Orbital profile and orbit algebra of oligomorphic permutation groups

Conjecture of Macpherson

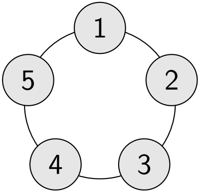
Justine Falque
joint work with Nicolas M. Thiéry

Laboratoire de Recherche en Informatique
Université Paris-Sud (Orsay)

February 22nd of 2018

Age and profile : example on a finite group (1)

Action of the cyclic group $G = C_5$ on the five pearl necklace
$\rightarrow$ induced action on subsets of pearls



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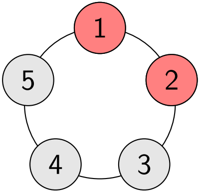

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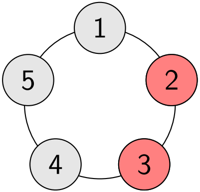

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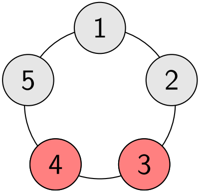

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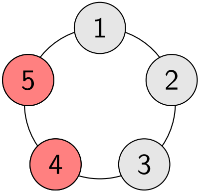

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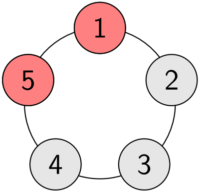

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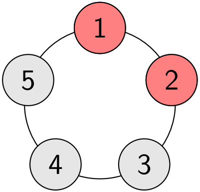

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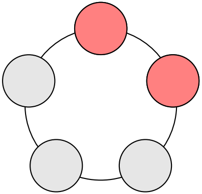

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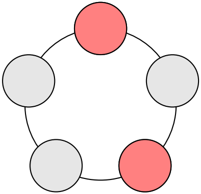

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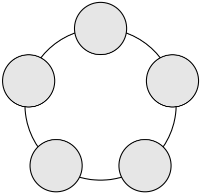

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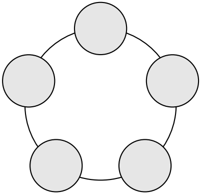

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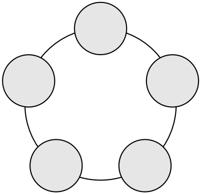

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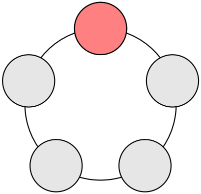

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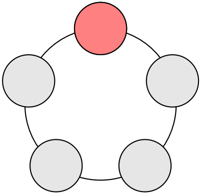

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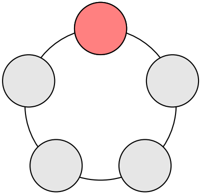

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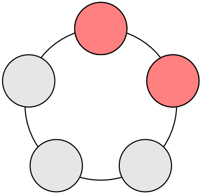

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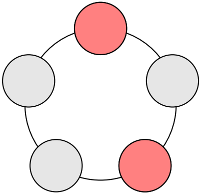

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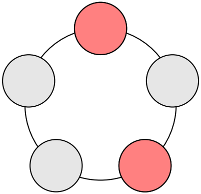

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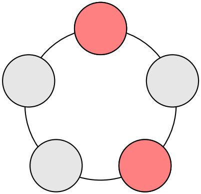

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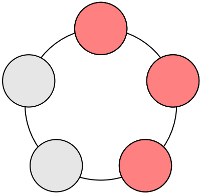

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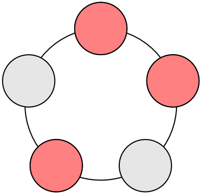

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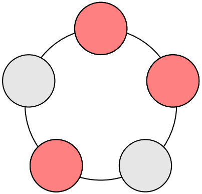

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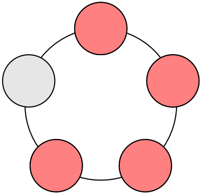

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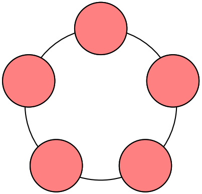

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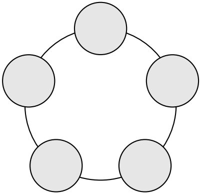

Age and profile : example on a finite group (2)

Generating polynomial of the profile :

$$\mathcal{H}_G(z) = \sum_{n\geq 0} \varphi_G(n) z^n = 1 + z + 2z^2 + 2z^3 + z^4 + z^5$$

Can be calculated using Pólya's theory

Age and profile of infinite permutation groups

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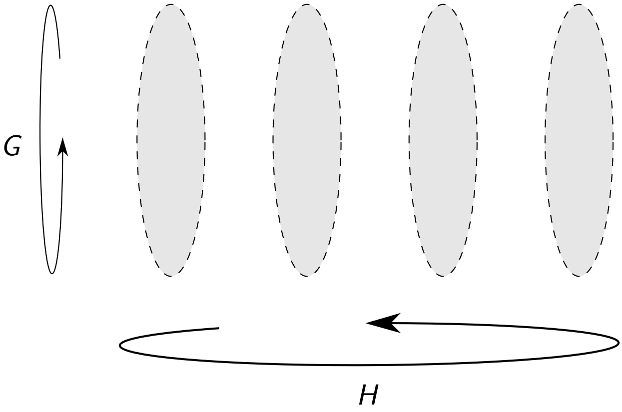
$\rightarrow$ **Oligomorphic groups**:

$$\varphi_G(n) < \infty \quad \forall n \in \mathbb{N}$$

Wreath product of two permutation groups

$G \leq \mathfrak{S}_M$, $H \leq \mathfrak{S}_N$

$G \wr H$ has a natural action on $E = \sqcup_{i=1}^{N} E_i$, with $\mathrm{card} E_i = M$.



Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)

  An orbit of degree $n \longleftrightarrow$ a partition of n

  $\varphi_G(n) = \mathscr{P}(n)$, the number of partitions of n

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- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$

  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into parts of size $\leq m$

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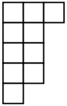

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_m$

  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into at most m parts

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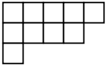

Conjecture of Cameron

Conjecture (Cameron, 70s)

If a profile is bounded by a polynomial it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

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Note

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})} \implies \varphi_G \text{ quasi-polynomial of degree at most } k-1$$

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \ldots, x_m]$ is a graded algebra.
A_n: homogeneous polynomials of degree n

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Proposition

A is finitely generated $\implies$ $\text{Hilbert}(A) = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

Example

$\text{Hilbert}(\mathbb{Q}[x, y, t^3]) = \frac{1}{(1-z)^2(1-z^3)}$

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$ such that

$$\mathcal{H}_G = \text{Hilbert}\left(\mathbb{Q}\mathcal{A}(G)\right)$$

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- Try to show that $\mathbb{Q}\mathcal{A}(G)$ is finitely generated
- Deduce:

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$$

and thus the quasi-polynomiality of $\varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

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Vector space structure

- finite formal linear combinations of orbits (ex: $2o_1 + 5o_2 - o_3$)
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Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$
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Product?

- Defined on subsets:

$$ef = \begin{cases} e \cup f & \text{if } e \cap f = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- $o = \{e_1, e_2, \ldots\} \quad \longleftrightarrow \quad e_1 + e_2 + \cdots$

Example of product on a finite case

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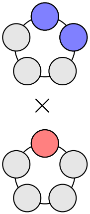

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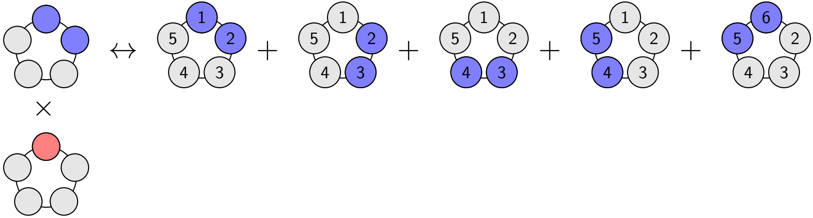

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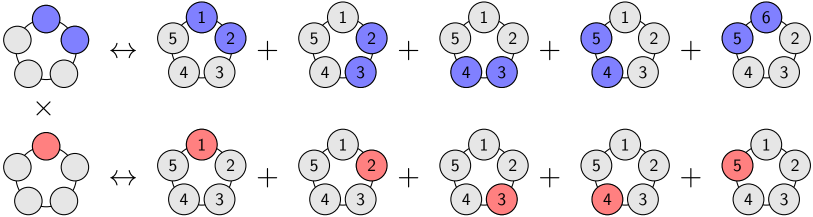

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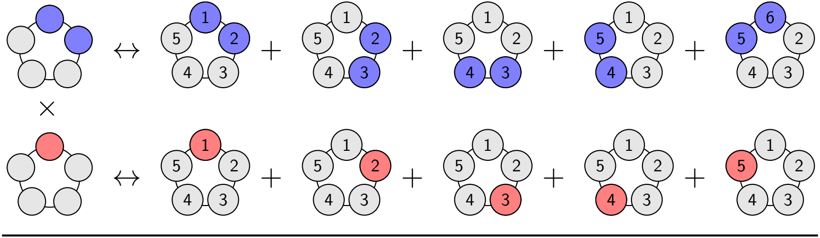

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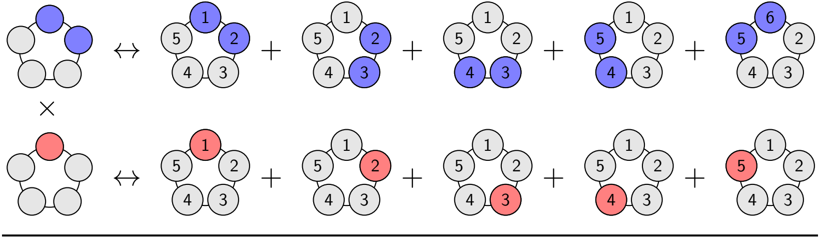

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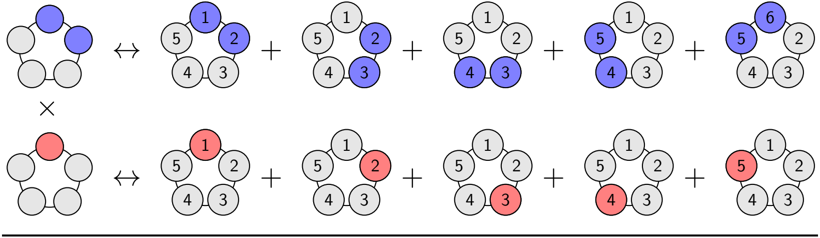

$$= \quad 0 \quad + \quad 0$$

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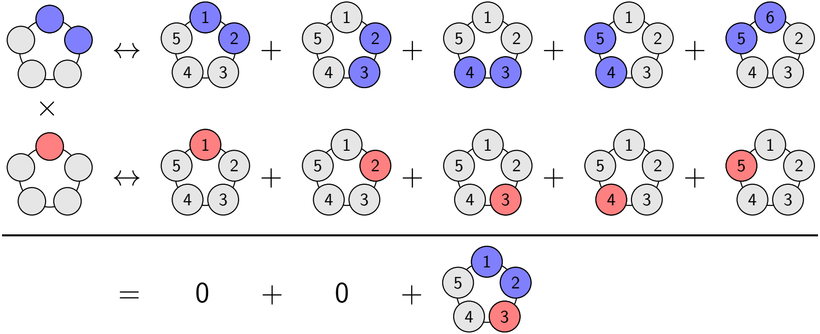

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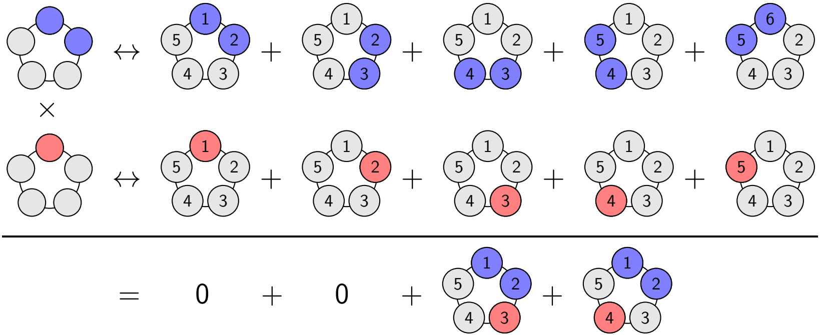

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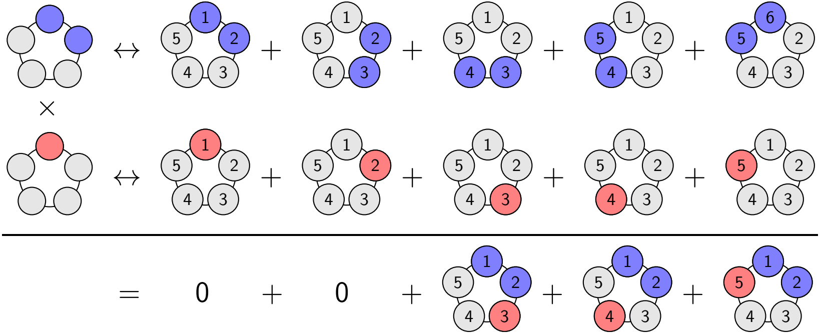

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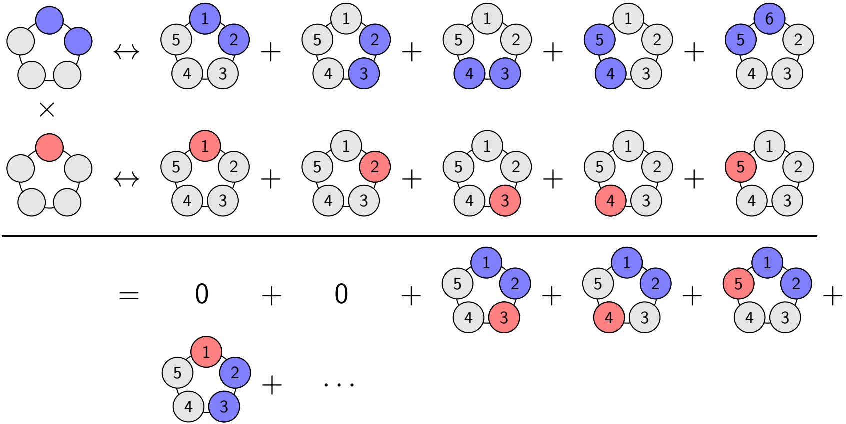

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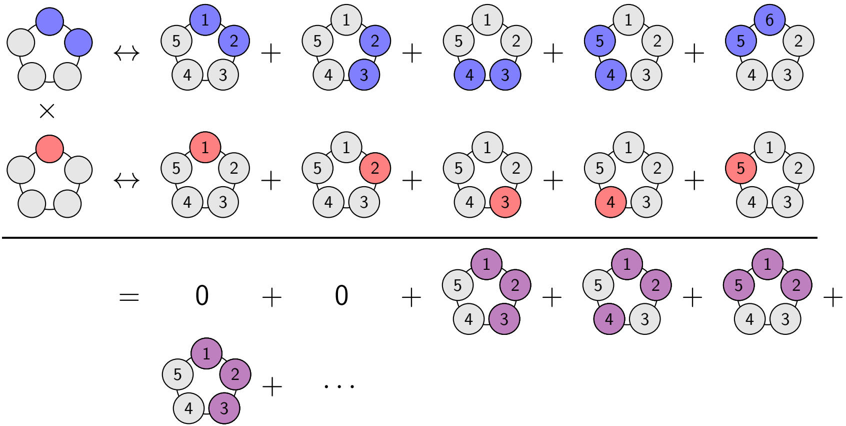

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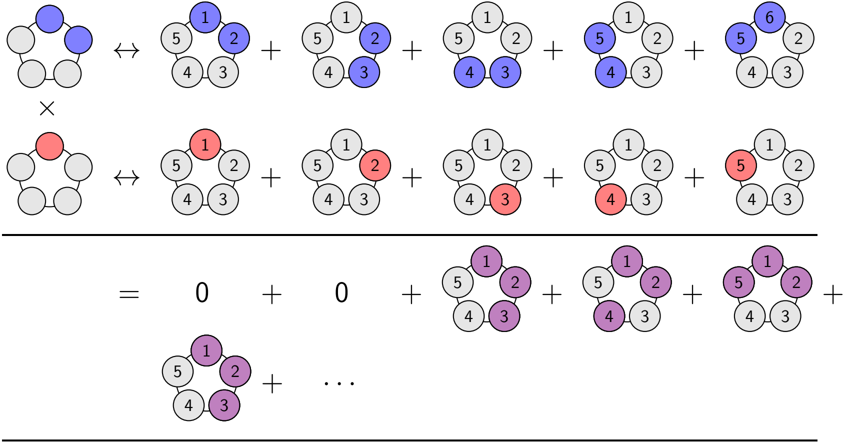

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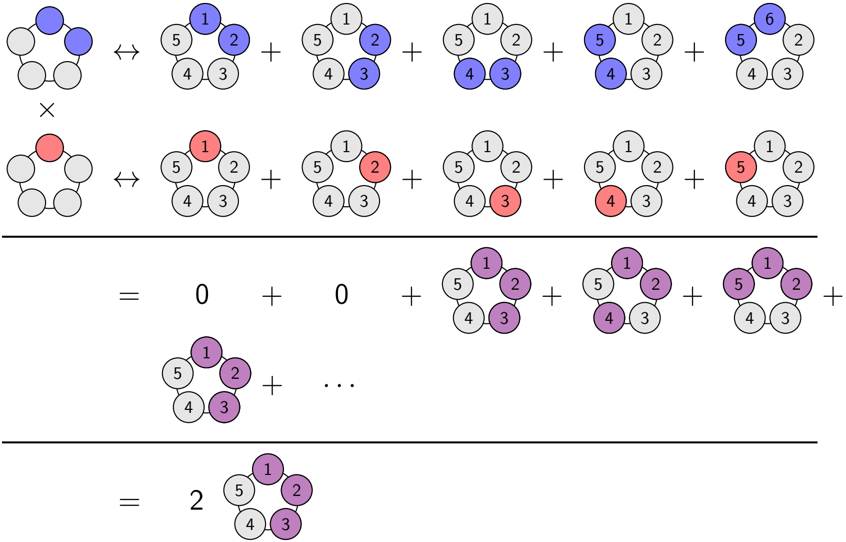

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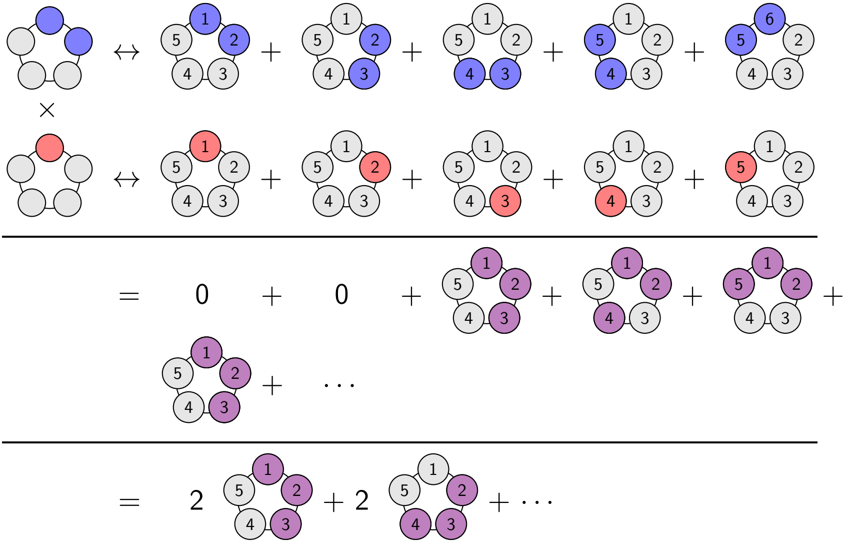

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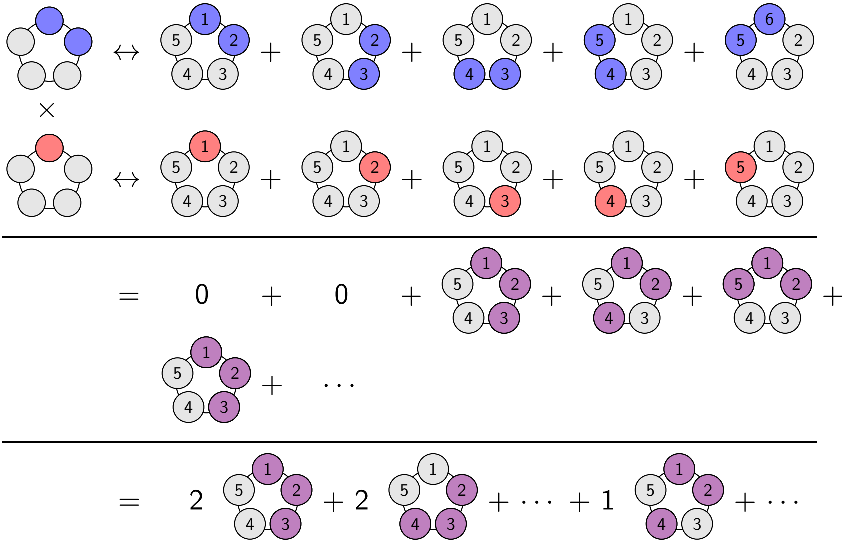
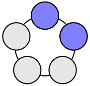
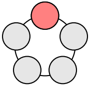
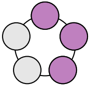
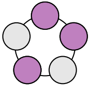
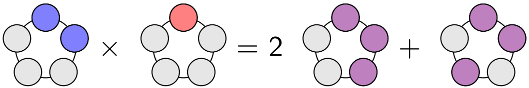


In the end:  $\times$  $= 2$  $+$ 

In the end: 

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Product well defined (and graded) on the space of orbits.

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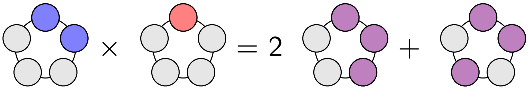

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$\longrightarrow$ **The orbit algebra of a permutation group**

Examples of orbit algebras (1)

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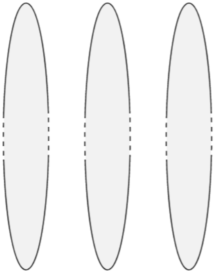

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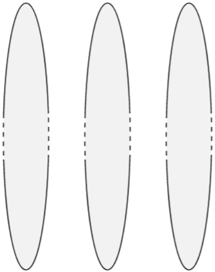

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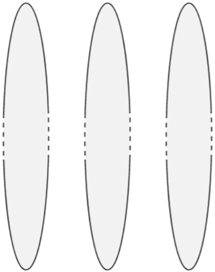

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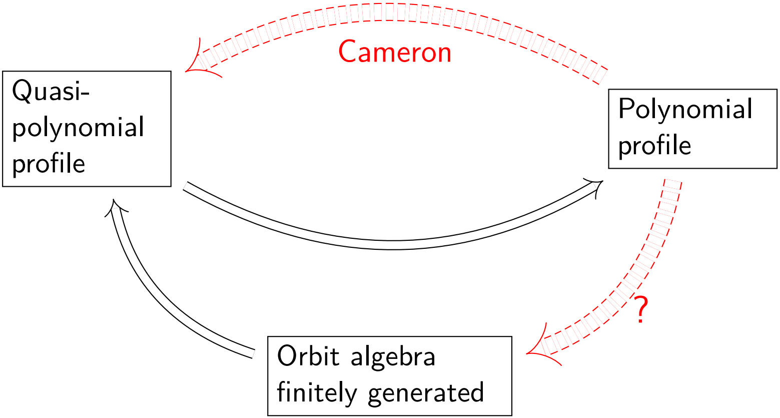
More generally, for H subgroup of $\mathfrak{S}_m$, $\mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr H) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

Overview and conjecture of Macpherson



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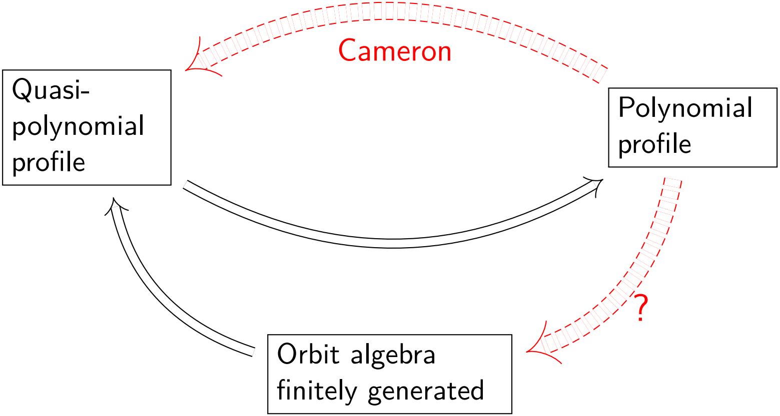
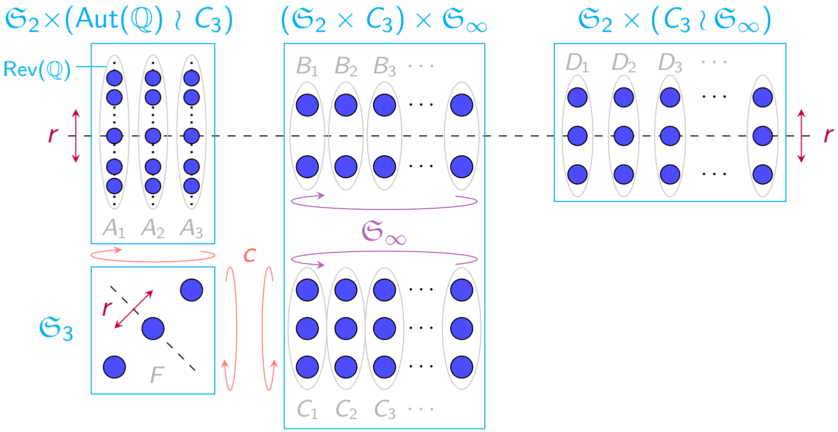


Conjecture (Macpherson, 1985)

Profile of G polynomial $\Longleftrightarrow$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated


$\mathfrak{S}_2 \times (\mathrm{Aut}(\mathbb{Q}) \wr C_3)$
$(\mathfrak{S}_2 \times C_3) \times \mathfrak{S}_\infty$
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$\mathrm{Rev}(\mathbb{Q})$
r
A_1 A_2 A_3
B_1 B_2 B_3 $\cdots$
D_1 D_2 D_3 $\cdots$
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Theorem

Let H be a finite index subgroup of G.

- The profiles of G and H are asymptotically equivalent
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Application : reduction of Macpherson's conjecture

Without loss of generality, we may assume for instance that G has no finite orbit.

But there will be more...

Block systems

Definition : Block system

Partition of E such that each part is globally mapped onto another one (or itself) by every element of G

(see previous examples)

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→ The groups we are interested in have a constantly equal to 1 profile or have a block system.

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Theorem (Classification, Cameron)

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Well known, nice groups.
In particular, their orbit algebra is finitely generated.

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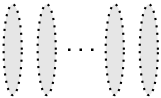

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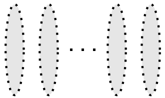

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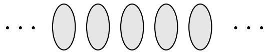

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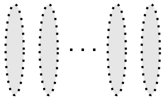

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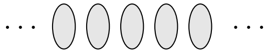

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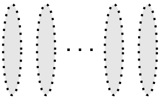

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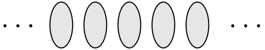

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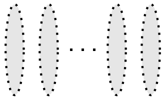

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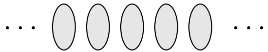

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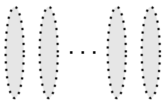

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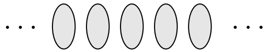

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Actually, G acts on the blocks as $\mathfrak{S}_\infty$

Synchronization

Case of 2 infinite orbits

$E_1 \sqcup E_2$, $\quad G_{|E_1} = G_1,\ G_{|E_2} = G_2$

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Example

If $G_1 = G_2 = \mathfrak{S}_\infty$, the actions are either independant or totally synchronized. One may assume safely, for our purposes, the same about the other four groups.

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Convenient fact if $E = E_1 \sqcup E_2$

G_1 and G_2 not synchronized $\Longrightarrow \mathbb{Q}\mathcal{A}(G) = \mathbb{Q}\mathcal{A}(G_1) \otimes \mathbb{Q}\mathcal{A}(G_2)$

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Only essential remaining case : infinite number of finite blocks

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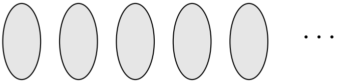
- General case ?

The "hard case" : transitive block system of finite blocks



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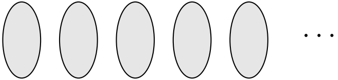

Definition : Tower of G

H_0 H_1 H_2 $\ldots$ where H_i is the restriction to the block $i+1$ of the subgroup of G that stabilizes all the blocks and acts trivially on the first i blocks.

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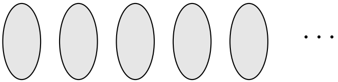

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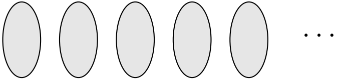

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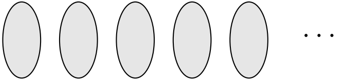

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The tower of G must be of shape : $H_0 \ H \ H \ H \ \ldots$
Thus, G has the same orbit algebra as $\frac{H_0}{H} \times H \wr \mathfrak{S}_\infty$,
which is of finite index over $H \wr \mathfrak{S}_\infty$.

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
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- Solves the issue of possible finite synchronizations between different orbits of blocks

Recap : proof of the conjecture of Macpherson

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$\longrightarrow$ *The conjectures of Macpherson and Cameron hold !*

Stronger result : Cohen-Macauley algebra

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- Once again, it is possible to adapt a proof of invariant theory to obtain that the orbit algebra is indeed a Cohen-Macauley algebra

Thank you for your attention !

Context

- G permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- **Hypothesis** : $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron) : quasi-polynomiality of φ_G
- Conjecture (Macpherson) : finite generation of the orbit algebra

Results

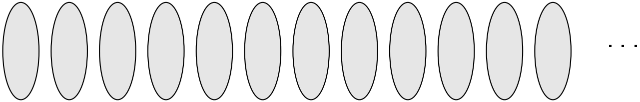
- Both conjectures hold
- The orbit algebra is a Cohen-Macauley algebra

Question

- On what algebra ?

Direct product in the case of finite blocks

"Speak, friend..."



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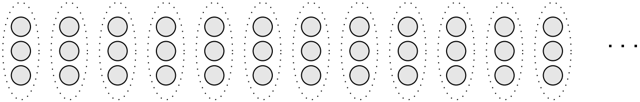
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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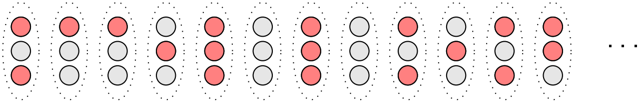

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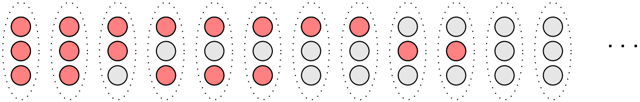

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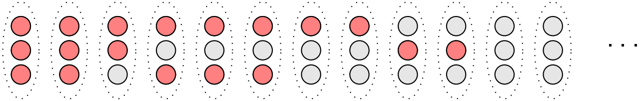

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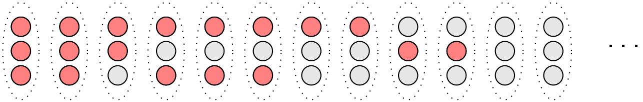

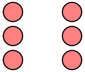

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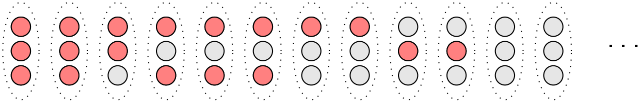

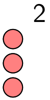

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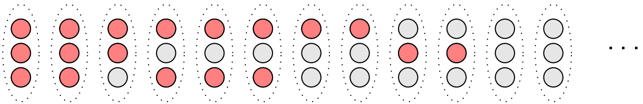

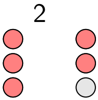

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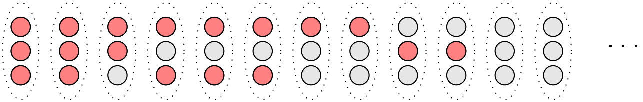

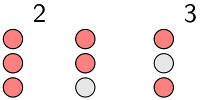

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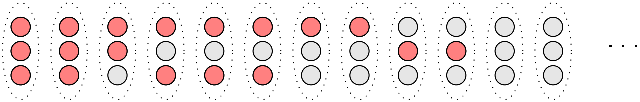

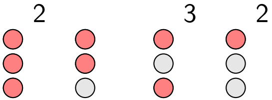

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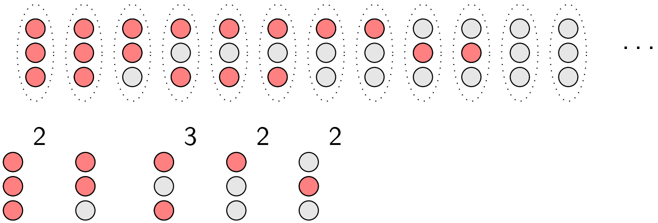

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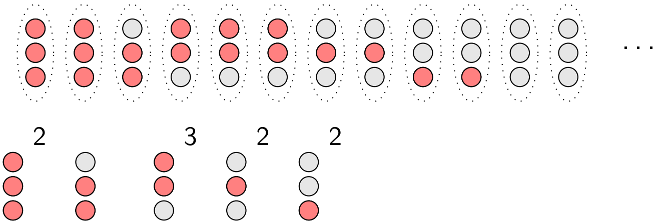

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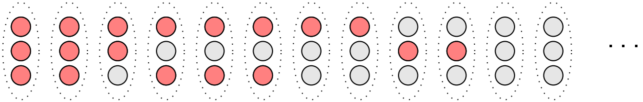

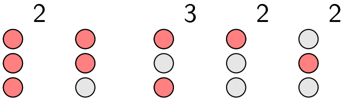

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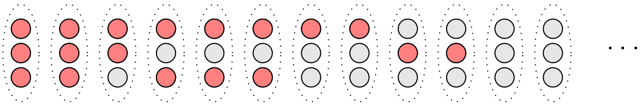

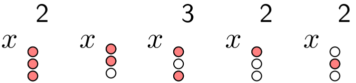

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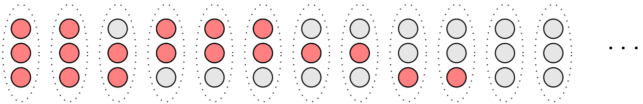

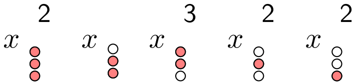

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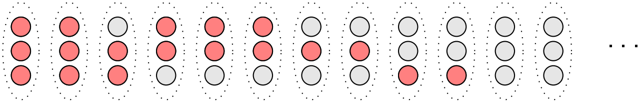

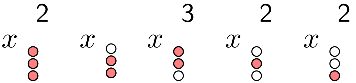

$\rightarrow$ C_3 acts on monomials

Direct product in the case of finite blocks

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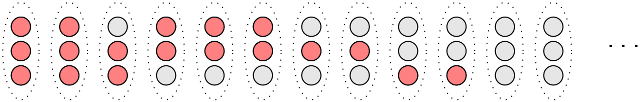

$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

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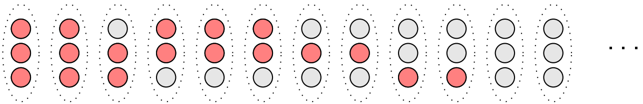

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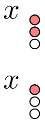

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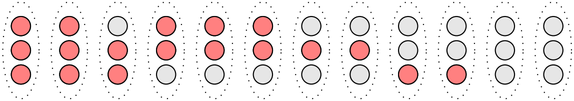

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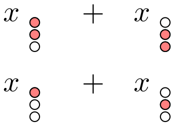

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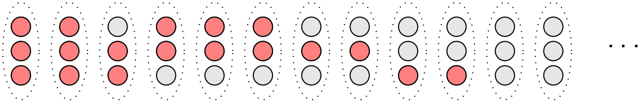

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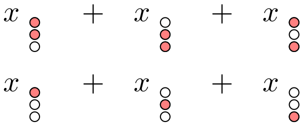

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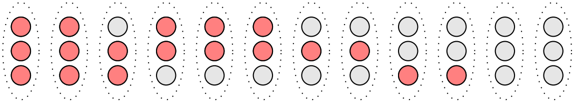

. . .

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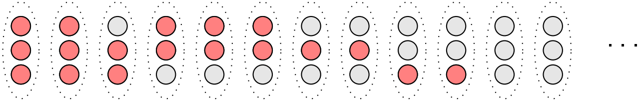

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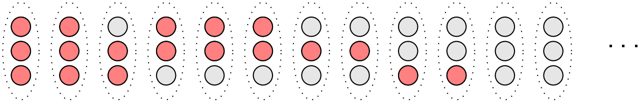

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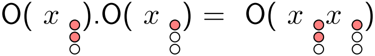

Direct product in the case of finite blocks

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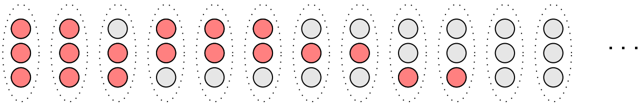
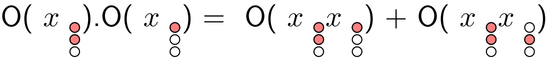
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?



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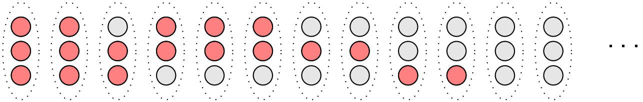

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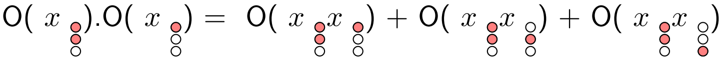

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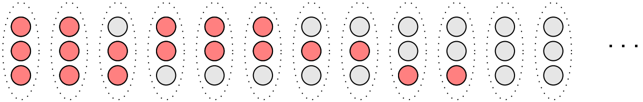

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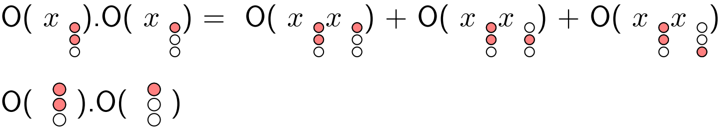

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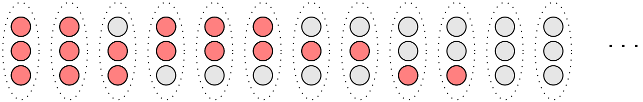

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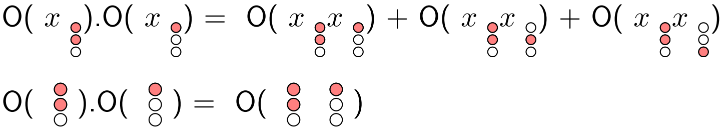

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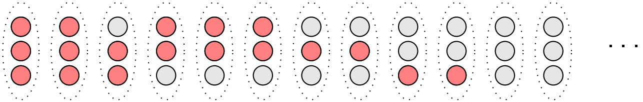

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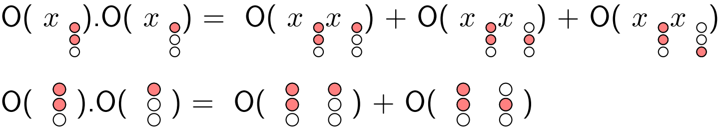

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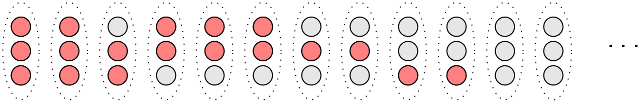

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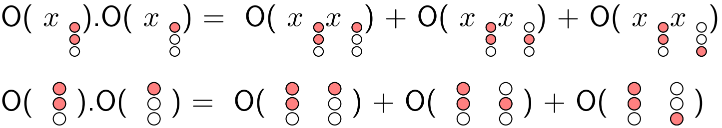

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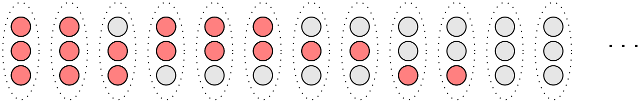

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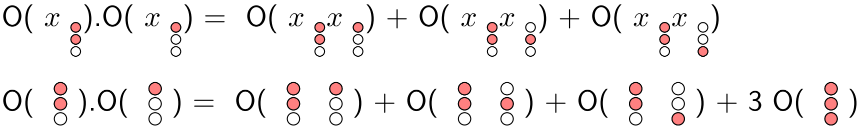
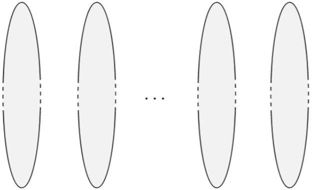


Examples of orbit algebras (2)

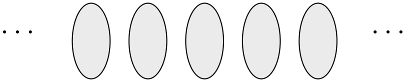
More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.



- $G = H \wr \mathfrak{S}_\infty$:
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



The "hard" case : case of four blocks

Lemma to prove

G has tower H_0 H_1 H_2 $H_3 \Rightarrow H_1 = H_2$

Lemma

In the general case :
$Fix_G(B_1, \ldots, B_n)$ acts on the remaining blocks as $\mathfrak{S}_\infty$
(due to the absence of normal subgroup of finite index of $\mathfrak{S}_\infty$).

Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists\ (1, g, h, k), \quad h, k \in H_1$.
Let σ be an element of G that permutes the first two blocks and fixes the other two.
Conjugation of x by σ in $G \quad \rightarrow \quad y = (g', 1, h, k)$
Then: $x^{-1}y = (g', g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, g^{-1} and therefore g are in H_2.