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Abstract: In this paper, we consider the problem of symbolic model design for the class of incrementally stable switched systems. Contrarily to the existing results in the literature where switching is considered as periodically controlled, in this paper, we consider aperiodic time sampling resulting either from uncertain or event-based sampling mechanisms. Firstly, we establish sufficient conditions ensuring that usual symbolic models computed using periodic time-sampling remain approximately bisimilar to a switched system when the sampling period is uncertain and belongs to a given interval; estimates on the bounds of the interval are provided. Secondly, we propose a new method to compute symbolic models related by feedback refinement relations to incrementally stable switched systems, using an event-based approximation scheme. For a given precision, these event-based models are guaranteed to enable transitions of shorter duration and are likely to allow for more reactiveness in controller design. Finally, an example is proposed in order to illustrate the proposed results and simulations are performed for a Boost dc-dc converter structure.

Keywords: Approximate bisimulation, feedback refinement relation, switched systems, aperiodic sampling, incremental stability.

1. INTRODUCTION

Switched systems represent a popular class of hybrid systems (Antsaklis (1998), Goebel et al. (2012), Liberzon (2003), Lin and Antsaklis (2009)). Due to the complexity of switched systems, the literature gives a particular attention to the stability and stabilization problem of this class of systems. Recent technology advances demand that different and more complex control objectives like safety properties, obstacle avoidance, language and logic specifications be considered. This motivates several studies based on the use of symbolic models, also called discrete abstractions, for controller design, see for instance (Tabuada (2009), Belta et al. (2017)). Indeed, in the case where the obtained symbolic model is finite, the problem of controller design can be efficiently solved using the existing methods for supervisory control design for discrete-event systems.

In the present work, we are interested in the problem of symbolic models design for incrementally stable switched systems. Various methods for the design of symbolic models for this class of systems have been already proposed in the literature. In (Girard et al. (2010)), a symbolic model has been designed using both state and time discretization. The obtained symbolic model is related by an approximate bisimulation relation to the original one. This approach has been extended to the case of multirate symbolic models in (Saoud and Girard (2017)). This method allows the minimization of the number of transitions in the symbolic models obtained in (Girard et al. (2010)). Nevertheless, these approaches consider that the switching occurs periodically. The time sampling period is then fixed and considered as exactly known.

Here, we consider the problem of symbolic models construction with aperiodic time sampling. Aperiodic sampling is often considered in the area of sampled-data systems. The aperiodicity can be considered as a disturbance (see e.g. Hetel et al. (2017)) or exploited for control purpose using an event-based scheme (see e.g. Heemels et al. (2012)). In this paper, we present constructive approaches for symbolic models design for incrementally stable switched systems with aperiodic time sampling. We first show that symbolic models computed with a periodic time sampling remain ε-approximately bisimilar to the original system presenting uncertainties in the sampling instants. Estimates on the bounds of the uncertain sampling period are provided. We then provide a novel construction for symbolic models using an event-based time sampling. In this case the designed symbolic abstraction is related to the original switched system by a feedback refinement relation (Reissig et al. (2017)) and is thus also suitable for control applications. While in the first case the aperiodicity of sampling is considered as a disturbance, in the second case it is exploited to design symbolic models with similar precision but with transitions of

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smaller durations, and thus likely to allow for more reactiveness in controller design. Approximately bisimilar switched systems under aperiodic sampling have been recently considered in (Kido et al. (2017)). However, in that work, the comparison is between two switched systems (one periodic and another aperiodic), while we compare switched systems and symbolic models. This allows us to use numerical estimates of admissible uncertain sampling periods (computed on the symbolic model), which are less conservative than analytical estimates. Moreover, event-based sampling is not considered in (Kido et al. (2017)).

The paper is structured as follows: Section 2 gives some preliminary notions and definitions necessary for our study. In Section 3, we propose a constructive method of symbolic models when the time sampling parameter is uncertain. A design approach for symbolic models construction for switched systems with aperiodic sampling time parameter is proposed in Section 4 using an event-based approach. In Section 5, a numerical example that illustrates the proposed results is provided. A brief conclusion ends the paper.

Notations. In this paper we use the notations $\mathbb{R}$, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ to refer to the set of real, non-negative real, and positive real numbers, respectively. $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}^+$ refer to the sets of integers, of non-negative integers and of positive integers, respectively. $\text{card}(\mathscr{S})$ refers to the cardinal of a set $\mathscr{S}$. $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$ and $x_{(i)}$ refers to its i-th row. A continuous function γ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if γ is $\mathscr{K}$ and $\gamma(r)$ goes to infinity as r tends to infinity. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathscr{KL}$ if : for any fixed r, the map $\beta(.,s)$ belongs to the class $\mathscr{K}$, and for each fixed s the map $\beta(r,.)$ is strictly decreasing and $\beta(r,.)$ goes to zero as s tends to infinity.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Incrementally stable switched systems

In this paper we consider the class of switched systems defined as follows:

Definition 1. A switched system is a quadruple $\Sigma = (\mathbb{R}^n, P, \mathscr{P}, F)$, where:

- $\mathbb{R}^n$ is the state space;
- P is the finite set of modes $P = \{1, \ldots, m\}$;
- $\mathscr{P}$ is a subset of $\sigma(\mathbb{R}_0^+, P)$ which denotes the set of piecewise constant and right continuous functions $\mathbf{p}$ from $\mathbb{R}_0^+$ to the finite set of modes P, with a finite number of discontinuities on every bounded interval of $\mathbb{R}_0^+$. This guarantees the absence of Zeno behaviours.
- $F = \{f_1, \ldots, f_m\}$ is a collection of vector fields indexed by P.

Σ_p will denote the continuous subsystems of the switched system Σ defined by the following differential equation:

$$\dot{\mathrm{x}}(t) = f_p(\mathrm{x}(t)), \forall p \in P. \quad (1)$$

We assume that for all $p \in P$ the vector field $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous map and forward complete. In this case the solutions of (1) are unique and defined for all $t \in \mathbb{R}_0^+$. Necessary and sufficient conditions for forward completeness of a system have been provided in (Angeli and Sontag (1999)).

In the rest of our paper we will denote by $\mathrm{x}(t, x, \mathbf{p})$ the point reached by the trajectory of Σ at time $t \in \mathbb{R}_0^+$ from the initial state x under the switching signal $\mathbf{p}$.

This paper deals with the construction of symbolic models for switched systems. This problem rely on the incremental stability notion that have been presented first for nonlinear systems in (Angeli (2002)). An extension of this results to the case of switched systems has been provided in (Girard et al. (2010)) and recalled hereafter.

Definition 2. A switched system Σ is said to be incrementally globally uniformly asymptotically stable (δ-GUAS) if there exists a $\mathscr{KL}$ function β such that for all $t \in \mathbb{R}_0^+$, for all $x, y \in \mathbb{R}^n$ and for all switching signal $\mathbf{p} \in \mathscr{P}$, the following condition holds:

$$\|\mathrm{x}(t, x, \mathbf{p}) - \mathrm{x}(t, y, \mathbf{p})\| \le \beta(\|x - y\|, t). \quad (2)$$

Roughly speaking, incremental stability means that all the trajectories induced by the same switching signal converge to the same reference trajectory independently of their initial states. As for the case of general nonlinear systems, incremental stability of switched systems can be characterised using Lyapunov function as follows:

Definition 3. A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a *common δ-GUAS Lyapunov function* for system Σ if there exist $\mathscr{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ and $\kappa \in \mathbb{R}^+$ such that

$$\underline{\alpha}(\|x - y\|) \le V(x, y) \le \overline{\alpha}(\|x - y\|); \quad (3)$$

and

$$\frac{\partial V}{\partial x}(x, y) f(x) + \frac{\partial V}{\partial y}(x, y) f(y) \le -\kappa V(x, y), \quad (4)$$

for all $x, y \in \mathbb{R}^n$.

Theorem 1. (Girard et al. (2010)). Consider a switched system $\Sigma = (\mathbb{R}^n, P, \mathscr{P}, F)$ with a common δ-GUAS Lyapunov function, then Σ is δ-GUAS.

In order to construct symbolic models for switched systems, we consider an additional assumption on the Lyapunov function:

- there exists a $\mathscr{K}_\infty$ function γ such that
$$\forall x, y, z \in \mathbb{R}^n, |V(x, y) - V(x, z)| \le \gamma(\|x - y\|). \quad (5)$$

It has been shown in ((Girard et al., 2010)) that this assumption is satisfied provided that the dynamics of the switched system are considered on a compact set $\mathscr{S} \subset \mathbb{R}^n$ and the Lyapunov function V is of class $\mathscr{C}^1$ on $\mathscr{S}$. In this case, we have

$$\forall x, y, z \in \mathscr{S},\ |V(x, y) - V(x, z)| \le c\|x - y\|, \quad (6)$$

with $c = \max_{x, y \in \mathscr{S}} \|\frac{\partial V}{\partial y}(x, y)\|$. Thus, (5) is verified for linear $\mathscr{K}_\infty$ function given by $\gamma(s) = cs$. Moreover, note that for all $x \in \mathbb{R}^n$ we have $V(x, x) = 0$, then

$$\forall x, y \in \mathbb{R}^n, V(x, y) \le |V(x, y) - V(x, x)| \le \gamma(\|x - y\|). \quad (7)$$

Therefore, there is no loss of generality in considering that the right inequality in (3) holds with $\overline{\alpha} = \gamma$.

2.2 Transition systems

In this paper, we are interested in providing symbolic models for switched systems. In what follows, we present the concept of transition systems allowing the description of both switched systems and symbolic models in the same framework:

Definition 4. A transition system is a tuple $T = (Q, U, Y, \mathscr{T}, I)$ where:

- Q is a set of states ;
- U is a set of inputs ;
- Y is a set of outputs ;
- $\mathscr{T} \subseteq Q \times U \times Q \times Y$ is a transition relation;
- $I \subseteq Q$ is a set of initial states .

T is said to be *metric* if the set of outputs Y is equipped with a metric d such that $d(y_1,y_2) = \|y_1 - y_2\|$, *symbolic* if Q and U are finite or countable sets.

$(x^{'},y) \in \mathscr{T}(x,u)$ will refer to the transition $(x,u,x^{'},y) \in \mathscr{T}$. This means that by applying the input u the trajectory of the transition system will evolve from the state x to the state $x^{'}$ while providing the output y. Given a state $x \in Q$, an input $u \in U$ is said to belong to the set of *enabled* inputs, denoted by $Enab(x)$, if $\mathscr{T}(x,u) \neq \emptyset$. A state $x \in Q$ is said to be blocking if $Enab(x) = \emptyset$, it is said non-blocking otherwise. T is said to be *deterministic* if for all $x \in Q$ and for all $u \in Enab(x)$, $\text{card}(\mathscr{T}(x,u)) = 1$.

Definition 5. Let $T_1 = (Q_1,U,Y,\mathscr{T}_1,I_1)$, $T_2 = (Q_2,U,Y,\mathscr{T}_2,I_2)$ be two metric transition systems with the same input set U and the same output set Y equipped with the metric d. Let $\varepsilon \geq 0$ be a given precision. A relation $\mathscr{R} \subseteq Q_1 \times Q_2$ is said to be an ε-approximate bisimulation relation between T_1 and T_2 if for all $(x_1,x_2) \in \mathscr{R}$, $Enab(x_1) = Enab(x_2)$ and for all $u \in Enab(x_1)$:

$$\forall (x_1^{'},y_1) \in \mathscr{T}_1(x_1,u), \exists (x_2^{'},y_2) \in \mathscr{T}_2(x_2,u) \text{ such that}$$
$$d(y_1,y_2) \leq \varepsilon \text{ and } (x_1^{'},x_2^{'}) \in \mathscr{R};$$
$$\forall (x_2^{'},y_2) \in \mathscr{T}_2(x_2,u), \exists (x_1^{'},y_1) \in \mathscr{T}_1(x_1,u) \text{ such that}$$
$$d(y_1,y_2) \leq \varepsilon \text{ and } (x_1^{'},x_2^{'}) \in \mathscr{R}.$$

The transition systems T_1 and T_2 are said to be ε-*approximately bisimilar*, denoted $T_1 \sim_\varepsilon T_2$, if and only if:

$$\forall x_1 \in I_1, \exists x_2 \in I_2, \text{ such that } (x_1,x_2) \in \mathscr{R};$$
$$\forall x_2 \in I_2, \exists x_1 \in I_1, \text{ such that } (x_1,x_2) \in \mathscr{R}.$$

Definition 6. Let $T_1 = (Q_1,U,Y,\mathscr{T}_1,I_1)$, $T_2 = (Q_2,U,Y,\mathscr{T}_2,I_2)$ be two metric transition systems with the same input set U and the same output set Y equipped with the metric d. Let $\varepsilon \geq 0$ be a given precision. A relation $\mathscr{R} \subseteq Q_1 \times Q_2$ is said to be an approximate feedback refinement relation from T_1 to T_2 if for all $(x_1,x_2) \in \mathscr{R}$

$$Enab(x_2) \subseteq Enab(x_1);$$

and for all $u \in Enab(x_2)$:

$$\forall (x_1^{'},y_1) \in \mathscr{T}_1(x_1,u), \exists (x_2^{'},y_2) \in \mathscr{T}_2(x_2,u)$$
$$\text{such that } d(y_1,y_2) \leq \varepsilon \text{ and } (x_1^{'},x_2^{'}) \in \mathscr{R};$$
$$\forall x_1 \in I_1, \exists x_2 \in I_2, \text{ such that } (x_1,x_2) \in \mathscr{R}.$$

3. UNCERTAIN SAMPLING

3.1 Symbolic model construction

In (Girard et al. (2010)) the problem of symbolic models construction for incrementally stable switched systems has been studied. In that paper, interest has been given to switched systems $\Sigma = (\mathbb{R}^n, P, \mathscr{P}, F)$ for which the switching is periodically controlled by a microprocessor with a clock of period $\tau^* \in \mathbb{R}^+$. In this case, given a switched system $\Sigma = (\mathbb{R}^n, P, \mathscr{P}, F)$ with $\mathscr{P} = \sigma(\mathbb{R}_0^+, P)$ and a sampling time period τ^*, the associated transition system has been defined by $T_{\tau^*} = (Q,U,Y,\mathscr{T}_{\tau^*},I)$ where:

- the set of states is $Q = \mathbb{R}^n$;
- the set of labels (inputs) is $U = P$;
- the set of outputs is $Y = \mathbb{R}^n$;
- the transition relation $\mathscr{T}_{\tau^*} \subseteq Q \times U \times Q \times Y$ is given as follows: $\forall x, x^{'} \in Q$, $\forall u \in U$, $\forall y \in Y$, $(x^{'},y) \in \mathscr{T}_{\tau^*}(x,u)$ if and only if
$$x^{'} = \mathbf{x}(\tau^*,x,p) \text{ and } y = x;$$
- the set of initial states is $I = \mathbb{R}^n$.

The state space is then approximated by the lattice:

$$[\mathbb{R}^n]_\eta = \left\{ q \in \mathbb{R}^n | q_i = k_i \frac{2\eta}{\sqrt{n}},\ k_i \in \mathbb{Z},\ i = 1,\ldots,n \right\},$$

where $\eta \in \mathbb{R}^+$ is the state space sampling parameter.

The quantizer $\mathscr{Q}_\eta : \mathbb{R}^n \to [\mathbb{R}^n]_\eta$ is defined by $\mathscr{Q}_\eta(x) = q$ if and only if

$$\forall i = 1,\ldots,n,\ q_{(i)} - \frac{\eta}{\sqrt{n}} \leq x_{(i)} < q_{(i)} + \frac{\eta}{\sqrt{n}}. \tag{8}$$

It can be easily verified that for all $x \in \mathbb{R}^n$, $\|\mathscr{Q}_\eta(x) - x\| \leq \eta$.

The symbolic model $T_{\eta,\tau^*} = (Q_\eta, U, Y, \mathscr{T}_{\eta,\tau^*}, I_\eta)$ has been then defined as follows:

- the set of states is $Q_\eta = [\mathbb{R}^n]_\eta$;
- the set of labels (inputs) is $U = P$;
- the set of outputs is $Y = \mathbb{R}^n$;
- the transition relation $\mathscr{T}_{\eta,\tau^*} \subseteq Q_\eta \times U \times Q_\eta \times Y$ is given as follows: $\forall q, q^{'} \in Q_\eta$, $\forall u \in U$, $\forall y \in Y$, $(q^{'},y) \in \mathscr{T}_{\eta,\tau^*}(q,u)$ if and only if
$$q^{'} = \mathscr{Q}_\eta(\mathbf{x}(\tau^*,q,p)) \text{ and } y = q;$$
- the set of initial states is $I_\eta = [\mathbb{R}^n]_\eta$.

In this section, we are interested in symbolic models for switched systems with uncertain sampling period $\tau \in [\underline{\tau}, \overline{\tau}] \subseteq \mathbb{R}^+$ described by transition system $T_{[\underline{\tau},\overline{\tau}]} = (Q,U,Y,\mathscr{T}_{[\underline{\tau},\overline{\tau}]},Y,I)$ associated to Σ and defined as follows:

- the set of states is $Q = \mathbb{R}^n$;
- the set of labels (inputs) is $U = P$;
- the set of outputs is $Y = \mathbb{R}^n$;
- the transition relation $\mathscr{T}_{[\underline{\tau},\overline{\tau}]} \subseteq Q \times U \times Q \times Y$ is given as follows: $\forall x, x^{'} \in Q$, $\forall u \in U$, $\forall y \in Y$, $(x^{'},y) \in \mathscr{T}_{[\underline{\tau},\overline{\tau}]}(x,u)$ if and only if
$$\exists \tau \in [\underline{\tau}, \overline{\tau}] \text{ such that } x^{'} = \mathbf{x}(\tau,x,p) \text{ and } y = x;$$
- the set of initial states is $I = \mathbb{R}^n$.

The goal of this study is to provide conditions on the sampling time parameters $\underline{\tau}$, $\overline{\tau}$ such that the transition systems $T_{[\underline{\tau},\overline{\tau}]}$ and T_{η,τ^*} are ε-*approximately bisimilar*. This result is presented in the following.

Theorem 2. Consider a switched system Σ. Assume that there exists a common δ-GUAS Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ for subsystems Σ_p satisfying (5) for some $\mathscr{K}_\infty$ function γ. Consider time and state sampling parameters τ^* and η and a chosen precision $\varepsilon \geq 0$ such that

$$\eta \leq \gamma^{-1}((1 - e^{-\kappa\tau^*})\underline{\alpha}(\varepsilon)). \tag{9}$$

Let us assume that for all $\tau \in [\underline{\tau}, \overline{\tau}] \subset \mathbb{R}^+$, $p \in P$ and for all $q, q^{'} \in Q_\eta$ such that $(q^{'},y) \in \mathscr{T}_{\eta,\tau^*}(q,p)$

$$g(\tau,q,p,q^{'}) := \gamma(\|\mathbf{x}(\tau,q,p) - q^{'}\|) - (1 - e^{-\kappa\tau})\underline{\alpha}(\varepsilon) \leq 0. \tag{10}$$

Then, systems $T_{[\underline{\tau},\overline{\tau}]}$ and T_{η,τ^*} are ε-approximately bisimilar with precision ε.

Proof. Following the statement of Definition 5, we start the proof by showing that the relation $\mathscr{R}$ defined by:

$$\mathscr{R} = \{(x,q) \in Q \times Q_\eta | V(x,q) \le \underline{\alpha}(\varepsilon)\} \tag{11}$$

is an ε-approximate bisimulation relation. Let $(x,q) \in \mathscr{R}$ and let $(x',y) \in \mathscr{T}_{[\underline{\tau},\overline{\tau}]}(x,p)$ then there exists $\tau \in [\underline{\tau},\overline{\tau}]$ such that $x' = \mathrm{x}(\tau,x,p)$. There exists $q' \in [\mathbb{R}^n]_\eta$ such that we have $(q',y) \in \mathscr{T}_{\eta,\tau^*}(q,p)$. Then, let verify that $(x',q') \in \mathscr{R}$.

From (5), we have

$$V(x',q') \le V(x',\mathrm{x}(\tau,q,p)) + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|). \tag{12}$$

This leads to

$$\begin{aligned} V(x',q') &\le V(\mathrm{x}(\tau,x,p),\mathrm{x}(\tau,q,p)) + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\le V(x,q)e^{-\kappa\tau} + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\le \underline{\alpha}(\varepsilon)e^{-\kappa\tau} + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\le \underline{\alpha}(\varepsilon), \end{aligned} \tag{13}$$

where the second inequality comes from (4), the third inequality is verified because of the fact that $(x,q) \in \mathscr{R}$, and the last inequality comes from (10). Therefore, $(x',q') \in \mathscr{R}$. Now let $(x,q) \in \mathscr{R}$, we have that

$$\underline{\alpha}(\|x-q\|) \le V(x,q) \le \underline{\alpha}(\varepsilon), \tag{14}$$

which leads to

$$\|x-q\| \le \underline{\alpha}^{-1}(V(x,q)) \le \varepsilon. \tag{15}$$

Likewise, we can show that for all $(q',y) \in \mathscr{T}_{\eta,\tau^*}(q,p)$ there exists $(x',y) \in \mathscr{T}_{[\underline{\tau},\overline{\tau}]}(x,p)$ such that $\|x-q\| \le \varepsilon$ and $(x',q') \in \mathscr{R}$. Thus, $\mathscr{R}$ is an ε-approximate bisimulation relation between the transition systems $T_{[\underline{\tau},\overline{\tau}]}$ and T_{η,τ^*}.

Now let us show that the transition systems $T_{[\underline{\tau},\overline{\tau}]}$ and T_{η,τ^*} are ε-approximately bisimilar. By construction of $I_2 = [\mathbb{R}^n]_\eta$, for all $x \in \mathbb{R}^n$, there exists $q \in I_2$ such that $\|x-q\| \le \eta$. Assume that the right inequality in (3) holds for $\overline{\alpha} = \gamma$, then we obtain

$$V(x,q) \le \gamma(\|x-q\|) \le \gamma(\eta). \tag{16}$$

Using (9), we have $\gamma(\eta) \le \underline{\alpha}(\varepsilon)$. Therefore, (16) becomes

$$V(x,q) \le \gamma(\eta) \le \underline{\alpha}(\varepsilon). \tag{17}$$

Thus $(x,q) \in \mathscr{R}$. Moreover, for all $q \in [\mathbb{R}^n]_\eta$, considering $x \in I_1$ such that $x = q$ leads to $V(x,x) = 0$. Then, $(x,q) \in \mathscr{R}$, which ends the proof. ■

Remark 1. The result in Theorem 2 can be interpreted as a robust version of the symbolic model proposed in (Girard et al. (2010)). Indeed, by (9), (10) is always satisfied for $\tau = \tau^*$. Hence, in the case where $\underline{\tau} = \tau^* = \overline{\tau}$ we can recover the result obtained in (Girard et al. (2010)). In addition, we show that if (10) holds, then even though the sampling time period τ is uncertain, transition systems $T_{[\underline{\tau},\overline{\tau}]}$ and T_{η,τ^*} remain ε-approximately bisimilar.

Remark 2. Note that the result of Theorem 2 can be used to evaluate the bounds $\underline{\tau}$, $\overline{\tau}$, numerically. Indeed, the lower and upper bounds $\underline{\tau}$ and $\overline{\tau}$ of the sampling time interval satisfying (10) can be computed for each transition $(q',y) \in \mathscr{T}_{\eta,\tau^*}(q,p)$, when computing the symbolic model. $\underline{\tau}$ and $\overline{\tau}$ corresponds to the instants of change of sign of the function g.

3.2 Estimates of $\underline{\tau}$ and $\overline{\tau}$

In the following we provide analytic estimates of $\underline{\tau}$ and $\overline{\tau}$.

Corollary 1. Consider the function g as in (10), $p \in P$ and $q,q' \in Q_\eta$ such that $(q',y) \in \mathscr{T}_{\eta,\tau^*}(q,p)$. Then,

(1) if $q \ne q'$, then there exists $\underline{\theta} > 0$ such that for all $\tau \in (0,\underline{\theta}]$, $g(\tau,q,p,q') > 0$.
(2) if Σ_p has an equilibrium x^* (i.e. $f_p(x^*) = 0$) such that $(x^*,q') \notin \mathscr{R}$, then there exists $\overline{\theta} > 0$ such that for all $\tau \in [\overline{\theta},+\infty)$, $g(\tau,q,p,q') > 0$.
(3) if there exists $b > 0$ such that for all $\tau > \tau^*$,

$$\|f(\mathrm{x}(\tau,p,q))\| < b$$

then there exists

$$\theta = \tau^* + \frac{1}{b}\left(\gamma^{-1}(\underline{\alpha}(\varepsilon)(1-e^{-\kappa\tau^*})) - \eta\right)$$

such that for all $\tau \in [\tau^*,\theta]$

$$\|\mathrm{x}(\tau,q,p) - \mathrm{x}(\tau^*,q,p)\| \le \left(\gamma^{-1}(\underline{\alpha}(\varepsilon)(1-e^{-\kappa\tau^*})) - \eta\right), \tag{18}$$

and thus,

$$\forall\tau \in [\tau^*,\theta],\ g(\tau,q,p,q') \le 0. \tag{19}$$

Proof. We prove each item separately:

(1) If $q \ne q'$, then $g(0,q,p,q') = \gamma(\|q-q'\|) > 0$. The result follows by continuity.

(2) If Σ_p has an equilibrium x^* we have for all $\tau \in \mathbb{R}_0^+$

$$V(x^*,\mathrm{x}(\tau,q,p)) = V(\mathrm{x}(\tau,x^*,p),\mathrm{x}(\tau,q,p)) \le e^{-\kappa\tau}V(x^*,q) \tag{20}$$

Then, $\lim_{\tau\to+\infty} \mathrm{x}(\tau,q,p) = x^*$ and it follows that

$$\lim_{\tau\to+\infty} g(\tau,q,p,q') = \gamma(\|x^*-q'\|) - \underline{\alpha}(\varepsilon).$$

Since $(x^*,q') \notin \mathscr{R}$, we have $\underline{\alpha}(\varepsilon) < V(x^*,q') \le \gamma(\|x^*-q'\|)$. Then, $\lim_{\tau\to+\infty} g(\tau,q,p,q') > 0$ and the result follows.

(3) Using the triangular inequality

$$\|\mathrm{x}(\tau,q,p) - q'\| \le \|\mathrm{x}(\tau,q,p) - \mathrm{x}(\tau^*,q,p)\| + \eta, \tag{21}$$

we obtain for all $\tau \in [\tau^*,\theta]$,

$$\begin{aligned} g(\tau,q,p,q') &\le \gamma(\|\mathrm{x}(\tau,q,p) - \mathrm{x}(\tau^*,q,p)\| + \eta) \\ &\quad -\underline{\alpha}(\varepsilon)(1-e^{-\kappa\tau}) \\ &\le \gamma(\|\mathrm{x}(\tau,q,p) - \mathrm{x}(\tau^*,q,p)\| + \eta) \\ &\quad -\underline{\alpha}(\varepsilon)(1-e^{-\kappa\tau^*}) \end{aligned}$$

Then (18) gives for all $\tau \in [\tau^*,\theta]$ $g(\tau,q,p,q') \le 0$. ■

The implications of the previous result on admissible bounds $\underline{\tau}$, $\overline{\tau}$ are as follows. Item (1) implies that $\underline{\tau} = 0$ is not a suitable choice and that a strictly positive lower bound is needed. Similarly, item (2) implies that $\overline{\tau} = +\infty$ is not a suitable choice and that a finite upper bound is needed. Item (3) can help us to choose suitable values $\underline{\tau}$, $\overline{\tau}$ by taking $\underline{\tau} = \tau^*$, $\overline{\tau} = \theta$.

4. EVENT-BASED SAMPLING

In this section, we are interested in the symbolic models construction for switched systems for which the switching does not occur periodically. In this context, we associate to the switched system $\Sigma = (\mathbb{R}^n,P,\mathscr{P},F)$ with $\mathscr{P} = \sigma(\mathbb{R}_0^+,P)$, the transition system $T = (Q,U,Y,\mathscr{T},I)$ where:

- $Q = \mathbb{R}^n$ is the set of states;
- $U = P \times \mathbb{R}_0^+$ is the set of inputs;
- $Y = \mathbb{R}^n$ is the set of outputs;

- $\mathscr{T} \subseteq Q \times U \times Q \times Y$ is the transition relation defined by: $\forall x, x' \in Q$, $\forall (p,\tau) \in U$, $\forall y \in Y$, $(x', y) \in \mathscr{T}(x,u)$ if and only if
$$\mathrm{x}(\tau, x, p) = x', \ y = x;$$
- $I = \mathbb{R}^n$ is the set of initial states.

Let $\varepsilon \in \mathbb{R}^+$ be the desired precision of the symbolic model. As in the previous section we approximate the state space by the lattice $[\mathbb{R}^n]_\eta$ where $\eta \in \mathbb{R}^+$, and we define the transition system $T^e_{\eta,\varepsilon}(\Sigma) = (Q^e_\eta, U, Y, \mathscr{T}^e_{\eta,\varepsilon}, I^e_\eta)$ where:

- $Q^e_\eta = [\mathbb{R}^n]_\eta$ is the set of states;
- $U = P \times \mathbb{R}^+_0$ is the set of inputs;
- $Y = \mathbb{R}^n$ is the set of outputs;
- $\mathscr{T}^e_{\eta,\varepsilon} \subseteq Q^e_\eta \times U \times Q^e_\eta \times Y$ is the transition relation defined as follows: $\forall q, q' \in Q^e_\eta$, $\forall u = (p,\tau) \in U$, $\forall y \in Y$, $(q', y) \in \mathscr{T}^e_{\eta,\varepsilon}(q,u)$ if and only if
$$\tau = \inf\{\theta \in \mathbb{R}^+ \mid h(\theta, q, p) \leq 0\} \quad (22)$$
where
$$h(\theta,q,p) := \gamma(\|\mathrm{x}(\theta,q,p) - \mathscr{Q}_\eta(\mathrm{x}(\theta,q,p))\|) - (1 - e^{-\kappa\theta})\underline{\alpha}(\varepsilon), \quad (23)$$
and
$$q' = \mathscr{Q}_\eta(\mathrm{x}(\tau,q,p)), \ y = q;$$
- $I^e_\eta = [\mathbb{R}^n]_\eta$ is the set of initial states.

Theorem 3. Consider a switched system Σ. Assume that there exists a common δ-GUAS Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+_0$ for subsystems Σ_p satisfying (5) for some $\mathscr{K}_\infty$ function γ. Consider a state sampling parameter $\eta > 0$ and a precision $\varepsilon > 0$ such that
$$\eta < \gamma^{-1}(\underline{\alpha}(\varepsilon)). \quad (24)$$
Then, the relation
$$\mathscr{R} = \{(q,x) \in Q \times Q^e_\eta | V(x,q) \leq \underline{\alpha}(\varepsilon)\} \quad (25)$$
is a feedback refinement relation from T to $T^e_{\eta,\varepsilon}$.

Proof. We start by showing that for all $(x,q) \in \mathscr{R}$, $Enab(q) \in Enab(x)$. Let $u = (\tau,p) \in Enab(q)$. There exists $(q',y) \in \mathscr{T}^e_{\varepsilon,\eta}(u,q)$ such that $\|\mathrm{x}(\tau,q,p) - q\| \leq \eta$. Then, there exists $x' = \mathrm{x}(\tau,x,p)$ such that $(x',y) \in \mathscr{T}(u,x)$. Thus, $u \in Enab(x)$, which proves the first condition of Definition 6.

Let $u \in Enab(q)$. Let $(x',y) \in \mathscr{T}(x,u)$. There exists $q' \in [\mathbb{R}^n]_\eta$ such that $\|\mathrm{x}(\tau,q,p) - q'\| \leq \eta$. Let show that $(x',q') \in \mathscr{R}$.

From (5), we have
$$V(x',q') \leq V(x', \mathrm{x}(\tau,q,p)) + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|). \quad (26)$$
This last inequality leads to
$$\begin{aligned} V(x',q') &\leq V(x', \mathrm{x}(\tau,q,p)) + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\leq V(\mathrm{x}(\tau,x,p), \mathrm{x}(\tau,q,p)) + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\leq V(x,q)e^{-\kappa\tau} + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\leq \underline{\alpha}(\varepsilon)e^{-\kappa\tau} + \gamma(\|\mathrm{x}(\tau,q,p) - q'\|) \\ &\leq \underline{\alpha}(\varepsilon). \end{aligned} \quad (27)$$
The third inequality comes from the incremental stability of Σ ((4) is verified), the fourth inequality holds thanks to the fact that $(x,q) \in \mathscr{R}$, and the last inequality comes from (23). Therefore, $(x',q') \in \mathscr{R}$.

Now let $(x,q) \in \mathscr{R}$, we have
$$\underline{\alpha}(\|x - q\|) \leq V(x,q) \leq \underline{\alpha}(\varepsilon). \quad (28)$$
This leads to
$$\|x - q\| \leq \underline{\alpha}^{-1}(V(x,q)) \leq \varepsilon. \quad (29)$$
Therefore, $d(x,q) \leq \varepsilon$. Finally, by construction of the state space approximation $[\mathbb{R}^n]_\eta$ we have that for all $x \in I$ there exists $q \in I^e_\eta$ given by $q = \mathscr{Q}(x)$ such that $\|x - q\| \leq \eta$. Under the assumption that the right inequality of (3) holds for $\overline{\alpha} = \gamma$ we obtain
$$V(x,q) \leq \gamma(\|x - q\|) \leq \gamma(\eta). \quad (30)$$
Using (24), we obtain
$$V(x,q) \leq \gamma(\|x - q\|) \leq \gamma(\eta) \leq \underline{\alpha}(\varepsilon). \quad (31)$$
Thus, $(x,q) \in \mathscr{R}$, which ends the proof. ■

Note that the result of Theorem 3 is constructive. The sampling time instants τ can be computed while computing the symbolic model. They correspond to the instants at which the function h changes sign. An analytic condition for the existence of this instants is provided in the following.

Corollary 2. Consider the function h as it is defined in (23), $p \in P$, $q \in Q^e_\eta$ and let $(q',y) \in \mathscr{T}^e_{\eta,\varepsilon}(q,u)$ with $u = (p,\tau)$. Then,

(1) $\tau \leq \tau^*$ where $\tau^* = -\frac{1}{\kappa}\ln(1 - \frac{\gamma(\eta)}{\underline{\alpha}(\varepsilon)})$.
(2) if $q \neq q'$, then $\tau > 0$.

Proof. The items are proved separately.

(1) Let $(q',y) \in \mathscr{T}^e_{\eta,\varepsilon}(q,u)$. Then there exists $x' = \mathrm{x}(\tau,q,p)$ such that $\|\mathrm{x}(\tau,q,p) - q'\| \leq \eta$. Therefore, we have
$$h(\tau,q,p) \leq \gamma(\eta) + (1 - e^{-\kappa\tau})\underline{\alpha}(\varepsilon) \leq 0. \quad (32)$$
For $\tau = \tau^*$ we have
$$h(\tau,q,p) \leq \gamma(\eta) + (1 - e^{-\kappa\tau})\underline{\alpha}(\varepsilon) = 0 \quad (33)$$
and the result follows from the definition of τ in (22).
(2) if $q \neq q'$, then $h(0,q,p) = \gamma(\|q - q'\|) > 0$. Thus, the result fellows by continuity and from the definition of τ in (22).

■

The implications of the previous corollary are as follows. Item (1) implies that the duration of transition in the event-based symbolic model $T^e_{\eta,\varepsilon}$ are always shorter than those in the periodic symbolic model T_{η,τ^*}, for identical precision. Item (2) implies that the transitions in $T^e_{\eta,\varepsilon}$ have nonzero duration.

One may remark that the result provided in this section is constructive. Therefore, in the case where the sampling time enabled is constrained, one can easily take into acount the constraints in the construction of the symbolic model.

5. ILLUSTRATIVE EXAMPLE

Consider the Boost dc-dc converter modelled as a switched affine system with two modes as follows:
$$\dot{\mathrm{x}}(t) = A_{\mathbf{p}(t)}\mathrm{x}(t) + b \quad (34)$$
where $x(t) = [i_l(t) \ \ v_c(t)]^T$ with $i_l(t)$ is the current in the inductor and v_c is the voltage in the capacitor, $b = \begin{bmatrix} \frac{v_s}{x_l} & 0 \end{bmatrix}^T$,
$$A_1 = \begin{bmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} -\frac{1}{x_l}\left(r_l + \frac{r_0 r_c}{r_0+r_c}\right) & -\frac{1}{x_l}\frac{r_0}{r_0+r_c} \\ \frac{1}{x_c}\frac{r_0}{r_0+r_c} & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{bmatrix}.$$
Here, we consider the numerical values of the parameters that have been provided in (Beccuti et al. (2005)) in the per-unit

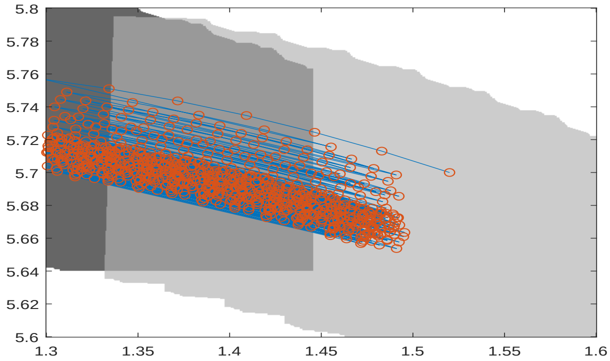

Fig. 1. Safety controller for the symbolic model(dark gray: mode 1, light gray: mode 2, medium gray: both modes are active, white: uncontrollable states) and the trajectory of the closed-loop switched system.

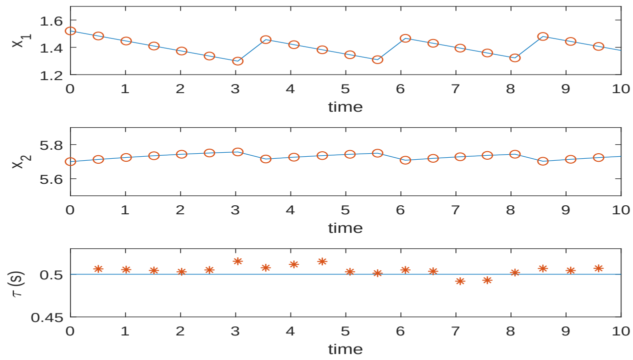

Fig. 2. Trajectory of the closed-loop switched system with the symbolic safety controller starting at $x = [1.59\, 5.6]^T$; The sampling time instants τ taken in the intervals $[\underline{\tau}, \overline{\tau}]$ generated while computing the symbolic abstraction.

system: $x_c = 70$, $x_l = 3$, $r_c = 0.005$, $r_l = 0.05$, $r_0 = 1$, and $v_s = 1$. In order to ensure a better numerical conditioning, the slower dynamic v_c is rescaled and the state vector becomes $x(t) = [i_l(t)\ 5v_c(t)]^T$. The matrices A_1, A_2, and b are modified accordingly. It has been shown in (Girard et al. (2010)) that the function $V(x) = \sqrt{(x-y)^T M (x-y)}$ with

$$M = \begin{bmatrix} 1.0224 & 0.0084 \\ 0.0084 & 1.0031 \end{bmatrix}$$

is a common δ-GUAS Lyapunov function for the switched affine system (34). Thus, inequalities (3), (4) and (5) are verified for $\underline{\alpha}(s) = s$, $\overline{\alpha}(s) = 1.0127s$, $\kappa = 0.014$ and $\gamma(s) = 1.0127s$. We restrict the dynamics of the system to the compact set $\mathscr{C} = [1.3\ \ 1.6] \times [5.6\ \ 5.8]$. The sampling time parameter $\tau^* = 0.5$ and the precision $\varepsilon = 0.1$.

5.1 Uncertain time sampling

First, we design the symbolic model proposed in Section 3. The state space sampling parameter η is fixed and given by $\eta = \gamma^{-1}((1 - e^{-\kappa\tau^*})\underline{\alpha}(\varepsilon))$. For each symbolic states q, q' we are able to estimate an upper and a lower bound $\underline{\tau}$, $\overline{\tau}$ satisfying (10).

In addition, we design a safety controller (see Tabuada (2009)), which allows to keep the output of the symbolic model inside the compact set $\mathscr{C}$. Figure 1, shows the symbolic controller with the sampling period τ^*. Figure 2 shows a trajectory of the closed-loop system with the safety controller and the variations of the sampling instant. We can observe that the trajectory of the closed-loop system with the safety symbolic controller remains in the safe region $\mathscr{C} = [1.3\ \ 1.6] \times [5.6\ \ 5.8]$. The sampling instants τ are randomly chosen in the computed intervals $[\underline{\tau}\ \overline{\tau}]$ and represented together with the sampling period τ^*. We can see that the sampling instant may change its value and may be different from the fixed sampling period τ^*.

5.2 Event-based sampling

In this part, we design the symbolic model proposed in Section 4. The state space sampling parameter η is fixed such that $\eta = 10^{-3}$. Then, the sampling time parameter is computed while designing the symbolic model.

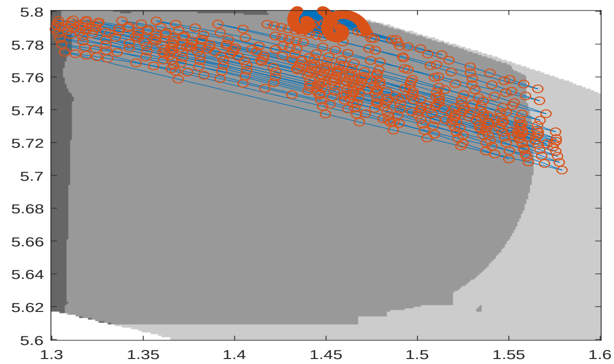

Fig. 3. Safety controller for the symbolic model(dark gray: mode 1, light gray: mode 2, medium gray: both modes are active, white: uncontrollable states) and the trajectory of the closed-loop switched system.

Note that the maximum sampling period that we obtain for this example is $\tau = 0.3s < \tau^*$.

In addition, we design a safety controller, which keeps the output of the symbolic model inside the compact set $\mathscr{C}$. The symbolic controller is shown in Figure 3. The trajectory of the closed-loop system with the safety controller and the sampling time instants are represented in Figure 4. We can observe from Figure 4 that the trajectory of the closed-loop system with the safety symbolic controller remains in the safe set $\mathscr{C}$. Moreover, we can see that the sampling time instants computed with the event-based approach remains smaller than $\tau^* = 0.5$ which is consistent with the theoretical result.

Comparing Figures 1 and 3 we can see that the symbolic controller computed using the event-based approach allows more reactiveness comparably to the controller with the fixed period τ^*. We may equally remark that the set of controllable states in the event based abstraction is much larger than that with the classical one.

6. CONCLUSION

This paper has provided methods for symbolic models design for the class of incrementally stable switched systems. The proposed methods considers aperiodic time sampling resulting

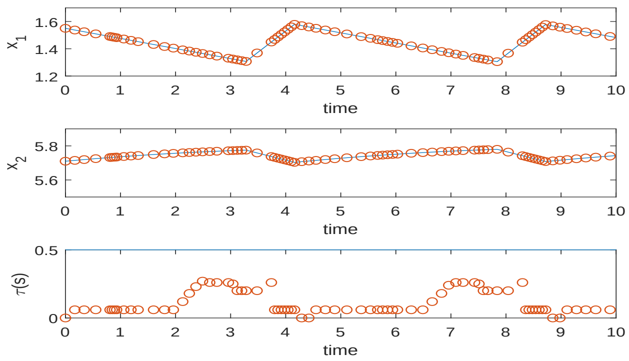

Fig. 4. Trajectory of the closed-loop switched system with the safety controller designed for the symbolic model starting at the initial state $x = [1.55, 5.71]^T$; The sampling instants generated while computing the symbolic abstraction.

either from uncertain or event-based sampling mechanisms. Sufficient conditions guaranteeing that usual symbolic models computed using periodic time-sampling remain approximately bisimilar to a switched system with uncertain sampling time period belonging to a given interval are provided. Estimations of the bounds of the sampling time interval are equally given. In addition, a new design approach of symbolic models related by feedback refinement relations to incrementally stable switched systems, using an event-based approximation scheme are developed. For a given precision, these event-based models are guaranteed to enable transitions of shorter duration. An estimation of the maximum duration is equally provided. Finally, simulations are performed for a Boost dc-dc converter structure in order to assess the efficiency of the proposed approaches.

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