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Some deterministic structured population models which are limit of stochastic individual based models

Philippe Carmona

Laboratoire de Mathématiques Jean Leray UMR 6629
Université de Nantes, 2 Rue de la Houssinière
BP 92208, F-44322 Nantes Cedex 03, France
e-mail: philippe.carmona@univ-nantes.fr

Abstract: * The aim of this paper is to tackle part of the program set by Diekmann et al. in their seminal paper Diekmann et al. (2001). We quote

"It remains to investigate whether, and in what sense, the nonlinear deterministic model formulation is the limit of a stochastic model for initial population size tending to infinity"

We set a precise and general framework for a stochastic individual based model : it is a piecewise deterministic Markov process defined on the set of finite measures. We then establish a law of large numbers under conditions easy to verify. Finally we show how this applies to old and new examples.

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1. Introduction

We shall focus on linear structured population model , as defined in Diekmann et al. (1998, 2001) : the evolution of individuals (i-state evolution) depends on the individual state, and of the rest of the population, but not on environmental variables. We shall consider nonlinear structured population model in a forthcoming paper Carmona.[1] We shall furthermore restrict ourselves to structured population model whose building blocks for the i-model are solutions of ODE (see e.g. Ackleh and Ito (2005); Diekmann, Gyllenberg and Metz (2007); Kooijman (2000)).
We shall establish a law of large number: when initial population goes to infinity, the measure valued stochastic process converges in distribution to a deterministic function, measure valued, that satisfies an integrodifferential equation which is the equation for a deterministic structured population model equation.
There has already been considerable work done in this direction : when the individuals have discrete traits, for SIR and compartmental models Kurtz (1970), when the individuals have continuous traits Fournier and Méléard (2004); Champagnat, Ferrière and Méléard

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[1] In fact what we call linear model are not linear in the strict sense of Diekmann et al. (1998, 2001) since the evolution may depend say on the proportion of individuals of a certain type

(2008a,b), for age structured population model Ferrière and Tran (2009); Tran (2008); Solomon (1987). The probabilistic toolbox used consists mainly of a representation in terms of Poisson random measures, and an identification of martingale problems with subtle limit theorems.
We introduce in this paper a significantly different toolbox. The stochastic structured population model is a PDMP, a *Piecewise Deterministic Markov Process*, on the space of finite measures over E, with E the state space for i-model.[2]
The first advantage of our framework is that you do not have to introduce explicit Poisson random measures, nor to perform a clever labelling of individuals (see e.g. Fournier and Méléard (2004); Metz and Tran (2013)) ; the martingale problem also is implicit since it is the one associated to the PDMP.
A second advantage of our approach is that we are able to introduce fairly complex i-model evolutions. (This in fact is the starting point of our work : we wanted to model norovirus evolution in individuals). An example is given in section 6.3 where we obtain a new, non trivial, integro differential equation with transport terms.
Another new feature of our approach is that we can incorporate, e.g. when we have reproduction, mean numbers of descendants.

Some Notations :

- $\nu \ll \mu$ means that the measure ν is absolutely continuous with respect to μ.
- $\varphi \sharp m$ is the image of measure m by the measurable function φ.

2. The linear stochastic structured population model

2.1. Definitions

The stochastic structured population model , abbreviated SSPM, is a PDMP (see Appendix A) on the state $\mathscr{M}_F(E)$ of finite measures over a measurable state space $(E, \mathscr{E})$: E is metrisable, separable and locally compact ; $\mathscr{E}$ is the Borel σ-field.

The *deterministic dynamic* is driven by a continuous time dynamical system on E , a measurable map

$$\begin{aligned} \varphi : \mathbb{R}^+ \times E &\to E \\ (t, x) &\to \varphi_t(x) \end{aligned}$$

such that $\varphi_t \circ \varphi_s = \varphi_{t+s}$ and $\varphi_0 = id$, and $t \to \varphi_t(x)$ continuous for any x.
We lift this dynamical system to the space $\mathscr{M}_F(E)$, $\phi : \mathbb{R}^+ \times \mathscr{M}_F(E) \to \mathscr{M}_F(E)$ by the prescription $\phi_t := \varphi_t \sharp m$ is the image of the measure m by φ_t, that is for every bounded measurable h:

$$\langle \phi_t(m), h \rangle := \int h(y) \phi_t(m)(dy) = \int h(\varphi_t(x))\, m(dx).$$

[2]Let us observe that in former works (see e.g. Metz and Tran (2013)) the deterministic evolution of the population between jump times has already been made clear, even if they do not explicitly introduce a PDMP

Then for any $h \in \bar{D}_\varphi$ we have $f(m) = \langle m, h\rangle \in \mathscr{D}_\phi$ and $A_\phi f(m) = \left\langle m, A_\varphi h\right\rangle$.

The *jump dynamic* is a measurable kernel $\mu : \mathscr{M}_F(E) \to \mathscr{M}_F(\mathscr{M}_F(E))$. A *population*, a p-state in the terminology of Diekmann et al. (1998), is a finite measure on E : $m \in \mathscr{M}_F(E)$.
Here is the basic construction step:

- Given a transition rate function $\alpha : \mathscr{M}_F(E) \times E \to \mathbb{R}_+$, $\alpha(m, x)$ is the transition rate of the individual x in the population m.
- Given a reproduction kernel $k : \mathscr{M}_F(E) \times E \to \mathscr{M}_S(E)$, so that is $k(m, x)$ is a signed finite measure on E
- We define a kernel

$$\mu(m, dm') := \int_E m(dx)\alpha(m, x)\delta_{m+k(m,x)}(dm') \tag{2.1}$$

Of course, since we want that $\mu(m)$ has support on $\mathscr{M}_F(E)$ we require that if the measure $m + k(m, x)$ is not positive, then $\alpha(m, x) = 0$.
Usually, to build complex models, we add a finite number of kernels : given α_i, k_i we let

$$\mu(m, dm') = \sum_i \mu_i(m, dm') = \int_E m(dx) \sum_i \alpha_i(m, x)\delta_{m+k_i(m,x)}(dm').$$

The transition rate of this kernel is

$$q(m) = \int_{\mathscr{M}_F(E)} \mu(m, dm') = \sum_i \int_E m(dx)\alpha_i(m, x).$$

Definition 2.1. *The PDMP ν driven by such (μ, φ) is called a (μ, φ) SSMP.*

It is evidently *stable*, that is does not explode, in finite time if the rate function $q(m)$ is bounded. We shall however use another non explosion criterion more suitable for complex models.

Eventually, a specific transition rate function, of interaction type, may be built by integrating a measurable function mutation kernel

$$\alpha(m, x) := \int \bar{\alpha}(m, x, y)\mu(dy).$$

2.2. Properties

Let us be more precise on the generator of this PDMP (see the Appendix).

Lemma 2.2. *1) Assume that $h : E \to \mathbb{R}$ is path-continuous for φ and bounded. Then the function $f : \mathscr{M}_F(E) \to \mathbb{R}$ defined by $f(m) = \langle m, h\rangle$ is path-continuous for ϕ and bounded.*
2) Assume furthermore that h is path-differentiable for φ with $A_\varphi h$ bounded. Then f is path-differentiable for ϕ with

$$A_\phi f(m) = \left\langle m, A_\varphi(h)\right\rangle = \int m(dx) A_\varphi(h)(x).$$

Therefore, we have the formula

$$\begin{aligned} L^\nu f(m) &= A_\phi f(m) + \int \mu(m, dm')(f(m+m') - f(m)) \\ &= \left\langle m, A_\varphi(h)\right\rangle + \sum_i \int m(dx)\alpha_i(m,x)\langle k_i(m,x), h\rangle. \end{aligned}$$

Proof. 1) Fix $m \in \mathscr{M}_F(E)$. We have:

$$f(\phi_t(m)) = \langle \phi_t(m), h\rangle = \langle \varphi_t \sharp m, h\rangle = \int h(\varphi_t(x))\, m(dx).$$

By assumption, for any x, $t \to h(\varphi_t(x))$ is continuous. Since it is bounded, we infer by dominated convergence that $t \to f(\phi_t(m))$ is continuous.
2) The proof is similar : we differentiate under the integral sign and use $\frac{d}{dt} h(\varphi_t(x)) = A_\varphi h(\varphi_t(x))$. □

2.3. Examples

Let us see now how versatile this SSMP framework is by obtaining classical and non classical models.

2.3.1. The Basic stochastic SIR

The state space is made of three compartments $E = \{S, I, R\}$ with σ-field $\mathscr{E} = \mathscr{P}(E)$. There is no deterministic evolution $\varphi_t = id$: individual stay in their compartment. Therefore the process is just a continuous time Markov chain on $\mathscr{M}_F(E)$.
The first kernel models the prescription "infected people recover at rate $\gamma > 0$":

$$\alpha_1(m,x) = \mu_1(x, dz) = \gamma\, \mathbf{1}_{(x=I)}, \qquad k_1(m,x) = \delta_R - \delta_I.$$

At rate γ an infected individual recovers, that is it is removed from the population, the term $-\delta_I$, and a recovered is added, the term $+\delta_R$.
The second kernel is an interaction kernel, that models the prescription "susceptible are infected by infected people at per capita rate $\beta > 0$ ":

$$\bar{\alpha}_2(m,x,y) = \beta\, \mathbf{1}_{(x=S, y=I)}, \quad k_2(m,x) = \delta_I - \delta_S.$$

which yields:

$$\alpha_2(m,x) = \beta I(m)\,\mathbf{1}_{(x=S)}\,,$$

with $I(m) = m(\{I\})$ the number of infected people. The total rate function is, with $S(m) = m(\{S\})$,

$$q(m) = q_1(m) + q_2(m) = \gamma I(m) + \beta I(m) S(m)\,.$$

2.3.2. Age since infection structured SIR

This model has also been introduced by Kermack and McKendrick, see Kermack and McKendrick (1927), (Perthame, 2007, section 1.5.2), (Martcheva, 2015, section 13.2).
The infection rate and recovery rate depend on the age since infection , i.e. are two measurable functions $\gamma, \beta : \mathbb{R}_+ \to \mathbb{R}_+$. The state space is

$$E = \{S\} \cup \{I\} \times [0, +\infty[\cup \{R\}\,, \quad \mathscr{E} = \sigma(\{S\}, \{R\}, \{I\} \times B, B \in \mathscr{B}(\mathbb{R}))\,.$$

We let $c : E \to \{S, I, R\}$ be the compartment function and $a(I,t) = t$ be the age function. Then the recovery mechanism is described by

$$\alpha_1(m,x) = \gamma(a(x))\,\mathbf{1}_{(c(x)=I)}\,, \quad k_1(m,x) = \delta_R - \delta_x\,.$$

And the infection mechanism is induced by

$$\bar{\alpha}_2(m,x,y) = \beta(a(y))\,\mathbf{1}_{(c(x)=S, c(y)=I)}\,, k_2(m,x) = \delta_{(I,0)}(dz) - \delta_x$$

so that

$$\alpha_2(m,x) = \lambda(m)\,\mathbf{1}_{(c(x)=S)}\,,$$

with $\lambda(m)$ the total rate of infection of population m:

$$\lambda(m) := \int \beta(a(y))\,\mathbf{1}_{(c(y)=I)} m(dy)\,.$$

The rate functions are

$$q_1(m) = \int \gamma(a(y))\,\mathbf{1}_{(c(y)=I)} m(dy)\,, \quad q_2(m) = S(m)\lambda(m)$$

and the driving dynamical system is

$$\varphi_t(S) = S, \varphi_t(R) = R, \varphi_t(I,s) = (I, t+s)\,.$$

2.3.3. A simple host/pathogen interaction with immigration of pathogen

This example cannot be considered as an age structured epidemiological model, and therefore is totally new.
The host pathogen interaction model comes from Gilchrist and Sasaki, see e.g. (Martcheva, 2015, section 14.2.2, equation (14.3)) and the references therein.

The continuous dynamical system $(\varphi_t)_{t\geq 0}$ is the flow of the ODE

$$\begin{cases} \dfrac{dP}{dt} = rP - cBP \\ \dfrac{dB}{dt} = aBP\,. \end{cases} \tag{2.2}$$

with state space $E = (0,+\infty)^2$. B is the quantity of immune cells, P of pathogen cells ; $r > 0$ is the pathogen reproduction rate, $c > 0$ the pathogen clearance rate by the immune cells, a the stimulation of immune cells production by the pathogen. The pathogen is always cleared, since every trajectory converges to $(P^* = 0, B^*)$ with B^* depending on the initial conditions.
We shall see that if we impose some absolute continuity on a mutation kernel, then the large population limit is an integro differential *transport equation*.

2.3.4. The Bell-Anderson model

This model is described in (Metz and Diekmann, 1986, Section 3) and Jagers (2001). A cell with size x dies with intensity $d(x)$ and splits into two cells of equal size $x/2$ with intensity $b(x)$. An individual cell growths with rate $g(x) > 0$.
There is a minimal size $a > 0$ and maximal cell size $4a$. Initially all cells have size in $[a, 2a]$, no cell with size smaller than 2a can divide.
The state space is therefore $E = [a, 4a]$ with its Borel sigma field. Cell-growth is modelled by the flow $(\varphi_t)_{t\geq 0}$ of the ODE

$$\dot{x} = g(x)\,.$$

We assume thus that E is stable by the flow.
Death is modelled by

$$\alpha_1(m,x) = d(x) \geq 0\,, \quad k_1(m,x) = -\delta_x\,,$$

and reproduction by

$$\alpha_2(m,x) = b(x) \geq 0\,, \quad k_2(m,x) = -\delta_x + 2\delta_{x/2}\,.$$

We assume that $b(x) = 0$ is $x < 2a$.
The total rate of population m is thus

$$q(m) = q_1(m) + q_2(m) = \int m(dx)(b(x) + d(x))\,.$$

3. Large Population Limit : Law of Large Numbers

Let $(\nu_t)_{t\geq 0}$ be a (μ, φ) SSMP process. Our first concern is to ensure that the size of the population $\nu_t(E) = \langle \nu_t, 1\rangle$ does not explodes in finite time. This shall not only ensure non explosion but also yield useful bounds on the size.

Assumption 1 (growth control). *1. The variation norms of the reproduction kernels k_i are uniformly bounded : $C_k := \sup_{i,m,x} \|k_i(m,x)\|_{VT} < +\infty$.*

2. Let $\bar{I}$ be the set of i such that there exists m, x with $\alpha_i(m,x) > 0$ and $k_i(m,x,E) > 0$. Then there exists a constant C_q such that

$$q_i(m) \leq C_q(1 + \langle m, 1\rangle) \qquad (i \in \bar{I}, m \in \mathscr{M}_F(E)).$$

Remark 3.1. *The first assumption ensures that a jump cannot increase or decrease the population of more than C_k unit. The second assumption controls the rate of the jumps.*

Proposition 3.2 (mass control). *Assume that for some $p \geq 1$ we have*

$$\mathbb{E}[\langle \nu_0, 1\rangle^p] < +\infty .$$

Then, under the Assumption 1,

1. *The (μ, ϕ)-SSMP process is defined on $[0, +\infty[$.*
2. *There exists a constant $C > 0$ such that for any $t > 0$,*

$$\mathbb{E}[\langle \nu_t, 1\rangle^p] \leq Ce^{Ct} .$$

When the size of the initial population is approximately n, e.g. if $\nu_0 = \sum_{i=1}^n \delta_{x_i}$ is deterministic, the x_i being the individuals, we can renormalize by considering $(\frac{1}{n}\nu_t)_{t\geq 0}$. The following theorem yields sufficient conditions for a law of large number, i.e. the convergence of the renormalization to a deterministic limit.

Assumption 2 (regular kernel). *Assume that the kernel μ satisfy on the space $\mathscr{M}_S(E)$ of bounded variation signed measure on E, that for any $m \in \mathscr{M}_S(E)$ and any $r > 0$, there exists constants C_1, C_2 such that with $B(m,r) = \{m' \in \mathscr{M}_S(E) : \|m - m'\|_{TV} \leq r\}$,*

$$\sup_{i,x\in E, m'\in B(m,r)} \left\|\alpha_i(m,x)k_i(m,x) - \alpha_i(m',x)k_i(m',x)\right\|_{TV} \leq C_1 \left\|m - m'\right\|_{TV} \tag{3.1}$$

$$\sup_{i,m'\in B(m,r)x\in E} \alpha_i(m',x) \leq C_2. \tag{3.2}$$

Remark 3.3. *This assumption entails some uniform Lipschitz bound in the total variation norm that is necessary not only to establish uniqueness in the limiting integro differential equation (Proposition 5.1) but also to prove tightness (compactness) of processes ad thus the existence of limits (see Step 1 of the proof of Theorem 3.4).*

Theorem 3.4. *Let $(\nu_t^n)_t \geq 0$ be a $(\mu^{(n)}, \varphi)$ SSMP process with the scaling*

$$\alpha_i^{(n)}(nm,x) = \alpha_i(m,x), \quad k_i^{(n)}(nm,x) = k_i(m,x),$$

with μ a fixed jump dynamic satisfying Assumptions 1,2 and the bound on the total rate function

$$q(m) \leq C_q'(1 + \langle m, 1\rangle + \langle m, 1\rangle^2).$$

Let $X_t^n = \frac{1}{n}\nu_t^n$ be the renormalized measure valued process. Assume that

1. *For some* $p \geq 3$, $\sup_n \mathbb{E}\left[\left\langle X_0^n, 1\right\rangle^p\right] < +\infty$.
2. *There exists* $\xi_0 \in \mathscr{M}_F(E)$ *such that* X_0^n *converges to* ξ_0 *in probability.*
3. *The set* $\bar{D}_\varphi$ *of functions* h *bounded, path-differentiable, such that* $A_\varphi h$ *is bounded and path-continuous is dense in* $C_0(E)$.

Then $(X_t^n)_{t\geq 0}$ *converges in probability in* $\mathbb{D}([0,T], \mathscr{M}_F(E))$ *to a deterministic continuous function* $(\xi_t)_{0\leq t\leq T}$ *which satisfies: for all* $h : \mathbb{R}_+ \times E \to \mathbb{R}$, *such that for fixed* x, $t \to h(t,x)$ *is* C^1 *and for every* t, $x \to h(t,x) \in \bar{D}_\varphi$,

$$\frac{d}{dt}\langle \xi_t, h\rangle = \left\langle \xi_t, \partial_t h(t,.) + A_\varphi(h)(t,.)\right\rangle + \sum_i \int \xi_t(dx)\alpha_i(\xi_t, x)\langle h(t,.), k_i(\xi_t, x, .)\rangle\,. \tag{3.3}$$

Remark 3.5. *Of course, the scaling assumption may be replaced by a convergence assumption such as*

$$\lim_{n\to+\infty} \alpha_i^{(n)}(nm, x) = \alpha_i(m,x)\,, \quad \lim_{n\to+\infty} k_i^{(n)}(nm,x) = k_i(m,x)\,.$$

We have encountered no need for such a generality in our studies, and thus we leave the generalisation to an interested reader.

4. Large Population Limit : the associated PDE

In order to show that equation (3.3) may be seen as a PDE, we need some absolute continuity assumption. We shall need more structure on the state space (what we require looks a lot like the framework used in Benaïm et al. (2015)).
The state space E is the union $E = E_1 \times E_2$, with E_1 a finite set (of compartments) and $E_2 = \cup_{i\in I_2}\{i\} \times \bar{\mathscr{X}}_i$ the finite union of compartments with a continuous trait : $\bar{\mathscr{X}}_i$ is the closure of an open connected set $\mathscr{X}_i$ of $\mathbb{R}^{d_i}$.
Let $(F^i)_{i\in I_2}$ be smooth vector fields $F^i : \bar{\mathscr{X}}_i \to \mathbb{R}^{d_i}$ and $\varphi^i = (\varphi_t^i)_{t\in\mathbb{R}}$ be the flow induced by them : $t \to \varphi_t^i(x)$ is the solution of

$$\dot{x} = F^i(x)$$

with initial condition $x(0) = x$.
The dynamical system on E is then $\varphi_t(x) = x$ if $x \in E_1$ and $\varphi_t(i,x) = \varphi_t^i(x)$ for $(i,x) \in E_2$. The domain $\bar{D}_\varphi$ contains bounded functions $h : E \to \mathbb{R}$ such that for any $i \in E_2$, $x \to h(i,x) \in C^1(\bar{\mathscr{X}}_i)$, and $x \to \nabla h(i,x)F^i(x)$ bounded. For these functions

$$A_\varphi h(x) = 0 \text{ if } x \in E_1 \quad \text{ and } A_\varphi(i,x) = \nabla h(i,x)F^i(x) \text{ for } (i,x) \in E_2\,.$$

Hence $\bar{D}_\varphi$ is dense in $C_0(E)$ since it contains constant functions and functions h s.t. for any $i \in I_2$, $x \to h(i,x) \in C_K^\infty(\mathscr{X}_i)$ (functions C^∞ with compact support).
The reference measure on E will be

$$\lambda = \sum_{x\in E_1} \delta_x + \sum_{i\in I_2} \delta_i \otimes Leb(\bar{\mathscr{X}}_i)$$

It should be clear then that for any t,the image $\varphi_t \sharp \lambda$ is absolutely continuous with respect to λ:

$$\varphi_t \sharp \lambda \ll \lambda \qquad (\forall t \in \mathbb{R}).$$

Indeed this is true on every compartment in E_1 and in $\{i\} \times \bar{\mathscr{X}}_i$ we just need to use the jacobian of the flow. Furthermore, the dual of A_φ in the sense of distributions is

$$A^*_\varphi(m) = -\sum_{i \in I_2} div_i(m(i,x)F^i(x))$$

with div_i the divergence in the sense of distribution on each $\mathscr{X}_i$.

Theorem 4.1. *Let $(\xi_t)_{i\in[0,T]}$ be a solution of (3.3) with (μ, φ) satisfying the Assumptions 1, 2.*
Assume furthermore that

- $\xi_0 \ll \lambda$.
- *for every i, m, x $k_i(m,x)$ is absolutely continuous with respect to λ with density say $k_i(m,x,z)$ a measurable positive function defined on $\mathscr{M}_F(E) \times E \times E$.*

Then for any $t \in [0,T]$, $\xi_t \ll \lambda$ and the density $\xi(t,x)$ satisfy in the weak sense the PDE

$$\partial_t \xi(t,x) - A^*_\varphi(\xi_t)(x) = -\int \xi(t,x') \sum_i \alpha_i(\xi_t, x') k_i(\xi_t, x', x) \lambda(dx').$$

5. Proofs of large population limits

5.1. Proof of Proposition 3.2

Proof. We shall follow closely the proof of (Fournier and Méléard, 2004, Theorem 3.1). Given $a > 0$ we let $\tau_a := \inf\{t \geq 0 : \langle \nu_t, 1\rangle \geq a\}$ and $f(m) = \langle m, 1\rangle^p$. Observe that

$$A_\phi f(m) = p\langle m, 1\rangle^{p-1}\big\langle m, A_\varphi(1)\big\rangle = 0.$$

Therefore,

$$\begin{aligned} 0 \leq L^\nu f(m) &= \sum_i \int m(dx)\alpha_i(m,x)((\langle m + k_i(m,x), 1\rangle)^p - \langle m, 1\rangle^p) \\ &= \sum_{i\in\bar{I}} \int m(dx)\alpha_i(m,x)((\langle m + k_i(m,x), 1\rangle)^p - \langle m, 1\rangle^p) \\ &\leq \sum_{i\in\bar{I}} \int m(dx)\alpha_i(m,x) C_p \langle m, 1\rangle^{p-1}, \end{aligned}$$

where $C_p > 0$ satisfies

$$(x + C_k)^p - x^p \leq C_p(1 + x^{p-1}) \qquad (x \geq 0).$$

Hence,

$$\begin{aligned} 0 \le L^{\nu} f(m) &\le C_p \sum_{i\in\bar{I}} \langle m,1\rangle^{p-1} q_i(m) \\ &\le C_p C_q \sum_{i\in\bar{I}} \langle m,1\rangle^{p-1}(1+\langle m,1\rangle) \\ &\le C(1+\langle m,1\rangle^p)\,. \end{aligned}$$

Therefore the martingale M_t^f in the decomposition (A.6) is such that $M_{t\wedge\tau_a}^f$ is a true martingale, and

$$\begin{aligned} \mathbb{E}\big[f(\nu_{t\wedge\tau_a})\big] &= \mathbb{E}[f(\nu_0)] + \mathbb{E}\left[\int_0^{t\wedge\tau_a} L^{\nu} f(\nu_s)\,ds\right] \\ &\le \mathbb{E}[\langle \nu_0,1\rangle^p] + C' \int_0^t \mathbb{E}\big[1+\big\langle \nu_{s\wedge\tau_a},1\big\rangle^p\big]\,ds\,. \end{aligned}$$

Gronwall's Lemma ensures then the existence of a constant C, not depending on a, such that for every $t>0$

$$\mathbb{E}\big[1+\big\langle \nu_{t\wedge\tau_a},1\big\rangle^p\big] \le Ce^{Ct}\,. \tag{5.1}$$

This implies that $\lim_{a\to+\infty}\tau_a = +\infty$ almost surely, that is non explosion. Then, taking limits in (5.1), yields the desired upper bound.

□

5.2. *Study of equation* (3.3)

Proposition 5.1 (Uniqueness)**.** *Assume that the kernel μ satisfy Assumption* (2) *and that bounded path-continuous functions are dense in bounded functions. Given $m_0 \in \mathscr{M}_F(E)$, there is at most only one solution $(\xi_t)_{t\in[0,T]}$ of* (3.3) *that satisfies $\xi_0 = m_0$.*

Proof. Assume that ξ,ξ' are two solutions with the same initial value $\xi_0=\xi_0'=m_0$. Fix $t>0$. Let $g:E\to\mathbb{R}$ measurable bounded path-continuous for φ, and let $h(s,x)=g(\varphi_{t-s}(x))$. Then, by definition of the generator A_φ of the dynamical system φ, h is a solution of

$$\partial_s h + A_\varphi(h) = 0\,, \quad h(t,x)=g(x)\,.$$

Therefore, injecting this into equation (3.3) yields

$$\langle \xi_t, g\rangle = \langle m_0, h(0,.)\rangle + \sum_i \int_0^t F_i(s,\xi_s)\,ds \tag{5.2}$$

with

$$F_i(s,m) = \int m(dx)\alpha_i(m,x)\langle k_i(m,x), h(s,.)\rangle\,.$$

By continuity of $t \to \xi_t$ and $t \to \xi'_t$, we can chose $r > 0$ such that for all $t \in [0,T]$, ξ_t and ξ'_t are in $B(m_0, r)$.
Observe that Assumptions (1) and (2) implies that for m, m' in $B(m_0, r)$

$$\begin{aligned}
\left|F_i(s,m) - F_i(s,m')\right| &\leq \int m(dx)\left|\int (\alpha_i(m,x)k_i(m,x,dz) - \alpha_i(m',x)k_i(m',x,dz))h(s,z)\right| \\
&\quad + \int (m(dx) - m'(dx))\alpha_i(m',x)\left|\int h(s,z)k_i(m',x,dz)\right| \\
&\leq \|g\|_\infty \int m(dx)\left\|\alpha_i(m,x)k_i(m,x) - \alpha_i(m',x)k_i(m',x)\right\|_{TV} \\
&\quad + \|g\|_\infty \left\|m - m'\right\|_{TV} \sup_x \alpha_i(m',x)\left\|k_i(m',x)\right\|_{TV} \\
&\leq \|g\|_\infty \left\|m - m'\right\|_{TV}(C_1\langle m, 1\rangle + C_k C_2) \\
&\leq C\|g\|_\infty \left\|m - m'\right\|_{TV}. \qquad (5.3)
\end{aligned}$$

We obtain that for a constant C that does not depend on g nor T, for all $t \in [0,T]$,

$$\left|\langle \xi_t, g\rangle - \langle \xi'_t, g\rangle\right| \leq C\|g\|_\infty \int_0^t \left\|\xi_s - \xi'_s\right\|_{TV} ds.$$

Since g is arbitrary, this implies that

$$\left\|\xi_t - \xi'_t\right\|_{TV} \leq C \int_0^t \left\|\xi_s - \xi'_s\right\|_{TV} ds$$

and we conclude by Gronwall's Lemma that for all $t \in [0,T]$, $\xi_t = \xi'_t$. □

5.3. *Proof of Theorem 3.4*

We shall follow closely the lines (and the arguments) of the proof of (Fournier and Méléard, 2004, Theorem 5.3). Since we have already established uniqueness of the limit equation in Proposition 5.1, we shall establish

Step 1 Tightness of the family of distributions of X^n in $\mathbb{D}([0,T],(\mathscr{M}_F(E),v))$ (that is when $\mathscr{M}_F(E)$ has the vague topology).
Step 2 Show any limit in distribution satisfies the limiting equation
Step 3 Convergence in distribution in $\mathbb{D}([0,T],(\mathscr{M}_F(E),w))$ (with the topology of weak convergence) to the unique solution of the limiting equation.

Step 1

Since $\bar{D}_\varphi$ is dense in $C_0(E)$ the set of continuous functions with a limit at infinity, according to Roelly-Coppoletta (1986), it is enough to prove that for any $h \in \bar{D}_\varphi$ the process $\langle X^n, h\rangle$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. We shall show first that

$$\sup_n \mathbb{E}\left[\sup_{t\in[0,T]} |\langle X^n, h\rangle|\right] < +\infty$$

This is a direct consequence of the boundedness of h and of Lemma B.1.
Hence, according to Aldous criterion Aldous (1989) and the Rebolledo criterion of (Joffe and Métivier, 1986, Corollary 2.3.3), it suffices to prove the tightness of the martingale part and the drift part of $\langle X^n, h\rangle$. More precisely, we only need to prove that for the decomposition of $f(X_t^n) = \left\langle X_t^n, h\right\rangle$:

$$f(X_t^n) = f(X_0^n) + M_t^{f,n} + U_t^{f,n}$$

we have for a constant $C_{h,T}$ depending only on h and $T > 0$, that for every stopping times $0 \le S \le S' \le S + \delta \le T$, and every $n \ge 1$, we have

$$\mathbb{E}\left[\left|U_{S'}^{f,n} - U_S^{f,n}\right|\right] \le C_{h,T}\delta\,,$$
$$\mathbb{E}\left[\left|\left\langle M^{f,n}\right\rangle_{S'} - \left\langle M^{f,n}\right\rangle_S\right|\right] \le C_{h,T}\delta\,.$$

Observe first that $U_t^{f,n} = \int_0^t L^{X^n} f(X_s^n)\,ds$ and that, thanks to scaling,

$$\begin{aligned}
L^{X_n} f(m) &= L^{\nu_n} f(\frac{1}{n}.)(nm) \\
&= \left\langle m, A_\varphi(h)\right\rangle + \int nm(dx)\sum_i \alpha_i^{(n)}(nm,x)\left\langle k_i^{(n)}(nm,x), \frac{1}{n}h\right\rangle \\
&= \left\langle m, A_\varphi(h)\right\rangle + \int m(dx)\sum_i \alpha_i(m,x)\langle k_i(m,x), h\rangle\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|L^{X_n} f(m)\right| &\le \langle m,1\rangle \left\|A_\varphi(h)\right\|_\infty + \|h\|_\infty \sum_i \int m(dx)\alpha_i(m,x)\|k_i(m,x)\|_{TV} \\
&\le \langle m,1\rangle \left\|A_\varphi(h)\right\|_\infty + \|h\|_\infty C_k q(m) \\
&\le \langle m,1\rangle \left\|A_\varphi(h)\right\|_\infty + \|h\|_\infty C_q'(1 + \langle m,1\rangle + \langle m,1\rangle^2) \\
&\le C_h(1 + \langle m,1\rangle^2)\,.
\end{aligned}$$

Consequently, by Lemma B.1, since $p \ge 3$,

$$\mathbb{E}\left[\left|U_{S'}^{f,n} - U_S^{f,n}\right|\right] \le C_h\delta\mathbb{E}\left[\sup_{s\le T}(1 + \left\langle X_s^n, 1\right\rangle^2)\right] \le C_{h,T}\delta\,.$$

Similarly,

$$\left\langle M_t^{f,n}\right\rangle = \int_0^t (L^{X^n} f^2 - 2f L^{X^n} f)(X_s^n)\,ds\,,$$

and similarly we have the bound

$$\begin{aligned}
(L^{X^n} f^2 - 2f L^{X^n} f)(m) &= \int n\,m(dx)\sum_i \alpha_i(m,x)\left\langle k_i(m,x), \frac{1}{n}h\right\rangle^2 \\
&\le \frac{1}{n}C_k^2\|h\|_\infty^2 q(m) \\
&\le \frac{1}{n}C_h(1 + \langle m,1\rangle^2)\,.
\end{aligned}$$

Therefore we obtain,

$$\mathbb{E}\Big[\Big|\big\langle M^{f,n}\big\rangle_{S'} - \big\langle M^{f,n}\big\rangle_{S}\Big|\Big] \le \delta C_h \mathbb{E}\Big[\sup_{s\le T}(1+\big\langle X^n_s, 1\big\rangle^2)\Big] \le C_{h,T}\delta\,.$$

Step 2

Let $(X_t)_{t\in[0,T]}$ be the limit in distribution in $\mathbb{D}([0,T],(\mathscr{M}_F(E),v))$ of a subsequence $X^{\kappa(n)}$. By construction, almost surely,

$$\sup_{t\in[0,T]}\sup_{h\in L^\infty(E),\|h\|_\infty\le 1}\Big|\big\langle X^n_t, h\big\rangle - \big\langle X^n_{t-}, h\big\rangle\Big| \le \frac{2}{n}\,.$$

Therefore X is almost surely strongly continuous. Let $h:\mathbb{R}_+\times E\to\mathbb{R}$, such that for fixed x, $t\to h(t,x)$ is C^1 and for every t, $x\to h(x,t)\in\bar{D}_\varphi$.
For any measured valued function $m\in C([0,T],\mathscr{M}_F(E))$ we let

$$\psi(m) = \langle m_t, h\rangle - \langle m, h\rangle - \int_0^t ds\big\langle m_s, \partial_s h(s,.) + A_\varphi(h)(s,.)\big\rangle$$
$$- \int_0^t ds \int m_s(dx)\sum_i \alpha_i(m_s,x)\langle k_i(m_s,x,.), h(s,.)\rangle\,.$$

We are going to show that

$$\mathbb{E}\,[|\psi(X)|] = 0\,.$$

Observe that for $f(s,m) = \langle m, h(s,.)\rangle$ we have

$$M^{f,n}_t = \psi(X^n)\,,$$

where we have the semi martingale decomposition

$$f(t,X^n_t) = f(0,X_0) + M^{f,n}_t + U^{f,n}_t$$

with

$$U^{f,n}_t = \int_0^t (\partial_s f(s,X^n_s) + L^{X^n} f(s,.)(X^n_s))\,ds\,.$$

The preceding computations applied to stopping times $S=0$ and $S'=t$ yield

$$\mathbb{E}\big[\psi(X^n)^2\big] = \mathbb{E}\Big[(M^{f,n}_t)^2\Big] = \mathbb{E}\Big[\big\langle M^{f,n}\big\rangle_t\Big] = \int_0^t \big(L^{X_n}(f^2(s,.)) - 2f\,L^{X_n}(f(s,.))\big)(X^n_s)\,ds \le C_{h,T}\frac{1}{n} \to 0\,.$$

We can prove as in the proof of Proposition 5.1 (see inequality (5.3)) that thanks to Assumption 2, ψ is Lipschitz in the total variation norm:

$$\big|\psi(m) - \psi(m')\big| \le C(\big\|A_\varphi(h)\big\|_\infty + \|\partial_s h\|_\infty + \|h\|_\infty)\big\|m - m'\big\|_{TV}\,.$$

Since X is a.s. strongly continuous, this implies that ψ is a.s. continuous at X. Since $\psi(X^n)$ is bounded in L^2 it is Uniformly Integrable, and we have

$$\mathbb{E}\,[|\psi(X)|] = \lim_{n\to\infty}\mathbb{E}\big[\big|\psi(X^{\kappa(n)})\big|\big] = \lim_{n\to\infty}\mathbb{E}\Big[\Big|M^{f,\kappa(n)}_t\Big|\Big] = 0\,.$$

Therefore, $X=\xi$ the unique solution of equation (3.3).

Step 3

The previous steps imply that X^n converges in distribution in $\mathbb{D}([0,T],(\mathscr{M}_F(E),v))$ to $(\xi_t)_{t\in[0,T]}$ the unique solution of equation (3.3).
If we apply Step 2 to the function $h = 1 \in \bar{D}_\varphi$, we obtain that $(\langle X_t^n, 1\rangle)_{0\le t\le T}$ converges in distribution to $(\langle \xi_t, 1\rangle)_{0\le t\le T}$. Since this limiting process is continuous, a criterion proved in Méléard and Roelly (1993) implies that this convergence holds in $\mathbb{D}([0,T],(\mathscr{M}_F(E),w))$.

5.4. Proof of Theorem 4.1

Let $g = 1_A$ with $\lambda(A) = 0$ and let $h(s,x) = k(\varphi_{t-s}(x))$.
Then for every $t \ge 0$, $k(\varphi_t(x)) \ge 0$ and since $\varphi_t\sharp\lambda \ll \lambda$, we have

$$\int g(\varphi_t(x))d\lambda(x) = \int g(y)d\varphi_t\sharp\lambda(y) = 0\,.$$

Therefore,$g(\varphi_t(x)) = 0$, for λ almost every x and this implies that $h(s,x) = 0$ for λ almost every x.
Let us examine equation (5.2). We obtain, since $\xi_0 \ll \lambda$ and $k_i(m,x) \ll \lambda$,

$$\begin{aligned} 0 \le \langle \xi_t, g\rangle &\le \langle \xi_0, g\circ\varphi_t\rangle \\ &\quad + \int_0^t ds \sum_i \int \xi_s(dx)\alpha_i(\xi_s,x)\int_E k_i(\xi_s,x,dz)h(s,z) = 0\,. \end{aligned}$$

Therefore $\xi_t \ll \lambda$. We let $\xi_t(dx) = \xi(t,x)\lambda(dx)$.
Now given a function $h : E \to \mathbb{R}$, such that for every $i \in I_2$, $x \to h(i,x) \in C_K^\infty(\mathscr{X}_i)$, we manipulate (3.3) and use the adjoint operator to get

$$\begin{aligned} \frac{d}{dt}\int \xi(t,x)h(x)d\lambda(x) &= \int A_\varphi^*(\xi(t,.))(x)h(x)d\lambda(x) \\ &\quad + \sum_i \int d\lambda(x)\xi(t,x)\alpha_i(\xi_t,x)\int k_i(\xi_t,x,z)h(z)\,d\lambda(z) \\ &= \int A_\varphi^*(\xi(t,.))(x)h(x)d\lambda(x) \\ &\quad + \int d\lambda(z)h(z))\int \sum_i \xi(t,x)\alpha_i(\xi_t,x)\int k_i(\xi_t,x,z)d\lambda(x)\,, \end{aligned}$$

which is exactly the desired weak sense PDE.

6. Applications and Examples

In this section we review the examples introduced in section 2.3 and show how to verify the assumptions of Theorem 3.4.

6.1. Basic SIR (continuation of 2.3.1)

The rate function is

$$q(m) = \gamma I(m) + \beta I(m)S(m) \leq C(\langle m,1\rangle + \langle m,1\rangle^2 .$$

Therefore Assumption 1 is satisfied. Since $I(m) = m(\{I\})$ and $S(m) = m(\{S\})$, Assumption 2 is also satisfied with $C_k = 2$.
The scaling relation requires that $\mu^{(n)}$ is associated to $\gamma_n = \gamma$ and $\beta_n = \frac{\beta}{n}$. We consider $X_t^n = \frac{1}{n}\nu_t^n$ where ν_n is the SSMP driven by $\mu^{(n)}$. We shall just assume that $X_0^n = \frac{1}{n}\nu_0^n$ converges in probability to ξ_0.
Therefore the law of large numbers yields that the limiting deterministic process $(\xi_t)_{t\geq 0}$ satisfies for every h

$$\begin{aligned}\frac{d}{dt}\langle \xi_t, h\rangle &= \int \xi_t(dx)(\gamma\, \mathbf{1}_{(x=I)}(h(R)-h(I)) + \beta I(\xi_t)\, \mathbf{1}_{(x=S)}(h(I)-h(S)) \\ &= \gamma\, I(\xi_t)(h(R)-h(I)) + \beta\, I(\xi_t)S(\xi_t)(h(I)-h(S)) .\end{aligned}$$

Therefore if $S(t) = S(\xi_t)$, $I(t) = I(\xi_t)$ and $R(t) = R(\xi_t)$ taking $h(x) = \mathbf{1}_{(x=a)}$ for $a \in \{S, I, R\}$ yields that (S, I, R) satisfy the system

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\ \dfrac{dI}{dt} = \beta SI - \gamma I \\ \dfrac{dR}{dt} = \gamma I . \end{cases} \tag{6.1}$$

This is the classical system of ODE introduced by Kermack and McKendrick (1927).

6.2. Age structured SIR (Continuation of Example 2.3.2)

We assume that the functions γ and β are bounded so that the total rate is quadratic at most:

$$q(m) = q_1(m) + q_2(m) \leq \|\gamma\|_\infty \langle m,1\rangle + \|\beta\|_\infty \langle m,1\rangle^2 .$$

The scaling relation impose that $\mu^{(n)}$ is associated to the functions $\gamma_n(a) = \gamma(a)$ and $\beta_n(a) = \frac{1}{n}\beta(a)$.
The dynamical system has generator $A_\varphi h(R) = A_\varphi h(S) = 0$ and $A_\varphi h(I,s) = \frac{d}{ds}h(I,s)$ and

$$\bar{D}_\varphi = \left\{h \text{ bounded } : t \to h(I,t) \in C_b^1\right\} .$$

The limiting process $(\xi_t)_{t\geq 0}$ satisfies for $h \in \bar{D}_\varphi$

$$\begin{aligned}\frac{d}{dt}\langle \xi_t, h\rangle = &\int \xi_t(dx)\,\mathbf{1}_{(c(x)=I)}\frac{d}{dt}h(I,t)\\ &+\int \xi_t(dx)\gamma(a(x))\,\mathbf{1}_{(c(x)=I)}(h(R)-h(x))\\ &+\int \xi_t(dx)\lambda(\xi_t)\,\mathbf{1}_{(c(x)=S)}(h(I,0)-h(x))\,.\end{aligned}$$

Therefore, if $S = S(t) = \xi_t(S)$ and $R(t) = \xi_t(\{R\})$, taking $h(x) = \mathbf{1}_{(c(x)=S)}$ yields

$$\frac{dS}{dt} = -\lambda(\xi_t)S_t$$

and with $h(x) = \mathbf{1}_{(c(x)=R)}$, we get

$$\frac{dR}{dt} = \int \gamma(a(x))\,\mathbf{1}_{(c(x)=I)}\xi_t(dx)\,.$$

If $h(x) = \mathbf{1}_{(c(x)=I)}g(a(x))$ and if κ_t is the image of the restriction of ξ_t to $\{i\} \times [0,+\infty[$ by the function $a(x)$, that is

$$\int \xi_t(dx)\,\mathbf{1}_{(c(x)=I)}f(a(x)) = \int_{\mathbb{R}_+} \kappa_t(da)f(a)\,,$$

then we get

$$\frac{d}{dt}\int \kappa_t(da)g(a) = \int g'(a)\kappa_t(da) + g(0)S_t\lambda(\xi_t)$$

Assuming the absolute continuity of the initial conditions with respect to Lebesgue measure $\kappa_0(da) = i(0,a)\,da$, then we obtain, following the proof of Theorem4.1, that $\kappa_t(da) = i(a,t)\,da$ and therefore, since $i(a,t) = 0$ for $a < 0$,

$$\frac{d}{dt}\int g(a)i(a,t)\,da = \int_0^{+\infty} g'(a)i(t,a)\,da + g(0)S_t\lambda(\xi_t)$$

with

$$\lambda(\xi_t) = \int_0^{\infty} \beta(a)i(t,a)\,da\,.$$

A simple integration by parts, for $g \in C^1$ with support in a compact $[0,M]$ proves that in a weak sens $i(t,a)$ is solution of

$$\partial_t i(t,a) + \partial_a i(t,a) = 0\,, \qquad i(t,0) = S_t\lambda(\xi_t)\,.$$

This is exactly the PDE system with boundary conditions derived by Kermack and McKendrick, see (Perthame, 2007, section 1.5.2) or (Martcheva, 2015, section 13.2).

6.3. A simple host/pathogen interaction (continuation of 2.3.3)

We assume that at a constant rate $\gamma > 0$ an individual $x = (b, p)$ mutates to and individual $x' = (b', p')$ with a density $\psi_x(x')$ with respect to Lebesgue measure on $E = (0, +\infty)^2$. This mutation may just be an injection of a random quantity of pathogens and a destruction of a random quantity of immune cells. The kernel is defined through

$$\alpha(m, x) = \gamma\,, \quad k(m, x, dz) = \psi_x(z)\,dz$$

with dz the Lebesgue measure. It has constant rate function $q(m) = \gamma$. Then $\mu^{(n)}$ is also associated to γ and ψ. Then, Theorem 4.1 implies that the limit process has a density $\xi(t, x)$ that satisfy in a weak sense the PDE:

$$\partial_t \xi(t, x) + div(\xi(t, x)F(x)) = -\gamma\xi(t, x) + \int_{(0,+\infty)^2} \psi_{x'}(x)\xi(t, x')\,dx'\,,$$

with F the smooth vector field

$$F(x)F(p, b) = \begin{pmatrix} rp - bcp \\ abp \end{pmatrix}.$$

6.4. The Bell Anderson model

We assume boundedness of the rates so $q(m) \le C\langle m, 1\rangle$ and Assumption 1 is satisfied. The regularity Assumption 2, is then also satisfied since the rates do not depend on the population m and the total variation of k_i is bounded. Scaling is also trivially satisfied so that we obtain the limit equation

$$\frac{d}{dt}\langle \xi_t, h\rangle = \int_E \xi_t(dx)\big(g(x)h'(x) - d(x)h(x) + b(x)(-h(x) + 2h(x/2))\big). \quad (6.2)$$

This is exactly the weak form of the PDE (Metz and Diekmann, 1986, section 3.4)

$$\frac{\partial n}{\partial t}(t, x) + \frac{\partial}{\partial x}(g(x)n(t, x)) = -d(x)n(t, x) - \beta(x)n(t, x) + 4\beta(2x)n(t, 2x)\,.$$

Of course one may object that this PDE may only be inferred from (6.2) if we prove that $\xi_t(dx)$ has a density $n(t, x)$ with respect to Lebesgue measure on E. With some additional assumptions on g this is however possible to establish. For example, if one assumes that g satisfies

$$2g(x) = g(2x)\,.$$

Comparison of ODE solutions yield that if $h(s, x) = \gamma(\phi_{t-s}(x))$ we have $h(s, x) = 2h(s, x/2)$ and thus if $\xi_0 \ll \lambda$, and $\gamma(x) = 1_A(x)$ with $\lambda(A) = 0$, we have, as in the proof of Theorem 4.1

$$0 \le \langle \xi_t, \gamma\rangle \le \langle \xi_0, g \circ \varphi_t\rangle - \int_0^t ds \int \xi_s(dx)d(x)h(s, x) \le \langle \xi_0, g \circ \varphi_t\rangle = 0\,.$$

And therefore $\xi_t \ll \lambda$.

Another way to obtain rigorously a limit PDE is to change the splitting mechanism as proposed in (Metz and Diekmann, 1986, section 3.2):

$$k_2(m,x,dy) = -\delta_x(dy) + 2\pi(x,y)\,dy$$

with $\pi(x,y) = \pi(x,x-y)$ and thus $\int \pi(x,y)y\,dy = x/2$.

Appendix A: Definitions and basic properties of PDMP

A PDMP (Piecewise Deterministic Markov Process) is a Markov process when its randomness is only given by a jump mechanism : between the jump times the trajectories are deterministic (see e.g. Davis (1984, 1993); Jacobsen (2006)) Let us collect here some facts and results on PDMP from the literature, mainly from Jacobsen (2006).
The state space is a measurable space $(G,\mathscr{G})$ (usually a Borel space or a Polish space). We are given three ingredients:

- a *rate function* $q : G \to \mathbb{R}_+$ measurable.
- a *probability transition kernel* on G, that is a measurable function $r : G \to \mathscr{P}(G)$ the set of probabilities on G endowed with the weak convergence topology. We shall write $r(x,C) = r_x(C)$ for $C \in \mathscr{G}$ and also write $r(x,dx')$ for the probability measure r_x. We assume $r(x,\{x\}) = 0$. Sometimes these two ingredients are joined by considering a *kernel* $\mu(x,C) := q(x)r(x,C)$, that is a measurable map $\mu : G \to \mathscr{M}_F(G)$ into the space of finite measures over G.
- A *continuous time dynamical system* on G, that is a map

$$\begin{aligned} \phi :\mathbb{R}^+ \times G \quad &\to G \\ (t,x) \quad &\to \phi_t(x) \end{aligned}$$

  such that $\phi_t \circ \phi_s = \phi_{t+s}$ and $\phi_0 = id$. We assume that for each x, $t \to \phi_t(x)$ is continuous.

Given $x_0 \in G$ we construct a two sequences $(T_n)n \geq 0$ and $(Y_n)_{n\geq 0}$ by specifying the conditional laws:

- $T_0 = 0$ and $Y_0 = x_0$.
- The law of T_{n+1} given $(T_k,Y_k) = (t_k,y_k), 0 \leq k \leq n$ is given by

$$\mathbb{P}(T_{n+1} > t + t_n \mid (T_k,Y_k) = (t_k,y_k), 0 \leq k \leq n) = \exp\left(-\int_0^t q(\phi_s(y_n))\,ds\right).$$

- The law of Y_{n+1} given $(T_k,Y_k) = (t_k,y_k), 0 \leq k \leq n$ and $T_{n+1} = t_{n+1}$ is given by

$$\mathbb{P}(Y_{n+1} \in C \mid (T_k,Y_k) = (t_k,y_k), 0 \leq k \leq n, T_{n+1} = t_{n+1}) = r(\phi_{t_{n+1}-t_n}(y_n), C).$$

We assume *stability* that is $T_n \to +\infty$ a.s. This is the case if the rate is bounded : for all $y \in G$, $q(y) \leq \bar{q} < +\infty$. Indeed, then we have stochastic domination $\tau_i \prec T_{i+1} - T_i$ where τ_i is IID exponential of parameter $\bar{q}$.
Eventually we let

$$X_t = \phi_{t-T_n}(Y_n) \quad \text{for} \quad T_n \leq t < T_{n+1}\,. \tag{A.1}$$

Then $(X_t)_{t\geq 0}$ is a strong homogeneous Markov process with respect to its natural filtration $\mathscr{F}_t^X$ (see (Jacobsen, 2006, Theorem 7.2.1)), called a PDMP of parameter (q, r, ϕ) starting from x_0. We let $\mathbb{P}_{x_0}$ denote the law of X.
Observe that if ϕ is the constant flow, then $(X_t)_{t\geq 0}$ is an homogeneous continuous time Markov chain on G with transition kernel $p(x, dx') = q(x)r(x, dx')$.
Observe also that if the rate function q is constant along the flow of the dynamic system, that is $q(\phi_t(x)) = q(x)$, then the construction of the sequence may be simplified as (if $\mathscr{E}(a)$ denotes the law of an exponential random variable of parameter a)

$$\mathscr{L}(T_{n+1} - T_n \mid T_1, Y_1, \ldots, T_n, Y_n) = \mathscr{E}(q(Y_n)) \tag{A.2}$$

$$\mathscr{L}(Y_{n+1} \mid T_1, Y_1, \ldots, T_n, Y_n, T_{n+1}) = r(\phi_{T_{n+1}-T_n}(Y_n), .) \tag{A.3}$$

We also have an Itô formula and the definition of an associated infinitesimal generator (see (Jacobsen, 2006, Theorem 7.6.1)).
A measurable function $h : G \to \mathbb{R}$ is *path-continuous* (resp. *path-differentiable*) is for all x the function

$$t \to h(\phi_t(x))$$

is continuous (resp. differentiable). If h is continuous it is of course path-continuous. If it is path-differentiable, we define

$$A_\phi h(y) := \lim_{s\to 0^+} \frac{1}{s}(h(\phi_s(y)) - h(y))\,,$$

and see that

$$\frac{d}{dt} h(\phi_t(y)) = A_\phi h(\phi_t(y))\,.$$

If $t \to \phi_t(y)$ is differentiable, h is C^1 and

$$a(y) := \lim_{s\to 0^+} \frac{1}{s}(\phi_s(y) - y)\,,$$

then $A_\phi h = \nabla h.a$.
We assume that for any bounded path-continuous function h the function

$$t \to q(\phi_t(y)) \int r(\phi_t(y), dz) h(z) \tag{A.4}$$

is continuous for any y.
The *full infinitesimal generator* for the PDMP is the linear operator L given by (A.5) acting on the domain $\mathscr{D}(L)$ of bounded measurable functions $h : G \to \mathbb{R}$ such that

h is path-differentiable and $A_\phi h$ is path-continuous, and the function $Lh : G \to \mathbb{R}$ given by

$$Lh(x) = A_\phi h(x) + q(x) \int r(x, dx')(h(x') - h(x)), \tag{A.5}$$

is bounded (this function is then path-continuous thanks to (A.4)).
We say that a function $f : [0, +\infty[\times G \to \mathbb{R}$ is *bounded on finite time intervals* if it is bounded on all sets $[0, t] \times G$.
If f is measurable, bounded on finite time intervals with $t \to f(t, x)$ continuously differentiable for all x and $x \to f(t, x)$ path-differentiable for all t, then the process $f(t, X_t)$ has the decomposition

$$f(t, X_t) = f(0, X_0) + M_t^f + U_t^f \tag{A.6}$$

where M_t^f is a local martingale reduced by the sequence $(T_n)_{n \geq 1}$, and U_t^f is the continuous predictable process

$$U_t^f = \int_0^t \mathscr{L} f(s, X_s)\, ds, \tag{A.7}$$

with

$$\mathscr{L} f(t, x) = \partial_t f(t, x) + L(f(t, .))(x). \tag{A.8}$$

If, in addition, the function $\mathscr{L} f$ is bounded, then M^f is a true martingale and

$$\mathbb{E}_{x_0}[f(t, X_t)] = f(0, x_0) + \int_0^t \mathbb{E}_{x_0}[\mathscr{L} f(s, X_s)]\, ds. \tag{A.9}$$

Furthermore, if f^2 satisfies the same assumptions, then the predictable quadratic variation of the local martingale M^f is

$$\left\langle M^t \right\rangle_t = \int_0^t (\mathscr{L}(f^2) - 2f \mathscr{L} f)(s, X_s)\, ds. \tag{A.10}$$

A straightforward computation yields that $A_\phi(f^2)(x) = 2f(x) A_\phi f(x)$ and therefore

$$(\mathscr{L}(f^2) - 2f \mathscr{L} f)(t, x) = q(x) \int r(x, dx') \big(f(t, x') - f(t, x)\big)^2. \tag{A.11}$$

We shall use the decomposition (A.6) when f is not bounded on finite intervals, but $f(t, X_t)$ is locally bounded and thus write $Lf(x)$ (resp. $\mathscr{L} f(t, x)$) even when f is not in the domain of the generator.

Appendix B: Additional Lemmas and Propositions

Lemma B.1 (Uniform mass control). *Let $(\nu_t)_{t\geq 0}$ be a (μ,φ) SSMP satisfying assumption 1. Assume that for some $p \geq 1$,*

$$\mathbb{E}[\langle \nu_0, 1\rangle^p] < +\infty .$$

Then for any $q \in \left[1, \frac{p+1}{2}\right]$, and any $T > 0$

$$\mathbb{E}\left[\sup_{t\leq T} \langle \nu_t, 1\rangle^q\right] < +\infty .$$

Proof. The proof of Proposition 3.2 yields that with $f(m) = \langle m, 1\rangle^q$, we have

$$f(\nu_t) \leq f(\nu_0) + C' \int_0^t (1 + f(\nu_s))\, ds + M_t^f .$$

Therefore, if $Y_t = \sup_{s\leq t} f(\nu_s)$, we have for any $t \leq T$

$$Y_t \leq Y_0 + \sup_{t\leq T} M_t^f + C't + C' \int_0^t Y_s\, ds .$$

We shall be able to apply Gronwall's Lemma, once we show that $\mathbb{E}\left[\sup_{t\leq T} M_t^f\right] < +\infty$. We shall use Cauchy-Schwarz inequality and the quadratic variation process. Indeed,

$$\begin{aligned}
Lf^2(m) - f(m)Lf(m) &= \sum_i \int m(dx)\alpha_i(m,x)(\langle m + k_i(m,x,.), 1\rangle^q - \langle m, 1\rangle^q)^2 \\
&= \sum_{i\in\bar{I}} \int m(dx)\alpha_i(m,x)(\langle m + k_i(m,x,.), 1\rangle^q - \langle m, 1\rangle^q)^2 \\
&\leq C(1 + \langle m, 1\rangle^{q-1})^2 \int \sum_{i\in\bar{I}} m(dx)\alpha_i(m,x) \\
&\leq C(1 + \langle m, 1\rangle)(1 + \langle m, 1\rangle^{q-1})^2 \\
&\leq C(1 + \langle m, 1\rangle^{2q-1}) \leq C'(1 + \langle m, 1\rangle^p) .
\end{aligned}$$

Therefore, by Doob's L^2 maximal inequality

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leq T} M_t^f\right]^2 &\leq C\mathbb{E}\left[\left\langle M^f \right\rangle_T\right] \\
&\leq C \int_0^T \mathbb{E}\left[(Lf^2 - 2fLf)(\nu_s)\right] ds \\
&\leq C \int_0^T (1 + \mathbb{E}[\langle \nu_s, 1\rangle^p])\, ds < +\infty ,
\end{aligned}$$

where in the last bound we used Proposition 3.2. □

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