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Irene Votsi

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Conditional failure occurrence rates for semi-Markov chains

Irene Votsi and Mohamed Hamdaoui

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Abstract

In the present paper, we aim at providing plug-in-type empirical estimators based on empirical processes to quantify the contribution of each operational or/and non-functioning state to the failures of a system described semi-Markov model. In particular we focus on an important reliability measure for random repairable systems, the rate of occurrence of failures and study different conditional versions in a discrete time and finite state space context. The aforementioned estimators of the conditional failure occurrence rates are caracterised by appealing asymptotic properties such as consistency and asymptotic normality. We further provide detailed analytical expressions for the covariance matrices of the random vectors describing the conditional failure occurrence rates. These results are illustrated on an academical numerical example based on simulated data. We further present an application to a real dataset that models earthquake occurrences in a semi-Markov framework.

1 Introduction

Semi-Markov models (SMMs) are state-of-the-art models that are widely used in many scientific fields such as reliability and DNA analysis (Barbu and Limnios (2008)), seismology (Votsi et al. (2012)) etc. One of the main distinguising features of SMMs is that contrary to Markov models, they enable us to describe systems that evolve based not only on their last visited state (Markov property) but also on the time elapsed since this state. Due to this feature, popular "memory-full" distributions, such as the Weibull distribution, could be employed to describe sojourn (or interevent) times between successive events. We refer the interested reader to Howard (1971) and

Mode and Sleeman (2000) for an introduction to homogeneous SMMs and to Vassiliou and Papadopoulou (1992, 1994) for non-homogeneous SMMs, respectively.

In the semi-Markov context, many reliability indicators have been introduced, including mean times to failure, hazard rates, availability functions etc. For recent advances in the topic concerning discrete time SMMs, see Barbu and Limnios (2008), Barbu et al. (2017) and Georgiadis et al. (2013); Georgiadis (2017). For continuous-time SMMs, we address the interested reader to Limnios and Oprişan (2001) and Limnios and Ouhbi (2006). For advances in estimation methods of nonparametric semi-Markov models, see Trevezas and Limnios (2011) and the references therein.

Here we focus on a fundamental reliability indicator, the rate of occurrence of failures (ROCOF). ROCOF represents the rate of failure occurrences for repairable, random systems subject to multiple failures. In the continuous time context, it represents a rate, whereas in the discrete time context it is rather a probability. Yeh (1997) was the first to investigate ROCOF for continuous-time Markov models defined in a finite state space. More recently, D'Amico (2015) investigated ROCOF for higher-order Markov processes with an application to financial credit ratings. Concerning continuous time SMMs in a finite or general state space, ROCOF was studied by Ouhbi and Limnios (2002) and Limnios (2012), respectively. The discrete time counterpart of ROCOF, was evaluated by Votsi et al. (2014) for SMMs and by Votsi and Limnios (2015) for hidden Markov renewal models, under the term "discrete time intensity hitting time" (DTIHT).

However, ROCOF is a "global" reliability indicator in the sense that it does not distinguish neither on the current up state nor on the ending down state. Here we introduce conditional versions of the ROCOF allowing us to quantify the impact of the current up state, the ending down state or both on the ROCOF. We further present empirical estimators of the conditional ROCOFs and study their asymptotic properties.

The organization of the paper is as follows. In Section 2 the notation and preliminaries of SMMs are presented. Section 3 describes the definition, evaluation and statistical estimation of the conditional rate of failure occurrences. Section 4 discusses numerical examples based on simulated and real data and finally, in Section 5, we give some concluding remarks.

2 Notation and Preliminaries in Semi-Markov Models

We briefly recall the main definitions from the theory of discrete-time SMMs (see, e.g., Barbu and Limnios (2008)). We consider a random system with finite state space $E = \{1, 2, ..., s\}$ described by an SMM. The stochastic evolution of the system is described by the following random sequences defined on a complete probability space:

1. The sequence $\mathbf{J} = (J_n)_{n\in\mathbb{N}}$ with state space E, where J_n is the state visited by the system at the $n-$th jump time, which forms a (time) homogeneous Markov chain, the embedded Markov chain (EMC);

2. The $\mathbb{N}$-valued sequence $\mathbf{S} = (S_n)_{n\in\mathbb{N}}$, where S_n is the $n-$th jump time. We suppose that $S_0 = 0$ and $0 < S_1 < S_2 < \ldots < S_n < S_{n+1} < \ldots$ almost surely (a.s.);

3. The $\mathbb{N}$-valued sequence $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ defined by $X_0 = 0$ a.s. and $X_n = S_n - S_{n-1}$ for all $n \in \mathbb{N}$. Thus for all $n \in \mathbb{N}$, X_n is the sojourn time in state J_{n-1}, before the $n-$th jump.

The stochastic process $(\mathbf{J}, \mathbf{S})$, called Markov renewal chain (MRC), is considered to be (time) homogeneous and is characterized by the semi-Markov kernel (SMK) $\mathbf{q} = (q_{ij}(k);\ i, j \in E, k \in \mathbb{N})$ defined by

$$q_{ij}(k) = P(J_{n+1} = j, X_{n+1} = k | J_n = i),$$

where $i, j \in E$ and $k, n \in \mathbb{N}$. The semi-Markov chain (SMC) $\mathbf{Z} = (Z_k)_{k\in\mathbb{N}}$ associated with the MRC $(\mathbf{J}, \mathbf{S})$ describes the system's state at each time $k \in \mathbb{N}$ and is defined by $Z_k = J_{N(k)}$, where $N(k) = \max\{n \geq 0 : S_n \leq k\}$. The transition kernel of the EMC is defined by $\mathbf{P} = (p_{ij};\ i, j \in E)$, where $p_{ij} = P(J_{n+1} = j | J_n = i)$, $i, j \in E$, $n \in \mathbb{N}$, and its initial distribution by $\boldsymbol{\alpha} = (\alpha(i);\ i \in E)$, where $\alpha(i) = P(J_0 = i)$. At the initial time, $t = 0$, the state of the SMC coincides with the state of the EMC, i.e. $Z_0 = J_0$. We assume that there are neither instantaneous transitions ($q_{ij}(0) = 0,\ i, j \in E$) nor self-transitions for the EMC ($p_{ii} = 0,\ i \in E$). The conditional distribution of the sojourn time in state $i \in E$ given that the next visited state is $j \in E$ is given by $f_{ij}(k) := P(X_{n+1} = k | J_n = i, J_{n+1} = j)$, $k \in \mathbb{N}$. Moreover we define the survival function of the sojourn time distribution in state $i \in E$ by $\overline{H}_i(\ell) = P(X_{n+1} > \ell | J_n = i)$, $\ell \in \mathbb{N}$, and by $m_i = \mathbb{E}(S_1 | Z_0 = i)$ the mean sojourn time of the SMC in state $i \in E$. The next assumptions have to be fulfilled in the following:

A1 The SMC is irreducible and aperiodic;

A2 The mean sojourn times are finite, i.e., $m_i < \infty$, for any $i \in E$.

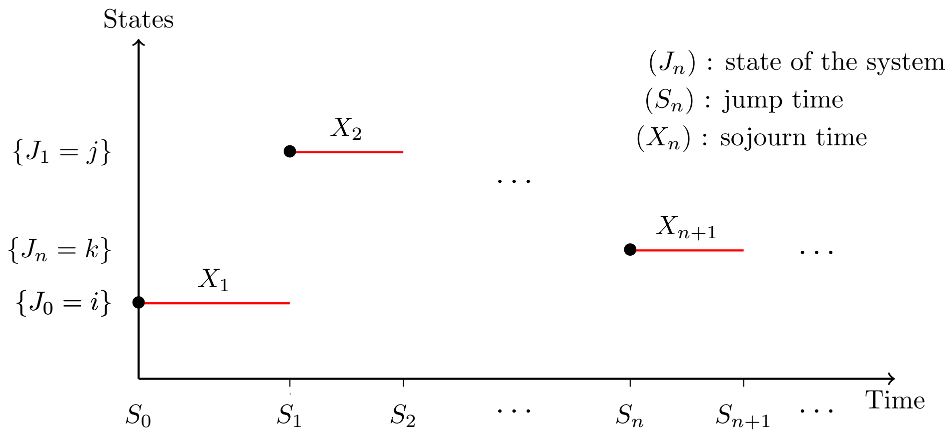

Figure 1: A typical trajectory of a Markov renewal chain.

3 Conditional Hitting Time Intensities

3.1 General definitions

The rate of occurrence of failures is one of the most significant indicators in reliability theory, in particular in the study of repairable systems. We first consider a subset of the state space E, $U \subset E$ ($U \neq \emptyset$, $U \neq E$), with up states $U = \{1, 2, ..., r\}$ and down states $D = \{r+1, ..., s\}$. Let $\mathbf{U} = (U_k)_{k\in\mathbb{N}}$ be the sequence of the backward recurrence times of $\mathbf{Z}$, that is, $U_k := k - S_{N(k)}$. Following Chryssaphinou et al. (2008), the stochastic process $(\mathbf{Z}, \mathbf{U}) = (Z_k, U_k)_{k\in\mathbb{N}}$ is a (time) homogeneous Markov chain governed by its initial distribution, $\widetilde{\boldsymbol{\alpha}} := (\widetilde{\alpha}(i, 0);\ i \in E)$, and its transition matrix $\widetilde{\mathbf{P}} := (\widetilde{P}\big((i, t_1), (j, t_2)\big);\ (i, t_1), (j, t_2) \in E \times \mathbb{N})$. In what follows, the matrix Λ, of dimension $\big(s(M+1)\big)^2 \times \big(s(M+1)\big)^2$, is defined by

$$\Lambda = \begin{pmatrix} \Lambda(1,0) & \underline{0} & \dots & \dots & \underline{0} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \Lambda(i,t_1) & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \dots & \dots & \dots & \Lambda(s,M) \end{pmatrix},$$

where the block $\Lambda(i,t_1)$ is the square matrix

$$\Lambda(i,t_1)=\Big(\delta_{(j,t_2)(r,t_3)}\widetilde{P}\big((i,t_1),(j,t_2)\big)-\widetilde{P}\big((i,t_1),(j,t_2)\big)\widetilde{P}\big((i,t_1),(r,t_3)\big)\Big),$$

and

$$\delta_{(j,t_2)(r,t_3)}=\begin{cases}1, & \text{if} \quad (j,t_2)=(r,t_3),\\ 0, & \text{elsewhere,}\end{cases}$$

for every $(i,t_1),(j,t_2),(r,t_3)\in E\times T_M$. Moreover, we define by Γ the matrix:

$$\Gamma=\begin{pmatrix}\frac{1}{\widetilde{\pi}(1,0)}\Lambda(1,0) & \underline{0} & \cdots & \cdots & \underline{0}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \frac{1}{\widetilde{\pi}(i,t_1)}\Lambda(i,t_1) & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \underline{0} & \cdots & \cdots & \cdots & \frac{1}{\widetilde{\pi}(s,M)}\Lambda(s,M)\end{pmatrix},$$

where $\widetilde{\boldsymbol{\pi}} := (\widetilde{\pi}(i,t_1);(i,t_1)\in E\times T_M)$, is the stationary distribution of $(\mathbf{Z},\mathbf{U})$. Consider now a sample path $\mathcal{H}(M)$ of an MRC with a fixed length M and denote by $T_M=\{0,1,\dots,M\}$. For all $i,j\in E$ and $1\leq k\leq M$, we further define the counting processes:

1. $N_i(M)=\sum_{n=1}^{N(M)-1}\mathbf{1}_{\{J_n=i\}}$ is the number of visits to state i of the EMC, before time M;
2. $N_{ij}(k,M)=\sum_{n=1}^{N(M)}\mathbf{1}_{\{J_{n-1}=i,J_n=j,X_n=k\}}$ is the number of transitions from i to j, up to time M, with sojourn time in state i equal to k.

Then, following Barbu and Limnios (2008), the empirical estimators of the semi-Markov kernel, q_{ij}, and the survival function, $H_i(k)$, are defined by $\widehat{q}_{ij}(k,M)=\frac{N_{ij}(k,M)}{N_i(M)}$, and $\widehat{\overline{H}}_i(l,M)=1-\sum_{j\in E}\sum_{k=0}^{l}\widehat{q}_{ij}(k,M)$, respectively. Additionally, the transition probabilities of the Markov chain $(\mathbf{Z},\mathbf{U})$ are estimated by (Chryssaphinou et al. (2008)):

$$\widehat{\widetilde{P}}_M\big((i,t_1),(j,t_2)\big)=\begin{cases}\widehat{q}_{ij}(t_1+1,M)/\widehat{\overline{H}}_i(t_1,M), & \text{if} \quad t_2=0,\\ \widehat{\overline{H}}_i(t_1+1,M)/\widehat{\overline{H}}_i(t_1,M), & \text{if} \quad i=j,t_2-t_1=1,\\ 0, & \text{elsewhere.}\end{cases}$$

3.2 Full conditional ROCOF

In the sequel we are interested in studying the impact of both the starting up and the ending down states on the ROCOF, simultaneously. In order to do that, we define the full conditional ROCOF as the expected number of transitions of the SMC to the set D at time k, given that it starts in state $i \in U$ and it ends in state $j \in D$, i.e.,

$$r^{\sharp}_{ij}(k) = \mathbb{E}[N^{\sharp}_{ij}(k) - N^{\sharp}_{ij}(k-1)],$$

where

$$N^{\sharp}_{ij}(k) = \textstyle\sum_{l=1}^{k} \mathbf{1}_{\{Z_{l-1}=i, Z_l=j\}}.$$

Theorem 1. *The full conditional ROCOF of the SMC at time k is given by*

$$r^{\sharp}_{ij}(k) \quad = \quad \sum_{m=0}^{k-1} [(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big), \quad k \in \mathbb{N}^*,$$

for any fixed arbitrary states $i \in U$, $j \in D$.

Proof. For fixed $i \in U$, $j \in D$ and $k \in \mathbb{N}^*$,

$$\begin{aligned} r^{\sharp}_{ij}(k) &= \mathbb{E}[N^{\sharp}_{ij}(k) - N^{\sharp}_{ij}(k-1)] \\ &= \mathbb{E}\mathbf{1}_{\{Z_{k-1}=i, Z_k=j\}} = P(Z_{k-1}=i, Z_k=j) \\ &= P(Z_{k-1}=i, Z_k=j). \end{aligned} \tag{1}$$

Moreover, following Votsi et al. (2014), we have that

$$P(Z_{k-1}=i, Z_k=j) = \sum_{m=0}^{k-1} [(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big), \tag{2}$$

for $i \neq j$. Therefore, from (1) and (2), we get the desired result. $\square$

For a given $i \in U$, $j \in D$, we estimate the full conditional ROCOF by means of the plug-in type estimator:

$$\widehat{r^{\sharp}_{ij}}(k,M) = \sum_{m=0}^{k-1} [(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)]\widehat{\widetilde{P}}_M\big((i,m),(j,0)\big).$$

Proposition 1. *For any state $i \in U$, $j \in D$ and any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the conditional intensity of the hitting time at time k is strongly consistent in the sense that*

$$\widehat{r^{\sharp}_{ij}}(k,M) \xrightarrow[M\to\infty]{a.s.} r^{\sharp}_{ij}(k).$$

Proof. Following Sadek and Limnios (2002) for the Markov chain $(\mathbf{Z}, \mathbf{U})$, we have $\widehat{\widetilde{P}}_M \xrightarrow[M\to\infty]{a.s.} \widetilde{P}$ and since we deal with finite sums, we can write directly:

$$\begin{aligned}
\widehat{r^\sharp_{ij}}(k, M) &= \sum_{m=0}^{k-1} [(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)]\widehat{\widetilde{P}}_M\big((i,m),(j,0)\big) \\
&\xrightarrow[M\to\infty]{a.s.} \sum_{m=0}^{k-1} [(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big) \\
&= r^\sharp_{ij}(k).
\end{aligned}$$

□

Proposition 2. *For any fixed $i \in U$, $j \in D$, $k \geq 1$, $\sqrt{M}\big(\widehat{r^\sharp_{ij}}(k, M) - r^\sharp_{ij}(k)\big)$ converges in distribution, as M tends to infinity, to a zero mean normal random variable with variance $\Phi'_{ij}\Gamma\Phi'^{\top}_{ij}$. Moreover $\Phi_{ij} : [0,1]^d \to \mathbb{R}^+$ ($d = (s(M+1))^2$) is the function*

$$\begin{aligned}
&\Phi_{ij}\Big(\widetilde{P}\big((i',m'),(j',t')\big);\ (i',m'),(j',t') \in E \times T_M\Big) \\
&= \sum_{m=0}^{k-1} \Big((\widetilde{a}\widetilde{P}^{k-1})(i,m)\Big)\widetilde{P}\big((i,m),(j,0)\big) \\
&= \sum_{m=0}^{k-1} \Big(\sum_{s\in E} \widetilde{a}(s,0)\widetilde{P}^{k-1}\big((s,0),(i,m)\big)\Big)\widetilde{P}\big((i,m),(j,0)\big)
\end{aligned}$$

and

$$\Phi'_{ij} = \Big(\frac{\partial \Phi_{ij}}{\partial \widetilde{P}\big((i',m'),(j',t')\big)};\ (i',m'),(j',t') \in E \times T_M\Big),$$

is the $d-$dimensional row vector of first derivatives of $r^\sharp_{ij}(k)$ with respect to $\widetilde{P}\big((i',m'),(j',t')\big)$.

Proof. Following the definition of the conditional intensity of the hitting time, we have

$$\begin{aligned}
&\sqrt{M}(\widehat{r^\sharp_{ij}}(k, M) - r^\sharp_{ij}(k)) \\
&= \sqrt{M} \sum_{m=0}^{k-1} \Big([(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)]\widehat{\widetilde{P}}_M\big((i,m),(j,0)\big) \\
&\quad - [(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big)\Big) \\
&= \sqrt{M}\Big(\Phi\big(\widehat{\widetilde{P}}_M\big((i',m'),(j',t')\big)\big) - \Phi\big(\widetilde{P}\big((i',m'),(j',t')\big)\big)\Big),
\end{aligned}$$

where Φ is a polynomial function and consequently differentiable. The vector function

$$\sqrt{M}\Big(\big(\widehat{\widetilde{P}}_M\big((i',m'),(j',t')\big)\big) - \big(\widetilde{P}\big((i',m'),(j',t')\big)\big)\Big)$$

converges, as M tends to infinity, to the normal distribution centred at the origin with covariance matrix Γ. Then by means of the delta method (see, e.g., van Der Vaart 1998), we obtain that

$$\sqrt{M}\Big(\Phi\big(\widehat{\widetilde{P}}_M\big((i',m'),(j',t')\big)\big) - \Phi\big(\widetilde{P}\big((i',m'),(j',t')\big)\big)\Big) \xrightarrow[M\to\infty]{\mathcal{L}} \mathcal{N}(0,\Phi'\Gamma\Phi'^{\top}).$$

$\square$

We now prove the asymptotic normality of the random vector $(\widehat{r^{\sharp}_{ij}})_{i,j\in E}$ with an explicit formula for the covariance matrix.

Lemma 1. *For $n \geq 2$, the random vector $F^n = \big(f^n_{(i',m'),(j',t')}\big)_{(i',m'),(j',t')\in E\times T_M}$ where*

$$f^n_{(i',m'),(j',t')} = \sqrt{M}\,\Big(\widehat{\widetilde{P}}^{\,n}\big((i',m'),(j',t')\big) - \widetilde{P}^n\big((i',m'),(j',t')\big)\Big)$$

has the same limit in distribution as the random vector

$$G^n = \big(g^n_{(i',m'),(j',t')}\big)_{(i',m'),(j',t')\in E\times T_M}$$

where

$$\begin{aligned} g^n_{(i',m'),(j',t')} = & \sum_{(j,t)\in E\times T_M} \Big(\widetilde{P}^{n-1}\big((i',m'),(j,t)\big) f_{(j,t),(j',t')} + \widetilde{P}^{n-1}\big((j,t),(j',t')\big) f_{(i',m'),(j,t)}\Big) \\ & + \mathbf{1}_{\{n\geq 3\}} \sum_{k=2}^{n-1} \sum_{(i_1,t_1)\in E\times T_M} \sum_{(i_2,t_2)\in E\times T_M} \widetilde{P}^{n-k}\big((i',m'),(i_1,t_1)\big)\, \widetilde{P}^{k-1}\big((i_2,t_2),(j',t')\big) f_{(i_1,t_1),(i} \end{aligned}$$

Moreover, F^n converges, as M tends to infinity, to a centered normal random vector with covariance matrix $\Sigma = \Sigma_f\,\Gamma\,\Sigma_f^T$. The matrix

$$\Sigma_f = \Sigma_f((i',m'),(j',t'),(u',v'),(s',t'))_{(i',m'),(j',t'),(u',v'),(s',w')\in E\times T_M}$$

of dimension $d\times d$ ($d = (s(M+1))^2$) is given by

$$\begin{aligned} \Sigma_f((i',m'),(j',t'),(u',v'),(s',w')) = & \ \delta_{(j',t'),(s',w')}\, \widetilde{P}^{n-1}\big((i',m'),(u',v')\big) \\ & + \delta_{(i',m'),(u',v')}\, \widetilde{P}^{n-1}\big((s',w'),(j',t')\big) \\ & + \sum_{k=2}^{n-1} \widetilde{P}^{n-k}\big((i',m'),(u',v')\big)\, \widetilde{P}^{k-1}\big((s',w'),(j',t')\big). \end{aligned}$$

Proof. Since $(\mathbf{Z},\mathbf{U})$ is a Markov chain and $\widetilde{P}\big((i',m'),(j',t')\big)$ $(i',m'),(j',t')\in E\times T_M$ is its transition matrix, we follow (Sadek and Limnios, 2002, Theorem 4) and obtain the desired result. $\square$

Theorem 2. *For any fixed $k\geq 3$,the random vector $\mathcal{R}(k)=(\mathcal{R}_{ij}(k))_{i,j\in E}$ where $\mathcal{R}_{ij}(k)=\sqrt{M}\big(\widehat{r^{\sharp}_{ij}}(k,M)-r^{\sharp}_{ij}(k)\big)$ converges in distribution, as M tends to infinity, to a centered normal random vector with variance $\Sigma\Gamma\Sigma^{\top}$.*

Proof. First, we notice that

$$\begin{aligned}\mathcal{R}_{ij}(k)=&\sqrt{M}\sum_{m=0}^{k-1}\sum_{s\in E}\widetilde{a}(s,0)\Big(\widehat{\widetilde{P}}^{k-1}\big((s,0),(i,m)\big)\widehat{\widetilde{P}}\big((i,m),(j,0)\big)\\&-\widetilde{P}^{k-1}\big((s,0),(i,m)\big)\widetilde{P}\big((i,m),(j,0)\big)\Big)\\=&\sum_{m=0}^{k-1}\sum_{s\in E}\widetilde{a}(s,0)\Big(f^{k-1}_{(s,0),(i,m)}\big(\widehat{\widetilde{P}}\big((i,m),(j,0)\big)-\widetilde{P}\big((i,m),(j,0)\big)\big)\\&+\widetilde{P}^{k-1}\big((s,0),(i,m)\big)f_{(i,m),(j,0)}+f^{k-1}_{(s,0),(i,m)}\widetilde{P}\big((i,m),(j,0)\big)\Big).\end{aligned}$$

Second, using Lemma 1 and following Sadek and Limnios (2002, Proposition 1), we obtain that

$$f^{k-1}_{(s,0),(i,m)}\xrightarrow[M\to\infty]{\mathcal{L}}Z,$$

where Z is a centered normal random variable, and

$$\widehat{\widetilde{P}}\big((i,m),(j,0)\big)-\widetilde{P}\big((i,m),(j,0)\big)\xrightarrow[M\to\infty]{\mathcal{P}}0.$$

Then from Slutsky's theorem, we deduce that $\mathcal{R}_{ij}(k)$ has the same limit in distribution as

$$\sum_{m=0}^{k-1}\sum_{s\in E}\widetilde{a}(s,0)\Big(\widetilde{P}^{k-1}\big((s,0),(i,m)\big)f_{(i,m),(j,0)}+f^{k-1}_{(s,0),(i,m)}\widetilde{P}\big((i,m),(j,0)\big)\Big),$$

which in turn has the same limit in distribution as

$$\sum_{m=0}^{k-1}\sum_{s\in E}\widetilde{a}(s,0)\Big(\widetilde{P}^{k-1}\big((s,0),(i,m)\big)f_{(i,m),(j,0)}+g^{k-1}_{(s,0),(i,m)}\widetilde{P}\big((i,m),(j,0)\big)\Big).$$

Since the last function is continuous and linear w.r.t. the vector $f_{(i',m'),(j',t')}$, we use the continuous mapping theorem and deduce that $\mathcal{R}_{ij}$ converges in distribution to a centered normal random vector with covariance matrix $\Sigma = \Sigma\,\Gamma\Sigma^T$. The rectangular matrix $\Sigma = \Sigma((i,j),(u',v'),(s',w'))_{i,j\in E,(u',v'),(s',w')\in E\times T_M}$ of dimension d ($d = s^2 \times (s(M+1))^2$) is given by

$$\begin{aligned}\Sigma((i,j),(u',v'),(s',w')) = {} & \delta_{(u',i)}\mathbf{1}_{(v'\leq k-1)}\,\delta_{(s',w'),(j,0)}\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-1}\big((s,0),(u',v')\big)\\ & + \delta_{(s',i)}\mathbf{1}_{(w'\leq k-1)}\,\widetilde{P}\big((i,w'),(j,0)\big)\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-2}\big((s,0),(u',v')\big)\\ & + \delta_{(u',v'),(s,0)}\,\widetilde{a}(u',0)\sum_{m=0}^{k-1}\widetilde{P}\big((i,m),(j,0)\big)\widetilde{P}^{k-2}\big((s',w'),(i,m)\big)\\ & + \sum_{m=0}^{k-1}\sum_{s\in E}\sum_{r=2}^{k-2}\widetilde{a}(s,0)\,\widetilde{P}\big((i,m),(j,0)\big)\,\widetilde{P}^{k-1-r}\big((s,0),(u',v')\big)\,\widetilde{P}^{r-1}\big((s',w'\end{aligned}$$

□

3.3 Left conditional ROCOF

Here we aim at studying the impact of the starting up state and the ending down state on the ROCOF. In order to do that, we first define the left conditional ROCOF as the expected number of transitions of the SMC to the set D at time k, given that it starts from the fixed state $i \in U$, i.e.,

$$\widetilde{r}_i(k) = \mathbb{E}[\widetilde{N}_i(k) - \widetilde{N}_i(k-1)],$$

where

$$\widetilde{N}_i(k) = \textstyle\sum_{l=1}^{k}\mathbf{1}_{\{Z_{l-1}=i,Z_l\in D\}}.$$

Following Votsi et al. (2014), we obtain an explicit expression of the left conditional ROCOF for a current up state $i \in U$.

Theorem 3. *The left conditional ROCOF of the SMC at time k is given by*

$$\widetilde{r}_i(k) \;=\; \sum_{j\in D}\sum_{m=0}^{k-1}[(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big), \quad k\in\mathbb{N}^*,$$

for any fixed, arbitrary state $i \in U$.

Proof. For fixed $i \in U$ and $k \in \mathbb{N}^*$,

$$\begin{aligned}\widetilde{r}_i(k) &= \mathbb{E}[\widetilde{N}_i(k) - \widetilde{N}_i(k-1)]\\ &= \mathbb{E}1_{\{Z_{k-1}=i,Z_k\in D\}} = P(Z_{k-1}=i, Z_k\in D)\\ &= \sum_{j\in D}P(Z_{k-1}=i, Z_k=j). \qquad (3)\end{aligned}$$

Therefore, from (2) and (3), we get the desired result. □

For a given state $i \in U$, we estimate the left conditional ROCOF by means of the plug-in type estimator

$$\widehat{\widetilde{r}}_i(k,M) = \sum_{j\in D}\sum_{m=0}^{k-1}[(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)]\widehat{\widetilde{P}}_M\big((i,m),(j,0)\big),$$

where $(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)$ is the (i,m) element of the vector $\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1}$, for every $k \in \mathbb{N}^*$.

Proposition 3. *For any fixed, arbitrary state $i \in U$ and any fixed, arbitrary positive integer $k \in \mathbb{N}$, the estimator of the left conditional ROCOF at time k is strongly consistent in the sense that*

$$\widehat{\widetilde{r}}_i(k,M) \xrightarrow[M\to\infty]{a.s.} \widetilde{r}_i(k).$$

Proof. To prove the consistency, we work as in Votsi and Limnios (2015). In particular, since we have $\widehat{\widetilde{P}}_M \xrightarrow[M\to\infty]{a.s.} \widetilde{P}$ and deal with finite sums, we obtain directly the desired result. □

Proposition 4. *For any fixed $i \in U$, $k \geq 1$, $\sqrt{M}\big(\widehat{\widetilde{r}}_i(k,M)-\widetilde{r}_i(k)\big)$ converges in distribution, as M tends to infinity, to a zero mean normal random variable with variance $\Phi_i'\Gamma\Phi_i'^{\top}$. Moreover $\Phi_i : [0,1]^d \to \mathbb{R}^+$ ($d = (s(M+1))^2$) is the function*

$$\begin{aligned}
&\Phi_i\Big(\widetilde{P}\big((i',m'),(j',t')\big);\ (i',m'),(j',t') \in E\times T_M\Big)\\
&= \sum_{j\in D}\sum_{m=0}^{k-1}[(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big)\\
&= \sum_{j\in D}\sum_{m=0}^{k-1}\Big(\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-1}\big((s,0),(i,m)\big)\Big)\widetilde{P}\big((i,m),(j,0)\big)
\end{aligned}$$

and

$$\Phi_i' = \Big(\frac{\partial\Phi_i}{\partial\widetilde{P}\big((i',m'),(j',t')\big)};\ (i',m'),(j',t') \in E\times T_M\Big),$$

is the $d-$dimensional row vector of first derivatives of $\widetilde{r}_i(k)$ with respect to $\widetilde{P}\big((i',m'),(j',t')\big)$.

Proof. First, we have that

$$\sqrt{M}(\widehat{\widetilde{r}}_i(k,M)-\widetilde{r}_i(k)) = \sqrt{M}\Big(\Phi_i\big(\widehat{\widetilde{P}}_M\big((i',m'),(j',t')\big)\big)-\Phi_i\big(\widetilde{P}\big((i',m'),(j',t')\big)\big)\Big).$$

The function Φ_i is a polynomial function of its arguments, it is then differentiable. Moreover, following (Sadek and Limnios, 2002, Corollary 1) for the double Markov chain, the vector function

$$\sqrt{M}\Big(\big(\widehat{\widetilde{P}}_M\big((i',m'),(j',t')\big)\big) - \big(\widetilde{P}\big((i',m'),(j',t')\big)\big)\Big)$$

converges, as M tends to infinity, to the normal distribution centered at the origin with covariance matrix Γ. Then by means of the delta method (see, e.g., van der Vaart (1998)), we obtain the desired result. □

Theorem 4. *For any fixed $k \geq 3$, the random vector $\sqrt{M}\big(\widetilde{r}_i(k,M) - r_i(k)\big)_{i\in U}$ converges in distribution, as M tends to infinity, to a centred normal random vector with variance $\Sigma_U \Gamma {\Sigma_U}^\top$ with $\Sigma_U = \Sigma_U(i,(u',v'),(s',w'))_{i\in U,(u',v'),(s',w')\in E\times T_M}$ of dimension d ($d = r \times (s(M+1))^2$) is given by*

$$\begin{aligned}\Sigma_U(i,(u',v'),(s',w')) = {}& \delta_{(u',i)}\mathbf{1}_{(v'\leq k-1)}\,\mathbf{1}_{(s'\in D)}\delta_{(w',0)}\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-1}\big((s,0),(u',v')\big)\\ &+ \delta_{(s',i)}\mathbf{1}_{(w'\leq k-1)}\sum_{j\in D}\widetilde{P}\big((i,w'),(j,0)\big)\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-2}\big((s,0),(u',v')\big)\\ &+ \delta_{(u',v'),(s,0)}\,\widetilde{a}(u',0)\sum_{j\in D}\sum_{m=0}^{k-1}\widetilde{P}\big((i,m),(j,0)\big)\widetilde{P}^{k-2}\big((s',w'),(i,m)\big)\\ &+ \sum_{j\in D}\sum_{m=0}^{k-1}\sum_{s\in E}\sum_{r=2}^{k-2}\widetilde{a}(s,0)\widetilde{P}\big((i,m),(j,0)\big)\widetilde{P}^{k-1-r}\big((s,0),(u',v')\big)\,\widetilde{P}^{r-1}\big((s',w'\end{aligned}$$

Proof. Working as in Theorem 2, we get the desired result. □

3.4 Right conditional ROCOF

We go one step further and aim at studying the impact of the ending down state to the ROCOF. In order to do that, we define the right conditional ROCOF as the expected number of transitions of the SMC to the set D at time k, given that it ends in state $j \in D$, i.e.,

$$\overline{r}_j(k) = \mathbb{E}[\overline{N}_j(k) - \overline{N}_j(k-1)],$$

where

$$\overline{N}_j(k) = \textstyle\sum_{l=1}^{k}\mathbf{1}_{\{Z_{l-1}\in U, Z_l = j\}}.$$

Following Votsi and Limnios (2015), we obtain an explicit expression of the right conditional ROCOF for any ending state $j \in D$.

Theorem 5. *The right conditional ROCOF of the SMC at time k is given by*

$$\overline{r}_j(k) \;=\; \sum_{i\in U}\sum_{m=0}^{k-1}[(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big), \quad k\in\mathbb{N}^*,$$

for any state $j \in D$.

Proof. For fixed $j \in D$ and $k \in \mathbb{N}^*$,

$$\begin{aligned}\overline{r}_j(k) &= \mathbb{E}[\overline{N}_j(k) - \overline{N}_j(k-1)]\\ &= \mathbb{E}\mathbf{1}_{\{Z_{k-1}\in U, Z_k=j\}} = P(Z_{k-1}\in U, Z_k = j)\\ &= \sum_{i\in U} P(Z_{k-1}=i, Z_k=j). \end{aligned} \tag{4}$$

Therefore, from (2) and (4), we get the desired result.

□

For a fixed state $j \in D$, we estimate the right conditional ROCOF by means of the plug-in type estimator :

$$\widehat{\overline{r}}_j(k,M) = \sum_{i\in U}\sum_{m=0}^{k-1}[(\widehat{\widetilde{a}}\widehat{\widetilde{P}}_M^{k-1})(i,m)]\widehat{\widetilde{P}}_M\big((i,m),(j,0)\big).$$

Proposition 5. *For any fixed arbitrary state $j \in D$ and any fixed arbitrary positive integer $k \in \mathbb{N}$, the estimator of the conditional ROCOF at time k is strongly consistent in the sense that*

$$\widehat{\overline{r}}_j(k,M) \xrightarrow[M\to\infty]{a.s.} \overline{r}_j(k).$$

Proof. Working as in Proposition 1, we obtain the desired result.

□

Proposition 6. *For any fixed $j \in D$, $k \geq 1$, $\sqrt{M}\big(\widehat{\overline{r}}_j(k,M) - \overline{r}_j(k)\big)$ converges in distribution, as M tends to infinity, to a zero mean normal random variable with variance $\Phi_j'\Gamma\Phi_j'^{\top}$ $\Phi_j : [0,1]^d \to \mathbb{R}^+$ ($d = (s(M+1))^2$) is the function*

$$\begin{aligned}&\Phi_j\Big(\widetilde{P}\big((i',m'),(j',t')\big);\;(i',m'),(j',t')\in E\times T_M\Big)\\ &= \sum_{i\in U}\sum_{m=0}^{k-1}[(\widetilde{a}\widetilde{P}^{k-1})(i,m)]\widetilde{P}\big((i,m),(j,0)\big)\\ &= \sum_{i\in U}\sum_{m=0}^{k-1}\Big(\sum_{s\in E}\widetilde{a}(s,0)\widetilde{P}^{k-1}\big((s,0),(i,m)\big)\Big)\widetilde{P}\big((i,m),(j,0)\big)\end{aligned}$$

and

$$\Phi'_j = \Big(\frac{\partial \Phi_j}{\partial \widetilde{P}\big((i', m'), (j', t')\big)};\ (i', m'), (j', t') \in E \times T_M \Big),$$

is the $d-$*dimensional row vector of first derivatives of* $\overline{r}_j(k)$ *with respect to* $\widetilde{P}\big((i', m'), (j', t')\big)$.

Proof. Working as in Proposition 2, we obtain the desired result. □

Theorem 6. *For any fixed* $k \geq 3$*, the random vector* $\sqrt{M}\big(\widetilde{r}_j(k, M) - r_j(k)\big)_{j \in D}$ *converges in distribution, as* M *tends to infinity, to a centred normal random vector with variance* $\Sigma_D \Gamma {\Sigma_D}^\top$ *with* $\Sigma_D = \Sigma_D(j, (u', v'), (s', w'))_{j \in D, (u',v'),(s',w') \in E \times T_M}$ *of dimension* d $(d = (s-r) \times (s(M+1))^2)$ *is given by*

$$\begin{aligned} \Sigma_D(j, (u', v'), (s', w')) = {} & \delta_{((s',w'),(j,0))} \mathbf{1}_{(u' \in U)} \mathbf{1}_{(v' \leq k-1)} \sum_{s \in E} \widetilde{a}(s, 0) \widetilde{P}^{k-1}\big((s, 0), (u', v')\big) \\ & + \mathbf{1}_{(s' \in U)} \mathbf{1}_{(w' \leq k-1)}\, \widetilde{P}\big((i, w'), (j, 0)\big) \sum_{s \in E} \widetilde{a}(s, 0) \widetilde{P}^{k-2}\big((s, 0), (u', v')\big) \\ & + \delta_{(u',v'),(s,0)}\, \widetilde{a}(u', 0) \sum_{i \in U} \sum_{m=0}^{k-1} \widetilde{P}\big((i, m), (j, 0)\big) \widetilde{P}^{k-2}\big((s', w'), (i, m)\big) \\ & + \sum_{i \in U} \sum_{m=0}^{k-1} \sum_{s \in E} \sum_{r=2}^{k-2} \widetilde{a}(s, 0) \widetilde{P}\big((i, m), (j, 0)\big) \widetilde{P}^{k-1-r}\big((s, 0), (u', v')\big)\, \widetilde{P}^{r-1}\big((s', w' \end{aligned}$$

Proof. Working as in Theorem 2, we get the desired result. □

4 Numerical Example

4.1 Simulated Data

To validate our findings and see how we can employ the conditional ROCOF to quantify the effects of the up state, we adopt simulation methods. In particular, we use the algorithm presented by Barbu and Limnios (2008), which generates trajectories of an MRC in a fixed time interval $[0, M]$. The algorithm, in its general form, takes as input the transition probability matrix of the EMC and the conditional sojourn time distributions.

We generate a trajectory of the MRC with length equal to $M = 100000$. The initial law $\mathbf{a}$ and the transition kernel $\mathbf{P}$ are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.5 & 0.5 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.7 & 0 & 0.3 \\ 0.5 & 0.5 & 0 \end{pmatrix},$$

Initialization;
Set $k = 0$, $S_0 = 0$ and sample J_0 from the initial distribution $\boldsymbol{\alpha}$;
Iteration;
while $S_k \leq M$ **do**
  Sample the random variable $J \sim \mathbf{P}(J_k, \cdot)$ and set $J_{k+1} = J(\omega)$;
  Sample the random variable $X \sim f_{J_k J_{k+1}}(\cdot)$;
  Set $S_{k+1} = S_k + X$;
  Set $k = k + 1$;
end

Algorithm 1: Monte-Carlo Algorithm

respectively. The sojourn times follow the discrete time Weibull distribution

$$f_{ij}(k) := \begin{cases} q^{k-1^b} - q^{k^b}, & \text{if} \quad k \geq 1, \\ 0, & \text{if} \quad k = 0, \end{cases}$$

with parameters $(q, b) = (0.1, 0.9)$ for the transitions $1 \longrightarrow 2$ and $2 \longrightarrow 1$, $(q, b) = (0.1, 2.0)$ for the transition $2 \longrightarrow 3$ and $(q, b) = (0.6, 0.9)$ for the remaining transitions. We further denote by $U = \{1, 2\}$ the subset of up states and by $D = \{3\}$ the subset of down states. Our objective is to estimate the DTIHT and quantify the impact of the current up state on the estimator.

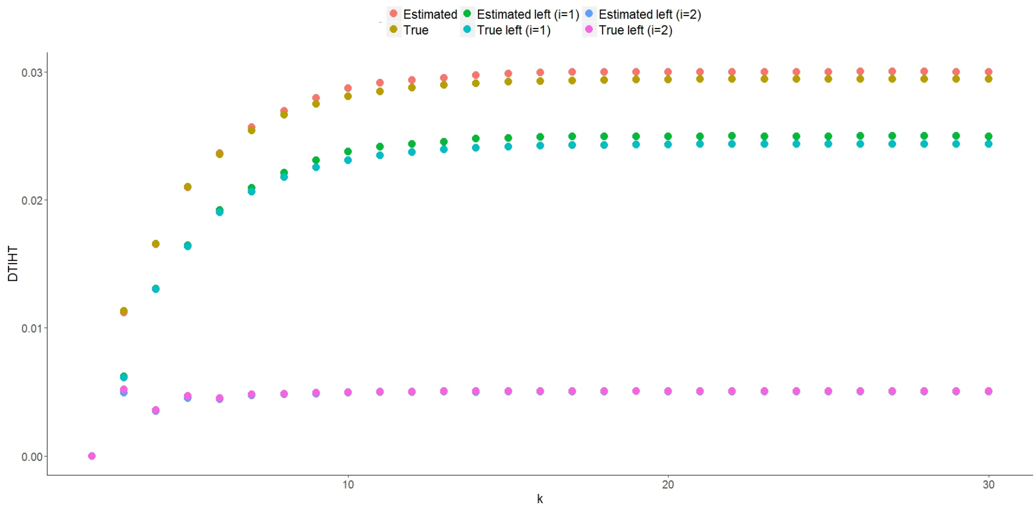

Figure 2: DTIHT and conditional hitting time intensities versus time.

We first notice that the DTIHT tends to increase for k smaller than 13, which means that the system is deteriorating, and then it becomes constant due to the stationarity of the semi-Markov chain. The estimated and true values of the DTIHT seem to be quite close, which validates our numerical approach. Moreover, we observe that the DTIHT is mostly affected by the up state $i = 1$ as the contribution of this state represents 83 % of its total value. The right conditional ROCOF in this example is equal to the DTIHT as there is only one down state.

4.2 Real Data

We further estimate the conditional ROCOF in a dataset that contains earthquake records with magnitudes $M \geq 5.5$ that occurred in Northern Aegean Sea (Greece) during the period $[1953, 2007]$ (see Votsi et al. (2012) for more informations). Following Votsi et al. (2012), three states are defined corresponding to earthquake magnitudes: State 1: $[5.5, 5.6]$, State 2: $[5.7, 6.1]$ and State 3: $[6.1, 7.2]$. Since we are more interested in the occurrence of high magnitude earthquakes, we consider down states are visited when earthquakes with magnitudes belonging to the third state occur, whereas up states are visited when earthquakes with lower magnitudes occur.

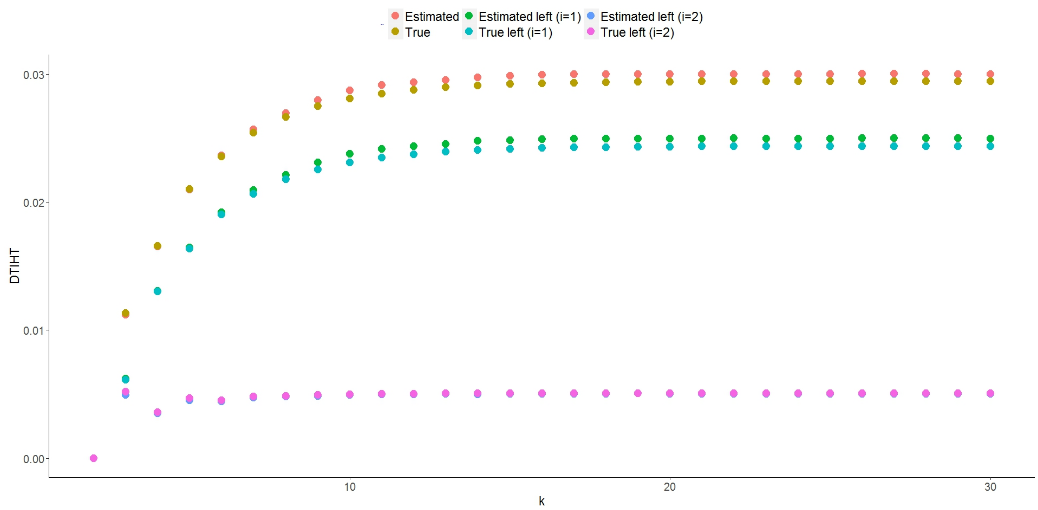

Figure 3: DTIHT and conditional hitting time intensities versus time.

We first notice that the DTIHT tends to increase for k smaller than 13, which means that the system is deteriorating, and then it becomes constant due to the stationarity of the semi-Markov chain. The estimated and true values of the DTIHT seem to be quite close, which validates our numerical approach. Moreover, we observe that the DTIHT is mostly affected by the up state $i = 1$ as the contribution of this state represents 83 % of its total

value. The right conditional ROCOF in this example is equal to the DTIHT as there is only one down state.

5 Concluding Remarks

In the present paper we study the sensitivity of the ROCOF in the current up and ending down states in a semi-Markov context. This is the first attempt to arise sensitivity issues of reliability indicators for SMMs. The results highlight that defining more precise reliability indicators could lead to more exact results for practitioners and researchers. Due to the flexibility and universality of the framework provided by SMMs, there exists a variety of extensions that could be adapted. Further research includes the extension of the results to higher order or non-homogeneous SMMs. Another topic for exploration is the study of sensitivity issues for reliability indicators concerning hidden Markov renewal models. DTIHT could further be investigated when independent and identical copies of the process are observed, each over a fixed duration, instead of one single copy over a fixed time interval. Modelling the reliability of mechanical systems subjected to earthquake-like loads is a subject of major concern and SMMs can be used to achieve this goal. An application to non-linear oscillators under seismic forces is a topic of further work.

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