t, x)$ may not be necessary. We might also expect that the pointwise bounds could be replaced with an $L^p_t(L^q_x)$ bound for E, M and H, similar to the Prodi-Serrin condition for Navier-Stokes.

It is also conceivable that some version of Theorem 1.2 (especially part (3)) should hold in the cutoff case. Because of our program, we are mostly interested in the non-cutoff model, so we have not investigated this problem.

1.6. **Organisation of the article.** The article is organised as follows. In Section 2 are gathered results from previous works that are used in the proof of the main result. Section 3 is concerned with the presentation of the strategy to get Theorem 1.2. The collision operator is divided into different pieces which are estimated successively in Section 4. Section 5 contains the proof of our main result. Finally in Section 6, we discuss to what extent we can relax the assumption that the solution considered have infinite polynomial decay at infinity (in order to obtain contact points in v). In a short appendix, we give explain in an informal way how to propagate and generate integral $L^1_{x,v}$ moments in the case $\gamma > 0$ when hydrodynamic quantities stay under control.

1.7. **Notation.** For two real numbers $a, b \in \mathbb{R}$, we write $a \wedge b$ for their minimum. Moreover, $a \lesssim b$ means that $a \leq Cb$ with C only depending on dimension, γ, s and hydrodynamic quantities m_0, M_0, E_0, H_0. $a \lesssim_q b$ means that C may additionnally depends on the parameter q. Constants C_q, R_q also depend on q, and can be large. The constant c_q is "explicit" in Proposition 4.1.

2. Preliminaries

2.1. **The collisional operator in Carleman coordinates.** Following [45, 42], we write the collisional operator $Q(f,f)$ using Carleman coordinates. Using (1.2), we have from the classical cancellation lemma [1, 45] and [42, Corollary 4.2] that $Q(f,f) = Q_1(f,f) + Q_2(f,f)$ with

$$\begin{aligned} Q_1(f,f) &= \mathrm{PV} \int (f(v') - f(v)) K_f(v,v') \, \mathrm{d}v' \\ &= \mathrm{PV} \int_{\mathbb{R}^d} (f(v') - f(v)) \left\{ \int_{w \perp (v'-v)} f(v+w) |w|^{\gamma+2s+1} A(|v'-v|, |w|) \, \mathrm{d}w \right\} \frac{\mathrm{d}v'}{|v'-v|^{d+2s}}, \\ &= \mathrm{PV} \int_{\mathbb{R}^d} f(v+w) |w|^{\gamma+2s} \left\{ \int_{(v'-v) \perp w} (f(v') - f(v)) A(|v'-v|, |w|) \frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}} \right\} \mathrm{d}w \\ Q_2(f,f) &= f(v) c_2 \int_{\mathbb{R}^d} f(v+w) |w|^{\gamma} \, \mathrm{d}w \end{aligned}$$

with $c_2 = \int_{\mathbb{S}^{d-1}} \left\{ \frac{2^{(d+\gamma)/2}}{(1+\sigma\cdot e)^{(d+\gamma)/2}} - 1 \right\} b(\sigma \cdot e) \, \mathrm{d}\sigma$ for any $e \in \mathbb{S}^{d-1}$. The second form of Q_1 is obtained after exchanging the order of integration in Q_1, since the associated Jacobian is $|w||u|^{-1}$.

In order to study $Q_1(f,f)$ in the framework of integro-differential equations, sometimes it is useful to write

$$Q_1(f,f) = \mathrm{PV} \int_{\mathbb{R}^d} (f(v') - f(v)) K_f(v,v') \, \mathrm{d}v',$$

where

$$K_f(v,v') := \left\{ \int_{w \perp (v'-v)} f(v+w) |w|^{\gamma+2s+1} A(|v'-v|, |w|) \, \mathrm{d}w \right\} \frac{1}{|v'-v|^{d+2s}}. \tag{2.1}$$

2.2. **Cone of non-degeneracy.** It is classical that the control of mass, energy and entropy, away from vacuum, implies that the solution is bounded from below in a small set.

Lemma 2.1 (Lower bound on f – [42])**.** *Under the assumption (1.3), there exists a set $\mathcal{D} = \mathcal{D}(t,x) \subset B_{R_0}$ for some $R_0 > 0$ such that*

$$\forall v \in \mathcal{D}, \ f(t,x,v) \geq c_0 \quad \text{and} \quad |D| \geq \mu > 0$$

for R_0 and μ only depending on M_0, m_0, E_0, H_0 and dimension.

We recall from [42, 33] the following more subtle result.

Lemma 2.2 (Cone of non-degeneracy – [42, 33])**.** *Assume (1.3) holds true. Then for any $v \in \mathbb{R}^d$, there exists a cone of directions $\Xi = \Xi(v)$, so that*

- *for all $h \in \Xi$, $K_f(v, v+h) \geq c_0 (1+|v|)^{1+2s+\gamma} |h|^{-d-2s}$;*
- *Ξ is symmetric: $\Xi = -\Xi$;*
- *$\Xi \cap \partial B_1$ is contained on a strip of width $\lesssim (1+|v|)^{-1}$ around the equator perpendicular to v. In other words, there exists a constant C_0 independent of v such that for all $h \in \Xi$, we have $|v \cdot h| \leq C_0 |h|$;*

- $|\Xi \cap \partial B_1| \geq \mu(1+|v|)^{-1}$ *for some* $\mu > 0$ *only depending on dimension,* γ*,* s*,* m_0*,* M_0*,* E_0 *and* H_0*.*

Remark 2.3. The reader may notice that $\Xi[v]$ has a different meaning in [42, 33].

2.3. **The L^∞ bound.** We recall the upper bound obtained by the third author in a previous work. We state a slightly refined version.

Theorem 2.4 (Upper bound – [42])**.** *Let* $\gamma \in \mathbb{R}$ *and* $s \in (0,1)$ *satisfy* $\gamma + 2s \in [0,2]$ *and* f *be a non-negative solution of the Boltzmann equation* (1.1) *such that* (1.3) *holds true for some positive constants* m_0, M_0, E_0, H_0*. Then*

$$\forall t > 0, \qquad \|f(t,\cdot)\|_{L^\infty} < N_\infty(1 + t^{-\frac{d}{2s}})$$

for positive constant N_∞ *only depending on* m_0, M_0, E_0, H_0*, dimension,* γ *and* s*.*

Moreover, if $\|f_0\|_{L^\infty} < N$*, for some* $N \geq N_\infty$*, then* $\|f(t,\cdot)\|_{L^\infty} < N$ *for all* $t > 0$*.*

Theorem 2.4 is essentially the same as in [42]. Here, we also include the borderline case $\gamma + 2s = 0$ and the fact that if the initial data is bounded, the upper bound holds for all times. We provide a proof below, applying estimates from the literature, in order to show how to incorporate these two new features.

Sketch proof of Theorem 2.4. Without loss of generality, it is enough to show the inequality holds for $t \in (0,1]$.

We want to show that for all $t \in (0,1]$ we have

$$f(t,x,v) < N_\infty t^{-\frac{d}{2s}}.$$

Indeed, if this inequality was invalidated for the first time at a time $t_0 > 0$, there would exist points $x_0 \in \mathbb{R}^d$ and $v_0 \in \mathbb{R}^d$ such that

$$f(t_0,x_0,v_0) = N_\infty t_0^{-\frac{d}{2s}}, \qquad \text{and} \qquad f(t,x,v) \leq N_\infty t^{-\frac{d}{2s}} \text{ for all } t \in (0,t_0], x \in \mathbb{R}^d, v \in \mathbb{R}^d.$$

In particular $f_t(t_0,x_0,v_0) \geq -\frac{d}{2s} N_\infty t_0^{-\frac{d}{2s}-1}$ and $\nabla_x f(t_0,x_0,v_0) = 0$. We find a contradiction (and thus finish the proof), if we can ensure that for N_∞ large enough $Q(f,f)(t_0,x_0,v_0) < -\frac{d}{2s} N_\infty t_0^{-\frac{d}{2s}-1}$.

We write $Q(f,f) = Q_1(f,f) + Q_2(f,f)$ as in [42]. Using [42, Lemma 7.2] (see also Proposition 4.1 below), we get

$$Q_1(f,f)(t_0,x_0,v_0) \lesssim -(1+|v|)^{\gamma+2s}\left(N_\infty t^{-\frac{d}{2s}}\right)^{1+\frac{2s}{d}} = -(1+|v|)^{\gamma+2s} N_\infty^{1+\frac{2s}{d}} t^{-\frac{d}{2s}-1}.$$

In case $\gamma + 2s > 0$, we see that (see [42, Eq. (7.8)])

$$Q_2(f,f) \leq \begin{cases} (1+|v|)^\gamma N_\infty t^{-\frac{d}{2s}} & \text{if } \gamma \geq 0, \\ \left(N_\infty t^{-\frac{d}{2s}}\right)^{1+\frac{\gamma}{d}} & \text{if } \gamma \leq 0. \end{cases}$$

In this case, we can choose N_∞ large enough so that $Q_1 + Q_2 < Q_1/2 < -\frac{d}{2s} N_\infty t_0^{-\frac{d}{2s}-1}$ for $t_0 \in (0,1]$, and the contradiction follows. This is the proof given in [42].

The case $\gamma + 2s = 0$ is slightly more delicate because the estimates above are not enough to assure that $Q_1 + Q_2 < Q_1/2$. Instead, we use [43, Lemma 3.5] to estimate Q_2 as

$$Q_2 \lesssim \left(N_\infty t^{-\frac{d}{2s}}\right)^{1+\frac{\gamma}{d}} \log\left(1 + N_\infty t^{-\frac{d}{2s}}\right)^{\frac{d+\gamma}{2\gamma}}.$$

With this inequality, we get $Q_1 + Q_2 < Q_1/2$ for N_∞ sufficiently large also in the case $\gamma + 2s = 0$ and the contradiction follows.

If $\|f_0\|_{L^\infty} < N$ for some $N \geq N_\infty$, we pick $t_0 \in (0,1)$ such that $N_\infty t_0^{-\frac{d}{2s}} = N$. By the same reasoning as before, we obtain

$$f(t,x,v) < N_\infty (t+t_0)^{-\frac{d}{2s}}.$$

In particular, $f(t,x,v) < N$ for $t \in (0, 1-t_0)$. This allows us to extend our upper bound for a fixed period of time. Iterating this, we extend it for all time. □

Remark 2.5. Here we present some further interpretation of the cone of non-degeneracy and the L^∞ bound. The cone of Lemma 2.1 is a cone of direction for $(v'-v)$, i.e. the so-called "ω" vector of the "ω-representation" (see [45], section 4.6):

$$A(v) := \left\{\omega \in \mathbb{S}^{d-1} \text{ s.t. } |\{v'_* \ : \ (v'_* - v_*)\cdot\omega = 0 \ \&\ f(v'_*) > \ell \ \&\ |v'_*| < r\}| > \delta\right\}.$$

(The variable v' remains to be integrated independently of this cone.) This is a set of directions where the kernel is bounded below in the Carleman representation. The fact that the set where f is bounded below can be some complicated Borel set in a ball near zero does not change fundamentally the argument, which would be very similar if $f \geq \ell\chi_{B_r}$. The set $A(v)$ is $|\omega \cdot \frac{v}{|v|}| \lesssim |v|^{-1}$ or a non-zero measure-proportion of this set, hence $|\omega \cdot v| \lesssim r$ or a non-zero proportion of this set of directions.

The goal of this cone of direction is to find configurations so that v'_* is brought back near 0 in a zone where f is bounded below, in order to bound from below the "coefficients" of the operator, i.e. the kernel.

Then this set of directions $A(v)$ creates a cone $v' \in \Xi(v)$ centered at v and of angles of order $r/|v|$ close to orthogonal to $v/|v|$. Then in [42] – see Theorem 2.4 above, the part of this cone where $f < (1/2)\max f$ is bounded below using the Chebycheff inequality and the mass and energy bounds. That is: the assumptions imply that f is, for a significant amount of the large velocities, far from its maximum, i.e. less than $(\max f)/2$. On this part of the cone, the coercivity of $Q_1(f,f)$ is recovered, and together with the bounds from above on $Q_2(f,f)$, gives the contradiction and the L^∞ barrier.

3. The breakthrough scenario

For each of the three estimates in our main theorem, we want to prove that $f(t,x,v) < N(t)\min(1,|v|^{-q})$, for different choices of the function $N : (0,+\infty) \to (0,+\infty)$ and $q \in (0,+\infty)$.

In the proof of the first two parts of the main theorem, we will choose $N(t) \to +\infty$ as $t \to 0$ so that the inequality $f(t,x,v) < N(t)\min(1,|v|^{-q})$ holds for sufficiently small values of t. Our strategy is to look for the first time t_0 when the inequality is invalidated. At this time t_0, we would pick a point (t_0,x_0,v_0) such that $f(t_0,x_0,v_0) = N(t_0)\min(1,|v_0|^{-q})$. We would obtain a contradiction evaluating the equation at this point. We know that a point (t_0,x_0,v_0) where the inequality is invalidated for the first time exists because of the rapid decay assumption in Definition 1.1, and we assume the solution to be periodic in x.

Let $g(t,v) = \min(1,|v|^{-q})$. We want to prove the following inequality

$$f(t,x,v) < g(t,v). \tag{3.1}$$

At the first time $t_0 > 0$ where the inequality (3.1) is invalidated, we know that we will have a first contact point with $x_0 \in \mathbb{R}^d$ (due to periodicity in x) and $v_0 \in \mathbb{R}^d$ (since the inequality always holds for large $|v|$) so that $f(t_0,x_0,v_0) = g(t_0,v_0)$. Since the right hand side does not depend on x, we must have $\nabla_x f(t_0,x_0,v_0) = 0$. We also deduce that $\partial_t f(t_0,x_0,v_0) \geq \partial_t g(t_0,v_0)$. Therefore, since f solves the equation (1.1), we would have

$$\partial_t g(t_0,v_0) \leq \partial_t f(t_0,x_0,v_0) = Q(f,f)(t_0,x_0,v_0). \tag{3.2}$$

The success of our proof depends on being able to estimate the right hand side as a very negative number and contradict this inequality.

The only purpose of the rapid decay assumption in Definition 1.1 is to obtain this first contact point (t_0,x_0,v_0). In Section 6 we explore the case in which we remove this qualitative property from the definition of solution. In that case we add a small correction term to the function $g(t,v)$ in order to ensure the inequality $f < g$ for large values of v. We recover a large part of Theorem 1.2, but we run into technical problems when $\gamma > 0$. See Theorem 6.2.

4. Estimate of the collision operator at the first contact point

In this section, we estimate $Q(f,f)$ from above at the first point of contact between f and g, where g is given by (6.2) or (6.3). We remove the subscript in (t_0,x_0,v_0) for clarity. That is, at a point (t,x,v) such that

$$f(t,x,v') \leq g(t,v') \text{ for all } v' \in \mathbb{R}^d \quad \text{and} \quad f(t,x,v) = g(t,v). \tag{4.1}$$

Because they play no role in the following computations, we forget about the (t,x) variables.

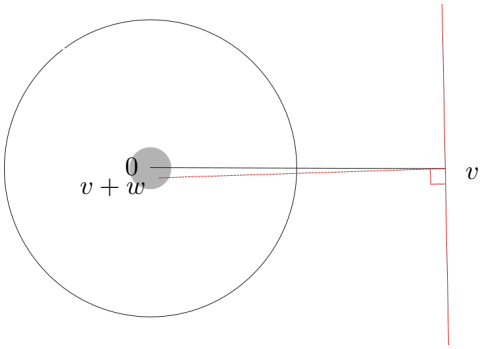


FIGURE 1. The good term: $\mathcal{G}(f,f)$ corresponds to the integration over the red (plain) line $(\{v' \in \mathbb{R}^d : v'-v \perp w\})$. The grey ball is of radius $c_q|v|$ while the larger ball is of radius $\frac{1}{2}|v|$.

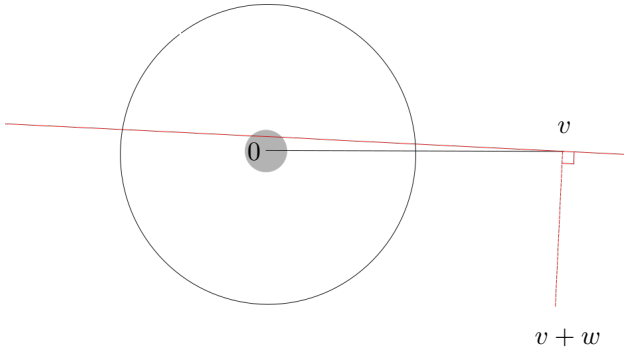


FIGURE 2. The bad terms: $\mathcal{B}_1(f,f)$ corresponds to the integration over the intersection of the line with the exterior of the balls, $\mathcal{B}_2(f,f)$ corresponds to the integration over the intersection of the line with the grey ball, and $\mathcal{B}_3(f,f)$ over the intersection of the line with the ring.

In order to estimate $Q(f,f)(v)$, we split it into several terms: a "good" term and four "bad" terms, including the lower order term $Q_2(f,f)$ defined above. Precisely, we write

$$Q(f,f) = \mathcal{G}(f,f) + \mathcal{B}_1(f,f) + \mathcal{B}_2(f,f) + \mathcal{B}_3(f,f) + Q_2(f,f) \tag{4.2}$$

with

$$\mathcal{G}(f,f) = \mathrm{PV} \int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s} \left\{ \int_{(v'-v)\perp w} (f(v')-f(v)) A(|v'-v|,|w|) \frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}} \right\} \mathrm{d}w,$$

$$\mathcal{B}_1(f,f) = \mathrm{PV} \int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s} \left\{ \int_{\substack{(v'-v)\perp w \\ |v'|>|v|/2}} (f(v')-f(v)) A(|v'-v|,|w|) \frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}} \right\} \mathrm{d}w,$$

$$\mathcal{B}_2(f,f) = \int_{|v'|<c_q|v|} (f(v')-f(v)) \left\{ \int_{\substack{w\perp(v'-v) \\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1} A(|v'-v|,|w|)\, \mathrm{d}w \right\} \frac{\mathrm{d}v'}{|v'-v|^{d+2s}},$$

$$\mathcal{B}_3(f,f) = \int_{c_q|v|<|v'|<|v|/2} (f(v')-f(v)) \left\{ \int_{\substack{w\perp(v'-v) \\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1} A(|v'-v|,|w|)\, \mathrm{d}w \right\} \frac{\mathrm{d}v'}{|v'-v|^{d+2s}},$$

and $Q_2(f,f)$ was defined above and c_q will be determined later.

4.1. **Estimates of the good term.** In this subsection, we estimate in two different ways our good term.

Proposition 4.1 (First estimate of $\mathcal{G}(f,f)$)**.** *Let $c_q \in (0, \frac{1}{4}q^{-\frac{1}{2}})$ and let f be such that (4.1) holds true with $g = N\min(1,|v|^{-q})$. There exists a radius $R_q \geq 1$ so that if $|v| \geq R_q$, then*

$$\mathcal{G}(f,f) \lesssim -q^s N|v|^{\gamma-q}.$$

We prove first the following preparatory lemma.

Lemma 4.2. *Let $q \geq 1$. For $c_q \in (0, \frac{1}{4}q^{-\frac{1}{2}})$ so that if $g(v) = N\min(1,|v|^{-q})$ and $v, w \in \mathbb{R}^d$ are such that $|v+w| < c_q|v|$, we have*

$$\int_{(v'-v)\perp w} (g(v')-g(v))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}} \lesssim -q^s N|v|^{-2s-q}.$$

Proof. We write $z = v'-v$ and divide the domain of integration between $|z| < r_q|v|$ and $|z| > r_q|v|$. The value of r_q will be determined below.

For $|z| < r_q|v|$, we proceed as follows. A Taylor expansion provides us with some $\theta \in (0,1)$ so that

$$\begin{aligned} 2\bigg(g(v+z)-g(v)-\nabla g(v)\cdot z\bigg) &= D^2 g(v+\theta z)z\cdot z \\ &= N|v+\theta z|^{-q-2}q\left(\frac{(q+2)}{|v+\theta z|^2}|(v+\theta z)\cdot z|^2 - |z|^2\right). \end{aligned} \tag{4.3}$$

Since $z \perp w$ and $|z| \leq r_q|v|$ and $|v+w| \leq c_q|v|$, we have

$$|(v+\theta z)\cdot z| \leq |v\cdot z| + |z|^2 = |(v+w)\cdot z| + |z|^2 \leq 2c_q|v||z| + |z|^2 \leq (2c_q + r_q)|v||z|.$$

Moreover,

$$|v+\theta z| \geq (1-r_q)|v|.$$

In particular,

$$\bigg(g(v+z)-g(v)-\nabla g(v)\cdot z\bigg) \leq \frac{qM}{2|v+\theta z|^{q+2}}\left((q+2)\left(\frac{2c_q+r_q}{1-r_q}\right)^2 - 1\right)|z|^2. \tag{4.4}$$

We choose $c_q, r_q \lesssim q^{-\frac{1}{2}}$ so as to make the right hand side negative. Therefore, taking into account the usual cancellation of the first order term,

$$\int_{\substack{z\perp w \\ |z|<c_q|v|}} (g(v+z)-g(v))A(|z|,|w|)\frac{\mathrm{d}z}{|z|^{d-1+2s}} \leq 0.$$

We now pick z such that $|z| \geq r_q|v|$. At this point we fix the value of $r_q = q^{-\frac{1}{2}}$. We may choose an even smaller value of c_q in the estimate of the rest of the integral if necessary. More precisely, we choose $c_q < r_q/4$ so that for $|z| \geq r_q|v|$ and $z \perp w$ and $|v+w| \leq c_q|v|$,

$$|v+z|^2 = |v|^2 + |z|^2 + 2(v+w)\cdot z \geq |v|^2 + |z|^2 - 2c_q|v||z| \geq (1 + r_q^2 - 2c_q r_q)|v|^2 \geq (1+r_q^2/2)|v|^2$$

and in particular,

$$g(v+z)-g(v) \leq -N\left(1-(1+r_q^2/2)^{-\frac{q}{2}}\right)|v|^{-q} \lesssim -N|v|^{-q}. \tag{4.5}$$

The last inequality holds because since $r_q = q^{-\frac{1}{2}}$ and $q \geq 1$, then $1-(1+r_q^2/2)^{-\frac{q}{2}} \approx 1$.

We deduce from (4.4) and (4.5) that, if $|v+w| \leq c_q|v|$ then

$$\begin{aligned} \int_{z\perp w} (g(v+z)-g(v))A(|z|,|w|)\frac{\mathrm{d}z}{|z|^{d-1+2s}} &\leq \int_{\substack{z\perp w \\ |z|\geq r_q|v|}} (g(v+z)-g(v))A(|z|,|w|)\frac{\mathrm{d}z}{|z|^{d-1+2s}} \\ &\lesssim -N|v|^{-q}\int_{z\in\mathbb{R}^{d-1},|z|\geq r_q|v|} \frac{\mathrm{d}z}{|z|^{d-1+2s}} \\ &\lesssim -Nr_q^{-2s}|v|^{-q-2s} \approx -q^s N|v|^{-q-2s}. \end{aligned}$$

This achieves the proof of the lemma. □

We can now prove Proposition 4.1.

Proof of Proposition 4.1. We estimate $\mathcal{G}(f,f)$ using Lemma 4.2.

$$\begin{aligned}\mathcal{G}(f,f) &= \mathrm{PV}\int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s}\left\{\int_{(v'-v)\perp w}(f(v')-f(v))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\right\}\mathrm{d}w,\\ &\lesssim -q^s N|v|^{-q-2s}\int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s}\,\mathrm{d}w,\\ &\lesssim -q^s N|v|^{-q+\gamma}.\end{aligned}$$

The last inequality holds provided that $|v|$ is sufficiently large so that $c_q|v|$ is larger than the radius R_0 of Lemma 2.1. The proof is now complete. □

We now get a second estimate for the good term, inspired by [42].

Proposition 4.3 (Second estimate of $\mathcal{G}(f,f)$)**.** *Assume f satisfies (4.1) for $g = N\min(1,|v|^{-q})$. There exists a radius $R_q \ge 1$ so that if $|v| \ge R_q$, then*

$$\mathcal{G}(f,f) \lesssim_q -g(v)^{1+\frac{2s}{d}}|v|^{\gamma+2s+\frac{2s}{d}}.$$

Proof. We first claim that the estimate for $\mathcal{G}(f,f)$ in Proposition 4.3 derives from Proposition 4.1 whenever $|v| \ge 1$ and

$$N|v|^{-q} \lesssim_q |v|^{-(d+1)}.$$

We justify the claim as follows. For such v, we have

$$N|v|^{-q-2s-\frac{2s}{d}}(N|v|^{-q})^{1+\frac{2s}{d}} \lesssim N|v|^{-q-2s-\frac{2s}{d}}$$

which in turn yields

$$g(v)^{1+\frac{2s}{d}} \lesssim N|v|^{-q-2s-\frac{2s}{d}}$$

and the claim is proved.

We next analyse $\mathcal{G}(f,f)$ in the case where $g(v) \gtrsim_q |v|^{-(d+1)}$. We proceed as in the proof of Proposition 4.1, except that now we estimate the difference $f(v') - g(v')$

$$\begin{aligned}\mathcal{G}(f,f) &= \mathrm{PV}\int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s}\left\{\int_{(v'-v)\perp w}(f(v')-f(v))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\right\}\mathrm{d}w,\\ &= \int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s}\Bigg\{PV\int_{(v'-v)\perp w}(g(v')-g(v))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\\ &\quad+\int_{(v'-v)\perp w}(f(v')-g(v'))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\Bigg\}\mathrm{d}w.\end{aligned}$$

The first of the two inner integral terms is negative because of Lemma 4.2. Thus

$$\mathcal{G}(f,f) \le \int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s}\left\{\int_{(v'-v)\perp w}(f(v')-g(v'))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\right\}\mathrm{d}w,$$

exchanging the order of integration

$$= \int_{\mathbb{R}^d}(f(v')-g(v'))K_{\bar f}(v,v')\,\mathrm{d}v'.$$

Here, $\bar f = f\mathbb{1}_{c_q|v|}$ and $K_{\bar f}$ is the kernel as in (2.1). If $|v|$ is sufficiently large, the estimates in Lemma 2.2 hold for $K_{\bar f}$ as well since f and $\bar f$ share comparable bounds on their hydrodynamic quantities.

Let us estimate the measure of points $h \in \Xi$, the cone from Lemma 2.2, such that $f(v+h) \ge g(v)/2$. Note that for sufficiently large $|v| > C_q$, whenever $h \in \Xi$, because of Lemma 2.2 we have $|v+h|^2 \ge |v|^2 + |h|^2 - 2C_0|h| \ge |v|^2/2$.

$$|\{h\in\Xi : f(v+h) \ge g(v)/2\}| \le \frac{2}{g(v)}\int_{h\in\Xi} f(v+h)\,\mathrm{d}h \le \frac{4E_0}{|v|^2 g(v)}.$$

Now pick $r > 0$ such that

$$|\Xi \cap B_r| = \frac{4^2 E_0}{|v|^2 g(v)}.$$

In particular,

$$|\Xi \cap B_r| \geq 4|\{h \in \Xi : f(v+h) \geq g(v)/2\}|.$$

In other words, three fourth of the h in $\Xi \cap B_r$ satisfy $f(v+h) \leq g(v)/2$. From Lemma 2.2, we have

$$\frac{\mu r^d}{|v|} \lesssim |\Xi \cap B_r| \lesssim \frac{C_0 r^d}{|v|}.$$

Thus, we get

$$r \approx \left(|v|^{-1} g(v)^{-1}\right)^{\frac{1}{d}}.$$

Going back to our estimate on $\mathcal{G}(f,f)$, we obtain

$$\mathcal{G}(f,f) \leq \int_{\Xi \cap B_r \cap \{f(v+h) \leq g(v)/2\}} (g(v)/2 - g(v+h)) K_{\bar{f}}(v,v')\, \mathrm{d}v'.$$

This is a useful estimate if $g(v+h) > g(v)/2$ for $h \in \Xi \cap B_r$. Recall that we assume that $g(v) \gtrsim C_q |v|^{-d-1}$ for an arbitrarily large constant C_q. Let us pick C_q large so that if $g(v) \gtrsim C_q |v|^{-d-1}$ then

$$r \lesssim \left(|v|^{-1} g(v)^{-1}\right)^{\frac{1}{d}} \leq \varepsilon_q |v| \quad \text{with } \varepsilon_q \text{ small},$$

so that $g(v+h) \geq \frac{3}{4} g(v)$ for $h \in B_r$. Therefore, we get

$$\begin{aligned}
\mathcal{G}(f,f) &\leq -\frac{g(v)}{4} \int_{\Xi \cap B_r \cap \{f(v+h) \leq g(v)/2\}} K_{\bar{f}}(v,v')\, \mathrm{d}v', \\
&\lesssim -g(v)|v|^{\gamma+2s+1} r^{-d-2s} |\Xi \cap B_r \cap \{f(v+h) \leq g(v)/2\}|,
\end{aligned}$$

recalling that $|\Xi \cap B_r \cap \{f(v+h) \leq g(v)/2\}| \geq \frac{3}{4}|\Xi \cap B_r| \approx r^d |v|^{-1}$,

$$\lesssim -g(v)|v|^{\gamma+2s} r^{-2s} = -g(v)^{1+2s/d} |v|^{\gamma+2s+2s/d}.$$

This achieves the proof of the second estimate of the good term. □

Remark 4.4. Here, we interpret in terms of the original angles v, v', v_*, v'_* the two estimates for the good term. In view of the proof of the first estimate given by Proposition 4.1, we see that the first estimate is generated for angles θ such that

$$|\sin(\theta/2)| = \frac{|v'-v|}{|v'_*-v|} = \frac{|v'-v|}{|w|} \geq \frac{r_q}{1+c_q} \approx q^{-\frac{1}{2}}$$

with $r_q = q^{-\frac{1}{2}}$ and $c_q \approx q^{-1}$. Hence, in some sense, the singularity is not used fundamentally. It is only used to get a constant larger and larger for $q \to +\infty$, because of the q^s factor coming for r_q^{-2s} in the proof of Lemma 4.2.

The second estimate given by Proposition 4.3 is genuinely non-cutoff in nature. Indeed, it is adapted from [42] where the nonlinear maximum principle for singular integral operators in the spirit of [17] is used. In particular, the constant r^{-2s} in the proof of Proposition 4.3 is crucial in order to conclude.

4.2. **Estimates of three bad terms.** Recall

$$\mathcal{B}_1(f,f) = \mathrm{PV} \int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s} \left\{ \int_{\substack{(v'-v)\perp w \\ |v'|>|v|/2}} (f(v') - f(v)) A(|v'-v|,|w|) \frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}} \right\} \mathrm{d}w.$$

Proposition 4.5 (Estimate of $\mathcal{B}_1(f,f)$)**.** *Let $g = N \min(1, |v|^{-q})$. If f satisfies (4.1), then for $|v| \geq 2$,*

$$\mathcal{B}_1(f,f) \lesssim_q |v|^{\gamma-2} g(v).$$

Proof. Since $|v|>2$, we have $1\wedge|v'|^{-q}=|v'|^{-q}$ for all $|v'|>|v|/2$ and $q>0$. This assumption makes the computations involving $g(v)$ and $g(v')$ simpler.

Estimate first

$$\begin{aligned}
I_1 :=& \,\mathrm{PV}\int_{\substack{(v'-v)\perp w\\ |v'|>|v|/2}}(f(v')-f(v))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\\
\le& \,\mathrm{PV}\int_{\substack{z\perp w\\ |v+z|>|v|/2}}(g(v+z)-g(v))A(|z|,|w|)\frac{\mathrm{d}z}{|z|^{d-1+2s}}\\
\le& \,\mathrm{PV}\int_{\substack{z\perp w,|z|<|v|/2\\ |v+z|>|v|/2}}(\dots)\frac{\mathrm{d}z}{|z|^{d-1+2s}}+\int_{\substack{z\perp w,|z|>|v|/2\\ |v+z|>|v|/2}}(\dots)\frac{\mathrm{d}z}{|z|^{d-1+2s}}\\
=:& I_{1,1}+I_{1,2}.
\end{aligned}$$

As far as $I_{1,1}$ is concerned, we use (4.3) and $|v+\theta z|\ge|v|/2$ and the fact that $\nabla g(v)\cdot z$ vanishes after integration (this is the classical cancellation). We obtain

$$\begin{aligned}
I_{1,1}=&\int_{\substack{z\perp w,|z|<|v|/2\\ |v+z|>|v|/2}}\left(g(v+z)-g(v)-\nabla g(v)\cdot z\right)A(|z|,|w|)\frac{\mathrm{d}z}{|z|^{d-1+2s}}\\
\lesssim&\int_{\substack{z\perp w,|z|<|v|/2\\ |v+z|>|v|/2}}C_qN|v|^{-q-2}|z|^2\frac{\mathrm{d}z}{|z|^{d-1+2s}}\\
\lesssim& \,C_qN|v|^{-q-2s}\lesssim_q|v|^{-2s}g(v).
\end{aligned}$$

As far as $I_{1,2}$ is concerned, we write

$$\begin{aligned}
I_{1,2}\le& N\int_{\substack{z\perp w,|z|>|v|/2\\ |v+z|>|v|/2}}|v+z|^{-q}\frac{\mathrm{d}z}{|z|^{d-1+2s}}\\
\lesssim& \,C_qN|v|^{-q-2s}.
\end{aligned}$$

Combining the estimates of $I_{1,1}$ and $I_{1,2}$, in any case we get

$$I_1\lesssim_q|v|^{-2s}g(v).$$

Finally, we compute

$$\begin{aligned}
\mathcal{B}_1(f,f)=&\,\mathrm{PV}\int_{|v+w|>c_q|v|}f(v+w)|w|^{\gamma+2s}I_1\,\mathrm{d}w\\
\lesssim_q&\, g(v)|v|^{-2s}\int_{|v+w|>c_q|v|}f(v+w)|w|^{\gamma+2s}\,\mathrm{d}w\\
\lesssim_q&\, g(v)|v|^{-2s}\int_{|v+w|>c_q|v|}f(v+w)|v+w|^2\frac{|w|^{\gamma+2s}}{|v+w|^2}\,\mathrm{d}w\\
\lesssim_q&\, g(v)|v|^{-2s}\int_{|v+w|>c_q|v|}f(v+w)|v+w|^2|v|^{\gamma+2s-2}\left\{\max_{\substack{e\in S^{d-1}\\ \bar w:|e+\bar w|>c_q}}\frac{|\bar w|^{\gamma+2s}}{|e+\bar w|^2}\right\}\,\mathrm{d}w\\
\lesssim_q&\, g(v)E_0|v|^{\gamma-2}.
\end{aligned}$$

This achieves the proof of the estimate of the first bad term. □

Recall

$$\mathcal{B}_2(f,f)=\int_{|v'|<c_q|v|}(f(v')-f(v))\left\{\int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}}f(v+w)|w|^{\gamma+2s+1}A(|v'-v|,|w|)\,\mathrm{d}w\right\}\frac{\mathrm{d}v'}{|v'-v|^{d+2s}}$$

$$\mathcal{B}_3(f,f)=\int_{c_q|v|<|v'|<|v|/2}(f(v')-f(v))\left\{\int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}}f(v+w)|w|^{\gamma+2s+1}A(|v'-v|,|w|)\,\mathrm{d}w\right\}\frac{\mathrm{d}v'}{|v'-v|^{d+2s}}.$$

Proposition 4.6 (Estimate of $\mathcal{B}_2(f,f)+\mathcal{B}_3(f,f)$)**.** *Assume that f satisfies (4.1) for $g = N\min(1,|v|^{-q})$. Then for $|v|\geq 2$,*

$$(\mathcal{B}_2+\mathcal{B}_3)(f,f) \lesssim \begin{cases} N|v|^{-d-1+\gamma} & \text{if } q>d-1,\\ N|v|^{-d-1+\gamma}\ln(1+|v|) & \text{if } q=d-1,\\ N|v|^{-q-2+\gamma} & \text{if } q<d-1. \end{cases}$$

Proof. For this estimate it is convenient to switch the order of integration. We compute

$$(\mathcal{B}_2+\mathcal{B}_3)(f,f) = \int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s}\left\{\int_{\substack{(v'-v)\perp w\\ |v'|<|v|/2}} (f(v')-f(v))A(|v'-v|,|w|)\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\right\}\mathrm{d}w.$$

We use that $f(v')-f(v)\leq g(v')-g(v)\leq N(1+|v'|)^{-q}$ in order to write

$$\begin{aligned}(\mathcal{B}_2+\mathcal{B}_3)(f,f) &\lesssim \int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s}\left\{\int_{\substack{(v'-v)\perp w\\ |v'|<|v|/2}} N(1+|v'|)^{-q}\frac{\mathrm{d}v'}{|v'-v|^{d-1+2s}}\right\}\mathrm{d}w,\\ &\lesssim N|v|^{-d+1-2s}\int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s}\left\{\int_{\substack{(v'-v)\perp w\\ |v'|<|v|/2}} (1+|v'|)^{-q}\,\mathrm{d}v'\right\}\mathrm{d}w.\end{aligned}$$

We get

$$\int_{\substack{(v'-v)\perp w\\ |v'|<|v|/2}} (1+|v'|)^{-q}\,\mathrm{d}v' \lesssim \begin{cases} 1 & \text{if } q>d-1,\\ \ln(1+|v|) & \text{if } q=d-1,\\ |v|^{d-1-q} & \text{if } q<d-1. \end{cases}$$

We continue the estimate in each case. If $q>d-1$, we get

$$\begin{aligned}(\mathcal{B}_2+\mathcal{B}_3)(f,f) &\lesssim N|v|^{-d+1-2s}\int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s}\,\mathrm{d}w,\\ &\leq N|v|^{-d+1-2s}\left(\max_{|v+w|>c_q|v|}\frac{|w|^\gamma}{|v+w|^2}\right)\int_{|v+w|>c_q|v|} f(v+w)|v+w|^2\,\mathrm{d}w,\\ &\lesssim_q N|v|^{-d-1+\gamma}.\end{aligned}$$

In the last inequality we use the energy estimate for f, and that is why we gain a decay factor $|v|^{-2}$.

If $q=d-1$, an identical computation gives us

$$\begin{aligned}(\mathcal{B}_2+\mathcal{B}_3)(f,f) &\lesssim N|v|^{-q-2s}\int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s}\,\mathrm{d}w,\\ &\leq N|v|^{-q-2s}\ln|v|\left(\max_{|v+w|>c_q|v|}\frac{|w|^\gamma}{|v+w|^2}\right)\int_{|v+w|>c_q|v|} f(v+w)|v+w|^2\,\mathrm{d}w,\\ &\lesssim_q N|v|^{-d-1+\gamma}\ln|v|.\end{aligned}$$

If $q<d-1$, an identical computation gives us

$$\begin{aligned}(\mathcal{B}_2+\mathcal{B}_3)(f,f) &\lesssim N|v|^{-q-2s}\int_{|v+w|>c_q|v|} f(v+w)|w|^{\gamma+2s}\,\mathrm{d}w,\\ &\leq N|v|^{-q-2s}\left(\max_{|v+w|>c_q|v|}\frac{|w|^\gamma}{|v+w|^2}\right)\int_{|v+w|>c_q|v|} f(v+w)|v+w|^2\,\mathrm{d}w,\\ &\lesssim_q N|v|^{-q-2+\gamma}.\end{aligned}$$

The proof is now complete. □

Recall

$$\mathcal{B}_2(f,f) = \mathrm{PV}\int_{|v'|<c_q|v|}(f(v')-f(v))\left\{\int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1}A(|v'-v|,|w|)\,\mathrm{d}w\right\}\frac{\mathrm{d}v'}{|v'-v|^{d+2s}}.$$

Proposition 4.7 (Estimate of $\mathcal{B}_2(f,f)$)**.** *Assume f satisfies (4.1) with $g = N\min(1,|v|^{-q})$, for $q > \gamma+2s+d$ and $c_q \lesssim 1/q$ (appearing in the definition of $\mathcal{B}_2(f,f)$). Assume $\gamma+2s \in [0,2]$. Then for $|v| \geq 2$, we have*

$$\mathcal{B}_2(f,f) \lesssim \frac{N}{q-(d+\gamma+2s)}|v|^{-q+\gamma}.$$

Proof. We first estimate from above

$$I_2 := \int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1}A(|v'-v|,|w|)\,\mathrm{d}w.$$

We pick c_q small enough so that

$$(1-c_q) \geq 1/\sqrt{2}.$$

This implies that for $|v'| < c_q|v|$, we have

$$\begin{aligned}|v+w| &\geq |v-v'+w| - c_q|v|\\ &\geq ((1-c_q)^2|v|^2+|w|^2)^{\frac{1}{2}} - c_q|v|\\ &\geq (1-\sqrt{2}c_q)\left(\frac{1}{2}|v|^2+|w|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Then for $|v'| < c_q|v|$ and g of the form (6.3), we can write

$$\begin{aligned}I_2 \lesssim& \int_{\substack{w\perp v'-v\\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1}\,\mathrm{d}w\\ \lesssim& \int_{w\perp v'-v} g(v+w)|w|^{\gamma+2s+1}\,\mathrm{d}w\\ \lesssim& N\int_{w\perp v'-v}|v+w|^{-q}|w|^{\gamma+2s+1}\,\mathrm{d}w\\ \lesssim& N(1-\sqrt{2}c_q)^{-q}|v|^{-q+\gamma+2s+d}\Gamma(q)\end{aligned}$$

with

$$\Gamma(\tilde{q}) = \int_{\mathbb{R}^{d-1}}\left(\frac{1}{2}+|\bar{w}|^2\right)^{-\frac{\tilde{q}}{2}}|\bar{w}|^{\gamma+2s+1}\,\mathrm{d}\bar{w}.$$

Remark that

$$\begin{aligned}\Gamma(\tilde{q}) =& \int_{\mathbb{R}^{d-1}}((1/2)+|\bar{w}|^2)^{-\frac{\tilde{q}}{2}}|\bar{w}|^{\gamma+2s+1}\,\mathrm{d}\bar{w}\\ =& \int_0^{+\infty}((1/2)+r^2)^{-\frac{\tilde{q}}{2}}r^{\gamma+2s+d-1}\,\mathrm{d}r\\ \leq& \int_0^{+\infty}((1/2)+r^2)^{-\frac{\tilde{q}-\gamma-2s-d+2}{2}}r\,\mathrm{d}r\\ =& \frac{1}{2}\int_{1/2}^{+\infty}s^{-\frac{\tilde{q}-\gamma-2s-d}{2}-1}\,\mathrm{d}s\\ \leq& \frac{1}{\tilde{q}-\gamma-2s-d} > 0.\end{aligned}$$

Hence

$$I_2 \lesssim N\frac{(1-\sqrt{2}c_q)^{-q}}{q-(d+\gamma+2s)}|v|^{-q+\gamma+2s+d} =: J_2.$$

Now choose $c_q \simeq \frac{1}{q}$ so that $(1-\sqrt{2}c_q)^{-q} \lesssim 1$. We plug our estimate on I_2 into the formula for $\mathcal{B}_2$,

$$\begin{aligned}\mathcal{B}_2(f,f) &= \int_{|v'|<c_q|v|}(f(v')-f(v))I_2\frac{\mathrm{d}v'}{|v'-v|^{d+2s}},\\ &\leq N|v|^{-d-2s}J_2 \lesssim \frac{N}{q-(d+\gamma+2s)}|v|^{-q+\gamma}.\end{aligned}$$

This achieves the proof of the estimate of the bad term $\mathcal{B}_2(f,f)$. □

Recall

$$\mathcal{B}_3(f,f) = \mathrm{PV}\int_{c_q|v|<|v'|<|v|/2} (f(v')-f(v))\left\{\int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1}A(|v'-v|,|w|)\,\mathrm{d}w\right\}\frac{\mathrm{d}v'}{|v'-v|^{d+2s}}.$$

Proposition 4.8 (Estimate of $\mathcal{B}_3(f,f)$)**.** *Assume f satisfies (4.1) with $g = N\min(1,|v|^{-q})$. For $q > \gamma+2s+d$, and for all $|v|\geq 2$,*

$$\mathcal{B}_3(f,f) \lesssim_q |v|^{\gamma-2}g(v).$$

Proof. We first estimate

$$I_3 := \int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}} f(v+w)|w|^{\gamma+2s+1}A(|v'-v|,|w|)\,\mathrm{d}w$$

for $|v'|<|v|/2$ as follows,

$$\begin{aligned} I_3 \lesssim & \int_{\substack{w\perp(v'-v)\\ |v+w|>c_q|v|}} g(v+w)|w|^{\gamma+2s+1}\,\mathrm{d}w \\ \lesssim & N|v|^{-q+\gamma+2s+d}\int_{\substack{\bar w\perp(v'-v)\\ \left|\frac{v}{|v|}+\bar w\right|>c_q}} \left|\frac{v}{|v|}+\bar w\right|^{-q}|\bar w|^{\gamma+2s+1}\,\mathrm{d}\bar w. \end{aligned}$$

Now estimate

$$\int_{\substack{\bar w\perp(v'-v)\\ \left|\frac{v}{|v|}+\bar w\right|>c_q}} \left|\frac{v}{|v|}+\bar w\right|^{-q}|\bar w|^{\gamma+2s+1}\,\mathrm{d}\bar w \lesssim \int_{\substack{\bar w\perp(v'-v),|\bar w|\leq 2\\ \left|\frac{v}{|v|}+\bar w\right|>c_q}} c_q^{-q}\,\mathrm{d}\bar w + \int_{\bar w\perp(v'-v),|\bar w|\geq 2}|\bar w|^{-q+\gamma+2s+1}\,\mathrm{d}\bar w.$$

The integral in the right hand side is convergent provided that $-q+\gamma+2s+1<-d+1$, i.e. $q>\gamma+2s+d$. We thus get,

$$\begin{aligned} \mathcal{B}_3(f,f) \lesssim & C_qN\int_{c_q|v|<|v'|<|v|/2} f(v')|v|^{-d-2s}|v|^{-q+\gamma+2s+d}\,\mathrm{d}v' \\ \lesssim & C_qN\int_{c_q|v|<|v'|<|v|/2} f(v')|v'|^2|v|^{-q+\gamma-2}\,\mathrm{d}v', \\ \lesssim & C_qN|v|^{-q+\gamma-2}. \end{aligned}$$

This achieves the proof of the estimate of the third bad term $\mathcal{B}_3(f,f)$. □

4.3. **Estimate of the lower order part of the collision operator.**

Proposition 4.9 (Estimate of $Q_2(f,f)$)**.** *Assume f satisfies (4.1) with $g = N\min(1,|v|^{-q})$. Then for $\gamma\geq 0$ and $|v|\geq 1$,*

$$Q_2(f,f) \lesssim |v|^\gamma g(v);$$

while for $\gamma<0$ and $|v|\geq 1$,

$$Q_2(f,f) \lesssim_q \begin{cases} |v|^{-2-\frac{2\gamma}{d}}g(v)^{1-\frac{\gamma}{d}} & \text{if } g(v)\geq 2^{-d}|v|^{-d-2}, \\ |v|^{\gamma-2}g(v) & \text{if } g(v)\leq 2^{-d}|v|^{-d-2}. \end{cases}$$

Proof. We first deal with the easier case $\gamma\geq 0$. We write

$$\begin{aligned} Q_2(f,f) = & f(v)\int f(v-w)|w|^\gamma\,\mathrm{d}w \\ \lesssim & g(v)\int f(v-w)(|v-w|^\gamma+|v|^\gamma)\,\mathrm{d}w \\ \lesssim & g(v)\int f(v-w)(|v-w|^2+1+|v|^\gamma)\,\mathrm{d}w, \\ \leq & g(v)\left(E_0+(1+|v|^\gamma)M_0\right). \end{aligned}$$

We now turn to the more delicate case $\gamma < 0$. We pick $r < |v|/2$ to be chosen later and we write

$$\begin{aligned} f^{-1}Q_2(f,f) &\leq \int_{|w-v|<r} g(w)|v-w|^\gamma \,\mathrm{d}w + \int_{|w-v|>r} f(w)|v-w|^\gamma \,\mathrm{d}w \\ &\lesssim C_q g(v) \int_{|w-v|<r} |v-w|^\gamma \,\mathrm{d}w + r^\gamma |v|^{-2} \int_{|w-v|>r} f(w)|w|^2 \,\mathrm{d}w \\ &\lesssim C_q g(v) r^{d+\gamma} + r^\gamma |v|^{-2}. \end{aligned}$$

This implies

$$Q_2(f,f) \lesssim_q g(v)^2 r^{d+\gamma} + r^\gamma |v|^{-2} g(v).$$

We now choose

$$r = \big(g(v)|v|^2\big)^{-\frac{1}{d}} \wedge \frac{|v|}{2}$$

and we estimate $Q_2(f,f)$ by distinguishing two cases.

We first assume that $r = \big(g(v)|v|^2\big)^{-\frac{1}{d}}$, i.e. $g(v) \geq 2^{-d}|v|^{-d-2}$, in which case we get

$$Q_2(f,f) \lesssim g(v)^{1-\frac{\gamma}{d}} |v|^{-2-\frac{2\gamma}{d}}.$$

If now $r = |v|/2$, i.e. $g(v) \leq 2^{-d}|v|^{-d-2}$, then

$$\begin{aligned} Q_2(f,f) &\lesssim g(v)^2 |v|^{d+\gamma} + |v|^{\gamma-2} g(v) \\ &\lesssim g(v)|v|^{\gamma-2}. \end{aligned}$$

The proof of the estimate of $Q_2(f,f)$ is now complete. □

5. Proof of the main result

In this section we analyze the breakthrough scenario described in Section 3. At the first contact point satisfying (4.1), we use the estimates from Section 4 in order find a contradiction to (3.2). In this way, we establish (3.1) for different choices of $g(t,v)$ in each case of Theorem 1.2.

The description of the proof given below is highly repetitive. We run the contradiction argument five times with slightly different assumptions and upper bounds $g(t,v)$. The first time is to prove part (1). The proofs of (2) and (3), each consists of two parts. We first establish the upper bounds of (2) and (3) for $q = d+1$. Once that upper bound is established, we run a similar argument again for large enough q. Each of these five proofs is very similar to the others, but applying a different subcollection of the estimates in Section 4.

Proof of Theorem 1.2-(1). We start with the case where $\gamma \leq 0$ and $q = d+1+\frac{\gamma d}{2s}$.

It is sufficient to prove that the conclusion holds for $t \in (0,1)$, since the estimate does not provide any further information if we let the equation flow longer.

We set

$$g(v) = N(t)\min(1, |v|^{-q}),$$

for $q = d+1+\frac{\gamma d}{2s}$ where $N(t) = N_0 t^{-\frac{d}{2s}}$ and $N_0 = N_0(m_0, M_0, E_0, H_0, \gamma, s, d)$ is a large constant to be determined below.

Note that since $\gamma \geq 0$ and $\gamma + 2s \geq 0$, we have $1 \leq q \leq d+1$.

Let (t,x,v) be the first contact point of f with g, i.e. consider the first time $t \in (0,1)$ such that there exists $(x,v) \in \mathbb{R}^{2d}$ such that (4.1) holds true. Thus, we have

$$N'(t)|v|^{-q} \leq f_t(t,x,v) = Q(f(t,x,\cdot), f(t,x,\cdot))(v). \tag{5.1}$$

Like we described in Section 3, our strategy is to find a bound for $Q(f,f)$ which contradicts (5.1).

Using Theorem 2.4, we can pick N_0 sufficiently large so as to ensure that the value of $|v|$ must be as large as we want. In that way, the estimates from Propositions 4.3, 4.5, 4.6 and 4.9 apply. Using these

propositions we get,

$$\begin{aligned}
\mathcal{G}(f,f) &\lesssim_q -|v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}} && \text{from Prop. 4.3,}\\
\mathcal{B}_1(f,f) &\lesssim_q |v|^{\gamma-2} g(v) && \text{from Prop. 4.5,}\\
(\mathcal{B}_2+\mathcal{B}_3)(f,f) &\lesssim_q g(v)|v|^{\gamma-2}\max(1,|v|^{q-d+1},\ln|v|) \le g(v)|v|^{\gamma} && \text{from Prop. 4.6,}\\
Q_2(f,f) &\lesssim |v|^{\gamma-2} g(v) + |v|^{-2-\frac{2\gamma}{d}} g(v)^{1-\frac{\gamma}{d}} && \text{from Prop. 4.9.}
\end{aligned}$$

The third inequality holds because $q \le d+1$.

Thus, for some constants $c_q > 0$ and $C_q > 0$, we get

$$Q(f,f) \le -c_q|v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}} + C_q\left(g(v)|v|^{\gamma} + |v|^{-2-\frac{2\gamma}{d}} g(v)^{1-\frac{\gamma}{d}}\right). \tag{5.2}$$

We will choose N_0 large enough to ensure that the negative term is larger than twice the two positive terms. Let us first compare the negative term with the first positive one. For N_0 large enough, since $q \le d+1$ and $s > 0$, we have

$$\begin{aligned}
-c_q|v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}} + C_q g(v)|v|^{\gamma} &\le |v|^{\gamma} g(v)\left(-c_q N(t)^{\frac{2s}{d}}|v|^{\frac{2s}{d}(d+1-q)} + C_q\right),\\
&\le -\frac{3}{4}c_q|v|^{\gamma} g(v) N(t)^{\frac{2s}{d}}|v|^{\frac{2s}{d}(d+1-q)}.
\end{aligned}$$

Let us now analyze the second positive term in (5.2). We claim

$$|v|^{-2-\frac{2\gamma}{d}} g(v)^{1-\frac{\gamma}{d}} \ll |v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}}.$$

Indeed, rearranging the terms, we see that this inequality is equivalent to

$$|v|^{-2-(\gamma+2s)\frac{d+1}{d}-\frac{\gamma}{d}} \ll \left(N(t)|v|^{-q}\right)^{\frac{\gamma+2s}{d}},$$

which is certainly true for large enough N_0 since $q \le d+1$ and $\gamma \ge -d$.

Going back to (5.2) and using that $q = d+1+d\gamma/s$, we see that

$$\begin{aligned}
Q(f,f) &\lesssim_q |v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}},\\
&\lesssim -N(t)^{1+\frac{2s}{d}}|v|^{-q} \approx -N(t)^{2s/d} N'(t)|v|^{-q}.
\end{aligned}$$

Thus, we find a contradiction with (5.1) picking a large enough constant N_0 and we finish the proof since $\varepsilon > 0$ is arbitrary. □

Proof of Theorem 1.2-(2). We only need to prove this theorem for large values of q. We can and do assume that q is large enough so that some inequalities below hold. We set $g(v) = N(t)\min(1,|v|^{-q})$ with $N(t) = N_0 t^{-\beta}$ and $\beta = \frac{q}{\gamma} + \frac{d}{2s}$.

Let (t,x,v) be the first contact point of f with g, where (4.1) holds true. At this point we have $N'(t)|v|^{-q} \le Q(f(t,x,\cdot),f(t,x,\cdot))(v)$. Moreover, using Theorem 2.4, we also have for N_0 large,

$$N_0 t^{-\beta}|v|^{-q} \le 2N_\infty t^{-\frac{d}{2s}}.$$

Thus,

$$|v| \gtrsim_q N_0^{\frac{1}{q}} t^{\frac{1}{q}(\frac{d}{2s}-\beta)} = N_0^{\frac{1}{q}} t^{-\frac{1}{\gamma}} \ge N_0^{\frac{1}{q}}. \tag{5.3}$$

Recall that $\beta = \frac{q}{\gamma} + \frac{d}{2s}$.

Note that for N_0 large, we will only touch at points (t,x,v) where $|v|$ is large. In particular Propositions 4.1, 4.5, 4.7 and 4.8 apply. In addition to Proposition 4.9, we get

$$\begin{aligned}
\mathcal{G}(f,f) &\lesssim -q^s|v|^{\gamma} g(v) && \text{from Prop. 4.1,}\\
\mathcal{B}_1(f,f) &\lesssim_q |v|^{\gamma-2} g(v) && \text{from Prop. 4.5,}\\
\mathcal{B}_2(f,f) &\lesssim \frac{1}{q}|v|^{\gamma} g(v) && \text{from Prop. 4.7,}\\
\mathcal{B}_3(f,f) &\lesssim_q |v|^{\gamma-2} g(v) && \text{from Prop. 4.8,}\\
Q_2(f,f) &\lesssim |v|^{\gamma} g(v) && \text{from Prop. 4.9.}
\end{aligned}$$

Combining Propositions 4.1, 4.7 and 4.9, we see that for q sufficiently large,

$$\mathcal{G}(f,f)+\mathcal{B}_2(f,f)+Q_2(f,f)\lesssim -q^s|v|^\gamma g(v).$$

For $|v|$ large enough, every term with a factor $|v|^{\gamma-2}$ is an order of magnitude smaller. Thus, for large values of $|v|$,

$$Q(f,f)=\mathcal{G}(f,f)+\mathcal{B}_1(f,f)+\mathcal{B}_2(f,f)+\mathcal{B}_3(f,f)+Q_2(f,f)\lesssim -q^s|v|^\gamma g(v).$$

Combining it with (5.3),

$$Q(f,f)\lesssim_q -N_0^{\gamma/q}t^{-1}g(v).$$

Choosing N_0 large, we contradict the inequality $Q(f,f)=f_t\geq g_t(t,v)\gtrsim -t^{-1}g(t,v)$ and we finish the proof. □

Proof of Theorem 1.2-(3). We now consider the case where $\gamma<0$ and q is large. We set $g=N_0\min(1,|v|^{-q})$.

Let (t,x,v) be the first contact point of f with g, i.e. consider the first time $t>0$ such that there exists $(x,v)\in\mathbb{R}^{2d}$ such that (4.1) holds true. In particular, at this point of contact we have the inequality

$$Q(f,f)=f_t\geq g_t=0 \tag{5.4}$$

The aim of this proof is to prove $Q(f,f)<0$ at the point of contact and contradicts (5.4).

Using Theorem 2.4, we also have that

$$N_0\min(1,|v|^{-q})\leq g(t,v)\leq N_\infty. \tag{5.5}$$

Therefore, by picking N_0 large, we can force $|v|$ to be arbitrarily large. In particular, we are able to apply Propositions 4.1, 4.3, 4.5, 4.7, 4.8 and 4.9.

Proposition 4.9 gives us two different upper bounds for $Q_2(f,f)$ depending on the size of $g(t,v)$ relative to $|v|^{-d-1}$. Let us consider the case $g(v)>2^{-d}|v|^{-d-2}$. In this case we get

$$Q_2(f,f)\lesssim |v|^{-2-\frac{2\gamma}{d}}g(v)^{1-\frac{\gamma}{d}}.$$

We claim this is a small fraction of the (negative) bound for $\mathcal{G}(f,f)$ given in Proposition 4.1. Indeed,

$$\frac{c_q|v|^{\gamma+2s+\frac{2s}{d}}g(t,v)^{1+\frac{2s}{d}}}{Q_2(f,f)}\gtrsim_q |v|^{2+\gamma+2s+\frac{2s+2\gamma}{d}}g(v)^{\frac{\gamma+2s}{d}}\geq |v|^{2-\frac{2s}{d}}.$$

This right hand side is arbitrarily large since $2>2s/d$. Therefore, when $g(t,v)\geq 2^{-d}|v|^{-d-2}$, the upper bound for $Q_2(f,f)$ is absorbed by the bound for $\mathcal{G}(f,f)$ from Proposition 4.1. In case $g(t,v)\leq 2^{-d}|v|^{-d-2}$, then we use the other upper bound for $Q_2(f,f)$, which is identical to the upper bounds for $\mathcal{B}_1(f,f)$ and $\mathcal{B}_3(f,f)$ from Propositions 4.5 and 4.8.

Therefore, we have

$$\begin{aligned}Q(f,f)&=\mathcal{G}(f,f)+\mathcal{B}_1(f,f)+\mathcal{B}_2(f,f)+\mathcal{B}_3(f,f)+Q_2(f,f),\\&\leq N_0|v|^{-q+\gamma}\left(-cq^s+C_q|v|^{-2}+\frac{C}{q}\right).\end{aligned}$$

We choose q large enough so that $cq^s\gg C/q$. Afterwards, we choose N large, so that $|v|$ is sufficiently large and $cq^s|v|^{-q+\gamma}\gg C_q|v|^{-q+\gamma-2}$). We get $Q(f,f)<0$ and achieve the desired contradiction. □

6. Solutions that may not decay rapidly

In this section we replace Definition 1.1 with a weaker notion of solution. Precisely,

Definition 6.1 (Solutions for the Boltzmann equation without rapid decay)**.** *We say that a function $f:[0,+\infty)\times\mathbb{T}^d\times\mathbb{R}^d\to[0,+\infty)$ is a solution to the Boltzmann equation (1.1) if*

- *the function f is differentiable in t and x and twice differentiable in v everywhere; the equation (1.1) holds classically at every point;*
- *The limit $\lim_{|v|\to\infty}f(t,x,v)=0$ holds uniformly in t and x.*

The difference with Definition 1.1 is that we eliminated the qualitative assumption of rapid decay as $|v|\to\infty$. This causes some technical difficulties for finding the first contact point between f and g, which were described at the end of Section 3.

For this class of solutions, we can prove the following weaker version of Theorem 1.2.

Theorem 6.2 (Decay upper bound)**.** *Let $\gamma \in \mathbb{R}$ and $s \in (0,1)$ satisfy $\gamma + 2s \in [0,2]$ and f be a solution of the Boltzmann equation (1.1) as in Definition 6.1 such that (1.3) holds. Then*

(1) *If $\gamma \leq 0$, then*
$$f(t,x,v) \leq N(1+t^{-\frac{d}{2s}}) \min\left(1, |v|^{-(d+1+\frac{\gamma d}{2s})}\right)$$
where $N > 0$ depends on m_0, M_0, E_0, H_0, d and s.

(2) *If $\gamma > 0$ there exists a constant N and a power $\beta > 0$ such that*
$$f(t,x,v) \leq N(1+t^{-\beta}) \min(1, |v|^{-d-1})$$
where $N > 0$ depends on m_0, M_0, E_0, H_0, d and s.

(3) *If $\gamma \leq 0$ and $\gamma + 2s < 1$, there exists q_0 such that for all $q \geq q_0$ and $f_{in} \leq C \min(1, |v|^{-q})$ then*
$$f(t,x,v) \leq N \min(1, |v|^{-q})$$
where q_0 depends on $d, s, \gamma, m_0, M_0, E_0, H_0$ and N depends on $C, m_0, M_0, E_0, H_0, q, d$ and s.

(4) *If $\gamma > 0$, there exists q_0 such that if*
$$\lim_{|v|\to\infty} |v|^{-q_0} f(t,x,v) = 0,$$
holds uniformly in t and x, then for all $q > 0$, there exists constants N and $\beta > 0$ such that
$$f(t,x,v) \leq N(1+t^{-\beta}) \min(1, |v|^{-q})$$
where $N > 0$ and β depend on m_0, M_0, E_0, H_0, q, d and s.

Remark 6.3. Note that, for $\gamma > 0$, we know from part (2) that for all positive time t,
$$\lim_{|v|\to\infty} |v|^{-q} f(t,x,v) = 0,$$
for any $q < d+1$. The assumption in part (4) would be automatically true if $q_0 < d+1$. Unfortunately, it is hard to compute q_0 explicitly from our proof.

Remark 6.4. The only purpose of the technical assumption $\gamma + 2s < 1$ in (3) is to handle the error term – see $\varepsilon|v|^{-d-1+\varepsilon}$ in (6.3) below. It is most likely not necessary. It is certainly not necessary for the a priori estimate if we knew that our solution decays faster than $|v|^{-d-2}$ at infinity.

Like in the proof of Theorem 1.2, we propagate an inequality $f(t,x,v) < g(t,v)$ for some appropriate barrier function g. Since we do not assume that f decays rapidly as $|v| \to \infty$, an uncomfortable difficulty arises in order to ensure the existence of a first crossing point (t_0, x_0, v_0) such that $f(t_0,x_0,v_0) = g(t_0,v_0)$. It is now possible that the inequality $f(t,x,v) < N(t)\min(1,|v|^{-q})$ is first broken asymptotically as $|v| \to \infty$. In order to work around this difficulty, we add an arbitrarily small term to $g(t,v)$ which ensures the inequality for large enough values of $|v|$.

Our precise strategy is the following. We want to prove the following inequality
$$f(t,x,v) < g(t,v). \tag{6.1}$$

The function $g(t,v)$ depends on an arbitrary parameter $\varepsilon > 0$ whose only purpose is to ensure that the inequality holds for large enough $|v|$. The choice of the function $g(t,v)$ changes slightly for the different parts of the theorem. We will use essentially the following forms.

For parts (1), (2) and (3), for powers $q \leq d+1$, we use
$$g(t,v) = N(t)(1 \wedge |v|^{-q}) + \varepsilon \qquad \text{with } q \leq d+1. \tag{6.2}$$

For part (3), when the power q is larger than $d+1$ and $\gamma \leq 0$, for certain choices of $N(t)$ and $\varepsilon(t)$ we use
$$g(t,v) = N(t)(1 \wedge |v|^{-q}) + \varepsilon(t)(1 \wedge |v|^{-d-1+\eta}). \tag{6.3}$$

For part (4), when $\gamma > 0$, it is enough to consider $q > q_0$ and we use
$$g(t,v) = N(t)(1 \wedge |v|^{-q}) + \varepsilon(t)(1 \wedge |v|^{-q_0}). \tag{6.4}$$

In each case, the extra correction term in g ensures that $f(t,x,v) < g(t,v)$ for large enough $|v|$, so that a first point of contact (t_0,x_0,v_0) must exists. In part (3), when $q > d+1$, we first use a function g as in (6.2) to ensure a decay of the form $N|v|^{-d-1}$ and then we use a function g as in (6.3).

The first time $t > 0$ where the inequality (6.1) is invalidated, in any case we know that we will have a first contact point with $x \in \mathbb{R}^d$ (due to periodicity in x) and $v \in \mathbb{R}^d$ (since the inequality always holds for large $|v|$) so that $f(t,x,v) = g(t,v)$. We deduce the same inequality (3.2) and we must estimate $Q(f,f)$ at this point in order to find a contradiction.

The following lemmas and propositions are relatively straight forward variations of the corresponding results in Section 4 taking into account the new form of the function $g(t,v)$.

Proposition 6.5. *Let $c_q > 0$ be such that $c_q \lesssim q^{-\frac{1}{2}}$ and let f be such that (4.1) holds true with g given by (6.2), (6.3) or (6.4). There exists a radius $R_q \geq 1$ so that if $|v| \geq R_q$, then*

$$\mathcal{G}(f,f) \lesssim \begin{cases} -q^s N|v|^{\gamma-q} & \text{if } g \text{ is as in (6.2)}, \\ -q^s N|v|^{\gamma-q} - \varepsilon(t)|v|^{\gamma-(d+1)+\eta} & \text{if } g \text{ is as in (6.3)}, \\ -q^s N|v|^{\gamma-q} - q_0^s \varepsilon(t)|v|^{\gamma-q_0} & \text{if } g \text{ is as in (6.4)}. \end{cases}$$

Proof. It is a straight forward modification of Proposition 4.1 adding an extra correction term. In the case g is as in (6.2), note that the extra terms $+\varepsilon$ will cancel out in the upper bound for $(f(v') - f(v))$. □

Proposition 6.6. *Assume f satisfies (4.1) for a function g of the form (6.2) or (6.3). There exists a radius $R_q \geq 1$ so that if $|v| \geq R_q$, then*

$$\mathcal{G}(f,f) \lesssim_q -g(v)^{1+\frac{2s}{d}}|v|^{\gamma+2s+\frac{2s}{d}}.$$

Proof. We follow the same ideas as in the proof of Proposition 4.3. We must first analyze the range of values of v where the inequality follows from Proposition 6.5.

If g is given by (6.2), then the estimate for $\mathcal{G}(f,f)$ in Proposition 6.6 derives from Proposition 6.5 for v, ε such that $|v| \geq 1$ and $\varepsilon \in (0,1)$ and

$$\frac{\varepsilon}{3} \leq N|v|^{-q} \lesssim_q |v|^{-(d+1)}.$$

We justify the claim as follows. If g is given by (6.2), then for such v, ε, we have

$$\begin{aligned} \varepsilon^{1+\frac{2s}{d}} &\lesssim N|v|^{-q-2s-\frac{2s}{d}} \\ (N|v|^{-q})^{1+\frac{2s}{d}} &\lesssim N|v|^{-q-2s-\frac{2s}{d}} \end{aligned}$$

which in turn yields

$$g(v)^{1+\frac{2s}{d}} \lesssim N|v|^{-q-2s-\frac{2s}{d}}$$

and then the proof would follow immediately applying Proposition 6.5.

If now g is given by (6.3) or (6.4), we claim that Proposition 6.6 derives from Proposition 6.5 as soon as $|v| \geq 1$, $\varepsilon \in (0,1)$ and

$$g(v) \lesssim_q |v|^{-(d+1)}.$$

Indeed, notice that in this case we have $g(v)^{\frac{2s}{d}}|v|^{2s+\frac{2s}{d}} \lesssim_q 1$ and the estimate follows easily.

We are left to do the proof in two cases: when $g(v) \gtrsim_q |v|^{-(d+1)}$ with g either given by (6.2), (6.3) or (6.4), or when $g(v) < \varepsilon/3$ for g of the form (6.2).

In either case, we proceed identically as in the proof of Proposition 4.1. We pick $r > 0$ such that

$$|\Xi \cap B_r| = \frac{4^2 E_0}{|v|^2 g(v)}.$$

Following the lines of the proof of Proposition 4.1, we obtain

$$\mathcal{G}(f,f) \leq \int_{\Xi \cap B_r \cap \{f(v+h) \leq g(v)/2\}} (g(v)/2 - g(v+h)) K_{\bar{f}}(v,v')\, \mathrm{d}v'.$$

As before, this is a useful estimate if $g(v+h) > g(v)/2$ for $h \in \Xi \cap B_r$. In the case $g(v) \gtrsim_q |v|^{-d-1}$, we proceed as in the proof of Proposition 4.1 (by choosing an appropriately large constant C_q). If g is given by (6.2) and $N|v|^{-q} \leq \frac{\varepsilon}{3}$, then we have for all $h \in \mathbb{R}^d$ that $g(v+h) \geq \frac{3}{4} g(v)$. In particular, we can argue as above and reach the desired conclusion. □

Proposition 6.7 (Estimate of $\mathcal{B}_1(f,f)$)**.** *Let g be of the form (6.2), (6.3) or (6.4). If f satisfies (4.1), then for $|v| \geq 2$,*

$$\mathcal{B}_1(f,f) \lesssim_q |v|^{\gamma-2} g(v).$$

Proof. It is a straight forward adaptation of the proof of Proposition 4.5. □

Proposition 6.8 (Estimate of $\mathcal{B}_2(f,f) + \mathcal{B}_3(f,f)$)**.** *Let g be a function of the form (6.2). Assume that f satisfies (4.1). Then for $|v| \geq 2$,*

$$(\mathcal{B}_2 + \mathcal{B}_3)(f,f) \lesssim \begin{cases} N|v|^{-d-1+\gamma} & \text{if } q > d-1, \\ N|v|^{-d-1+\gamma} \ln(1+|v|) & \text{if } q = d-1, \\ N|v|^{-q-2+\gamma} & \text{if } q < d-1. \end{cases}$$

Proof. The proof is identical to that of Proposition 4.7. Note that the extra $+\varepsilon$ terms cancel out in the estimate $f(v') - f(v) \leq g(v') - g(v)$. □

Proposition 6.9 (Estimate of $\mathcal{B}_2(f,f)$)**.** *Assume f satisfies (4.1) and either g is of the form (6.3) with $\gamma + 2s < 1 - \eta$, or g is of the form (6.4) with $q_0 > d + \gamma + 2s$. Assume further that $q > \gamma + 2s + d$ and $c_q \lesssim 1/q$ (appearing in the definition of $\mathcal{B}_2$). Then for $|v| \geq 2$, we have*

$$\mathcal{B}_2(f,f) \lesssim \begin{cases} \frac{1}{q-(d+\gamma+2s)} |v|^{-q+\gamma} + \varepsilon |v|^{-d-1+\eta+\gamma} & \text{if } g \text{ is as in (6.3)}, \\ \frac{1}{q-(d+\gamma+2s)} |v|^{-q+\gamma} + \varepsilon \frac{1}{q_0-(d+\gamma+2s)} |v|^{-q_0+\gamma} & \text{if } g \text{ is as in (6.4)}. \end{cases}$$

Proof. It is the result of the same computation as in the proof of Proposition 4.7 but with the extra correction terms. The purpose of the assumptions $\gamma + 2s < 1 - \varepsilon$ or $q_0 > d + \gamma + 2s$ is to make sure the tail of the integral

$$\int_{w \perp v'-v} g(v+w) |w|^{\gamma+2s+1} \, \mathrm{d}w$$

is convergent (which is the same purpose of the assumption $q > \gamma + 2s - d$). □

Proposition 6.10 (Estimate of $\mathcal{B}_3(f,f)$)**.** *Assume f satisfies (4.1) with g of the form (6.3) or (6.4). For $q > \gamma + 2s + d$, and for all $|v| \geq 2$,*

$$\mathcal{B}_3(f,f) \lesssim_q |v|^{\gamma-2} g(v).$$

Proof. The proof follows the same lines as in Proposition 4.8 but taking the extra correction terms into account. We define

$$I_3 := \int_{\substack{w \perp (v'-v) \\ |v+w| > c_q |v|}} f(v+w) |w|^{\gamma+2s+1} A(|v'-v|, |w|) \, \mathrm{d}w$$

for $|v'| < |v|/2$ as follows,

$$\begin{aligned} I_3 &\lesssim \int_{\substack{w \perp (v'-v) \\ |v+w| > c_q |v|}} g(v+w) |w|^{\gamma+2s+1} \, \mathrm{d}w \\ &\lesssim N|v|^{-q+\gamma+2s+d} \int_{\substack{\bar{w} \perp (v'-v) \\ \left|\frac{v}{|v|} + \bar{w}\right| > c_q}} \left| \frac{v}{|v|} + \bar{w} \right|^{-q} |\bar{w}|^{\gamma+2s+1} \, \mathrm{d}\bar{w} \\ &\quad + \varepsilon \begin{cases} |v|^{-(d+1)+\eta+\gamma+2s+d} \int_{\substack{\bar{w} \perp (v'-v) \\ \left|\frac{v}{|v|} + \bar{w}\right| > c_q}} \left| \frac{v}{|v|} + \bar{w} \right|^{-(d+1)+\eta} |\bar{w}|^{\gamma+2s+1} \, \mathrm{d}\bar{w} & \text{if } g \text{ is as in (6.3)}, \\ |v|^{-q_0+\gamma+2s+d} \int_{\substack{\bar{w} \perp (v'-v) \\ \left|\frac{v}{|v|} + \bar{w}\right| > c_q}} \left| \frac{v}{|v|} + \bar{w} \right|^{-q_0} |\bar{w}|^{\gamma+2s+1} \, \mathrm{d}\bar{w} & \text{if } g \text{ is as in (6.4)}. \end{cases} \end{aligned}$$

From this we deduce

$$\mathcal{B}_3(f,f) \lesssim \begin{cases} C_q N|v|^{-q+\gamma-2} + \varepsilon |v|^{-(d+1)+\eta+\gamma-2} & \text{when } g \text{ is as in (6.3)}, \\ C_q N|v|^{-q+\gamma-2} + C_{q_0} \varepsilon |v|^{-q_0+\gamma-2} & \text{when } g \text{ is as in (6.4)}. \end{cases}$$

And the proposition follows. □

Proposition 6.11 (Estimate of $Q_2(f,f)$)**.** *Assume f satisfies (4.1) with g of the form (6.2) or (6.3). Then for $\gamma \geq 0$ and $|v| \geq 1$,*

$$Q_2(f,f) \lesssim |v|^\gamma g(v);$$

while for $\gamma < 0$ and $|v| \geq 1$,

$$Q_2(f,f) \lesssim_q \begin{cases} |v|^{-2-\frac{2\gamma}{d}} g(v)^{1-\frac{\gamma}{d}} & \text{if } g(v) \geq 2^{-d}|v|^{-d-2}, \\ |v|^{\gamma-2} g(v) & \text{if } g(v) \leq 2^{-d}|v|^{-d-2}. \end{cases}$$

Proof. In the case $\gamma \geq 0$, the estimate $Q_2(f,f) \lesssim |v|^\gamma f(v)$ holds, and then the result follows trivially for any form of the function g.

In the more delicate case $\gamma < 0$, exploring the proof of Proposition 4.9 we observe that the proof holds as soon as $g(v') \leq C_q g(v)$ whenever $|v' - v| < |v|/2$. This property is satisfied for all the variants of the function g given by (6.2), (6.3) or (6.4). □

Proof of Theorem 6.2. The proof of **part (1)** is identical to the proof of part (1) in Theorem 1.2 but using $g(t,v) = N(t)\min(1,|v|^{-q}) + \varepsilon$ for $\varepsilon > 0$ arbitrarily small. We apply Propositions 6.5, 6.7, 6.8 and 6.11 instead of Propositions 4.1, 4.5, 4.6 and 4.9 and we arrive to the same set of inequalities that imply the contradiction.

We prove **part (2)** using the same estimates as for part (1), which are not the same as the ones used for part (2) in Theorem 1.2. We set $g(t,v) = N(t)\min(1,|v|^{-q}) + \varepsilon$ and $N(t) = Nt^{-\frac{d}{2s}}$, where N is a large constant depending on $m_0, M_0, E_0, H_0, \gamma, s$ and d, to be determined below. We apply Propositions 6.5, 6.7, 6.8 and 6.11, for $|v|$ large enough we get

$$\begin{aligned} \mathcal{G}(f,f) &\lesssim -|v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}} && \text{from Prop. 6.6,} \\ \mathcal{B}_1(f,f) &\lesssim |v|^{\gamma-2} g(v) && \text{from Prop. 6.7,} \\ (\mathcal{B}_2 + \mathcal{B}_3)(f,f) &\lesssim |v|^\gamma g(v) && \text{from Prop. 6.8,} \\ Q_2(f,f) &\lesssim |v|^\gamma g(v) && \text{from Prop. 6.11.} \end{aligned}$$

The L^∞ norm of Theorem 2.4 tells us that if N is large, the equality $f(t,x,v) = g(t,v)$ can only take place for $|v|$ large. Thus, it is safe to make this assumption.

Adding up these inequalities, we obtain

$$\begin{aligned} Q(f,f) &\lesssim -|v|^{(\gamma+2s)+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}}, \\ &\lesssim -N(t)^{1+\frac{2s}{d}} |v|^{-q}|v|^\gamma, \\ &\leq -N(t)^{1+\frac{2s}{d}} |v|^{-q}, \\ &= -N^{1+\frac{2s}{d}} t^{-\frac{d}{2s}-1} |v|^{-q}. \end{aligned}$$

We contradict the fact that $Q(f,f) = f_t \geq N'(t)|v|^{-q}$ if N is large enough. This finishes the proof of part (2).

We now turn to **part (3)**. We first consider the case $\gamma \leq 0$ and $q = d+1$. After that, we will redo the estimate for q large.

In case $q = d+1$, we consider a function g of the form (6.2): for $\varepsilon > 0$ arbitrarily small, we take

$$g(t,v) = N\min(1,|v|^{-d-1}) + \varepsilon.$$

We choose N sufficiently large so that $g(0,v) \geq f(t,x,v)$ among other conditions to be determined below.

Let (t,x,v) be the first contact point of f with g, i.e. consider the first time $t > 0$ such that there exists $(x,v) \in \mathbb{R}^{2d}$ such that (4.1) holds true. In particular, at this point of contact we have the inequality

$$Q(f,f) = f_t \geq g_t = 0. \tag{6.5}$$

Our aim is to show that $Q(f,f) < 0$ at the point of contact and contradict (6.5).

Under the assumptions of Theorem 1.2-(3), the initial data f_0 is bounded. Using Theorem 2.4, we have an L^∞ bound N_∞ for $f(t,x,v)$ that is uniform in time. Therefore,

$$N\min(1,|v|^{-d-1}) \leq g(t,v) \leq N_\infty. \tag{6.6}$$

Thus, by picking N large, we can force $|v|$ to be arbitrarily large. In particular, we are able to apply Propositions 6.6, 6.7, 6.8 and 6.11.

Proposition 6.11 gives us two different upper bounds for $Q_2(f,f)$ depending on the size of $g(t,v)$ relative to $|v|^{-d-1}$. Let us consider the case $g(v) > 2^{-d}|v|^{-d-2}$. In this case we get

$$Q_2(f,f) \lesssim |v|^{-2-2\gamma/d} g(v)^{1-\gamma/d}.$$

We claim this is a small fraction of the (negative) bound for $\mathcal{G}(f,f)$ given in Proposition 6.6. Indeed,

$$\frac{c_q |v|^{\gamma+2s+\frac{2s}{d}} g(t,v)^{1+\frac{2s}{d}}}{Q_2(f,f)} \gtrsim_q |v|^{2+\gamma+2s+\frac{2s+2\gamma}{d}} g(v)^{\frac{\gamma+2s}{d}} \geq |v|^{2-\frac{2s}{d}}.$$

This right hand side is arbitrarily large since $2 > 2s/d$. Therefore, when $g(t,v) \geq 2^{-d}|v|^{-d-2}$, the upper bound for Q_2 is absorbed by the bound for $\mathcal{G}(f,f)$ from Proposition 6.6. In case $g(t,v) \leq 2^{-d}|v|^{-d-2}$, then we use the other upper bound for Q_2, which is identical to the upper bounds for $\mathcal{B}_1$ from Proposition 6.7.

Therefore, we have

$$\begin{aligned} Q(f,f) &= \mathcal{G}(f,f) + \mathcal{B}_1(f,f) + \mathcal{B}_2(f,f) + \mathcal{B}_3(f,f) + Q_2(f,f), \\ &\leq -c|v|^{\gamma+2s+\frac{2s}{d}} g(v)^{1+\frac{2s}{d}} + C|v|^\gamma g(t,v), \\ &\leq |v|^\gamma g(t,v) \left(-cN^{\frac{2s}{d}} + C \right). \end{aligned}$$

We choose N large so that $-cN^{\frac{2s}{d}} + C < 0$ and achieve the desired contradiction. This finishes the proof of part (3) in the particular case $q = d+1$.

We now move to the case of q large in part (3). Having proved already that the statement holds when $q = d+1$, we are able to use a function g as in (6.3). For this type of function, we know that $f(t,x,v) < g(t,v)$ when v is sufficiently large and therefore there will always be a first crossing point.

We set g of the form (6.3) and $\varepsilon(t) = \varepsilon_0 e^{C_q t}$ for $\varepsilon_0 > 0$ arbitrarily small. We aim at contradicting the inequality $Q(f,f) = f_t \geq g_t = C\varepsilon(t)$ at the first contact point. Like in the other cases, using Theorem 1.2 we can reduce our analysis to large values of $|v|$ and apply Propositions 6.5, 6.6, 6.7, 6.9, 6.10 and 6.11. Following the same computations as in the proof of part (3) of Theorem, the principal terms cancel out and we are left with the terms derived from the correction term $+\varepsilon(t)|v|^{-d-1}$. We get

$$Q(f,f)(t,x,v) \leq C\varepsilon(t)|v|^{\gamma+d+1-\varepsilon}.$$

Since $\gamma \leq 0$ and $|v|$ is large, we have in particular that $Q(f,f) < C\varepsilon(t)$ for some constant C. This is how we choose the constant C in the exponential for $\varepsilon(t) = \varepsilon_0 e^{Ct}$ and achieve the contradiction. This finishes the proof of part (3).

We finally turn to **part (4)**. We set g as in (6.4), that is

$$g(t,v) = N(t)\min(1, |v|^{-q}) + \varepsilon \min(1, |v|^{-q_0}),$$

where $\varepsilon > 0$ is arbitrarily small and the exponents q and q_0 will be large.

As always, using Theorem 2.4 we can restrict our analysis to large values of $|v|$, so that we apply Propositions 6.5, 6.7, 6.9, 6.10 and 6.11. We obtain

$$\begin{aligned} \mathcal{G}(f,f) &\lesssim -q^s N(t)|v|^{-q+\gamma} - q_0^s \varepsilon |v|^{-q_0+\gamma} && \text{from Prop. 6.5,} \\ \mathcal{B}_1(f,f) &\lesssim_{q,q_0} N(t)|v|^{-q+\gamma-2} + \varepsilon|v|^{-q_0+\gamma-2} && \text{from Prop. 6.7,} \\ \mathcal{B}_2(f,f) &\lesssim \frac{1}{q} N(t)|v|^{-q+\gamma} + \frac{1}{q_0}\varepsilon|v|^{-q_0+\gamma} && \text{from Prop. 6.9,} \\ \mathcal{B}_3(f,f) &\lesssim_{q,q_0} N(t)|v|^{-q+\gamma-2} + \varepsilon|v|^{-q_0+\gamma-2} && \text{from Prop. 6.10,} \\ Q_2(f,f) &\lesssim N(t)|v|^{-q+\gamma} + \varepsilon|v|^{-q_0+\gamma} && \text{from Prop. 6.11.} \end{aligned}$$

We choose q and q_0 sufficiently large so that $-\mathcal{G}(f,f)/2$ is larger than $\mathcal{B}_2(f,f) + Q_2(f,f)$.

Adding these terms, we conclude that for q and q_0 sufficiently large, and large values of $|v|$, we have

$$Q(f,f) \lesssim -q^s N|v|^{-q+\gamma}.$$

Just like in the proof of part (2) of Theorem 1.2, we know that $|v| \gtrsim_q N^{1/q} t^{-1/\gamma}$. Choosing N large, we get

$$Q(f,f) \leq -Ct^{-\beta-1}|v|^{-q} = g_t(t,v).$$

This gives us the desired contradiction and finishes the proof. □

Appendix A. Integral $L^1_{x,v}$ moments

In this appendix, we explain in an informal way how integral $L^1_{x,v}$ moments can be studied under the assumptions that hydrodynamic fields stay under control in the case where $\gamma > 0$. We thus consider a solution f to (1.1) such that $M(t,x)$ bounded above and below (strict positive) and $E(t,x)$, $H(t,x)$ bounded above. Then consider the time evolution of a integral moments $\mathcal{M}_q[f] := \int_{x,v} f|v|^q \,\mathrm{d}x\,\mathrm{d}v$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_q[f] = \int_{x,v,v_*} ff_* \left(\int_{\sigma\in\mathbb{S}^{d-1}} [|v'|^q + |v'_*|^q - |v|^q - |v_*|^q]\, b(\cos\theta)\,\mathrm{d}\sigma \right) |v-v_*|^\gamma \,\mathrm{d}x\,\mathrm{d}v.$$

The Povzner inequality in the cutoff case states for instance [38] that (given $s_0 > 0$)

$$\int_{\sigma\in\mathbb{S}^{d-1}} [|v'|^q + |v'_*|^q - |v|^q - |v_*|^q]\, b(\cos\theta)\chi_{|\sin\theta|\geq s_0}\,\mathrm{d}\sigma \lesssim_{s_0} (q-2)|v|^{q/2}|v_*|^{q/2} - \kappa\left(|v|^q + |v_*|^q\right)$$

for some $\kappa > 0$ uniform as $s_0 \to 0$, and the singular part gives by Taylor expansion

$$\int_{\sigma\in\mathbb{S}^{d-1}} [|v'|^q + |v'_*|^q - |v|^q - |v_*|^q]\, b(\cos\theta)\chi_{|\sin\theta|\geq s_0}\,\mathrm{d}\sigma \lesssim \eta(s_0)\left(|v|^q + |v_*|^q\right)$$

where $\eta(s_0) \to 0$ as $s_0 \to 0$. Altogether it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_q[f] \leq C\mathcal{M}_q[f] \sup_x E(t,x) - K\mathcal{M}_{q+\gamma}[f] \inf_x M(t,x)$$

for some constant $K > 0$ (we recall that we assume $\gamma > 0$ here). This gives the propagation and appearance of all polynomial moments, with rate $O(t^{-(q-2)})$. It is not clear whether this integral arguments carries to the case of exponential moments (with the Bobylev spatially homogeneous argument) because of how to distribute the supremum norm in x in each q-moment when summing all moments.

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