Property (T) for locally compact groups and C*-algebras

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Abstract. Let G be a locally compact group and let $C^*(G)$ and $C^*_r(G)$ be the full group C^*-algebra and the reduced group C^*-algebra of G. We investigate the relationship between Property (T) for G and Property (T) as well as its strong version for $C^*(G)$ and $C^*_r(G)$ (as defined in [Ng–14]). We show that G has Property (T) if (and only if) $C^*(G)$ has Property (T). In the case where G is a locally compact IN-group, we prove that G has Property (T) if and only if $C^*_r(G)$ has strong Property (T). We also show that $C^*_r(G)$ has strong Property (T) for every non-amenable locally compact group G for which $C^*_r(G)$ is nuclear. Some of these groups (as for instance $G = SL_2(\mathbf{R})$) do not have Property T.

1. Introduction and statement of the results

Kazhdan's Property (T), introduced by Kazhdan, in [Kaz–67], is a rigidity property of unitary representations of locally compact groups with amazing applications, ranging from geometry and group theory to operator algebras and graph theory (see [BHV–08]). Property (T) was defined in the context of operator algebras by Connes and Jones, in [CoJ–85], for von Neumann algebras, and by the first named author, in [Bek–06], for *unital* C^*-algebras (in terms of approximate central vectors and central vectors for appropriate bimodules over the algebras).

This allowed to characterize Property (T) for a *discrete* group Γ in terms of various operator algebras attached to it: Γ has Property (T) if and only if A has Property (T), where $A = C^*_r(\Gamma)$ is the reduced group C^*-algebra, or $A = C^*(\Gamma)$ is the full group C^*-algebra of Γ. More generally, Property (T) for a pair of groups $\Lambda \subset \Gamma$ (also called relative Property (T)) is characterized by Property (T) for the pair of the corresponding C^*-algebras (as for instance $C^*_r(\Lambda) \subset C^*_r(\Gamma)$).

Property (T) for unital C^*-algebras was further studied by various authors (see e.g. [Bro–06], [LeN–09], [LNW–08] and [Suz–13]) and a stronger version of it, called the strong Property (T), was defined by the second named author *et al.* in [LeN–09].

Let G be a locally compact group. Recall that the reduced group C^*-algebra $C^*_r(G)$ or the full group C^*-algebra $C^*(G)$ of G is unital if and only if G is discrete (see [Mil–71]). It is important to study Property (T) in the context of non-discrete locally compact groups for its own sake or as a tool towards establishing this property for suitable discrete subgroups of them (this is for instance how usually lattices in higher rank Lie groups are shown to have Property (T); see [BHV–08]). So, it is of interest to study the relationship between Property (T) for G and suitable rigidity properties of $C^*_r(G)$ and $C^*(G)$.

There have been attempts to define a notion of Property (T) for *non-unital* C^*-algebras. As shown in [LNW–08], the most straightforward extension of the definition given in [Bek–06] leads to a disappointing result: no separable non-unital C^*-algebra has such Property (T). In [Ng–14], the second named author introduced a refined notion of Property (T) and strong Property (T) for general C^*-algebras (as well as their relative versions) which seems to be more sensible (see Section 2 below). For instance, it was shown in [Ng–14] that, if G has Property (T), then $C^*(G)$ has strong Property (T).

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The aim of the article is to further investigate Property (T) as defined in [Ng–14]. Our first theorem shows that Property (T) of a general locally compact group is indeed characterized by Property (T) of its full group C^*-algebra.

Theorem 1. *Let G be a locally compact group and H a closed subgroup of G.*
(a) The pair (G, H) has Property (T) if and only if the pair $(C^*(G), C^*(H))$ has Property (T).
(b) G has Property (T) if and only if $C^*(G)$ has Property (T) (equivalently, strong Property (T)).

The proof of Theorem 1 is based on an isolation property of one dimensional representations (see Proposition 10) of an arbitrary C^*-algebra with Property (T), which is of independent interest. This isolation property also allows us to give a sufficient condition for a locally compact quantum group to satisfy Property (T) (see, e.g., [ChN–15]). Notice that we do not know whether this sufficient condition is also a necessary condition.

Theorem 2. *If $\mathbb{G}$ is a locally compact quantum group such that its full group C^*-algebra $C_0^{\mathrm{u}}(\widehat{\mathbb{G}})$ has Property (T), then $\mathbb{G}$ has property (T).*

It is natural to ask whether Theorem 1 remains true when $C^*(G)$ is replaced by $C_r^*(G)$. As mentioned above, this is the case when G is discrete. This also holds when G is amenable: indeed, in this case, $C_r^*(G) \cong C^*(G)$ and Theorem 1 shows that $C_r^*(G)$ has Property (T) if and only if G has Property (T), that is (see Theorem 1.1.6 in [BHV–08]), if and only if G is compact.

However, as we now see, Property (T) of the reduced group C^*-algebra does not imply Property (T) of the original group, for a large class of locally compact groups related to nuclear C^*-algebras (see e.g. [BrO–08]).

Theorem 3. *Let G be a locally compact group such that $C_r^*(G)$ is a nuclear C^*-algebra. If G is non-amenable, then $C_r^*(G)$ has strong Property (T).*

Example 4. (i) The class of locally compact groups G for which $C_r^*(G)$ is a nuclear C^*-algebra includes not just all amenable groups but also all connected groups as well as all groups of type I (see [Pat–88]). Observe that a discrete group G has a nuclear reduced group C^*-algebra only if G is amenable (see [BrO–08, Theorem 2.6.8]).
(ii) Let G be a simple non-compact Lie group of real rank 1, as for instance $SO(n, 1)$ for $n \geq 2$ or $SU(n, 1)$ for $n \geq 1$. Then G does not have Property (T). However, as $C_r^*(G)$ is nuclear, it follows from Theorem 3 that $C_r^*(G)$ has strong Property (T).

In combination with a result from [Ng–15], Property (T) for $C_r^*(G)$ admits the following characterization when G is a non-compact group.

Corollary 5. *Let G be a non-compact locally compact group such that $C_r^*(G)$ is a nuclear C^*-algebra. The following statements are equivalent.*

(1) G is non-amenable;
(2) $C_r^*(G)$ admits no tracial state;
(3) $C_r^*(G)$ has Property (T) (equivalently, strong Property (T)).

Let G be a non-amenable locally compact group with a nuclear reduced C^*-algebra, as in Theorem 3. The proof that $C_r^*(G)$ has strong Property (T) relies simply on the fact that there is no non-degenerate Hilbert *-bimodule over $C_r^*(G)$ with approximate central vectors.

Apart from amenable groups, the only groups for which we are able to construct appropriate non-degenerate Hilbert *-bimodules over $C_r^*(G)$ with approximate central vectors are the so-called IN-groups. Recall that G is said to be an *IN-group* if there exists a compact neighborhood of the identity in G which is invariant under conjugation (for more detail on these groups, see [Pal–78]). For this class of groups, we indeed do have a positive result concerning the relation between Property (T) of G and strong Property (T) of $C_r^*(G)$.

Theorem 6. *Let G be an IN-group and $H \subseteq G$ a closed subgroup.*

(a) If $\big(C^*_r(G), C^*_r(H)\big)$ *has strong Property* (T)*, then* (G, H) *has Property* (T)*.*
(b) In the case where G *is* σ*-compact, the pair* (G, H) *has Property* (T) *if and only if* $\big(C^*_r(G), C^*_r(H)\big)$ *has strong Property* (T)*.*
(c) G *has Property* (T) *if and only if* $C^*_r(G)$ *has strong Property* (T)*.*

Finally, we point out that the classes of groups considered in Theorem 6 (IN-groups) and in Theorem 3 (non-amenable groups with a nuclear reduced C^*-algebra) are disjoint.

Proposition 7. *Let* G *be an IN-group with a nuclear reduced* C^**-algebra. Then* G *is amenable.*

This paper is organized as follows. In Sections 2, 4 and 5, we recall some preliminary facts and establish some tools which are crucial for the proofs of our results. In Section 6, we conclude the proofs of Theorems 1, 2, 3 and 6, as well as Corollary 5 and Proposition 7.

2. Property (T) for non-unital C^*-algebras

We recall from [Ng–14] the notion of Property (T) and of strong Property (T) for pairs (or inclusions) of general C^*-algebras.

Let A be a (not necessarily unital) C^*-algebra and let $M(A)$ denote its mutiplier algebra. Let $\mathfrak{H}$ be a non-degenerate Hilbert *-bimodule over A. Observe that the associated *-representation and *-anti-representation of A (on $\mathfrak{H}$) uniquely extend to a *-representation and a *-anti-representation of $M(A)$ on the same space.

A net $(\xi_i)_{i\in\mathfrak{I}}$ in $\mathfrak{H}$ is said to be

- *almost-A-central* if
$$\|a\cdot\xi_i - \xi_i\cdot a\| \to_i 0 \qquad \text{for every } a \in A;$$
- *almost-*$\mathfrak{K}_A$*-central* if
$$\sup_{x\in L} \|x\cdot\xi_i - \xi_i\cdot x\| \to_i 0$$
for every subset L of $M(A)$ which is compact for the *strict topology* on $M(A)$; recall that this is the weakest topology on $M(A)$ for which the maps $x \mapsto xa$ and $x \mapsto ax$ from $M(A)$ to A are continuous for every $a \in A$, when A is equipped with the norm topology.

Let B be a C^*-subalgebra of $M(A)$ which is *non-degenerate*, in the sense that an approximate identity of B converges strictly to the identity of $M(A)$. The pair (A, B) is said to have *Property* (T) (respectively, *strong Property* (T)) if for every non-degenerate Hilbert *-bimodule $\mathfrak{H}$ over A with an almost-$\mathfrak{K}_A$-central net $(\xi_i)_{i\in\mathfrak{I}}$ of unit vectors, the space
$$\mathfrak{H}^B := \{\xi \in \mathfrak{H} : b\cdot\xi = \xi\cdot b \quad \text{for every} \quad b \in B\}$$
of *central vectors* is non-zero (respectively, one has $\|\xi_i - P^B\xi_i\| \to_i 0$, where P^B is the orthogonal projection onto the closed subspace $\mathfrak{H}^B$).

We say that A has *Property* (T) if (A, A) has Property (T) and that A has *strong Property* (T) if (A, A) has strong Property (T).

Notice that, compared to the original definition of Property (T) for unital C^*-algebras in [Bek–06], we use here almost-$\mathfrak{K}_A$-central nets of unit vectors instead of almost-A-central ones. Taking $a_0 = 1$ in Item (b) of the following elementary lemma, we see that these two definitions coincide in the unital case. This elementary lemma will also be needed later on.

Lemma 8. *Let* K *be a non-empty strictly compact subset of* $M(A)$*.*
(a) The set K *is norm-bounded.*
(b) For any $a_0 \in A$ *and* $\epsilon > 0$*, one can find* $x_1, \dots, x_n \in K$ *such that for every* $x \in K$*, there exists* $k \in \{1, \dots, n\}$ *with* $\|xa_0 - x_ka_0\| < \epsilon$*.*

Proof. Item (a) follows from the uniform boundedness principle; Item (b) is a consequence of the fact that Ka_0 is norm-compact. ☐

3. Hilbert *-bimodules over nuclear reduced group C^*-algebras

Let G be a locally compact group. Recall that, given a unitary representation $(u, \mathfrak{H})$ of G, a net $(\xi_i)_{i\in\mathfrak{I}}$ of unit vectors in $\mathfrak{H}$ is *almost-invariant* if

$$\sup_{s\in Q} \|u(s)\xi_i - \xi_i\| \to_i 0$$

for every compact subset Q of G.

Identifying every element $s \in G$ with the Dirac measure δ_s, we view G as a subset of $M(C^*(G))$. Notice that this embedding $\delta : s \mapsto \delta_s$ is continuous, when $M(C^*(G))$ is equipped with the strict topology.

Let $\mathfrak{H}$ be a non-degenerate Hilbert *-bimodule over $C^*(G)$. Define a map $u_{\mathfrak{H}} : G \times G \to \mathcal{L}(\mathfrak{H})$ by

$$u_{\mathfrak{H}}(s,t)\xi := \delta_s \cdot \xi \cdot \delta_{t^{-1}} \qquad \text{for all} \quad s,t \in G, \xi \in \mathfrak{H}.$$

It is easily checked that $u_{\mathfrak{H}}$ is a unitary representation of the cartesian product group $G \times G$. Thus, it induces a representation

$$\tilde{u}_{\mathfrak{H}} : C^*(G \times G) \to \mathcal{B}(\mathfrak{H}).$$

As we now see, when $C_r^*(G)$ is nuclear and $\mathfrak{H}$ comes from a *-bimodule over $C_r^*(G)$, the representation $u_{\mathfrak{H}}$ factors through the regular representation of $G \times G$.

Lemma 9. *Let $\mathfrak{H}$ be a non-degenerate Hilbert *-bimodule over $C^*(G)$ and $u_{\mathfrak{H}}$ the associated unitary representation of $G \times G$.*
(a) If $(\xi_i)_{i\in\mathfrak{I}}$ is an almost $\mathfrak{K}_{C^*(G)}$-central net of unit vectors, then $(\xi_i)_{i\in\mathfrak{I}}$ is almost $u_{\mathfrak{H}}|_{\Delta(G)}$-invariant, where $\Delta(G) := \{(g,g) \mid g \in G\}$ is the diagonal subgroup of $G \times G$.
(b) Assume that $C_r^*(G)$ is nuclear. Then $\mathfrak{H}$ is a Hilbert *-bimodule over $C_r^*(G)$ (i.e., the associated *-representation and *-anti-representation of $C^*(G)$ both factorize through $C_r^*(G)$) if and only if $u_{\mathfrak{H}}$ is weakly contained in the left regular representation $\lambda_{G\times G}$ of $G \times G$.

Proof. (a) Let Q be a compact subset of G. Then Q can be viewed as a strictly compact subset of $M(C^*(G))$ because δ is continuous. The claim follows, since δ_s is a unitary operator for every $s \in G$.
(b) Assume that $\mathfrak{H}$ is a Hilbert *-bimodule over $C_r^*(G)$. By the universal property, there is a *-representation $\phi_{\mathfrak{H}}$ of the maximal tensor product $C_r^*(G) \otimes_{\max} C_r^*(G)^{\mathrm{op}}$ on $\mathfrak{H}$ compatible with the Hilbert *-bimodule structure over $C_r^*(G)$ (here, $C_r^*(G)^{\mathrm{op}}$ means the opposite C^*-algebra of $C_r^*(G)$; see e.g. [Ng–14]). However, since $C_r^*(G)$ is nuclear, $C_r^*(G) \otimes_{\max} C_r^*(G)^{\mathrm{op}}$ coincides canonically with the minimal tensor product $C_r^*(G) \otimes_{\min} C_r^*(G)^{\mathrm{op}}$.

Now, under the canonical identifications of $C_r^*(G)$ with $C_r^*(G)^{\mathrm{op}}$ (which, in the $L^1(G)$ level, sends f to $(\bar{f})^*$, with $\bar{f}$ being the complex conjugate of the function f) as well as the identification

$$C_r^*(G) \otimes_{\min} C_r^*(G) \cong C_r^*(G \times G),$$

one has $\phi_{\mathfrak{H}} \circ \lambda = \tilde{u}_{\mathfrak{H}}$, where λ is the canonical map from $C^*(G \times G)$ to $C_r^*(G \times G)$. In other words, $u_{\mathfrak{H}}$ is weakly contained in $\lambda_{G\times G}$.

Conversely, assume that $u_{\mathfrak{H}}$ is weakly contained in $\lambda_{G\times G}$. Then, by definition of $u_{\mathfrak{H}}$, we see that the associated *-representation and *-anti-representations of $C^*(G)$ on $\mathfrak{H}$ factor through $C_r^*(G)$. $\square$

4. On the spectrum of a C^*-algebra with Property (T)

Let A be a (not necessarily unital) C^*-algebra. As said in the above, every non-degenerate *-representation $\pi : A \to \mathcal{B}(\mathfrak{H})$ of A extends canonically to a non-degenerate *-representation of $M(A)$, which will again be denoted by π. Moreover, if B is a non-degenerate C^*-subalgebra of $M(A)$, the restriction $\pi|_B$ of π to B is a non-degenerated representation of B. Analogous statements are true for a non-degenerate *-anti-representation.

The following isolation property of one dimensional *-representations of A is the crucial tool for the proof of Theorems 1 and 2. Concerning general facts about the topology of the spectrum (or dual space) $\widehat{A}$ of A, see Chapter 3 in [Dix–69].

Proposition 10. *Let A be a C^*-algebra and B be a non-degenerate C^*-subalgebra of $M(A)$. Let $\chi : A \to \mathbb{C}$ be a non-zero *-homomorphism.*
(a) Suppose that (A, B) has Property (T). For any non-degenerate *-representation $(\pi, \mathfrak{H})$ of A which weakly contains χ, the representation $\chi|_B$ is contained in $(\pi|_B, \mathfrak{H})$.
(b) If A has Property (T), then χ an isolated point in the spectrum $\widehat{A}$ of A.

Proof. Notice that Item (b) follows from Item (a), by considering $A = B$ and $(\pi, \mathfrak{H}) \in \widehat{A}$. Hence, it suffices to prove Item (a).

By the assumption, there exists a net $(\xi_i)_{i\in\mathfrak{I}}$ of unit vectors in $\mathfrak{H}$ such that

$$\lim_i \|\pi(a)\xi_i - \chi(a)\xi_i\| = 0 \qquad \text{for all} \quad a \in A. \tag{1}$$

We have to prove that $\pi|_B$ actually contains $\chi|_B$.

Define a Hilbert *-bimodule structure (over A) on $\mathfrak{H}$ by

$$a \cdot \xi = \pi(a)\xi \qquad \text{and} \qquad \xi \cdot a = \chi(a)\xi \qquad \text{for all} \quad a \in A, \xi \in \mathfrak{H}.$$

Fix an element $a_0 \in A$ with $|\chi(a_0)| = 1$. It follows from Relation (1) that

$$\lim_i \|\pi(a_0)\xi_i\| = |\chi(a_0)| = 1. \tag{2}$$

Thus, without loss of generality, we may assume that $\pi(a_0)\xi_i \neq 0$ for all $i \in \mathfrak{I}$, and we set

$$\eta_i := \frac{\pi(a_0)\xi_i}{\|\pi(a_0)\xi_i\|}.$$

Let K be a non-empty strictly compact subset of $M(A)$ and let $\epsilon > 0$. By Lemma 8,

$$C := \sup_{y\in K} \|y\| < \infty$$

and there exist $x_1, \dots, x_n \in K$ such that for every $x \in K$, we can find $k \in \{1, \dots, n\}$ with

$$\|xa_0 - x_k a_0\| \leq \epsilon/12. \tag{3}$$

Moreover, Relations (1) and (2) produce $i_0 \in \mathfrak{I}$ such that for $i \geq i_0$ and $l \in \{1, \dots, n\}$, one has

$$\frac{1}{\|\pi(a_0)\xi_i\|} \leq 2, \quad \|\pi(a_0)\xi_i - \chi(a_0)\xi_i\| \leq \frac{\epsilon}{4C} \quad \text{and} \quad \|\pi(x_l a_0)\xi_i - \chi(x_l a_0)\xi_i\| \leq \frac{\epsilon}{12} \tag{4}$$

Let $x \in K$. Choose an integer k in $\{1, \dots, n\}$ such that Relation (3) holds. We then have, using Relations (3) and (4), that

$$\begin{aligned} \|x \cdot \eta_i - \eta_i \cdot x\| &= \frac{1}{\|\pi(a_0)\xi_i\|} \|\pi(x)\pi(a_0)\xi_i - \chi(x)\pi(a_0)\xi_i\| \\ &\leq 2\|\pi(xa_0)\xi_i - \pi(x_k a_0)\xi_i\| + 2\|\pi(x_k a_0)\xi_i - \chi(x_k a_0)\xi_i\| + \\ &\qquad 2\|\chi(x_k a_0)\xi_i - \chi(xa_0)\xi_i\| + 2|\chi(x)|\|\chi(a_0)\xi_i - \pi(a_0)\xi_i\| \\ &\leq \epsilon/6 + \epsilon/6 + \epsilon/6 + \epsilon/2 \end{aligned}$$

This shows that $(\eta_i)_{i\in\mathfrak{I}}$ is an almost $\mathfrak{K}_A$-central net of unit vectors in $\mathfrak{H}$.

Since, by the assumption, (A, B) has Property (T), it follows that $\mathfrak{H}^B \neq \{0\}$. Hence, there exists a non-zero vector $\xi \in \mathfrak{H}$ such that

$$\pi(b)\xi = \chi(b)\xi \qquad \text{for all} \qquad b \in B.$$

This means that $\chi|_B$ is contained in the representation $\pi|_B$. □

Remark 11. As is well-known (see Theorem 1.2.5 in [BHV–08]), Property (T) of a locally compact group G can be characterized by the fact that one (or equivalently, any) finite dimensional irreducible representation of G is an isolated point in the dual space of G. It is natural to ask whether Proposition 10 remains true when χ is an arbitrary finite dimensional representation of the C^*-algebra A. We will not deal with this question in the current paper.

5. A bimodule over $C_r^*(G)$ associated to a unitary representation

Let G be a locally compact group. We will need to associate to every unitary representation of G a bimodule of $C_r^*(G)$, via a standard procedure (see [CoJ–85], [Bek–06]).

As usual, we denote by $L^p(G)$ the Banach space $L^p(G, \mu)$ for a fixed Haar measure μ on G.

Lemma 12. *Let G be a locally compact unimodular group. Let $s \mapsto u_s$ be a unitary representation of G on a Hilbert space $\mathfrak{H}$.*
(a) The space $\mathfrak{H} \otimes L^2(G) = L^2(G; \mathfrak{H})$ is a non-degenerate Hilbert *-bimodule over $C_r^*(G)$ for the following left and right actions:

$$(g \cdot \eta)(t) := \int_G g(s)\eta(s^{-1}t)\, d\mu(s) \quad \textit{and} \quad (\eta \cdot g)(t) := \int_G g(r^{-1}t) u_{t^{-1}r}(\eta(r))\, d\mu(r),$$

where $g \in L^1(G), \eta \in L^2(G; \mathfrak{H})$ and $t \in G$.
(b) Let H be a closed subgroup of G and $\zeta \in (\mathfrak{H} \otimes L^2(G))^{C_r^*(H)}$. Then, for every $s \in H$, we have

$$\mu\big(\{t \in G : u_s(\zeta(t)) \neq \zeta(sts^{-1})\}\big) = 0.$$

Proof. (a) As the left-hand displayed equality is given by the left regular representation, it defines a *-representation of $C_r^*(G)$. On the other hand, the right-hand displayed equality is given by the conjugation of the right regular representation by the "Fell unitary" $U \in \mathcal{B}(L^2(G; \mathfrak{H}))$, where

$$U(\eta)(t) := u_t(\eta(t)) \qquad (\eta \in L^2(G; \mathfrak{H}); t \in G).$$

Thus, it defines a *-anti-representation of $C_r^*(G)$. It is easy to check that this *-representation and this *-anti-representations are non-degenerate, and they commute with each other.
(b) As noted in the above, $\mathfrak{H} \otimes L^2(G)$ is a unital Hilbert *-bimodule over $M(C_r^*(G))$. Moreover, ζ is a $M(C_r^*(H))$-central vector. If δ_s is the canonical image of s in $M(C_r^*(G))$, then

$$\delta_s \cdot \zeta = \zeta \cdot \delta_s,$$

and the claim follows. □

6. Proofs of the results

6.1. **Proof of Theorems 1 and 2.** To show Item (a) of Theorem 1, assume that the pair $(C^*(G), C^*(H))$ has Property (T). It follows from Proposition 10(a) that (G, H) has Property (T), by the definition of Property (T) for pairs of groups. Conversely, the fact that Property (T) for (G, H) implies Property (T) for $(C^*(G), C^*(H))$ was proved in [Ng–14, Proposition 4.1(a)].

Item (b) of Theorem 1 follows from Item (a) in combination with [Ng–14, Proposition 4.1(c)].

Theorem 2 follows from Proposition 10(b) and [ChN–15, Proposition 3.2].

6.2. **Proof of Theorem 3.** Let $\mathfrak{H}$ is a non-degenerate Hilbert *-bimodule over $C_r^*(G)$. Let $u_{\mathfrak{H}}$ and $\Delta(G)$ be as in Lemma 9. Then $u_{\mathfrak{H}}$ is weakly contained in $\lambda_{G\times G}$ (by Lemma 9(b)). Since $\lambda_{G\times G}|_{\Delta(G)}$ is weakly contained in the left regular representation $\lambda_{\Delta(G)}$ of $\Delta(G)$ (see [BHV–08, Proposition F.1.10]), it follows that there is no almost $u_{\mathfrak{H}}|_{\Delta(G)}$-invariant net of unit vectors, since $\Delta(G) \cong G$ is non-amenable. Hence, $\mathfrak{H}$ has no almost $\mathfrak{K}_{C_r^*(G)}$-central net of unit vectors, by Lemma 9(a). It follows that $C_r^*(G)$ has strong Property (T).

6.3. **Proof of Corollary 5.** Under the assumption that $C_r^*(G)$ is nuclear, the equivalence of (1) and (2) follows from [Ng–15, Theorem 8]. By Theorem 3, we know that (1) implies (3). Finally, assume that G is amenable. Since, by assumption, G is not compact, it follows from Theorem 1 (see also the comments after the statement of Theorem 2) that $C_r^*(G)$ does not have Property (T). Hence, (3) implies (1), and the proof is complete.

6.4. **Proof of Theorem 6.** We fix a conjugation invariant compact neighborhood V of the identity e of the IN-group G and choose a Haar measure μ on G. Let us recall the following two well-known facts:

- G is unimodular.
- the characteristic function χ_V is in the center of the algebra $L^1(G)$.

(a) Notice that as V is conjugation invariant, we have

$$\chi_V(s^{-1}t) = \chi_V(ts^{-1}) \qquad (s, t \in G).$$

Let $(u, \mathfrak{H})$ be a unitary representation of G and $(\xi_i)_{i\in\mathfrak{I}}$ be an almost u-invariant net of unit vectors in $\mathfrak{H}$. We consider $\mathfrak{H} \otimes L^2(G) = L^2(G; \mathfrak{H})$ to be a Hilbert *-bimodule over $C^*_r(G)$, as in Lemma 12(a).

By the assumption, for any $\epsilon > 0$ and any continuous function g on G with its support, $\operatorname{supp} g$, being compact, there exists $i_0 \in \mathfrak{I}$ such that for any $i \geq i_0$, one has $\sup_{s\in\operatorname{supp} g} \|\xi_i - u_{s^{-1}}\xi_i\| < \epsilon$. Hence, for any $i \geq i_0$,

$$\begin{aligned}
&\|g \cdot (\xi_i \otimes \chi_V) - (\xi_i \otimes \chi_V) \cdot g\|^2_{L^2(G;\mathfrak{H})} \\
&\qquad = \int_G \Big\| \int_G g(s)\big(\chi_V(s^{-1}t)\xi_i - \chi_V(ts^{-1})u_{s^{-1}}\xi_i\big)\, ds \Big\|^2_{\mathfrak{H}} dt \\
&\qquad \leq \int_G \Big(\int_{\operatorname{supp} g} |g(s)| \|\xi_i - u_{s^{-1}}\xi_i\| \chi_V(ts^{-1})\, ds \Big)^2 dt \\
&\qquad \leq \|g\|^2_{L^2(G)}\epsilon^2 \int_G \int_{\operatorname{supp} g} \chi_V(ts^{-1})\, ds\, dt \\
&\qquad \leq \mu(\operatorname{supp} g)\mu(V)\|g\|^2_{L^2(G)}\epsilon^2
\end{aligned}$$

Thus, by an approximation argument, it follows that the bounded net $(\xi_i \otimes \chi_V)_{i\in\mathfrak{I}}$ is almost-$C^*_r(G)$-central.

We consider $\epsilon > 0$ and denote $\kappa := \|\chi_V^2\|_{L^2(G)}$, where χ_V^2 is the convolution product of χ_V with itself. Suppose that $K \subseteq M(C^*_r(G))$ is a strictly compact subset. Then by Lemma 8(b) and the above, there exists $i_1 \in \mathfrak{I}$ such that for any $i \geq i_1$, one has

$$\sup_{y\in K} \|y\chi_V \cdot (\xi_i \otimes \chi_V) - (\xi_i \otimes \chi_V) \cdot y\chi_V\|_{L^2(G;\mathfrak{H})} < \kappa\epsilon. \tag{5}$$

Moreover, one can find $i_2 \geq i_1$ such that for every $i \geq i_2$,

$$\|\chi_V \cdot (\xi_i \otimes \chi_V) - (\xi_i \otimes \chi_V) \cdot \chi_V\|_{L^2(G;\mathfrak{H})} < \kappa\epsilon. \tag{6}$$

Set $h := \dfrac{\chi_V^2}{\kappa}$. Then

$$\chi_V \cdot \frac{\xi_i \otimes \chi_V}{\kappa} = \xi_i \otimes h.$$

Inequalities (5) and (6), together with the fact that χ_V is in the center of $C^*_r(G)$, imply that

$$\|y \cdot (\xi_i \otimes h) - (\xi_i \otimes h) \cdot y\|_{L^2(G;\mathfrak{H})} \leq (1 + \|y\|)\epsilon \qquad \text{for all} \quad y \in K.$$

As K is norm-bounded (by Lemma 8(a)), we conclude that $(\xi_i \otimes h)_{i\in\mathfrak{I}}$ is an almost-$\mathfrak{K}_{C^*_r(G)}$-central net of unit vectors. Therefore, strong Property (T) of $\big(C^*_r(G), C^*_r(H)\big)$ implies that

$$\big\|\xi_i \otimes h - P^{C^*_r(H)}(\xi_i \otimes h)\big\|_{L^2(G;\mathfrak{H})} \to_i 0. \tag{7}$$

For each $i \in \mathfrak{I}$, put

$$\eta_i := P^{C^*_r(H)}(\xi_i \otimes h) \in L^2(G; \mathfrak{H})^{C^*_r(H)}$$

and let ζ_i be the restriction of η_i on $V^2 := \{st : s, t \in V\}$. As h vanishes outside V^2, it follows from (7) that

$$\|\xi_i \otimes h - \zeta_i\|_{L^2(G;\mathfrak{H})} \to_i 0. \tag{8}$$

For a fixed $s \in H$, the condition $\eta_i \in L^2(G; \mathfrak{H})^{C^*_r(H)}$ implies that

$$u_s(\eta_i)(t) = \eta_i(sts^{-1}) \quad \text{for } \mu\text{-almost every} \quad t \in G,$$

by Lemma 12(b). Therefore, the invariance of V^2 under conjugation ensures that

$$u_s(\zeta_i(t)) = \zeta_i(sts^{-1}) \qquad \text{for } \mu\text{-almost all } t \in V^2.$$

This, together with the inequality

$$\int_{V^2} \|\zeta_i(t)\|\, dt \leq \mu(V^2)^{1/2} \|\zeta_i\|_{L^2(G;\mathfrak{H})},$$

implies that $\int_{V^2} \zeta_i(t)\, dt$ exists and lies inside the space, $\mathfrak{H}^{u|_H}$, of H-invariant vectors in $\mathfrak{H}$.

Finally, it follows from

$$\int_{V^2} \|(\xi_i \otimes h - \zeta_i)(t)\|\, dt \leq \mu(V^2)^{1/2} \|\xi_i \otimes h - \zeta_i\|_{L^2(G;\mathfrak{H})}$$

and from (8) that

$$\Big\| \frac{\mu(V)^2}{\kappa} \xi_i - \int_{V^2} \zeta_i(t)\, dt \Big\|_{\mathfrak{H}} = \Big\| \int_{V^2} \xi_i \otimes h(t) - \zeta_i(t)\, dt \Big\|_{\mathfrak{H}} \to_i 0.$$

This finishes the proof of Item (a).
(b) The claim follows from Item (a) and [Ng–14, Proposition 4.1(b)].
(c) The claim follows from Item (a) and [Ng–14, Proposition 4.1(c)].

Remark 13. (i) If one relaxes the assumption of Theorem 6 to $\big(C^*_r(G), C^*_r(H)\big)$ having Property (T), then the proof of Theorem 6 still produces a non-zero element $\eta \in L^2(G;\mathfrak{H})^{C^*_r(H)}$, but we do not know whether the H-invariant vector $\int_{V^2} \eta(t)\, dt$ is non-zero.
(ii) Let us say that a net $(\xi_i)_{i\in\mathfrak{I}}$ is *almost-$\mathfrak{K}^0_A$-central* if $\sup_{x\in K} \|x \cdot \xi_i - \xi_i \cdot x\| \to_i 0$ for any subset $K \subseteq M(A)$ such that Ka is norm-compact for every $a \in A$. If we define weaker versions of Property (T) and strong Property (T) with almost-$\mathfrak{K}_A$-central nets of unit vectors being replaced by almost-$\mathfrak{K}^0_A$-central nets of unit vectors, then - as the proofs show - all the results in the article remain true.

6.5. **Proof of Proposition 7.** Let V and χ_V be as in the proof of Theorem 6. Consider $L^2(G)$ as a Hilbert *-bimodule over $C^*_r(G)$ (see Lemma 12). Since χ_V is in the center of $L^1(G)$, we know χ_V is a $C^*_r(G)$-central vector in $L^2(G)$. Thus, $\rho \in C^*_r(G)^*_+$ defined by $\rho(x) := \langle x(\chi_V), \chi_V \rangle$ $(x \in C^*_r(G))$ is tracial. The conclusion now follows from Corollary 5 (note that compact groups are amenable).

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